Mahler's Unfinished Symphony: Etudes in Knots, Algebra and Geometry

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Part I

Prelude

Chapter 1

Introduction

This project investigates representation varieties of some knot groups, some of their geometric interpretations, and strolls in some pleasant fields along the way. The central investigations are of A-polynomials and parabolic representations of knots, and the geometric interpretation given to them in [3]; especially for twist knots, using the geometric insight provided by [43].

These topics are quite involved and require some background, which will take significant time. In this sense this project spends three chapters (2,3,4) establishing basic relevant propositions in knot theory, hyperbolic geometry and algebraic geometry respectively. This is the "prelude".

Essential to the geometric interpretation of A-polynomials in [3] is the notion of Mahler measure, which comprises the second part of this project. Mahler measure is related to conspiratorially many branches of mathematics, and deserves investigation in its own right. We digress in chapter 5 to investigate some number-theoretic applications as well — this chapter is largely a detour, but also includes some discussion of the geometric significance of the one-variable Mahler measure. In chapter 6, we establish a more useful form of the measure for subsequent purposes.

The remainder of the project investigates representations of knot groups. Chapter 7 introduces the A-polynomial and discusses some of its astounding properties. Chapter 8 is an attack on twist knots, whose simple fundamental groups admit quite explicit calculations. As far as I am aware, a number of results in this chapter are new. We then conclude with a consideration of a few aspects of representations of two-bridge knots.

In one sense this project is a series of etudes, dealing with discrete topics one after the other. But (except for chapter 5) all the streams eventually flow together in considering the Mahler measure of A-polynomials and interpretations in hyperbolic geometry. But the symphony of knots, algebra and geometry is not complete: there is much more to be understood in this area, and the data in the appendix may provide evidence for some further conjectures.

Chapter 2

Basic Knot Theory

We think of a knot as a closed loop of string lying in 3-dimensional space. The *ambient space* in which the loop sits will always be considered as $S^3 = \mathbb{R} \cup \{\infty\}$. Therefore a *knot* is defined as an embedding of S^1 in S^3. A *link* is an embedding of several copies of S^1 in S^3 without intersection.

Two knots or links L, L' are *equivalent* if there is an orientation-preserving homeomorphism $h : S^3 \to S^3$ with $h(L) = L'$. Two knots or links have the same *knot (or link) type* if there is a (not necessarily orientation-preserving) homeomorphism $h : S^3 \longrightarrow S^3$ with $h(L) = L'$.

We represent knots or links by diagrams (regular projections) with crossings (for a precise definition, see [45, p. 7]). It's well known that two diagrams represent equivalent knots if and only if one can be transformed into the other by a sequence of isotopies of the diagram and the three Reidemeister moves. In a diagram of a knot or link, each crossing involves an overpass and an underpass, which are drawn in an obvious way. An *arc* is a section of the knot which does not run through an underpass. A *maximal arc* is an arc which cannot be extended any further, i.e. it runs between two underpasses. It can be seen that the number of maximal arcs is equal to the number of crossings. A *maximal overpass* is a maximal arc which passes over at least one other arc.

If we remove the knot K from S^3 we obtain a 3-manifold $S^3 - K$. If we remove, along with the knot K, a tubular open neighbourhood $\nu(K)$ also, then $S^3 - \nu(K)$ is a compact 3-manifold with boundary a torus T. Provided K is non-trivial, T is incompressible.

There can exist tori other than T embedded in $S^3 - \nu(K)$. Uninteresting ones are compressible. If there is an incompressible torus T' which is not parallel to the boundary torus, then it partly 'swallows' and partly 'follows' the knot, and we have a *satellite knot*.

We are interested in both algebraic and geometric aspects of knots and their complements. The algebraic aspects derive from the fundamental group of $S^3 - K$ (and the associated peripheral subgroup). Many of the techniques used subsequently are not specific to knot complements, and could be generalised to larger classes of manifolds.

2.1 Fundamental Group, Peripheral Subgroup

The fundamental group of a knot complement $\pi_1(S^3 - K)$, often simply denoted $\pi_1(K)$, is perhaps its most important algebraic invariant. The boundary torus T of $S^3 - \nu(K)$ has its own fundamental group $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, which injects naturally into $\pi_1(K)$ if K is non-trivial. This $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is called the *peripheral subgroup* of $\pi_1(K)$. A *standard meridian* μ is a loop on T which is contractible in the solid torus $\overline{\nu(K)}$ but not in T. A *standard longitude* λ is a loop on T which intersects μ at one point and is nullhomologous in $S^3 - \nu(K)$.

