Some field-theoretic ideas out of contact geometry and elementary topology

Daniel V. Mathews

dan.v.mathews@gmail.com

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5 December 2012

Outline

Overview

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  - Intricate algebraic structures keeping track of analytic data
- However, *in the simplest cases* some of this structure reduces to some very physical-looking *combinatorics and algebra* which is interesting in its own right.

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- Discuss some of these algebraic and combinatorial results in their own right.
  (No symplectic geometry / contact topology / holomorphic curves assumed.)
- Indicate some of the connections to string topology and holomorphic invariants.

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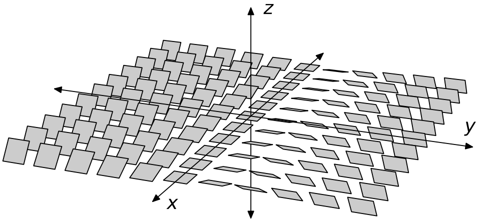
E.g. $\mathbb{R}^3$ with $\xi = \ker \alpha$, where $\alpha = dz - y\,dx$.



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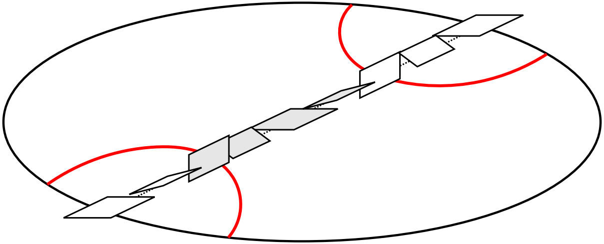
- Tangent to ∂D
- "Perpendicular" to D precisely along Γ



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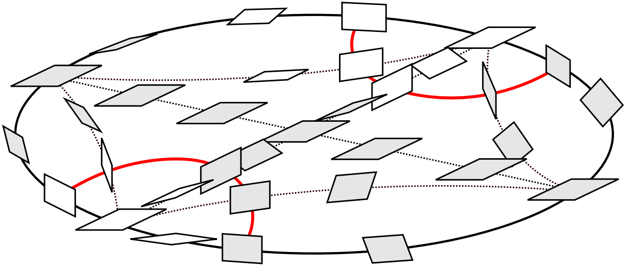
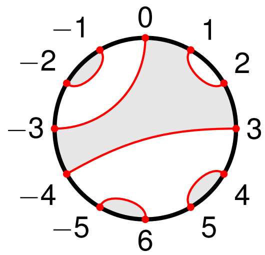


Chord diagrams

So a *chord diagram* on a disc describes a contact structure.



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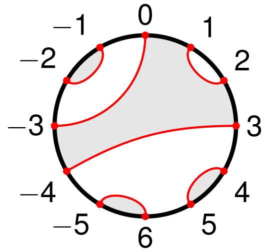

- Shading = visible side of contact planes.
- Similar to the structure of *sutures*.

Overtwisted contact structures

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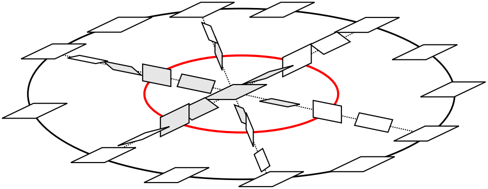

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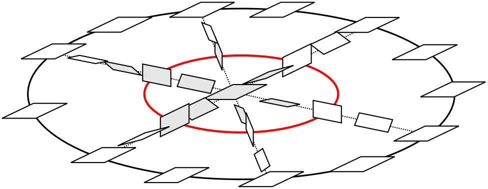


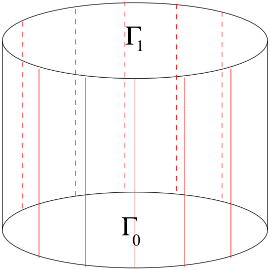
- Overtwisted contact geometry reduces to (well-understood) homotopy theory. Tight contact structures offer important topological information.

Outline

1. Introduction
2. Contact TQFT
   - "Inner product" on chord diagrams
   - Bypass surgery
   - Contact QFT = Quantum pawn dynamics
   - Quantum pawn dynamics
3. Strings, holomorphic curves, beyond

An “inner product” on chord diagrams

There’s a bilinear form on chord diagrams defined by *inserting into a cylinder*.



