Hyperbolic geometry

Daniel V. Mathews

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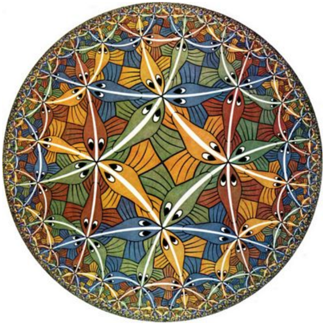
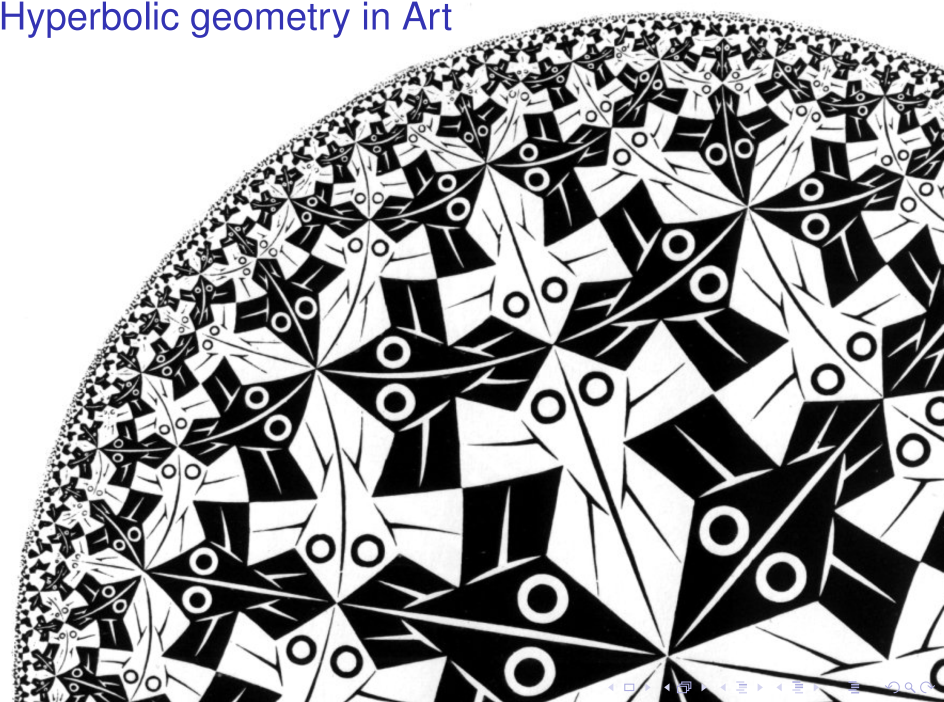
3 September 2013

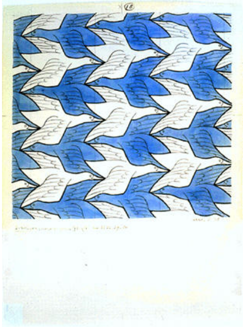


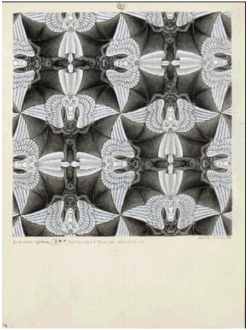
Hyperbolic geometry in Art

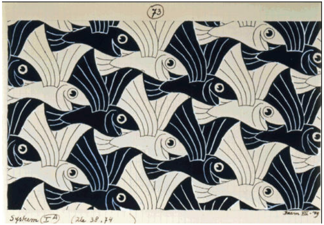


Hyperbolic geometry in Art

M.C. Escher made several drawings of “Symmetry” in the (Euclidean) plane.







However, he also made drawings of symmetry in *other* geometries...

