Mathellaneous by Daniel Mathews

Quadratic geography, algebraic extreme sports and magical Farey trees

1 Beauty and (quadratic) form

Part of the beauty of mathematics comes from the existence of unexpected connections between apparently simple concepts, concepts we thought we knew, until they leap out, surprise us, and fit themselves together in the most audacious ways.

I once spotted a book entitled "The Sensual Form" which, in a mathematics library, was a little too exciting. It is by John Conway, and typically entertaining and chock full of amazing mathematics.

Closer inspection reveals the title actually to be "The Sensual (Quadratic) Form", and the book involves rapid increases in difficulty after the first chapter (a predicament well known to mathematicians!). But there was no disappointment. Conway connects the theory of quadratic forms to pictures and concepts so simple that high-school students could use his methods to solve otherwise very difficult problems. This is no exaggeration: it has been verified by experiment. But while the mathematics involved is quite simple, we must offer some caution. The techniques of proof involved require knowledge of geomorphology and extreme sports, but only at a high-school level.

2 Quadratic Diophantine equations

We are interested in *quadratic Diophantine equations*: quadratic equations, for which we want integer solutions.

Sometimes solving these is not too difficult. Using high-school techniques we should be able to find all integer solutions (indeed, all real solutions) of

$$x^2 + 6x + 5 = 0.$$

But things become more complicated when more variables are involved. For instance, can we solve

$$3x^2 + 6xy - 5y^2 = 7$$

in integers?

The general problem we will consider is the following.

Problem. Given integers a, b, c, n, find all solutions to the equation

$$ax^2 + bxy + cy^2 = n$$

for integers x, y.

Conway's method allows us to "see" all the possible values of a quadratic form like $3x^2 + 6xy - 5y^2$, by following a simple procedure. Actually, our "picture" is a jagged rocky landscape, consisting of vast mountains, valleys, plateaus, and sheer cliff faces. We

base jump onto these inhospitable lands, then abseil down and climb up until we reach the altitude which defines a solution.

Our problem, in general, is quite a difficult one, but we should not assume this is always the case. I will leave the following Diophantine equations, which can be solved by methods far easier than Conway's, as puzzles for the reader. (Solutions are provided at the end.) Find a solution to each, or show that none exists.

(1) $5x^2 + 25xy + 10y^2 = 892$
(2) $3x^2 - 11xy + 9y^2 = 9$
(3) $x^2 - 6xy + 9y^2 = -2$
(4) $-100x^2 - 60xy + y^2 = 7$

3 Base jumping and laxity

Let's suppose we have our equation to solve:

$$ax^2 + bxy + cy^2 = n.$$

For ease of notation, let $f(x, y) = ax^2 + bxy + cy^2$ and condense (x, y) to a vector $\mathbf{v}$. So we solve $f(\mathbf{v}) = n$.

There are two initial considerations. First, note that $f(\mathbf{v}) = f(-\mathbf{v})$. So, we do not really care whether a vector is positive or negative — whatever. In such a state of algebraic permissiveness we'll refer to such a vector $\pm\mathbf{v}$ (we don't care which) as a *lax vector*.

Secondly, we can see that if k is an integer, then $f(k\mathbf{v}) = k^2 f(\mathbf{v})$. So it is sufficient to look at $\mathbf{v} = (x, y)$ where x, y have no common factor (other than ± 1), i.e. x, y are *relatively prime*. If we can solve the problem in this case, then the case where x, y do have a common factor follows easily. Call a vector $\mathbf{v} = (x, y)$ with x, y relatively prime a *primitive vector*. If we think of our vectors as lattice points in the plane, the primitive vectors are those which are visible from the origin.

Unfortunately, before we can get anywhere we need to define a few words. Words, words, words. Well, four of them. They are four types of objects which we call "bases". Before we can base jump, we will need to know what the bases are!

A *strict base* is a pair of vectors $\{\mathbf{v_1}, \mathbf{v_2}\}$ whose integral linear combinations cover all integer vectors. That is, *any* possible pair of integers can be obtained as $m\mathbf{v_1} + n\mathbf{v_2}$, where $m, n \in \mathbb{Z}$. (The knowledgeable reader will note that this is just the familiar notion of a basis in algebra.) It is not difficult to see that a strict base must consist of primitive vectors.