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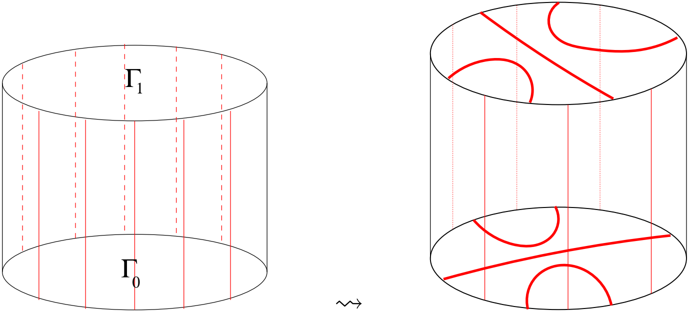

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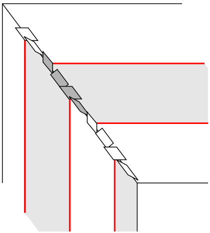

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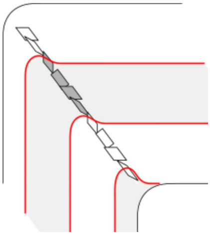

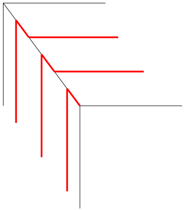

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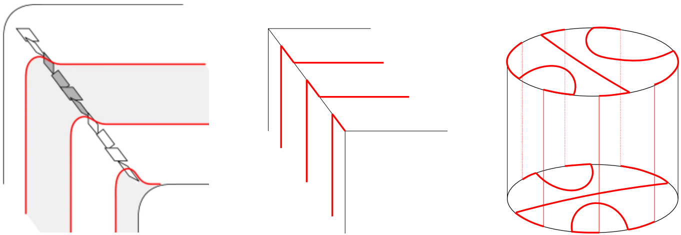

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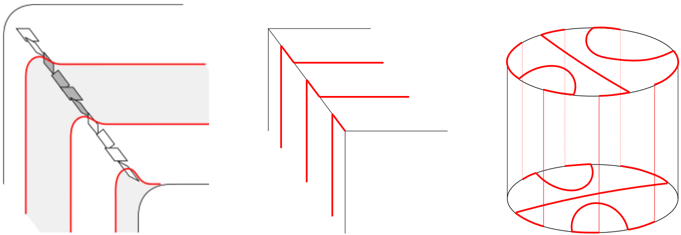
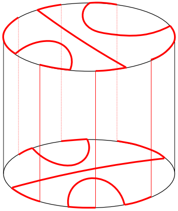


Definition (M.)

$$\langle \Gamma_0 | \Gamma_1 \rangle = \begin{cases} 1 & \text{if the resulting curves on the cylinder} \\ & \quad \text{form a single connected curve;} \\ 0 & \text{if the result is disconnected.} \end{cases}$$

NB: This "inner product" is not symmetric!

Contact meaning of the "inner product"



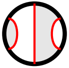
Proposition (Eliashberg)

Let Γ_0, Γ_1 be chord diagrams. The following are equivalent:

- $\langle \Gamma_0 | \Gamma_1 \rangle = 1$.
- *The solid cylinder with dividing set Γ_0 on the bottom and Γ_1 on the top has a tight contact structure.*

Bypass surgery

In a chord diagram on disc D, consider a sub-disc B as shown:



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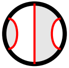
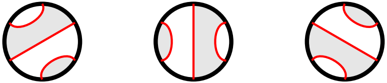


Two natural ways to adjust this chord diagram, consistent with the colours: *bypass surgeries*.



Γ' Γ Γ''

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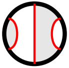

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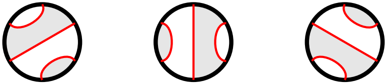


Γ' $\quad$ Γ $\quad$ Γ''

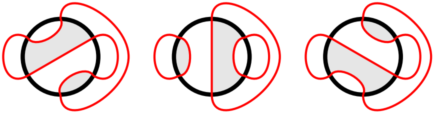
Proposition

With $\Gamma, \Gamma', \Gamma''$ *as above, for any* Γ_1*,*

$$\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma'' | \Gamma_1 \rangle = 0.$$

Bypass surgery

Idea of proof:



$1 + 0 + 1 = 0$

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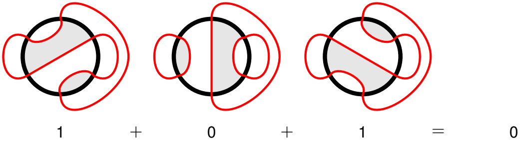

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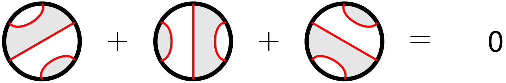

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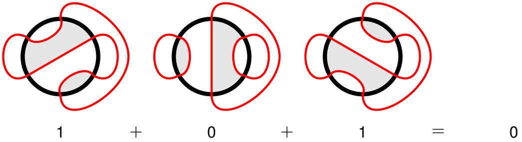

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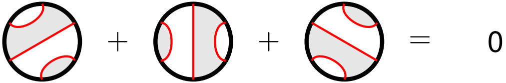

So we define a vector space

$$V_n = \frac{\mathbb{Z}_2 \langle \text{Chord diagrams with } n \text{ chords} \rangle}{\text{Bypass relation}}$$

Contact TQFT = Quantum pawn dynamics

These definitions give many of the properties of a (2+1)-dimensional *topological quantum field theory*.

- Contact structure near disc (2-dim) $\rightsquigarrow$ "states" in V_n
- Contact structure over cylinder (2+1-dim) $\rightsquigarrow$ element of $\mathbb{Z}_2$.
- "Possibility of a tight contact structure from one state to another" $\rightsquigarrow$ inner product $\langle \cdot | \cdot \rangle : V_n \otimes V_n \longrightarrow \mathbb{Z}_2$.

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(Also: chord diagrams / cylinders form a *category* with distinguished bypass triples — a *triangulated category*. V_n is its *Grothiendick group*.)

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Theorem (M.)

V_n has dimension 2^{n-1} and is isomorphic to 1-dimensional quantum pawn dynamics.

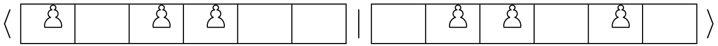
Quantum Pawn Dynamics

Consider pawns on a finite 1-dimensional chessboard.
Pawns move left to right.
"Inner product" describes the possibility of pawn moves.

Definition (Pawn "inner product")

$$\langle w_0 | w_1 \rangle = \begin{cases} 1 & \textit{if it is possible for pawns to move from } w_0 \textit{ to } w_1 \\ & \quad \textit{(this includes the case } w_0 = w_1\textit{);} \\ 0 & \textit{if not.} \end{cases}$$

Very asymmetric — in fact, "booleanization of a partial order".
E.g.



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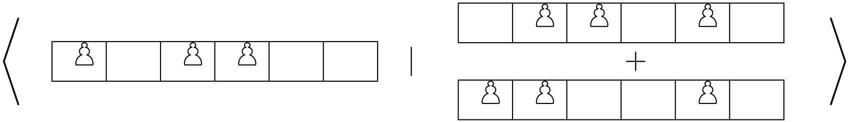
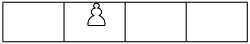


$= 1 + 0 = 1.$

Chord diagram of a chessboard

Construction of the *slalom skiing* chord diagram of a chessboard.

$w =$ 

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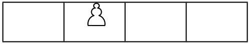
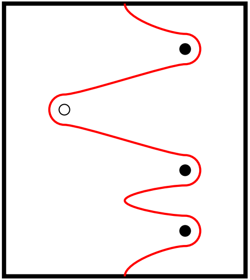
$w =$ 

$\leftrightarrow$ 

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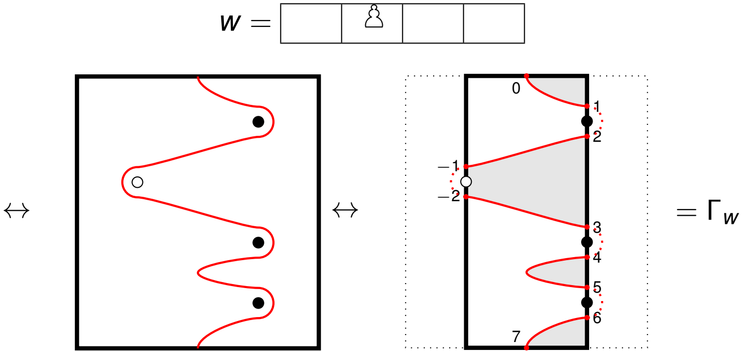
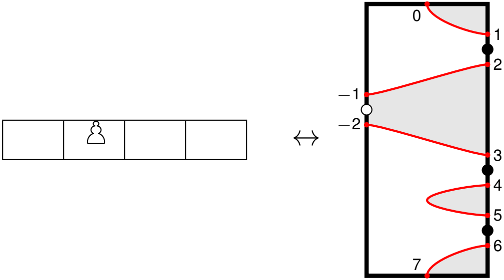


Square decomposition

Pawns correspond to a decomposition of the disc into *squares*.