Hyperbolic geometry in Art

M.C. Escher "Circle limit"' series

Circle Limit I

Hyperbolic geometry in Art

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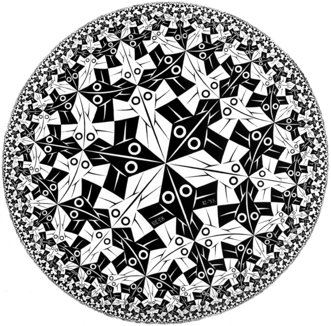

Circle Limit I

Circle Limit II

Hyperbolic geometry in Art

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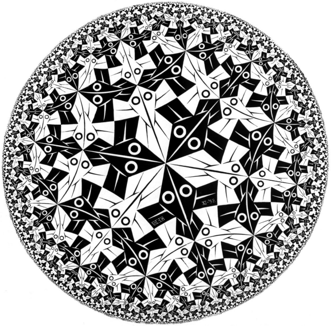

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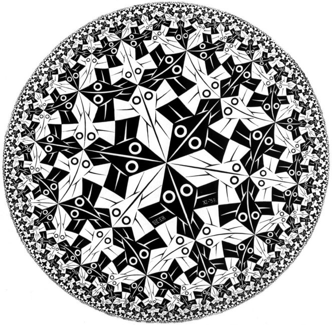

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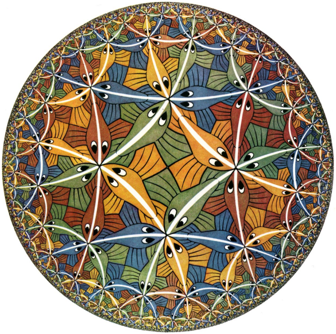

Circle Limit III

Hyperbolic geometry in Art

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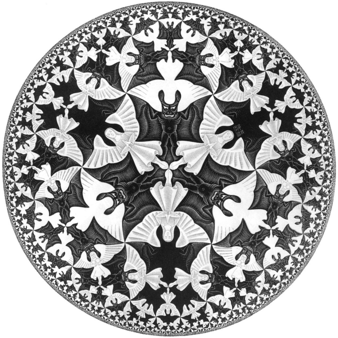
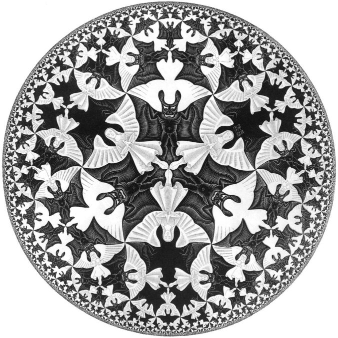


Circle Limit IV

Hyperbolic geometry in Art

M.C. Escher “Circle limit"’ series



Circle Limit IV

Circle Limit with Butterflies

Hyperbolic geometry in Literature

Shakespeare, *Hamlet*:

> *... I could be bounded in a nut shell and count myself a king of infinite space...*

Hyperbolic geometry in Literature

Shakespeare, *Hamlet*:

> *... I could be bounded in a nut shell and count myself a king of infinite space...*

Thomas Mann, *Little Herr Friedmann*:

> *Some of the men stood talking in this room, and at the right of the door a little knot had formed round a small table, the center of which was the mathematics student, who was eagerly talking. He had made the assertion that one could draw through a given point more than one parallel to a straight line; Frau Hagenström had cried out that this was impossible, and he had gone on to prove it so conclusively that his hearers were constrained to behave as though they understood.*

Euclid's axioms

The Elements (c. 300 BCE)

Euclid’s axioms

The Elements (c. 300 BCE)
Five “common notions”

1. Things that are equal to the same thing are also equal to one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things that coincide with one another are equal to one another.
5. The whole is greater than the part.

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"Good axioms":

- should be intuitively clear or "obviously true".
- "We hold these truths to be self-evident..."

Euclid's axioms

Five "postulates"

1. To draw a straight line from any point to any point.
2. To produce [extend] a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance [radius].
4. That all right angles are equal to one another.

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and one more...