As a knot invariant, the fundamental group is difficult to work with, encountering all the usual problems of combinatorial group theory; but it is almost a complete invariant. Two prime knots are equivalent if and only if they have isomorphic fundamental groups ([45, thm. 6.1.12]; [10, thm. 15.23]; [88]; [36]). The fundamental group and peripheral subgroup together form a complete knot invariant: [86], [38].

The fundamental group can be calculated from a diagram D of K using the *Wirtinger presentation*, as follows; the same method also works for links. First, place an orientation on the knot arbitrarily. Imagine the knot lying in a plane, except at intersections, where the overpass or underpass lifts off the plane. We consider a basepoint above this plane. Let the maximal arcs in D be $a_1, \ldots, a_n$ and the crossings be $c_1, \ldots, c_n$. (These two n's are equal.) For each maximal arc a_i in D, we let $g_i \in \pi_1(K)$ be represented by a loop running down from the basepoint to the knot, passing under a_i so as to create a positive crossing, then running back to the basepoint. For a positive (resp. negative) crossing c_i in D as illustrated in figure 2.1, we add the relation $r_i = g_j g_i g_k^{-1} g_i^{-1}$ (resp. $g_j g_i^{-1} g_k^{-1} g_i$). Here and throughout, group elements are to be multiplied from left to right. The word r_i represents a contractible loop under a crossing, so must be the identity in $\pi_1(K)$. It can be proven without too much trouble using van Kampen's theorem that $\pi_1(K) =< g_i \mid r_i >$. For details see [45, p. 78–80] or [10, thm. 3.4].

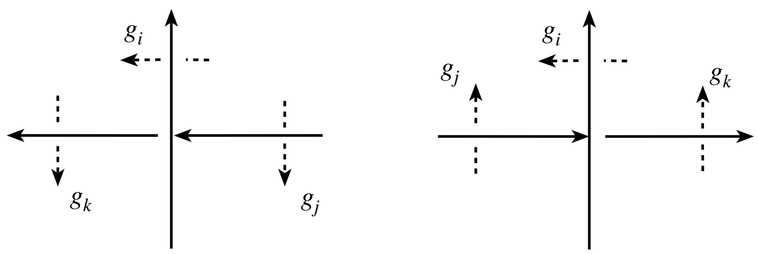


Figure 2.1: relators for positively and negatively oriented crossings

Any one relator is a consequence of all the others, and so can be omitted. This is because a loop under a crossing can be expressed in terms of loops under the other crossings, following the knot around.

As an example we compute the fundamental group of the Whitehead link complement, which we require later. With arcs labelled as shown in figure 2.2, the Wirtinger presentation is

$$< a,b,c,d,e,f \quad | \quad acb^{-1}c^{-1},\ bf^{-1}a^{-1}f,\ cad^{-1}a^{-1},\ ea^{-1}f^{-1}a,\ df^{-1}e^{-1}f,\ fd^{-1}c^{-1}d > .$$

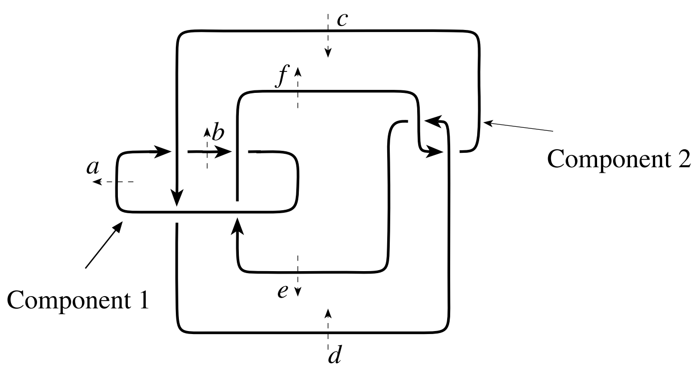


Figure 2.2: Whitehead link

We eliminate generators. For example $f = aea^{-1}$ so the group has presentation

$$< a,b,c,d,e \quad | \quad acb^{-1}c^{-1},\ bae^{-1}a^{-1}a^{-1}aea^{-1},\ cad^{-1}a^{-1},\ da^{-1}e^{-1}ae^{-1}aea^{-1},\ aea^{-1}d^{-1}c^{-1}d > .$$

