# Trinities, hypergraphs, and contact structures

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# Outline

# Overview

This talk is about

- Combinatorics involving various notions related to graph theory...
  - Trinities: Triple structures closely related to *bipartite planar graphs*.
  - Hypergraphs: Generalisations of graphs; also related to bipartite graphs.
  - Hypertrees: A notion related to spanning trees in hypergraphs.

# Overview

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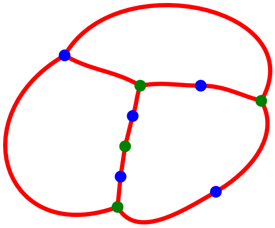
- Combinatorics involving various notions related to graph theory...
  - Trinities: Triple structures closely related to *bipartite planar graphs*.
  - Hypergraphs: Generalisations of graphs; also related to bipartite graphs.
  - Hypertrees: A notion related to spanning trees in hypergraphs.
- ... and some related discrete mathematics arising in 3-dimensional topology.
  - Formal knots: A notion developed by Kauffman in knot theory.
  - Contact structures: A type of geometric structure on 3-dimensional spaces.

# Outline

# Bipartite planar graphs

- Let $G$ be a bipartite planar graph.



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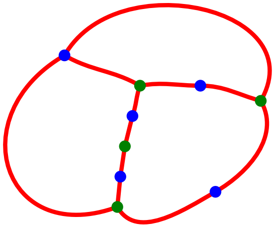

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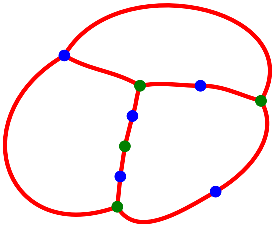

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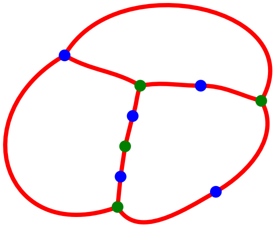
- Embedded in $\mathbb{R}^2 \subset S^2$.



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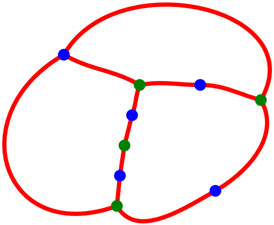

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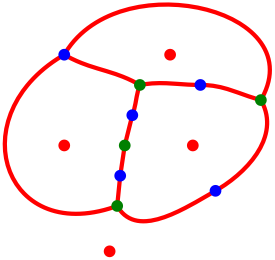

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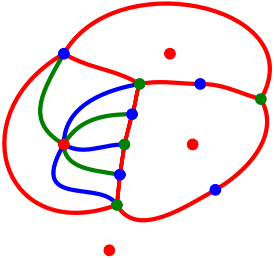

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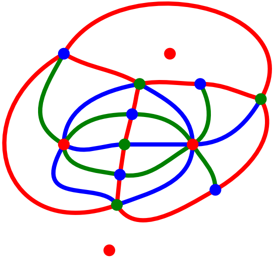

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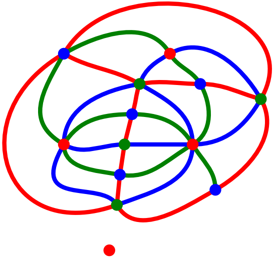

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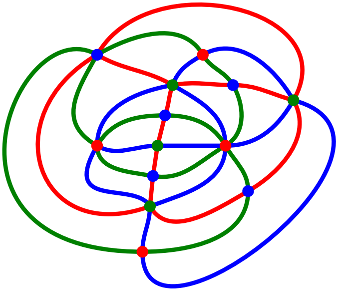
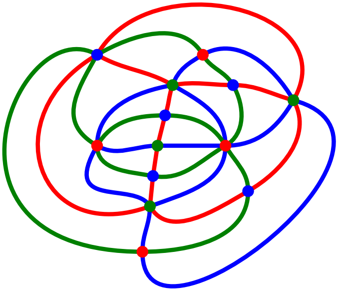


# Trinities from bipartite graphs

From a bipartite planar graph $G$...

- Add red vertices in complementary regions, and connect to blue and green vertices around the boundary of the region.
- This yields a *3-coloured* graph called a trinity. Each edge connects two vertices of distinct colours.
- We can colour each edge by the unique colour distinct from endpoints.