5. The *parallel postulate*: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

The parallel postulate

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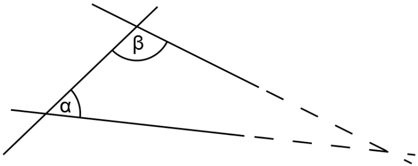

$$\alpha + \beta < \pi$$

Not quite so self-evident...

The parallel postulate

One can show that the parallel postulate is equivalent to *Playfair's axiom*:

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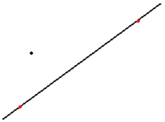
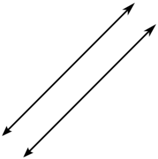
At most one line can be drawn through any point not on a given line parallel to the given line in a plane.



$\Longrightarrow$



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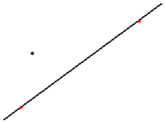

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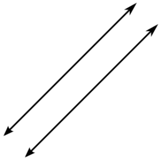

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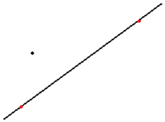

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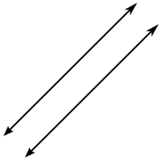

For 2,000 years mathematicians tried to prove the parallel postulate from the other axioms.

All failed.

Questioning the axiom

But if the parallel postulate doesn't follow from the others...

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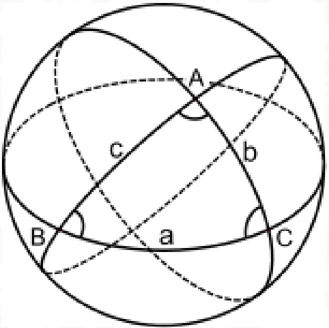

Given the line *BC* and the point *A* not on it, *no line can be drawn parallel to the given line*.

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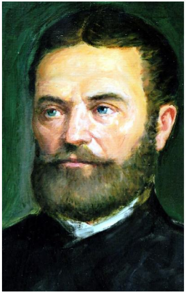

János Bolyai

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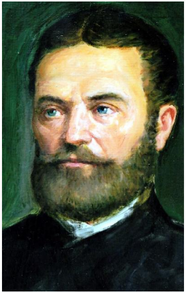


János Bolyai

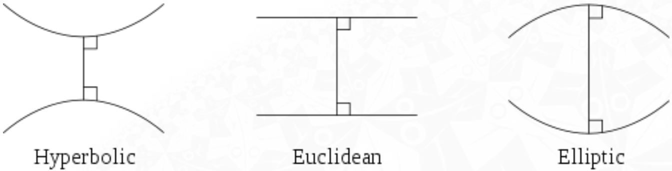
E.T. Bell, *Men of Mathematics* on Lobachevsky:

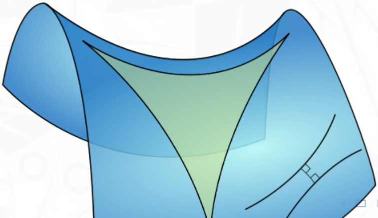
> *The boldness of his challenge inspired scientists to challenge other 'axioms' or accepted 'truths'... It is no exaggeration to call Lobatchewsky the Copernicus of Geometry.*

Hyperbolic musical interlude

There's a Tom Lehrer song about Lobachevsky.

From spherical to hyperbolic





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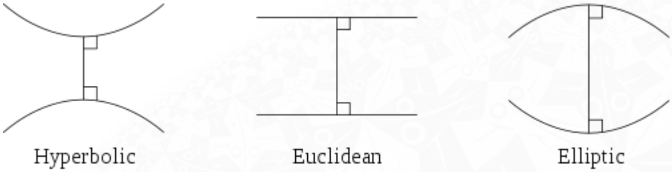
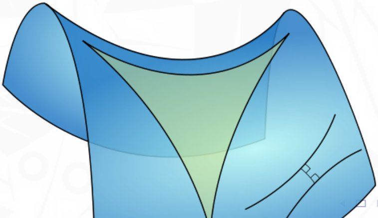


- In spherical/elliptic geometry (curvature > 0), straight lines/*geodesics* tend to converge — *no parallel lines*.