A *lax base* is a pair of lax vectors $\{\pm\mathbf{v_1}, \pm\mathbf{v_2}\}$, where $\{\mathbf{v_1}, \mathbf{v_2}\}$ form a strict base.

A *strict superbase* is a triple of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ such that $\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$ and $\{\mathbf{v_1}, \mathbf{v_2}\}$ form a (strict) base. Thus we obtain a superbase from a base by adding a third vector, so that the three sum to zero.

A *lax superbase* is a triple of lax vectors $\{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}\}$, where $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a strict superbase.

4 Tectonic construction

The key to constructing our landscape is the relationship between bases and superbases. We want to get from a base to a superbase by adding a vector; and we want to get from a superbase to a base by removing a vector.

From a strict superbase $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, there are three ways to get to a strict base: by removing any one of the three vectors. But from a strict base $\{\mathbf{v_1}, \mathbf{v_2}\}$, there is clearly only one way to get to a strict superbase: by adding $\mathbf{v_3} = -\mathbf{v_1} - \mathbf{v_2}$. And this will go back to the original superbase.

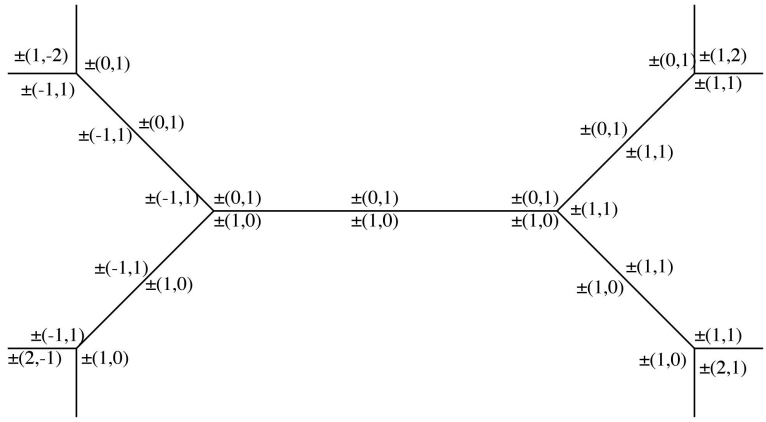
Things are more interesting if we consider the lax versions. (From now on, when referring to bases or superbases, we mean the lax versions.) From a lax superbase $\{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}\}$, there are again three ways to get to a lax base.

$$\begin{aligned} \{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}\} &\mapsto \{\pm\mathbf{v_1}, \pm\mathbf{v_2}\} \\ &\mapsto \{\pm\mathbf{v_1}, \pm\mathbf{v_3}\} \\ &\mapsto \{\pm\mathbf{v_2}, \pm\mathbf{v_3}\} \end{aligned}$$

But from a lax base $\{\pm\mathbf{v_1}, \pm\mathbf{v_2}\}$, there are *two* possibilities:

$$\begin{aligned} \{\pm\mathbf{v_1}, \pm\mathbf{v_2}\} &\mapsto \{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm(\mathbf{v_1} + \mathbf{v_2})\} \\ &\mapsto \{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm(\mathbf{v_1} - \mathbf{v_2})\} \end{aligned}$$

So, here is our construction. Let the *superbases* represent vertices, or junctures. Let the *bases* represent edges, or cliffs, or rifts, or fault lines. Thus we obtain a *graph*, which will serve as the map of our landscape. For this reason it is known as a *topograph*.



Note how, if we label our bases and superbases intelligently, then each region, or plateau, of the topograph corresponds to a single lax vector.

It is not yet clear that our topograph is connected: there might be several components, with different bases, superbases, and vectors. And our picture above seems to show the graph as a tree, but we do not know this yet either.

The perspicacious reader will note that we haven't looked at any quadratic forms yet! But now that we have constructed the tectonics of our world, this situation is easy to rectify. Each plateau corresponds to a single lax vector $\mathbf{v}$; so given a quadratic form f, $f(\mathbf{v})$ will represent the *altitude* of that plateau.