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Consider $G$,

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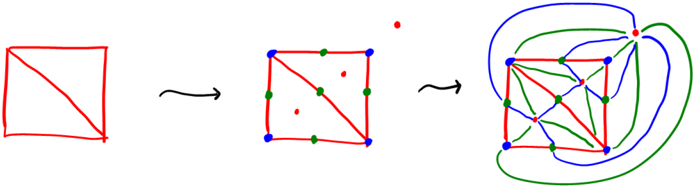

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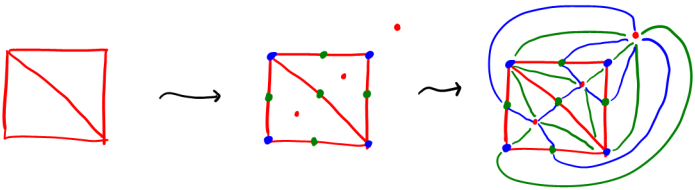
- Let the vertices of $G$ be blue.



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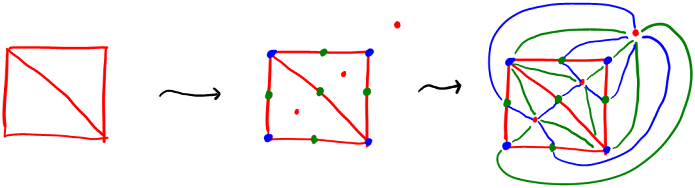

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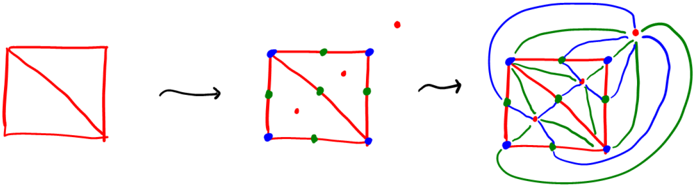
- Place a *green* vertex on each edge of $G$.
- Place a *red* vertex in each complementary region of $G$ and connect it to adjacent vertices. This yields a trinity.



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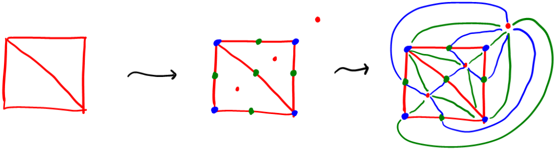

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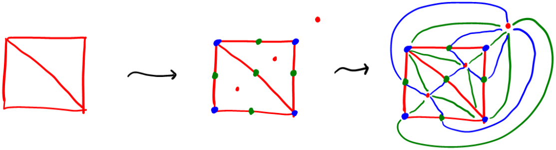

Notice correspondence between dimension and colour:

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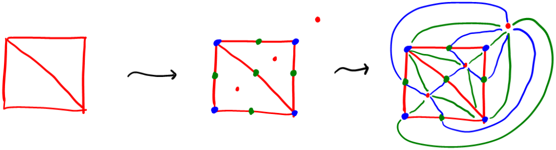

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| Dim | On $G$ | On $G'$ |
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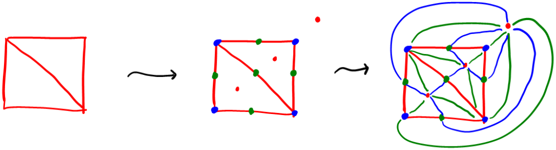

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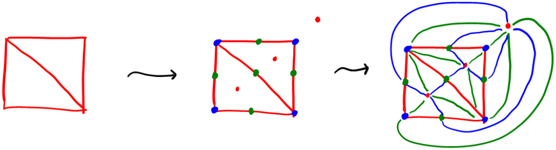

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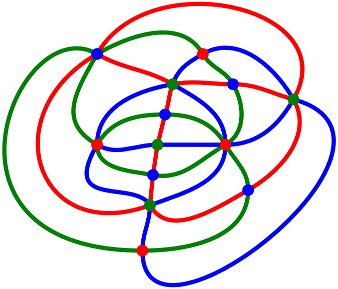
- We refer to blue, green red as *violet*, *emerald*, *red* instead.

# Bipartite graphs and trinities

A trinity naturally contains *three* bipartite planar graphs: take all edges of a single colour.



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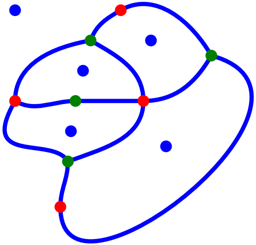

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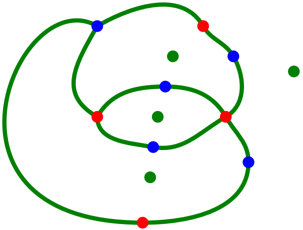

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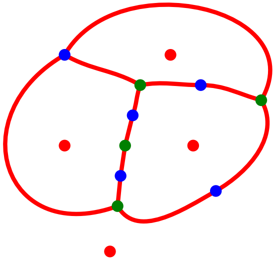