We continue, eliminating $d = a^{-1}ca$, $b = c^{-1}ac$ and $e = a^{-2}c^{-1}aca^{-1}ca^2$ (the foolhardy reader is encouraged to verify) until we obtain a presentation involving only two generators and two identical relators (we expect one to be redundant). Thus the group is

$$< a,c \mid a^{-1}c^{-1}aca^{-1}cac^{-1}aca^{-1}c^{-1}ac^{-1}a^{-1}c > .$$

We can simplify by taking $\lambda^{-1} = a^{-1}c^{-1}aca^{-1}cac^{-1}$, which is a longitude of component 1. (That it commutes with a, which is a meridian for component 1, is not surprising.) Then the presentation becomes

$$< a,c,\lambda \mid a^{-1}\lambda a\lambda^{-1} = 1,\ \lambda = ca^{-1}c^{-1}ac^{-1}a^{-1}ca > .$$

We now replace a with μ_1, c with μ_2, and λ with λ_1 to obtain a standard form for the fundamental group of Whitehead link complement (see e.g. [43]):

$$< \mu_1,\mu_2,\lambda_1 \mid [\mu_1,\lambda_1] = 1,\ \lambda_1 = \mu_2\mu_1^{-1}\mu_2^{-1}\mu_1\mu_2^{-1}\mu_1^{-1}\mu_2\mu_1 > .$$

One easy consequence of the Wirtinger presentation is the abelianization of a knot group.

Proposition 2.1.1 *Let K be a knot. The abelianization of $\pi_1(K)$ (or $H_1(S^3 - K)$ with coefficients in $\mathbb{Z}$) is $\mathbb{Z}$.*

Proof. Take a Wirtinger presentation for $\pi_1(K)$. If we require all generators to commute, then each relator collapses to $g_i = g_j$, where g_i, g_j correspond to adjacent maximal arcs. Thus the abelianization is

$$< g_1, \ldots, g_n \mid g_1 = g_2 = \cdots = g_n >= \mathbb{Z}.$$

□

There is therefore a homomorphism $w : \pi_1(K) \longrightarrow \mathbb{Z}$ given by $w(g_i) = 1$ for $i = 1, \ldots, n$. This can be thought of as giving the "winding number" of a loop about K.

Finally, we can perform *Dehn surgery* on a knot or link complement. Having removed $\nu(K)$ to obtain a 3-manifold with torus T boundary. We then glue a disc onto our manifold with its boundary glued to the loop $p\mu + q\lambda$ on T. We then glue a ball into the hole remaining in T. This process is called (p, q) *surgery* on K. It can be performed similarly on components of links. Algebraically it adds a relator $\mu^p\lambda^q = 1$ to $\pi_1(K)$.

2.2 Bridge number

The minimal number of maximal overpasses occurring in any projection of K is called the *bridge number* of K, $b(K)$. K is called a $b(K)$-*bridge knot.*

Proposition 2.2.1 ([45] p.82) *$\pi_1(K)$ has a presentation with $b(K)$ generators, all of which correspond to maximal overpasses.*

Proof. Taking a minimal diagram and Wirtinger presentation, we only need show that each generator which is not a maximal overpass can be expressed in terms of the generators for maximal overpasses. Take such a generator g, which corresponds to an arc running between two overpasses. We follow the knot, under the overpasses $y_1, y_2, \ldots, y_k$ (possibly $k = 0$) as shown in figure 2.3 until arriving at a maximal overpass h. Then

$$g = y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_k^{\epsilon_k} h y_k^{-\epsilon_k} \cdots y_1^{-\epsilon_1}$$

where ϵ_i is the sign of the crossing of y_i. □

We can see that having expressed obtained such a representation, relators arise from consecutive sequences of underpasses between maximal overpasses.

A very important and well-understood class of knots is the two-bridge knots, which we will define and classify.