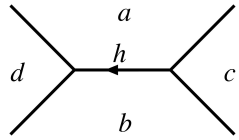
We have now fully constructed our landscape. As we can see, the tectonics are independent of the particular quadratic form, but the geomorphology — the altitudes of different plateaus — is determined by the quadratic form.

We have a jagged mountain landscape of interlocking plateaus. Plateaus meet at rifts, where cliffs have formed, and cliffs meet three at a time, at junctures. Each pattern of mountain ranges, and the particular heights of various cliffs, is the "landscape" of the quadratic form. Solving $f(\mathbf{v}) = n$ corresponds to asking the question: are there any plateaus of altitude n?

5 Mountain climbing

Intrepid mathematical adventurers would like to indulge in all their favourite extreme sports on quadratic landscapes. We start with mountain climbing. At first it appears that getting around the landscape will not be too hard.

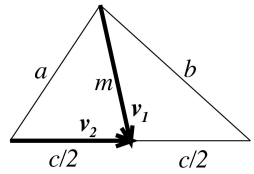
Theorem 1 (Apollonius' theorem) *Mountains are not too steep. More precisely, in the situation shown, with altitudes labelled a, b, c, d,*



we have $c + d = 2(a + b)$. That is, $d, a + b, c$ forms an arithmetic progression.

Denote the common difference h of the arithmetic progression by a label and put an arrow on the rift, pointing upwards, as shown. If the common difference is 0, then the edge is *neutral* and we draw no arrow. The arrow gives us an idea of which way is up, and how fast we are going up. If three arrows point outward from a juncture, then we are in a bit of a rut! We will call such a place, where three arrows radiate outward from a juncture (or are neutral), a *well*.

Why is this Apollonius' theorem? The usual form of the theorem is that, given a triangle as shown, with a median drawn,



$$a^2 + b^2 = 2\left(m^2 + \left(\frac{c}{2}\right)^2\right).$$

With vectors $\mathbf{v_1}, \mathbf{v_2}$ as shown, this becomes

$$|\mathbf{v_1} - \mathbf{v_2}|^2 + |\mathbf{v_1} + \mathbf{v_2}|^2 = 2\left(|\mathbf{v_1}|^2 + |\mathbf{v_2}|^2\right).$$

But note that $|\mathbf{v}|^2 = |(x, y)|^2 = x^2 + y^2$ is just one particular quadratic form. In fact the same holds for any quadratic form $f(\mathbf{v}) = ax^2 + bxy + cy^2$:

$$\begin{aligned} f(\mathbf{v_1} - \mathbf{v_2}) + f(\mathbf{v_1} + \mathbf{v_2}) &= a\,(x_1 - x_2)^2 + b\,(x_1 - x_2)\,(y_1 - y_2) + c\,(y_1 - y_2)^2 \\ &\quad + a\,(x_1 + x_2)^2 + b\,(x_1 + x_2)\,(y_1 + y_2) + c\,(y_1 + y_2)^2 \\ &= 2a\left(x_1^2 + x_2^2\right) + 2b\,(x_1 y_1 + x_2 y_2) + 2c\left(y_1^2 + y_2^2\right) \\ &= 2f(\mathbf{v_1}) + 2f(\mathbf{v_2}). \end{aligned}$$

If we notice that the four lax vectors corresponding to the four regions in our diagram are $\pm\mathbf{v_1}$, $\pm\mathbf{v_2}$, $\pm(\mathbf{v_1} - \mathbf{v_2})$ and $\pm(\mathbf{v_1} + \mathbf{v_2})$, then this generalised version of Apollonius' theorem gives us the result.

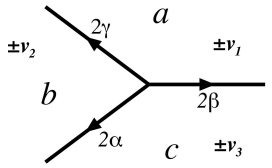
Apollonius' theorem tells us that, if we know the altitudes at a single juncture, then we may find other altitudes nearby. By repeating the process, we can figure out the altitude anywhere... well, almost anywhere. If the topograph has several components, we can only figure out altitudes in the "local" component.

Actually we can improve on Apollonius' theorem. The entire landscape (including all of its components) is determined by an "eruption" from a single juncture. We can see this directly. Suppose we know $f(0,1) = 3$, $f(1,0) = -6$, and $f(1,1) = 11$. It follows immediately from substitution that

$$f(x,y) = -6x^2 + 14xy + 3y^2.$$

Linear algebra tells us that the same is true, if we know the values of f at any superbase. In fact there is a formula.