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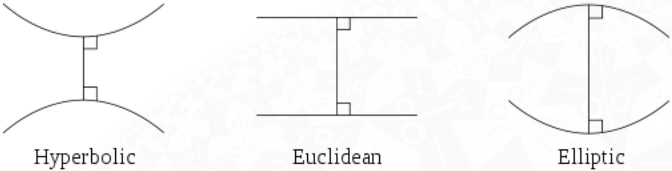

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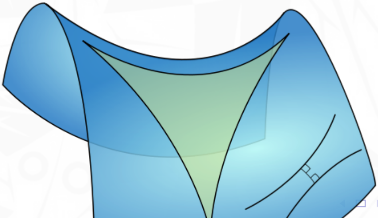

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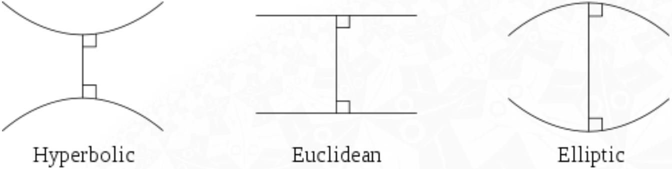

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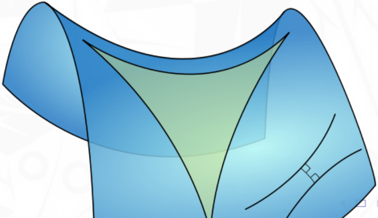
- In *hyperbolic geometry* (curvature < 0), "most" geodesics don't meet — *many parallel lines*.



Hyperbolic means constant negative curvature

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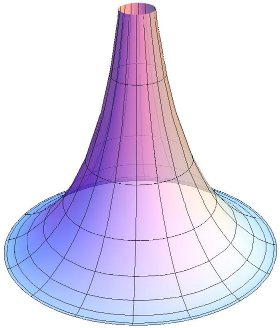
Does a constant curvature -1 surface exist in $\mathbb{R}^3$*?*

- Partially: *The Pseudosphere.*



(Obtained by rotating a *tractrix* about an axis.)

Image: `commons.wikimedia.org`

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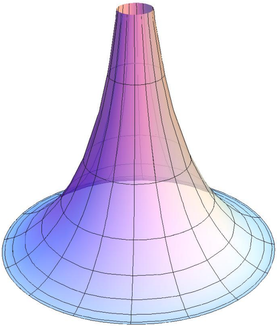

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- Runs out of space in $\mathbb{R}^3$!

Models of the hyperbolic plane

Instead of embedding the hyperbolic plane in $\mathbb{R}^3$, we take *models* of it. Define a different metric.

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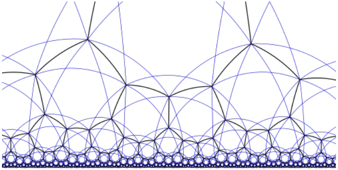
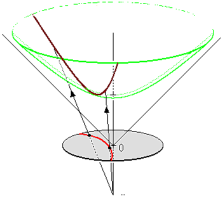
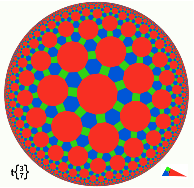
Three popular models:


Upper half plane model


Hyperboloid model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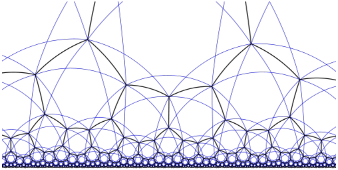

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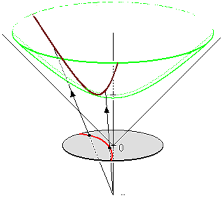

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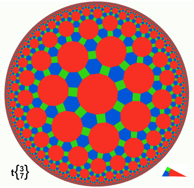

Poincaré disc model

- Escher's drawings use the disc model.
- Hyperboloid model closely related to relativity theory.
- Some computations are easier in upper half plane model.