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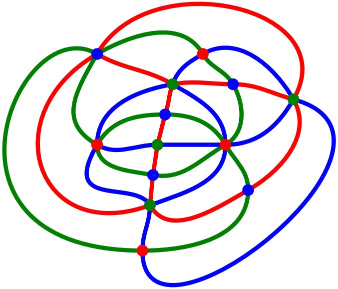
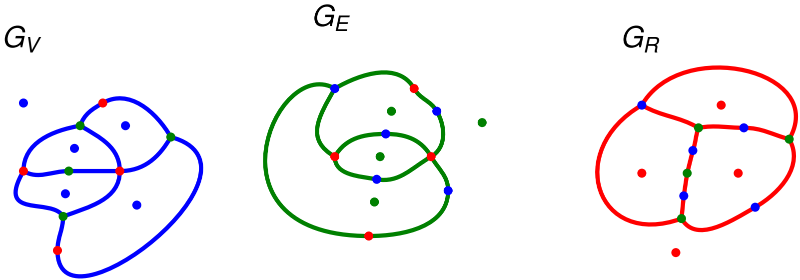


# Bipartite graphs and trinities

The *violet graph* $G_V$, *emerald graph* $G_E$, *red graph* $G_R$ are all bipartite planar graphs which yield (and are subsets of) the same trinity.



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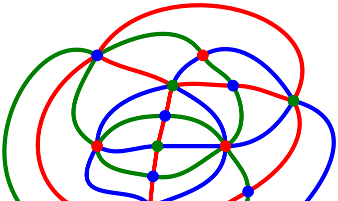

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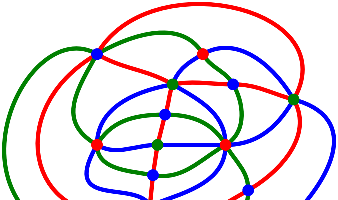

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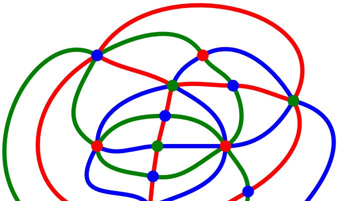

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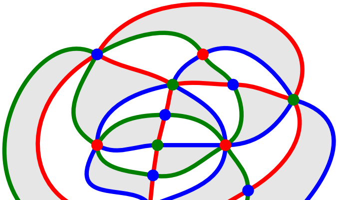

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  - *anticlockwise* — colour the triangle *white* — or
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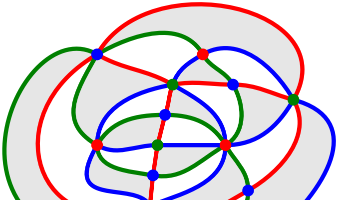
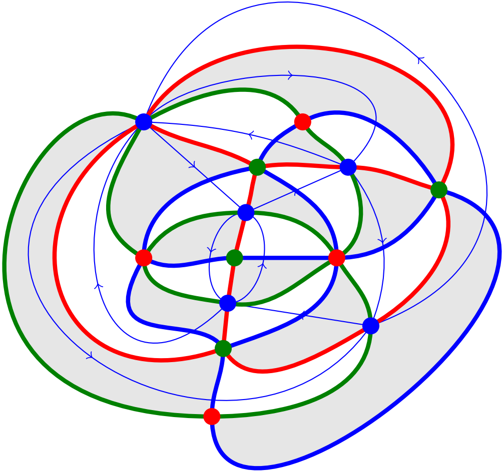
- Triangles sharing an edge must be *opposite* colours.
- Triangles are *2-coloured*. (Planar dual graph is bipartite.)



# Planar duals of trinities

Consider the planar dual $G_V^*$ of $G_V$ in a trinity.



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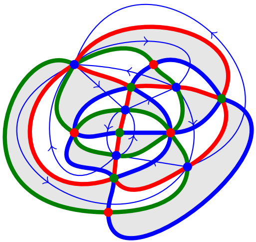


- $G_V^*$ has vertices $V$ and edges bijective with violet edges.
- Each edge of $G_V^*$ crosses precisely *two* triangles of the trinity and hence is naturally *oriented*, say black to white.
- Around each vertex of $G_V^*$, edges alternate in and out
- $G_V^*$ is a *balanced directed planar graph*.

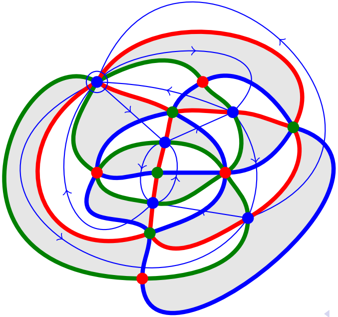
# Arborescences

Let $D$ be a directed graph $D$. Choose a *root* vertex $r$.