Theorem 2 (Eruption theorem) *Suppose we have a juncture with superbase* $\{\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}\}$ *where* $\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$*, as shown.*



Then the quadratic form is given by

$$f(m_1\mathbf{v_1} + m_2\mathbf{v_2} + m_3\mathbf{v_3}) = \alpha(m_2 - m_3)^2 + \beta(m_1 - m_3)^2 + \gamma(m_1 - m_2)^2.$$

(Note that since $\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$, there is more than one way to express a vector in the form $m_1\mathbf{v_1} + m_2\mathbf{v_2} + m_3\mathbf{v_3}$.)

From Apollonius' theorem, we know that

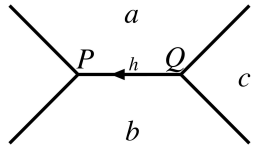
$$\alpha = \frac{b+c-a}{2}, \quad \beta = \frac{c+a-b}{2}, \quad \gamma = \frac{a+b-c}{2}.$$

Clearly our purported formula is a quadratic form, and it agrees with f at $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$, using the above expressions. So it is correct.

Thus, surveying a quadratic landscape from a given vantage point is a very precise and simple matter. But while mountain climbing is quite safe, the prospects of mountain climbers scaling summits are soon dashed.

Theorem 3 (Futility theorem) *Suppose we have a well with surrounding plateau altitudes positive. Then no matter how you traverse the landscape, without backtracking you must keep going up. There are no summits.*

To see why, consider the situation below around the well. All of a, b, c are positive and $h \geq 0$. Labelling altitudes and arrows as shown, because of the direction of the arrow, we have $c \geq a + b$. Therefore $a + c > b$ and $b + c > a$. So the other (undrawn) arrows at Q must point away from Q. Continuing this process, arrows must continually point outwards. So you keep going up.



6 Bushwalking and abseiling

We now turn to bushwalking; an expedition along a plateau would be a good idea. A circuit of a plateau might be quite an experience! Sadly, again this is not very feasible.

Theorem 4 (Tree theorem) *Each plateau continues out to infinity. The topograph has no loops.*

Pick an arbitrary juncture/superbase, and consider the quadratic form with surrounding altitudes $2, 2, 2$, i.e. with $\alpha = \beta = \gamma = 1$. Such a quadratic form exists: we could, if we wanted, explicitly work out its formula from the eruption theorem. Then we have a well with surrounding altitudes positive. So, by the futility lemma, all arrows radiate outwards from the well. As we attempt to circumnavigate the plateau, surrounding cliffs just get higher and higher. They cannot return to the starting altitude, so there cannot be a loop.

Note the rather amazing character of this proof. We have proved a fact about the topograph, which is an object consisting of connections between bases and superbases, completely independent of quadratic forms. But by imposing some geomorphology, through a particular quadratic form, we have proved something about it.

While a circuit of a plateau is impossible, if we are prepared to climb up and abseil down cliffs, it turns out that we can visit any part of our quadratic universe! That is...

Theorem 5 (Odyssey theorem) *All the primitive vectors occur on the one topograph. They are all connected.*

To see why, we will apply similarly shifty reasoning. Again choose a random lax superbase $V = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and the quadratic form which makes it a well, and associates altitudes $2, 2, 2$ to it, i.e. $\alpha = \beta = \gamma = 1$. From the eruption theorem, we have

$$f(m_1\mathbf{v_1} + m_2\mathbf{v_2} + m_3\mathbf{v_3}) = (m_2 - m_3)^2 + (m_1 - m_3)^2 + (m_1 - m_2)^2.$$

In particular, $f(\mathbf{v}) > 0$ for all primitive vectors $\mathbf{v}$.

It's not difficult to see that if $\mathbf{v} = m_1\mathbf{v_1} + m_2\mathbf{v_2} + m_3\mathbf{v_3}$ is a primitive vector distinct from $\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}$, then m_1, m_2, m_3 are all distinct. Therefore $f(\mathbf{v}) \geq 3$. It follows that the three minimum values of f are precisely $2, 2, 2$, and are achieved only at the three lax vectors $\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}$. That is, the plateaus around our well are strictly the lowest in the whole universe.