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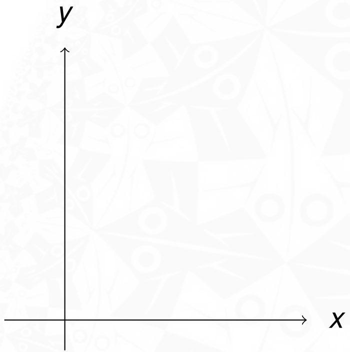
Take the upper half plane $\mathbb{H} = \{(x, y) \; : \; y > 0\} \subset \mathbb{R}^2$.



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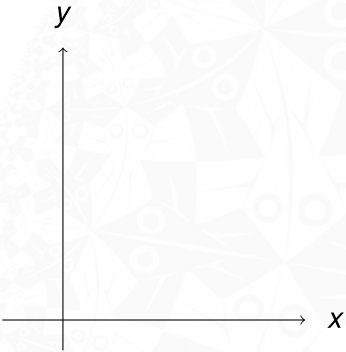
Take the upper half plane $\mathbb{H} = \{(x, y) \;:\; y > 0\} \subset \mathbb{R}^2$.
But distances get *longer* as y decreases.

$$\text{Hyperbolic distance} = \frac{\text{Euclidean distance}}{y}$$



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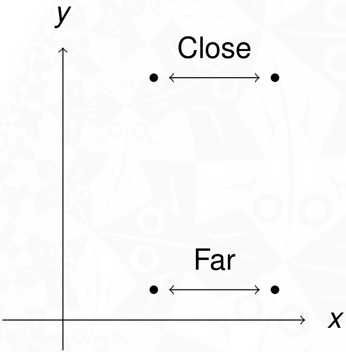

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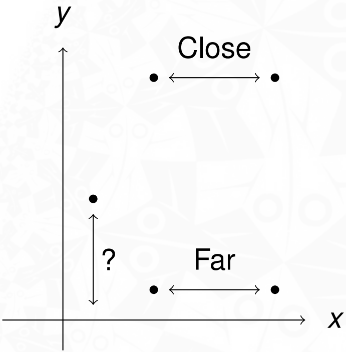

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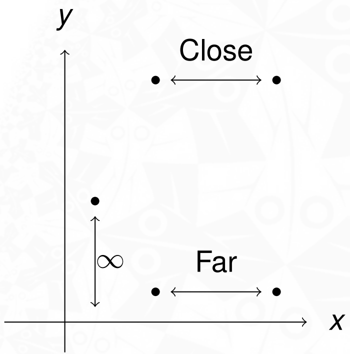
$$\int_0^1 \frac{1}{y}\, dy = [\log y]_0^1 = \infty$$



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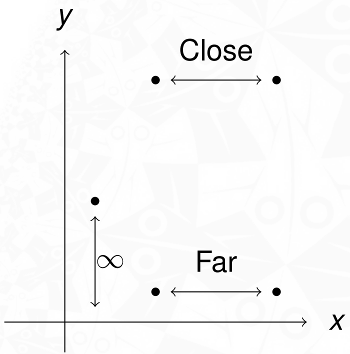

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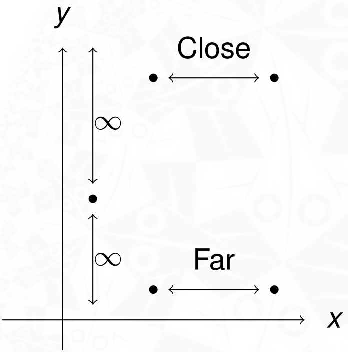

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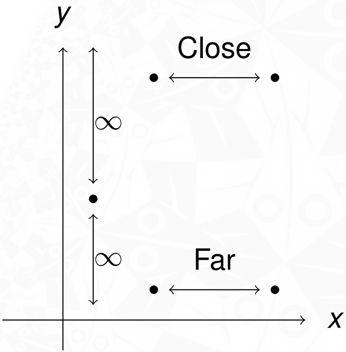
$$\text{Hyperbolic distance} = \frac{\text{Euclidean distance}}{y}$$

(“Swimming in a sea of chocolate pudding".)