## Definition

*A (spanning) arborescence of $D$ is a spanning tree $T$ of $D$ all of whose edges point away from $r$.*

- I.e. for each vertex $v$ of $D$ there is a unique directed path in $T$ from $r$ to $v$.



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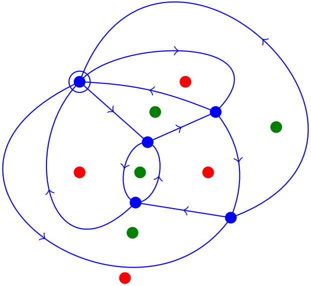

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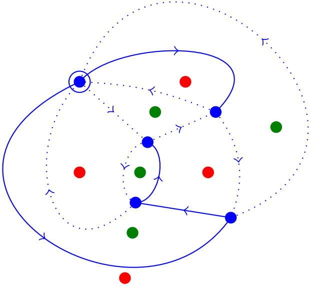


# Tutte's tree trinity theorem

**Theorem (Tutte, 1948)**

*Let $D$ is a balanced finite directed graph. Then the number of spanning arborescences of $D$ does not depend on the choice of root point.*

Hence we may define $\rho(D)$, the *arborescence number* of $D$, to be the number of spanning arborescences.

**Theorem (Tutte's tree trinity theorem, 1975)**

*Let $G_V^*, G_E^*, G_R^*$ be the planar duals of the coloured graphs of a trinity. Then*

$$\rho(G_V^*) = \rho(G_E^*) = \rho(G_R^*).$$

# Hypergraphs

A graph has *edges*.

- Each edge joins *two* vertices.

A *hypergraph* has *hyperedges*.

- Each *hyperedge* joins *many* vertices.

## Definition

*A hypergraph is a pair $\mathcal{H} = (V, E)$, where $V$ is a set of vertices and $E$ is a (multi-)set of hyperedges. Each hyperedge is a nonempty subset of $V$.*

- A hypergraph where each hyperedge contains 2 vertices is a graph (with multiple edges allowed).
- A hypergraph $\mathcal{H} = (V, E)$ naturally determines a bipartite graph $\operatorname{Bip} \mathcal{H}$ with vertex classes $V, E$. An edge connects $v \in V$ to $e \in E$ in $\operatorname{Bip} \mathcal{H}$ iff $v \in e$.

# Hypergraphs and trinities

A hypergraph $\mathcal{H} = (V, E)$ naturally has an *abstract dual* $\overline{\mathcal{H}} = (E, V)$.
A trinity naturally gives rise to *six* hypergraphs

$$\begin{array}{rlll} & \mathcal{H} = (V, E), & \mathcal{H}^* = (R, E), & \overline{\mathcal{H}*} = (E, R), \\ \overline{\mathcal{H}*}^* = & \overline{\overline{H}^*} = (V, R), & \overline{\mathcal{H}}^* = (R, V), & \overline{\mathcal{H}} = (E, V). \end{array}$$

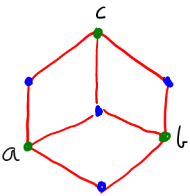
These are related by abstract and planar duality.

# Hypertrees in hypergraphs

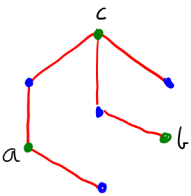
We now consider *spanning trees* in (the bipartite graph of) a hypergraph $\mathcal{H} = (V, E)$.

## Definition

*A hypertree in a hypergraph* $\mathcal{H}$ *is a function* $f \colon E \longrightarrow \mathbb{N}_0$ *such that there exists a spanning tree in* $\operatorname{Bip}\mathcal{H}$ *with degree* $f(e) + 1$ *at each* $e \in E$.

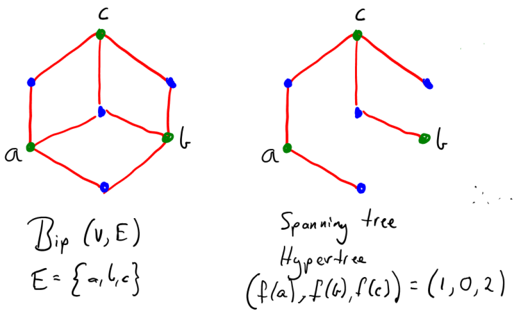



Bip $(V, E)$
$E = \{a, b, c\}$




Spanning tree
Hypertree
$(f(a), f(b), f(c)) = (1, 0, 2)$

# Hypertrees in hypergraphs




When $\mathcal{H}$ is a graph (i.e. $|e| = 2$ for all $e \in E$), a hypertree reduces to a tree.