Consider a random vertex (superbase) on the topograph. If it is not a well, then we can always climb "down" to lower levels, following the arrows along edges backwards. Because $f(\mathbf{v}) > 0$ always, we must eventually stop and come to a well $W = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

But now apply the same reasoning to W as we did to V. Let the three altitudes around W be a, b, c, so $a, b, c \geq 2$, and we obtain $f(m_1\mathbf{w_1} + m_2\mathbf{w_2} + m_3\mathbf{w_3}) = \alpha(m_2 - m_3)^2 + \beta(m_1 - m_3)^2 + \gamma(m_1 - m_2)^2$, where $\alpha + \beta + \gamma = (a + b + c)/2$. Away from $\pm\mathbf{w_1}, \pm\mathbf{w_2}, \pm\mathbf{w_3}$ we have m_1, m_2, m_3 all distinct and $f(\mathbf{v}) \geq \alpha + \beta + \gamma = (a + b + c)/2 \geq 3$. But $2, 2, 2$ must occur somewhere! It follows that $V = W$.

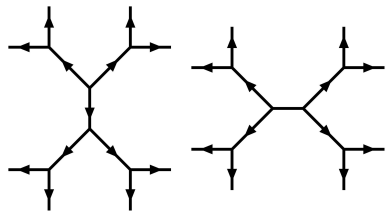
That is, from any random vertex, if we climb down, we come to our original well V. We conclude that our topograph is connected, and our quadratic "universe" consists only of one landscape.

This is another proof of a non-trivial fact purely about plate tectonics, which has nothing to do with any particular quadratic form. But applying the geomorphology of a particular form gives us a simple proof!

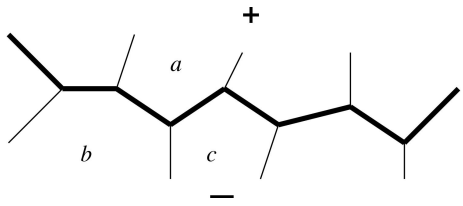
7 Exploring new worlds

We now have enough information to make a basic classification of the possible quadratic landscapes. We can classify based on whether a landscape goes above and/or below sea level. Thus there are 7 types: 0, $+$, $-$, $+-$, $0+$, $0-$ and $0+-$. We adventurous types wish to see what all these types look like.

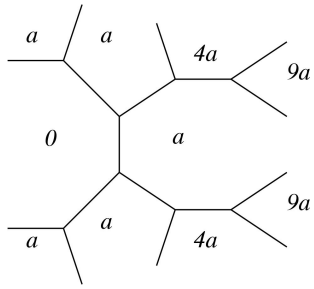
The least interesting world is the 0 world — "waterworld". As for + worlds, by similar methods to the previous section, we can show that the entire landscape radiates outwards and upwards from a well or *abyss*. If all $\alpha, \beta, \gamma > 0$ around the well, our abyss is a simple well. But if (say) $\gamma = 0$, then the abyss is a "double well". The − worlds are of course inverted underwater versions of the + worlds.



In a +− world, plateaus go from positive to negative altitude, without ever taking the value 0. Thus there is a "river" of edges separating positive from negative altitudes. Consider a portion of the river, i.e. an edge separating plateaux of altitudes $a > 0$ and $b < 0$. At the next juncture, the altitude c is either positive or negative; and so the river never stops, but continues infinitely. Apollonius' theorem shows us that, if we climb away from the river on the positive side, we go upwards indefinitely, while if we climb down on the negative side, we go downwards indefinitely. In fact it can be shown that the river is periodic — if you continue along, the pattern of altitudes will repeat over and over again.

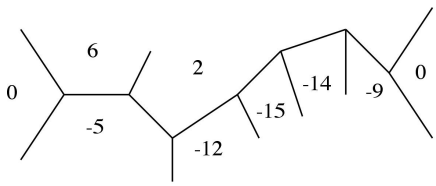


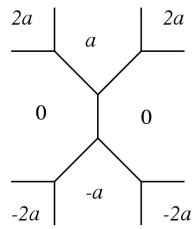
In a 0+ world, there must be a *lake* of altitude 0. Furthermore, Apollonius' theorem shows us that the plateaus around the lake form an arithmetic progression. As the lake has infinite circumference, the arithmetic progression will go negative somewhere (which is not allowed) unless it is actually constant. Walking away from the lake, we see that we must continually climb up. So the world consists of a single lake, from which hills radiate in a symmetric fashion. Obviously 0− worlds are inverted forms of the 0+ worlds.