Distance from $(x, 1)$ to $(x, 0)$:

$$\int_0^1 \frac{1}{y}\, dy = [\log y]_0^1 = \infty$$

The x-axis is infinitely far away from any point of $\mathbb{H}$!
Also, ∞ is infinitely far away from any point of $\mathbb{H}$.

- $\mathbb{R} \cup \{\infty\} =$ “circle at infinity"

Models of the hyperbolic plane

Shortest distance between two points?

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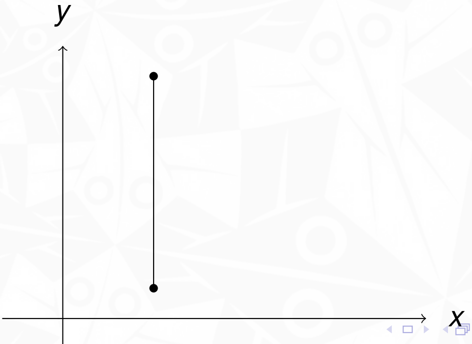

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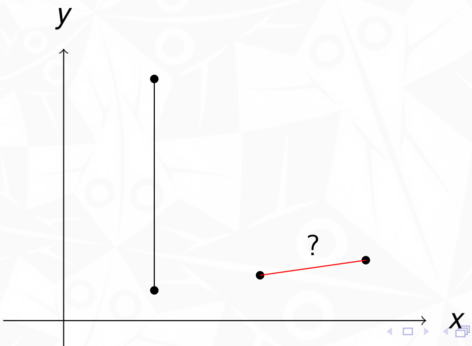

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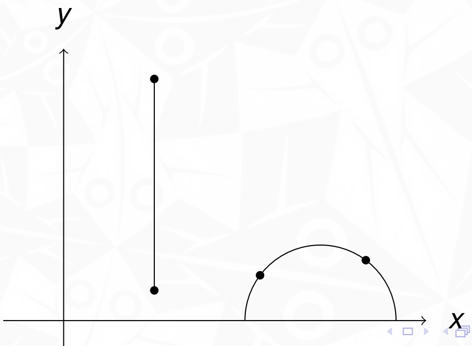

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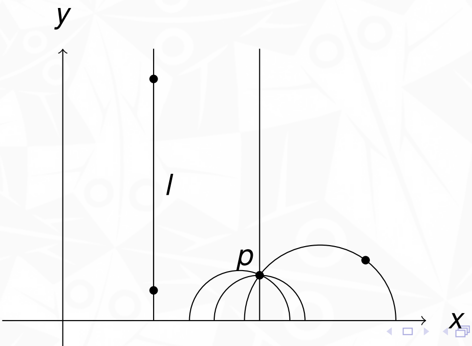
- So given point p and line l, *many* lines through p are parallel to l — parallel postulate fails!



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Turns out that all the *Möbius transformations* of the complex plane

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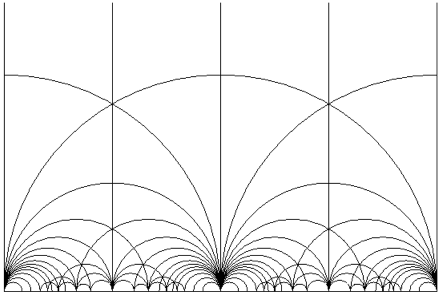

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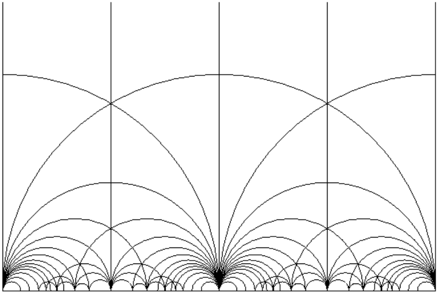

Image: `http://www.math.ethz.ch/~pink/ModularCabinet/`

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The disc model is similar...