- A tree is chosen by selecting edges with $f(e) = 1$.

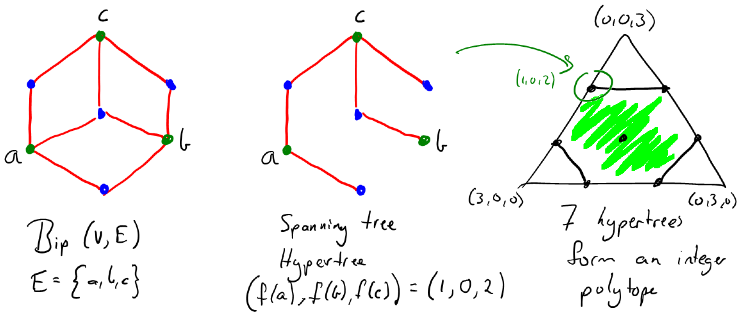
Since a hypertree is a function $f \colon E \longrightarrow \mathbb{N}_0 \subset \mathbb{Z}$, it can be regarded as an element of the $|E|$-dimensional integer lattice $\mathbb{Z}^E$.

# The hypertree polytope

Consider the set $Q_{\mathcal{H}} \subset \mathbb{Z}^E$ of hypertrees of a hypergraph $\mathcal{H} = (V, E)$.

**Theorem (Postnikov 2009, Kálmán 2013)**

*$Q_{\mathcal{H}}$ is the set of lattice points of a convex polytope in $\mathbb{R}^E$.*



# Hypertree polytopes of a trinity

Considered a planar bipartite $G$ with vertex classes $(V, E)$.

- Associated abstract dual hypergraphs, $\mathcal{H} = (V, E)$, $\overline{\mathcal{H}} = (E, V)$.
- $\operatorname{Bip}\mathcal{H} = \operatorname{Bip}\overline{\mathcal{H}} = G$

**Theorem (Kálmán 2013)**

*The number of hypertrees in $\mathcal{H}$ and $\overline{\mathcal{H}}$ are equal, and also equal to the arborescence number of $G^*$. I.e.*

$$|Q_{\mathcal{H}}| = |Q_{\overline{\mathcal{H}}}| = \rho(G^*).$$

**Corollary**

*In a trinity,*

$$\rho(G_V^*) = \rho(G_E^*) = \rho(G_R^*)$$
$$= |Q_{(V,E)}| = |Q_{(E,V)}| = |Q_{(E,R)}| = |Q_{(R,E)}| = |Q_{(R,V)}| = |Q_{(V,R)}|.$$

# Duality of polytopes

Postnikov related dual polytopes $Q_{(V,E)} \subset \mathbb{Z}^E$, $Q_{(E,V)} \subset \mathbb{Z}^V$.

$$Q_{(V,E)} = \left( \sum_{v \in V} \Delta_v \right) - \Delta_E = Q^+_{(V,E)} - \Delta_E,$$

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Essentially they are projections of $Q$, e.g.: $\pi_V : \mathbb{R}^V \oplus \mathbb{R}^E \to \mathbb{R}^V$

$$Q^+_{(V,E)} \cong |V| \left(Q \cap \pi_V^{-1}\left(\frac{1}{|V|}\sum_{v\in V} v\right)\right)$$

# Outline

# Plane bipartite graphs and links

Given a planar graph $G$, there is a natural way to construct a knot or link $L_G$: the median construction.

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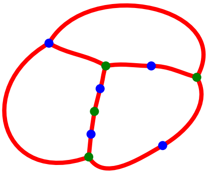
- Take a regular neighbourhood of $G$ in the plane (ribbon).
- Insert a negative half twist over each edge of $G$ to obtain a surface $F_G$. Then $L_G = \partial F_G$.



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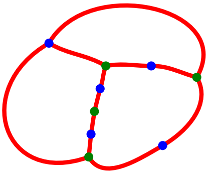
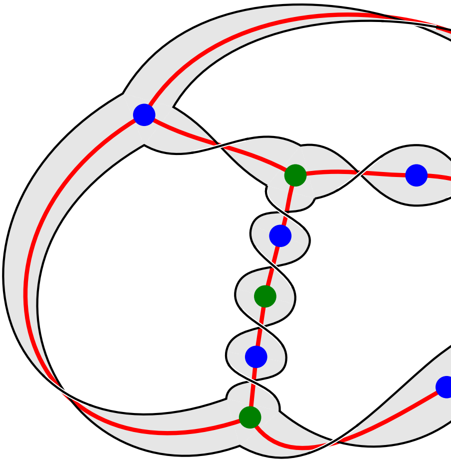