This only leaves 0+− worlds. Again there is a lake at altitude 0. Around its circumference there is again an arithmetic progression, which changes from positive to negative somewhere. So at this point a *river* flows out of the lake. Since infinite rivers eventually become periodic, this river must be finite, and run into another lake. Therefore, the world consists of two lakes, joined by a river, separating the world into a positive and negative part. There is

also the possibility of a "degenerate river", in which case the two lakes adjoin each other, forming a *weir*.





8 Extreme-sports solutions to Diophantine equations

Armed with our newfound knowledge of quadratic tectonics and geomorphology, we are ready for some high-speed Diophantine-equation-solving action! It is a four-step process.

Recall we are attempting to solve the equation $f(x,y) = ax^2 + bxy + cy^2 = n$, where a, b, c, n are integers. That is, we want to find all plateaux of altitude n.

(1) *Base jump.* Launch yourself randomly into the landscape — choose a random superbase and evaluate the quadratic form there. (Surprisingly, I always seem to land at the superbase $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$.)

(2) *Abseil.* From there, use Apollonius' theorem to abseil down (or climb up, if we land below see level) the landscape to a well, or river, lake, or weir.

(3) *Climb.* From this base point, climb up in all possible directions until you get up to n or past it. This can only take finitely long. The worst case is where there is a river, but rivers are periodic.

(4) *Check.* If we find an altitude of n exactly, we have a solution. If we find $f(\mathbf{v}) = \frac{n}{k^2}$ for some integer k, then we also have a solution also as $f(k\mathbf{v}) = n$. Otherwise, there is no solution.

Note that the only time we actually evaluate f is in step 1. From there, all calculations are done using Apollonius' theorem, and our knowledge of the landscape. Thus, we have an algorithm to solve the problem, and to "see" the values of f, quite literally.

9 The Farey tree

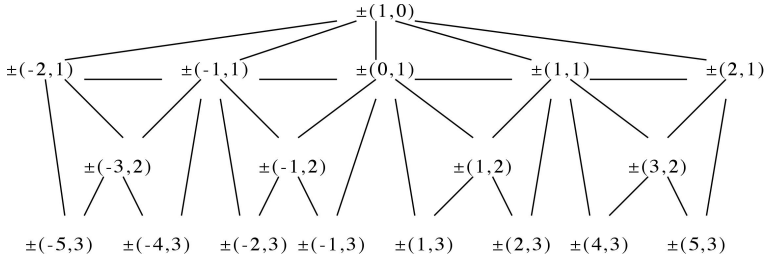
We have constructed the highly exciting and extreme topograph in the course of our adventures, which turned out to be a tree. In fact this tree turns out to have more magical properties, if we take its *dual*. Any planar graph has a *dual*, obtained by placing a vertex in every face, and joining two of these vertices if the faces are adjacent. (A tree only has one "face", usually, but we can consider every plateau of our topograph as a "face". We will call the dual of our topograph a *Farey graph*.

The Farey graph inherits a number of properties from the topograph. It is an infinite planar graph. The vertices correspond to the primitive lax vectors. Each vertex has infinite degree. The edges are precisely the lax bases. The faces are precisely the lax superbases. They are all triangles.

Now, around each vertex or juncture of the original graph, we had three lax primitive vectors $\pm\mathbf{v_1}, \pm\mathbf{v_2}, \pm\mathbf{v_3}$, where $\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$. So, we could have constructed the original graph by starting from the edge $\{\pm(1,0), \pm(0,1)\}$. Then the two superbases at either end would be $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$ and $\{\pm(1,0), \pm(0,1), \pm(-1,1)\}$. So, from two vectors $\pm\mathbf{v_1}$ and $\mathbf{v_2}$, we obtain two new vectors, one on either side: $\pm(\mathbf{v_1} + \mathbf{v_2})$ and $\pm(\mathbf{v_1} - \mathbf{v_2})$. This process could be continued to yield the entire graph.