Given a diagram for K involving the minimal number $b(K)$ of maximal overpasses, we consider $2b(K)$ points close to the end of the maximal overpasses and cut a plane intersecting the knot only in those $2b(K)$ points — the maximal overpasses coming out of the plane, and underpasses being under the plane. The

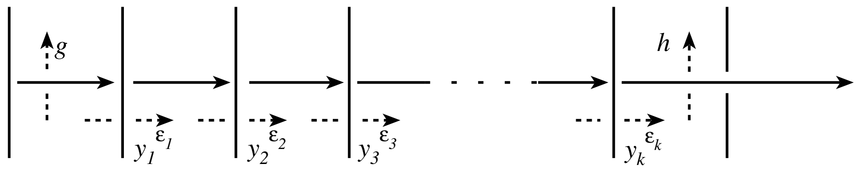


Figure 2.3: Underpasses in terms of overpasses

$b(K)$ maximal overpasses are themselves unknotted, since they do not intersect each other in our diagram. The $b(K)$ other strands of the knot under the plane are similarly unknotted, since they do not intersect each other either. Thus we can disentangle the maximal overpasses (which most likely involves making a mess of the other half of the knot). The plane can be closed up to form a sphere.

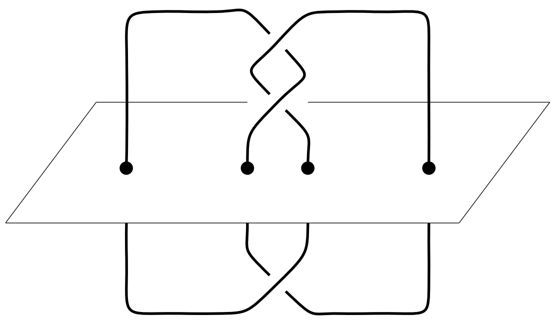


Figure 2.4: A 2-bridge knot

Conversely, if there exists such a plane or sphere, then K has a diagram with $\leq n$ maximal overpasses.

Thus a non-trivial knot K is a 2-bridge knot if and only if there exists a 2-sphere intersecting the knot transversely in 4 points, dividing the knot into 2 unknotted arcs on either side. (The only 1-bridge knot is the unknot.)

'Knots' which occur within the confines of a 2-sphere are called *tangles* and are the subject of a very beautiful theory.

2.3 All Tangled Up

Definition 2.3.1 *A* tangle *is the intersection of a knot or link in S^3 with a 3-ball B, and intersects the 2-sphere ∂B transversely in 4 points, the* endpoints.

Whereas in isotopies of the knot we may move the knot freely in 3-space (always without self-intersection), in isotopies of a tangle we keep the endpoints fixed — we call this *isotopy relative to endpoints*. Two tangles are *isotopic* *relative to endpoints* (written $P \sim Q$) if they are related by an isotopy relative to endpoints. Every 2-bridge knot corresponds to a tangle, but not every tangle corresponds to a 2-bridge knot.

Vaguely, since a 2-bridge knot has two unknotted arcs on both sides of ∂B, the tangle of a 2-bridge knot must be able to be unknotted by winding the endpoints around each other. This corresponds to the class of tangles known as 'rational' tangles.

To make this precise we need to define several notions. A *horizontal (vertical) integer tangle* h_a (v_a) is a twist of two horizontal (vertical) strands $|a|$ times in a signed direction, where $a > 0$ (resp. $a < 0$) if the overcrossings have positive (resp. negative) slope, as shown in figure 2.5. The *horizontal (vertical) sum* $P \overset{h}{+} Q$ ($P \overset{v}{+} Q$) of two tangles joins them horizontally (vertically). A *rational tangle* is essentially one that is obtained by repeating these processes. It is defined inductively as follows.