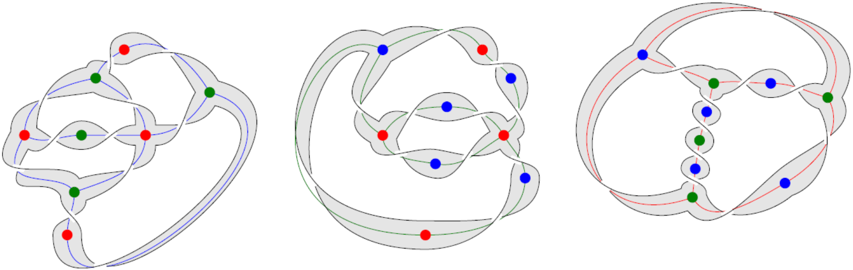


# Trinities of links

Note:

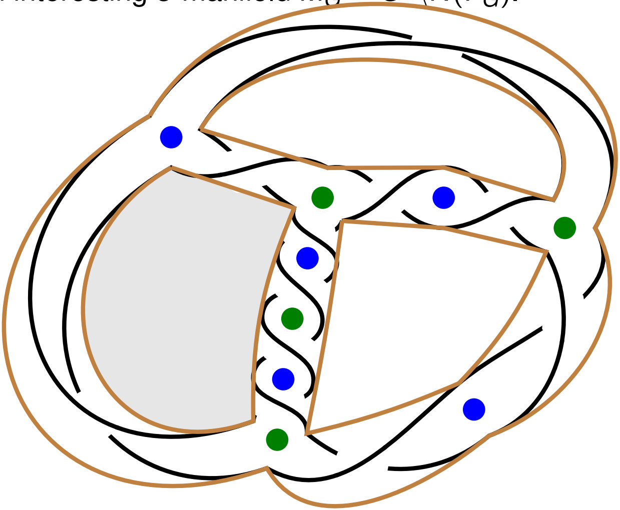
- The link $L_G$ is *alternating* (crossings are over, under, ...)
- The surface $F_G$ is oriented iff $G$ is bipartite.
- When $G$ is bipartite $L_G$ is naturally oriented.

A trinity yields 3 bipartite planar graphs and hence three alternating links $L_{G_V}, L_{G_E}, L_{G_R}$ with Seifert surfaces.



# Plane bipartite graphs and 3-manifolds

Given $L_G$ and $F_G$, remove a neighbourhood $N(F_G)$ of $F_G$ to obtain an interesting 3-manifold $M_G = S^3 \backslash N(F_G)$.



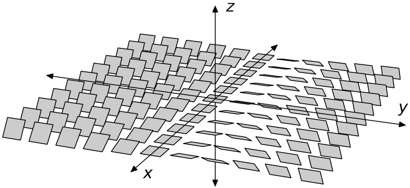
# Trinities of 3-manifolds

- Topologically, $N(F_G)$ and $M_G$ are handlebodies (solid pretzels).
- The boundary $\partial M_G$ naturally has a copy of $L_G$ on it.
- $L_G$ splits $\partial M_G$ into two surfaces, both equivalent to $F_G$.
- $(M_G, L_G)$ has the structure of a *sutured manifold*.
- From a trinity we obtain a triple of sutured manifolds

$$(M_{G_V}, L_{G_V}), \quad (M_{G_E}, L_{G_E}), \quad (M_{G_R}, L_{G_R}).$$

# Contact structures on 3-manifolds

A *contact structure* on a 3-manifold $M$ is a 2-plane field which is *non-integrable*.




A standard question in contact topology:

- Given a sutured 3-manifold $(M, \Gamma)$, how many (isotopy classes of tight) contact structures are there on $M$?

With Kálmán, we investigated this question for the sutured manifolds from bipartite planar graphs and trinities.

# Contact structures and trinities

For a bipartite planar graph $G$, let $cs(G)$ denote this number of contact structures on $(M_G, L_G)$.

**Theorem (Kálmán-M.)**

*Let $(V, E, R)$ form a trinity. Then $cs(G_R)$ is equal to the number of hypertrees in the hypergraph $(E, R)$.*

This number is also the arborescence number, and hence

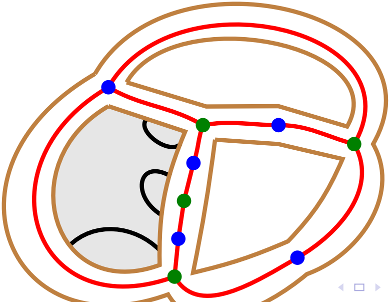
$$\begin{gathered} cs(G_V) = cs(G_E) = cs(G_R) \\ = \rho(G_V^*) = \rho(G_E^*) = \rho(G_R^*) \\ = |Q_{(V,E)}| = |Q_{(E,V)}| = |Q_{(E,R)}| = |Q_{(R,E)}| = |Q_{(R,V)}| = |Q_{(V,R)}|. \end{gathered}$$

Although contact structures are very differential-geometric, the proof boils down to a combinatorial game.