- The "boundary circle" is infinitely far away — *circle at infinity*.
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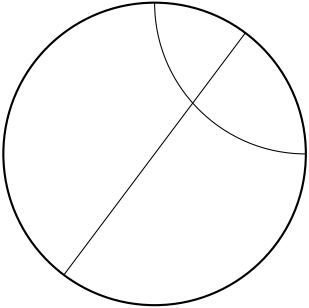

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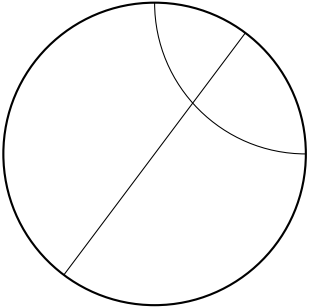

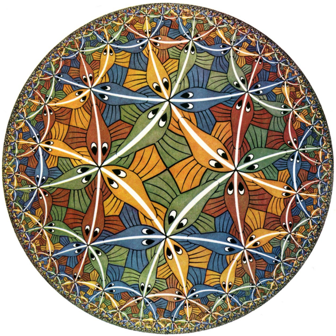

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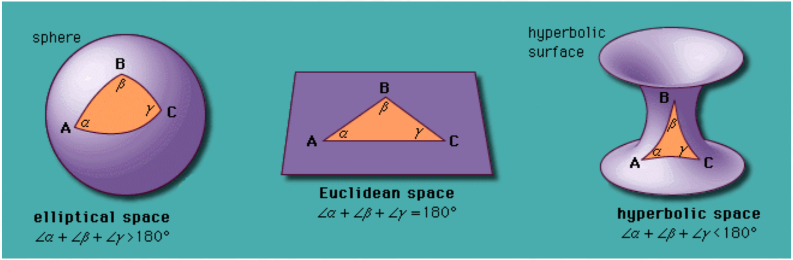

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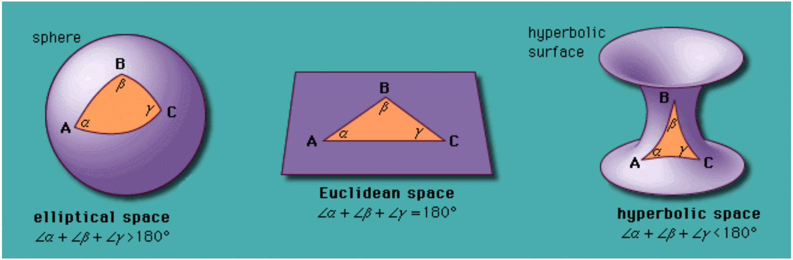

Image: `http://mathforum.org/`

Theorem (Amazing theorem!)

The area of a hyperbolic triangle is its angle defect from π:

$$\textit{Area} = \pi - (\alpha + \beta + \gamma)$$

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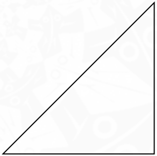

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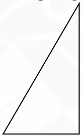

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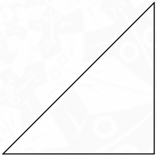

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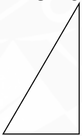


- But for any angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

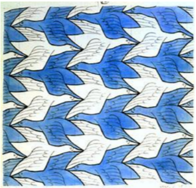
a hyperbolic triangle exists and tessellates the hyperbolic plane.

Hyperbolic tessellations

The program *Kaleidotile* by Jeff Weeks provides hours of hyperbolic tessellation fun.

Tessellations and surfaces

Suppose we have a tessellation of the Euclidean plane, and the tessellation's symmetry consists of *translations*.