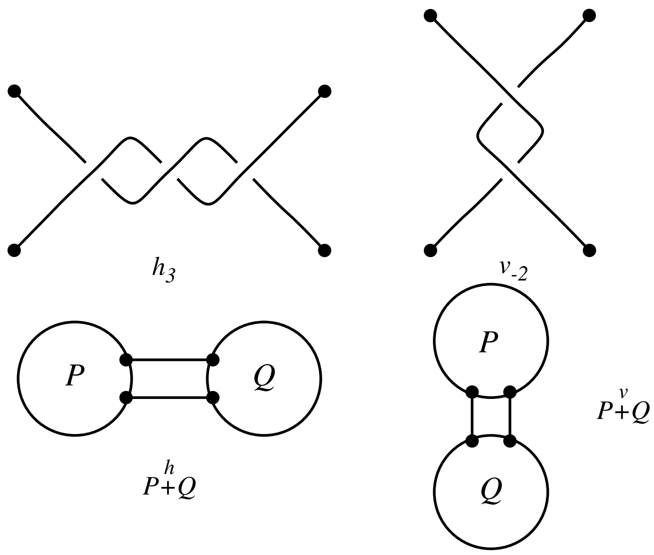


Figure 2.5: Horizontal and vertical intenger tangles and tangle addition

Definition 2.3.2 *A horizontal or vertical integer tangle is a rational tangle. If P is a rational tangle and $a \in \mathbb{Z}$ then $P \overset{h}{+} h_a$ and $h_a \overset{h}{+} P$ are rational tangles. If P is a rational tangle and $a \in \mathbb{Z}$ then $P \overset{v}{+} v_a$ and $v_a \overset{v}{+} P$ are rational tangles.*

A *basic horizontal tangle* is a rational tangle where constructed from a pair of horizontal strands and repeatedly alternately adding horizontal integer tangles to the right and vertical integer tangles to the bottom. That, is, a tangle of the form

$$\left(\cdots\left(\left(\left(h_{a_1} \overset{v}{+} v_{a_2}\right) \overset{h}{+} h_{a_3}\right) \overset{v}{+} v_{a_4}\right) \cdots\right) \overset{h}{+} h_{a_n}$$

where the tangles v_{a_i} or h_{a_i} are added in order of i. We denote this tangle $H(a_1, a_2, \ldots, a_n)$.

A *basic vertical tangle* is a rational tangle constructed from a pair of vertical strands and repeatedly alternately adding vertical integer tangles on the bottom and horizontal integer tangles on the left:

$$\left(\cdots\left(h_{a_4} \overset{h}{+} \left(\left(h_{a_2} \overset{h}{+} v_{a_1}\right) \overset{v}{+} v_{a_3}\right)\right) \cdots\right) \overset{v}{+} v_{a_n}$$

We denote it $V(a_1, a_2, \ldots, a_n)$.

Together basic vertical and horizontal tangles are called *basic tangles.*

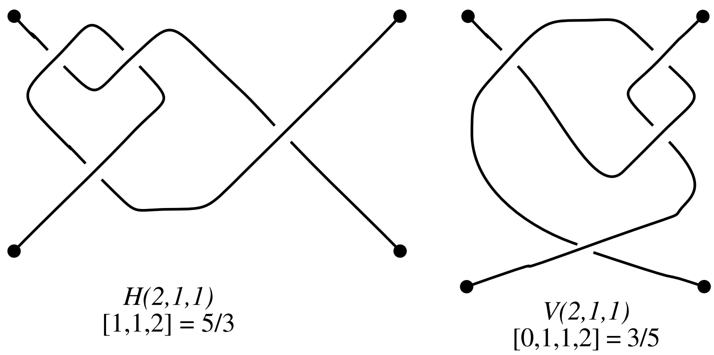


Figure 2.6: two basic tangles

To the basic horizontal tangle $H(a_1, \ldots, a_n)$ we can associate the continued fraction $\mathrm{Fr}\left(H(a_1, \ldots, a_n)\right)$, given by

$$[a_n, a_{n-1}, \ldots, a_1] = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{a_1}}}.$$

Similarly, to a basic vertical tangle $V(a_1, \ldots, a_n)$ we associate the continued fraction $\mathrm{Fr}\left(V(a_1, \ldots, a_n)\right)$:

$$[0, a_n, a_{n-1}, \ldots, a_1] = \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{a_1}}}}.$$

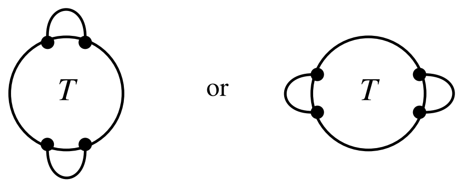
With this arsenal of definitions we come to the beautiful theorems classifying rational tangles, and hence, two-bridge knots.

Theorem 2.3.3 *Every rational tangle is isotopic, relative to endpoints, to a basic tangle.*

Theorem 2.3.4 (Conway's Theorem) *Let P and Q be two basic tangles. $Fr(P) = Fr(Q)$ if and only if $P \sim Q$.*

Conway's theorem thus gives a complete classification of rational tangles, placing them in 1-1 correspondence with $\mathbb{Q} \cup \infty$. There is an elegant elementary proof in [35].

A given two-bridge knot or link K has a representation of the form



where T is a rational tangle, and hence has a continued fraction. Clearly more than one rational tangle can be associated to K, if the tangle is rotated or (if we only consider knot type) flipped.