# Matchings on discs

It turns out that contact structures in $M_{G_R}$ can be described by certain curves in in the complementary regions of $G_R$.

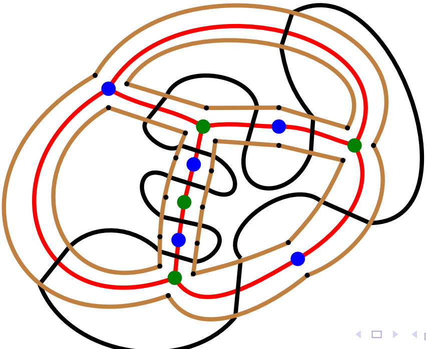
- Let $D_r$, $r \in R$, be the disc complementary regions of $G_R$.
- The boundary $\partial D_r$ consists of red edges.
- Consider non-crossing matchings on $D_r$ which join the midpoints of the red edges around $\partial D_r$.



# Configurations

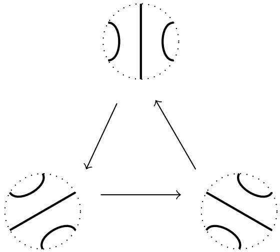
## Definition

*A configuration on $G_R$ consists of a non-crossing matching on each $D_r$, such that when the matchings are joined across the edges of $G_R$, a single closed curve is obtained.*



# State transitions

A *bypass surgery* is a local operation on a set of curves which takes 3 arcs and “rotates" them as shown.



## Definition

*A state transition is a bypass surgery in a disc $D_r$ which turns a configuration into another configuration.*

## Definition

*The configuration graph $\mathbb{G}$ is the graph with vertices given by configurations, and edges given by state transitions.*

# Contact structures via states and transitions

**Theorem (Honda, Kálmán–M.)**

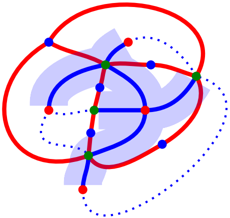
*(Isotopy classes of tight) contact structures on $M_G$ are in bijective correspondence with connected components of $\mathbb{G}$.*

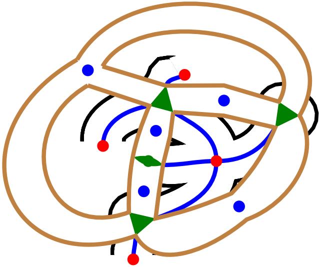
- So finding $cs(G)$ is reduced to a combinatorial problem: how many connected components does $\mathbb{G}$ have?
- I.e., which configurations can be reached from which others via transitions.

Proof uses *spanning trees* in $G_V$.

# Configurations from spanning trees

Idea: *spanning trees* in $G_V$ give configurations.





Such a configuration hugs the boundary of a neighbourhood of a tree: *tree-hugging configurations*.

# From configurations to spanning trees

**Proposition**

*Any configuration is connected via state transitions to a tree-hugging configuration.*

Thus it remains to find which tree-hugging configurations are related...

**Proposition**

*Two tree-hugging configurations are related iff they arise from spanning trees with the same degree at each red vertex, that is, if they arise from the same hypertree of $(E, R)$.*

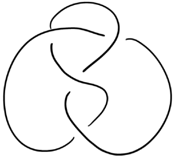
This gives a bijection $cs(G_R) \leftrightarrow Q_{(E,R)}$.

# Outline

# Knot theory and graphs

In *knot theory* one considers *knots*, which are knotted loops of string in 3-dimensional space.
(Precisely, embeddings $S^1 \hookrightarrow \mathbb{R}^3$.)
One often considers a *knot diagram*, which is a projection of a knot to a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ with no triple crossings.

- A knot diagram can be regarded as a graph which is
  - connected, planar
  - each vertex has degree 4, and
  - each vertex is decorated with *crossing data* (i.e. over/under).



Knot diagram

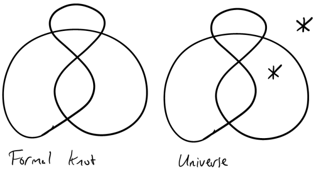
# Formal knot theory

Kauffman's *formal knot theory* studies knots by *forgetting crossing data* and considering the 4-valent graph only.

## Definition

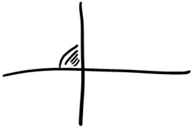
- *A* ***formal knot*** *is a connected planar graph, where each vertex has degree* 4*.*
- *A* ***universe*** *is a formal knot, where two adjacent complementary regions are labelled with stars.*

- I.e. a formal knot is a knot diagram without crossing data.