A similar process could be used to construct the dual graph. Start from vertex $\pm(1,0)$. Next add $\pm(0,1)$. Since these two vectors form a lax base, they are connected by an edge. Then we add two more vectors, corresponding to the two possible superbases. And so we continue, forming a triangle on each available edge.

Let us consider the graph after a few steps. I have arranged it in a certain way. In particular, each "row" corresponds to a particular value of y in (x, y).



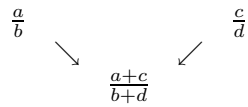
Alternatively, as suggested by the diagram, we can consider this graph as starting from the "highest" vector $\pm(1,0)$, and continually adding on lower and lower vectors ("lower" vectors have greater second coordinate). If we have two vectors $\pm(a,b)$ and $\pm(c,d)$, and we arrange them so that $b > 0$ and $d > 0$, then the "lower" vector which is eventually connected to them will be $\pm(a+c, b+d)$. (The other possibility, $\pm(a-c, b-d)$ will be on a "higher" level".)

10 Farey orienteering

The coup de gras comes when we notice that there is a perfect correspondence between *primitive lax vectors* and *rational numbers*. The "primitiveness" precisely removes the potential problem that, say, $\frac{8}{4} = \frac{2}{1}$, and "laxity" precisely removes the problem that $\frac{-2}{-1} = \frac{2}{1}$. So now label all the vertices with rational numbers. (For convenience always write rationals with positive denominator.)

We know that all the rational numbers occur exactly once in the graph, by the odyssey theorem. Therefore, if we consider the graph down to level n, it must contain all the rational numbers with denominator up to n.

Now, when we go down from level $n-1$ to n, the new vertices are all attached through the process discussed in the previous section.



Such new vertices can only be attached when $b + d = n$, and since each addition of a vertex completes a new triangle, $\frac{a}{b}$, $\frac{c}{d}$ must be joined by an edge. However, a base can only extend to a superbase in two ways. So if an edge (base) is already part of two triangles (superbases), we call that edge *spent* and it can no longer be used to form new triangles.

We claim that the rational numbers with denominator up to n, if "lined up" in order from left to right as in the diagram below, are all consecutively connected by edges. We'll call this the *level n road*. You can see from the diagram that, as n increases, we obtain progressively harder and harder orienteering courses for mathematical adventurers! After construcing the diagram to level n, all edges not on the level n road are spent. All of the new vertices are added by putting "detours" on this road, as in the diagram above, thereby forcing the original portion of the road to be spent. Furthermore, it is easy to check that

if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. So if the triangles are inserted with the lower fraction lying horizontally between the two upper ones, then the fractions are horizontally in order, and the level-n road connects them consecutively, and all other edges are spent. By induction we prove our claim.

Consider any two fractions $\frac{a}{b} < \frac{c}{d}$ connected by an edge. Then (a, b), (c, d) form a base. It is a well-known fact from linear algebra that this implies $ad - bc = \pm 1$. Hence we have proved:

Theorem 6 (Farey fractions) *Let S_n be the set of rational numbers with denominator $\leq n$, written in order. If $\frac{a}{b} < \frac{c}{d}$ are consecutive fractions in this sequence then $ad - bc = -1$. Further, S_{n+1} can be obtained from S_n by inserting into the sequence, between consecutive $\frac{a}{b}$ and $\frac{c}{d}$, the fraction $\frac{a+c}{b+d}$, wherever $b + d = n + 1$.*

This gives an unorthodox connection between the theory of quadratic forms, and this nugget of number theory known as Farey fractions.

Finally, for the interested reader, we point out that the topograph we have constructed actually occurs "in nature", in tiling the hyperbolic plane by fundamental domains of the group $PSL(2, \mathbb{Z})$, acting via isometries.

11 Solutions

(1) There is no solution as the left hand side is a multiple of 5 and the right hand side is not.
(2) $(0, \pm 1)$ is an obvious solution. I leave determination of all solutions to the interested reader!
(3) There is no solution as $x^2 - 6xy + 9y^2 = (x - 3y)^2 \geq 0$.
(4) Considered modulo 10 we have $y^2 \equiv 7$. But 7 is not a quadratic residue modulo 10, so there is no solution.

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