But given a rational tangle, there is a unique corresponding 2-bridge knot or link obtained by joining the endpoints in pairs. Either the top points and bottom points, or the left and right, are joined. The pairs are joined so that the last twist we made doesn't unravel immediately.

Note there is an alternative way to draw a 2-bridge knot or link derived from its associated tangle. Given a rational tangle with continued fraction $[a_1, a_2, \ldots, a_n]$, we may represent it in a diagram involving 4 plaited strands. For instance, the 2-bridge knot associated to $H(3, 1, 2)$ may be represented as shown in figure 2.6

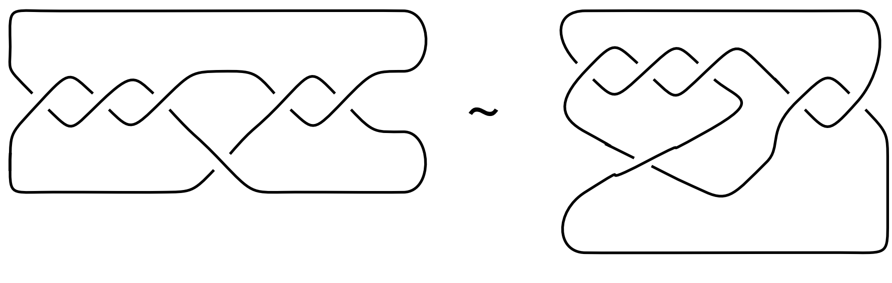


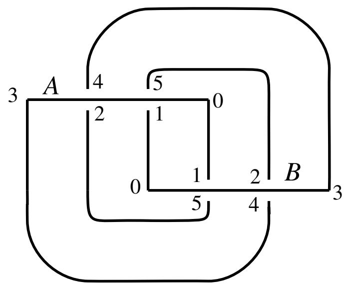
Figure 2.7: $H(3, 1, 2)$ in 4-plait form, and as a basic horizontal tangle

Note that the direction of crossings between the lower two plaits is the opposite to that in the basic tangle notation.

2.4 Classification of Two-Bridge Knots

Each two-bridge knot or link also has a description in *Schubert normal form.* We take a projection involving two maximal overpasses and by isotopy we assume they are parallel straight line segments A, B. Every other arc runs between two underpasses, and by isotopy if necessary we may assume that every other arc runs between A and B. For the under-arcs to be so arranged without intersecting, one finds that there must be a certain regularity, which can be described as follows.

Let the number of arcs passing under A (or B) be $\alpha - 1$. We consider these crossings, and the endpoints of A and B, as vertices. The endpoints of A and B are labelled $0, \alpha$, and the underpasses are labelled 1 to $\alpha - 1$ on one side and $\alpha + 1$ to $2\alpha - 1$ on the other, as shown. Starting from 0 on A and heading away from A, the arc continues to meet B at some number β (take the number on the side where the arc hits B). Following the knot, the next arc continues to meet A at 2β, taken modulo 2α. Then it meets B and 3β (mod 2α), and so on.



S(3,1)

It can now be seen that 2α and β must be coprime if a knot is to be realised. So (α, β) are coprime and β is odd. The process continues until the α'th intersection, at which point we reach some endpoint of A or B. If α is even then we end on A and, repeating the process on B in the same way we obtain a link. If α is odd then we arrive at an endpoint of B and continuing the process gives a 2-bridge knot.

Thus each 2-bridge knot has an associated pair of odd coprime integers (α, β) where α is positive and odd, and β is taken modulo 2α. Conventionally we take $-\alpha < \beta < \alpha$. We denote the knot by $S(\alpha, \beta)$. This is called the *Schubert normal form.*

Theorem 2.4.1 ([45] thm. 2.1.3) *$S(\alpha_1, \beta_1)$ and $S(\alpha_2, \beta_2)$ are equivalent if and only if $\alpha_1 = \alpha_2 = \alpha$ and either $\beta_1 \equiv \beta_2$ (mod α) or $\beta_1^{-1} \equiv \beta_2$ (mod α).*

If we only consider knots up to knot type, then also $S(\alpha, \beta) = S(\alpha, -\beta)$. We can then assume $\beta > 0$.

There is an extraordinary relationship between this (α, β) and the tangle form discussed above.