# Markers of a universe

A *marker* at a vertex $v$ of $\mathcal{U}$ is a choice of one of the four adjacent corners of regions at $v$.




One can show (via Euler's formula) that in any formal knot $K$,

$$\#\{\text{Vertices of } K\} + 2 = \#\{\text{Complementary regions of } K\}.$$

As two regions in a universe $\mathcal{U}$ contain stars,

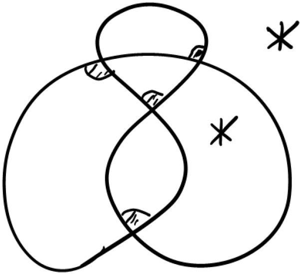
$$\#\{\text{Vertices of } \mathcal{U}\} = \#\{\text{Unstarred regions of } \mathcal{U}\}.$$

## Definition

*A state of a universe $\mathcal{U}$ is a choice of marker at each vertex of $\mathcal{U}$, so that each unstarred region contains a marker.*

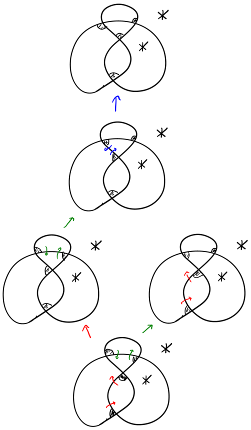
A state provides a bijection

$$\{\text{Vertices of } \mathcal{U}\} \to \{\text{Unstarred regions of } \mathcal{U}\}.$$



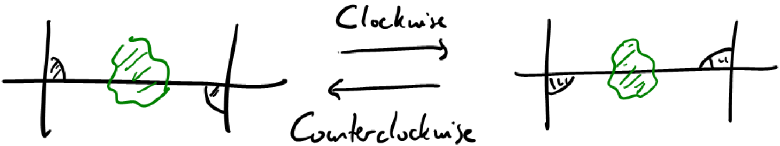
# States of a universe

For example, consider the following universe and its states.



# Transpositions between states

A *transposition* swaps two state markers as shown.

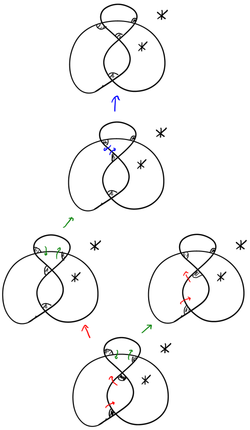



A transposition is naturally *clockwise* or *counterclockwise*.
We can form a directed graph (*the clock graph*) with

- Vertices given by states
- Directed edges given by clockwise transpositions

# The clock lattice of a universe

A clock graph:



# Kauffman's clock theorem

The clock graph has more structure than a mere directed graph.

**Theorem (Kauffman's clock theorem)**

*The clock graph is naturally a lattice.*

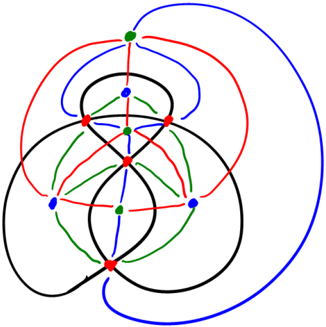
- *Directed edges provide a partial order on states.*
- *Any two states have a unique least upper bound and greatest lower bound, with respect to this order.*

In particular, there is a unique minimal *clocked* state and a unique maximal *counter-clocked* state.

# Trinities and universes

A universe $\mathcal{U}$ with formal knot $K$ naturally yields a trinity.

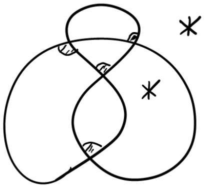
- Let the vertices of $K$ be the red vertices $R$.
- Complementary regions can be 2-coloured $(V, E)$ by the *checkerboard* colouring; place violet and emerald vertices.
- Connect violet and emerald edges across edges of $K$ by red edges $G_R$; so $G_R^* = K$.

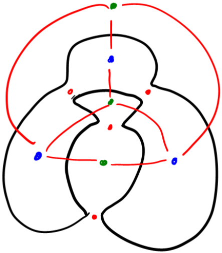


# Trinities and universes

Observation: there is a bijection

$$\{\text{States of } \mathcal{U}\} \leftrightarrow \{\text{Configurations of } G_R\}$$





**Theorem (Kálmán–M.)**

*The number of states of $\mathcal{U}$ is also equal to the magic number of the trinity.*

# Thanks for listening!

Happy $\pi$ day!

References: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