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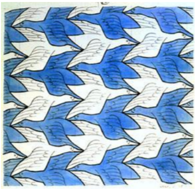

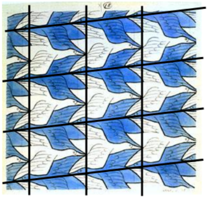

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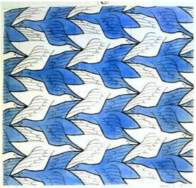

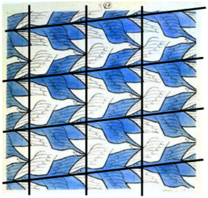

There's a square *fundamental domain* and its sides are *identified*. It's drawn on a *torus*.

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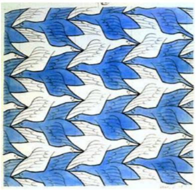

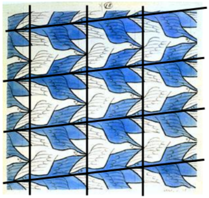

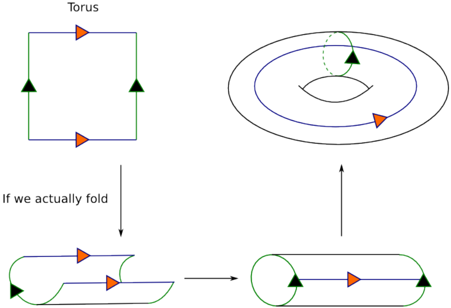

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Image: `topologygeometry.blogspot.com.au`

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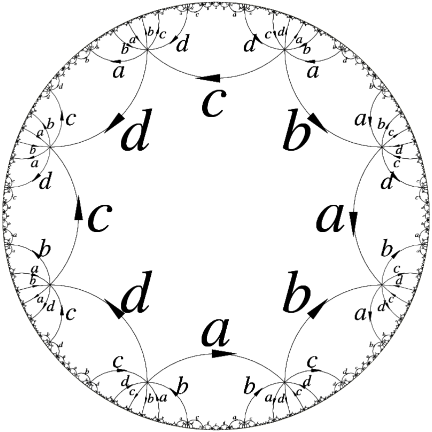

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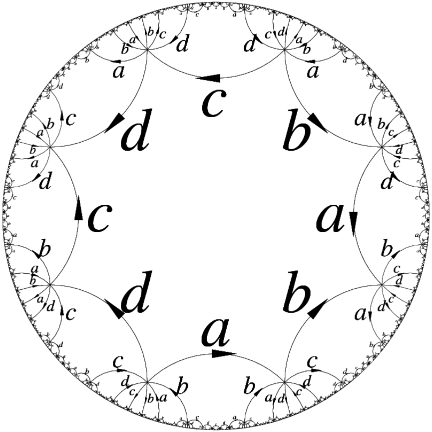

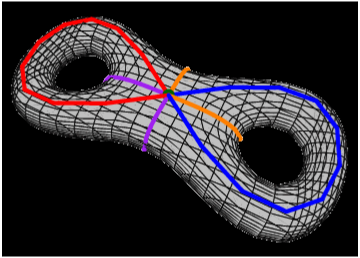

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This is a *hyperbolic surface*.

In 3 dimensions

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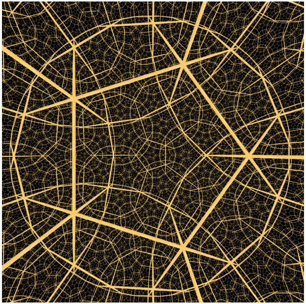

Image: bulatov.org, commons.wikimedia.org

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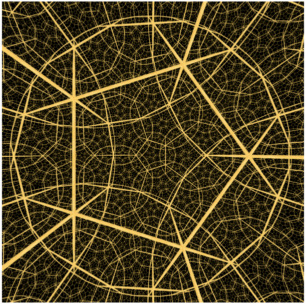

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We can obtain *hyperbolic structures* on 3-dimensional versions of surfaces, called *3-manifolds*.

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Conjectured by Thurston 1982, proved by Perelman 2003.
Still much active research...