Theorem 2.4.2 ([45] thm. 2.1.10; [10] thm. 12.1.3) *The 2-bridge knot or link $K = S(\alpha, \beta)$ has an associated rational tangle T with fraction $[a_n, \dots, a_1] = \frac{\alpha}{\beta}$.*

By rotating T, $S(\alpha, \beta)$ is also associated to the tangle $-\frac{\beta}{\alpha}$; from the result above, to $\frac{\alpha}{\beta'}$ where $\beta' \equiv \beta^{\pm 1} \bmod \alpha$; hence also to $-\frac{\beta'}{\alpha}$. If we are only interested up to knot type, $S(\alpha, \beta)$ is also associated to the mirror images (negatives) of these tangles, $-\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, -\frac{\alpha}{\beta'}$, and $\frac{\beta'}{\alpha}$.

There is a simple number-theoretic relationship between continued fractions which are reverses of each other, which shows that the associated tangles give equivalent 2-bridge knots. Its proof is not too hard.

Proposition 2.4.3 *The fractions in simplest form with $\alpha, \alpha' > 0$*

$$\frac{\alpha}{\beta} = [a_1, a_2, \dots, a_n], \qquad \frac{\alpha'}{\beta'} = [a_n, a_{n-1}, \dots, a_1]$$

satisfy $\alpha = \alpha'$ and $\beta\beta' \equiv (-1)^{n-1}$ (mod α).

Finally, we compute the fundamental group of $S(\alpha, \beta)$.

Theorem 2.4.4

$$\pi_1(S(\alpha, \beta)) = < g, h \mid gw = wh >$$

where

$$w = h^{\epsilon_1} g^{\epsilon_2} \cdots g^{\epsilon_{\alpha-1}}$$

and

$$\epsilon_j = (-1)^{\lfloor \frac{j\beta}{\alpha} \rfloor}, \qquad \text{for } j = 1, \dots, \alpha - 1.$$

($\lfloor x \rfloor$ is the greatest integer not exceeding x.)

Proof. Consider the Schubert normal form for $S(\alpha, \beta)$. Let g, h represent loops under the two maximal overpasses. From previous discussion, the Wirtinger presentation for $S(\alpha, \beta)$ can be simplified to one involving only g, h as generators, and with relators corresponding to the two series of underpasses, departing from 0 on one overpass and passing alternately under the overpasses at the points labelled $\beta, 2\beta, \dots, (\alpha - 1)\beta, \alpha$. We have a positive intersection at $j\beta$ when we approach the arc on the side labelled with the numbers $1, 2, \dots, \alpha - 1$ — that is, when $j\beta$ is congruent to one of these values, mod 2α. But this is precisely when $\lfloor \frac{j\beta}{\alpha} \rfloor$ is even. So the two relators obtained are

$$gh^{\epsilon_1} g^{\epsilon_2} \cdots g^{\epsilon_{\alpha-1}} = h^{\epsilon_1} g^{\epsilon_2} \cdots g^{\epsilon_{\alpha-1}} h, \text{ and } hg^{\epsilon_1} h^{\epsilon_2} \cdots h^{\epsilon_{\alpha-1}} = g^{\epsilon_1} h^{\epsilon_2} \cdots h^{\epsilon_{\alpha-1}} g.$$

We now use the fact that each relator is a consequence of all the others — this is still true after the elimination of other generators. So we only need take one of these relators; we take the first, which is $gw = wh$. □

Using the same approach, we find that the a standard longitude λ (with g the standard meridian) is given by $g^{-2\sigma} w\tilde{w}$, where $\tilde{w} = g^{\epsilon_{\alpha-1}} \cdots h^{\epsilon_1}$ is w written backwards, and σ is sum of the exponents in w. The leading $g^{-2\sigma}$ renders λ nullhomologous.