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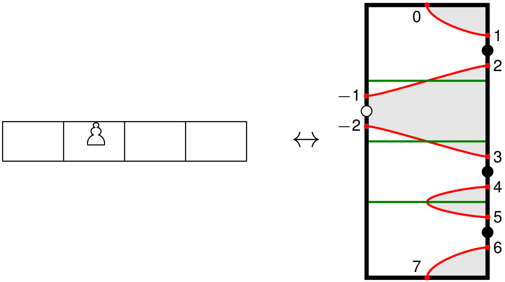

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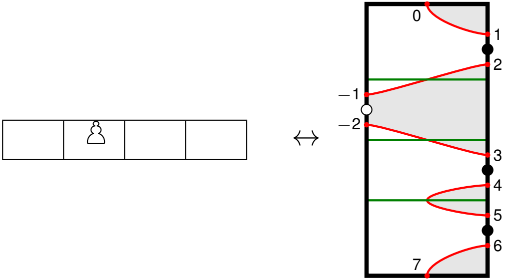
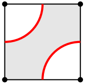


Note after decomposing along green lines, each square is



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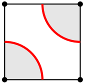

Behave partly like *particles* (can be created/annihilated) and partly like *qubits* (binary). Reminiscent of stat. mech...

Theorem (M.)

V_n has a basis given by the diagrams of chessboards:

$$V_n \cong \mathbb{Z}_2 \langle \textit{Chessboards with } n-1 \textit{ squares} \rangle.$$

For any two chessboards w_0, w_1,

$$\langle w_0 | w_1 \rangle = \langle \Gamma_{w_0} | \Gamma_{w_1} \rangle.$$

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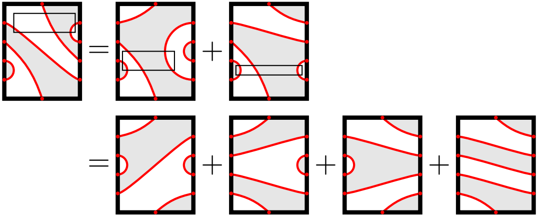

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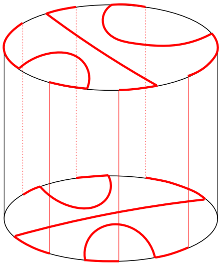
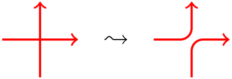


Outline

A "stringy" interpretation of V_n

Consider, instead of chord diagrams, a *string complex*:

- *oriented* curves on D which may intersect, between $2n$ fixed points on ∂D, up to homotopy;
- $\widehat{CS}(D^2, F_n) =$ free vector space generated by them;
- *differential* defined by *resolving crossings*:



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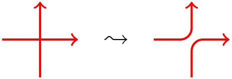

Theorem (M.–Schoenfeld)

$\partial^2 = 0$ *and the homology* $\widehat{HS}(D^2, F_n)$ *of the string complex is isomorphic to* V_n.

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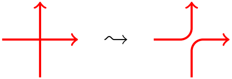

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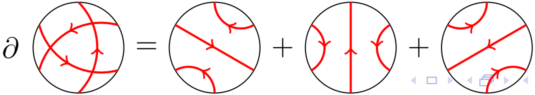

The holomorphic origin of V_n

We've seen V_n is:

- Chord diagrams modulo bypass relation
- $\mathbb{Z}_2\langle\text{Chessboards}\rangle$
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SFH is an invariant of *sutured manifolds* (M, Γ) defined by...

- Taking a Heegaard decomposition $\Sigma, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$
- Considering holomorphic curves in $\Sigma \times I \times \mathbb{R}$ as a symplectic manifold with an almost complex structure
- Chain complex generated by boundary conditions
- Differential counting index-1 holomorphic curves
- Homology of this complex is $SFH(M, \Gamma)$.

Thanks for listening!

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