2.5 Twist Knots

The twist knots are a subset of the two-bridge knots, but calculations of many invariants of these knots are easy to calculate, and will be used here. The twist knots K_m are defined for all $m \in \mathbb{Z}$ as the knots obtained from the basic horizontal tangle $H(2, m)$:

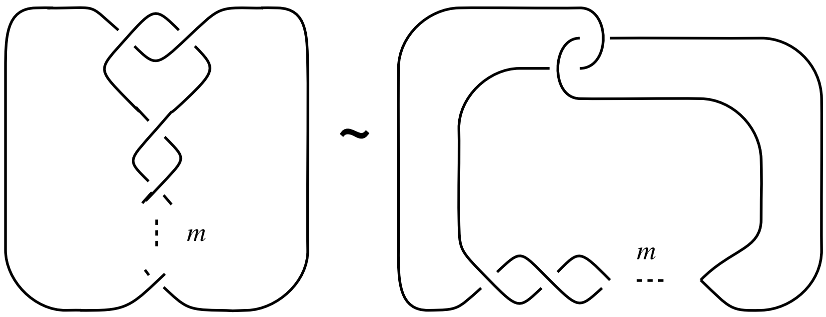


Figure 2.8: twist knot K_m

where the crossings are signed. By a flip in the projection plane and a twist of the "clasp", we can see that K_m and K_{-1-m} are mirror images of one another. Since these have homeomorphic complements, following [43], we will only consider the case where m is even, $m = 2n$.

The continued fraction associated to the tangle for K_{2n} is $[2n, 2] = \frac{4n+1}{2}$. By the discussion in section 2.4, K_{2n} is associated to the fraction $\frac{4n+1}{2} \sim \frac{4n+1}{2n+1} = \frac{4(-n)-1}{2(-n)-1}$. So for $n \geq 0$ we have so $K_{2n} = S(4n+1, 2n+1)$ and $K_{-2n} = S(4n-1, 2n-1)$.

We now calculate $\pi_1(K_{2n})$ using the Schubert normal form and the notation of section 2.4. First we calculate when $n \geq 0$. Then we obtain for $0 \leq 2i, 2i-1 \leq 4n$ (i.e. $0 \leq i \leq 2n$),

$$\begin{aligned}\left\lfloor \frac{2i\beta}{\alpha} \right\rfloor &= \left\lfloor \frac{2i(2n+1)}{4n+1} \right\rfloor = \left\lfloor i + \frac{i}{4n+1} \right\rfloor = i, \\ \left\lfloor \frac{(2i-1)\beta}{\alpha} \right\rfloor &= \left\lfloor \frac{(2i-1)(2n+1)}{4n+1} \right\rfloor = \left\lfloor i - 1 + \frac{2n+i}{4n+1} \right\rfloor = i - 1\end{aligned}$$

so that $\epsilon_j = (-1)^{\lfloor \frac{j\beta}{\alpha} \rfloor}$ alternates according to the pattern

$$+1, -1, -1, +1, +1, -1, \ldots, -1, +1.$$

A similar calculation for $n < 0$ gives ϵ_j following the pattern

$$+1, +1, -1, -1, +1, +1, \ldots, +1, +1.$$

Hence

$$\begin{aligned}\pi_1(K_{2n}) &= \left\langle g,h|g\left(hg^{-1}h^{-1}g\right)^n = \left(hg^{-1}h^{-1}g\right)^n h\right\rangle, \\ \pi_1(K_{-2n}) &= \left\langle g,h|g\left(hgh^{-1}g^{-1}\right)^{n-1} hg = \left(hgh^{-1}g^{-1}\right)^{n-1} hgh\right\rangle\end{aligned}$$

but the presentation for K_{-2n} can be rewritten so that in fact, for all $n \in \mathbb{Z}$

$$\pi_1(K_{2n}) = \left\langle g,h|g\left(hg^{-1}h^{-1}g\right)^n = \left(hg^{-1}h^{-1}g\right)^n h\right\rangle.$$

Alternatively, the fundamental group can be obtained by considering the Whitehead link and performing $(1,n)$ Dehn surgery on the first component (see [43]). This gives the complement of the knot K_{2n}.

The Whitehead link has fundamental group (see section 2.1):

$$< \mu_1, \mu_2, \lambda_1 \mid [\mu_1, \lambda_1] = 1,\ \lambda_1 = \mu_2\mu_1^{-1}\mu_2^{-1}\mu_1\mu_2^{-1}\mu_1^{-1}\mu_2\mu_1 > .$$

The Dehn surgery glues a disc to the curve $\mu_1\lambda_1^n$ so we add the relation $\mu_1\lambda_1^n = 1$, or equivalently substitute $\mu_1 = \lambda_1^{-n}$. Then we have

$$\pi_1(K_{2n}) = \left\langle \mu_2, \lambda_1 \mid \lambda_1 = \mu_2\lambda_1^n\mu_2^{-1}\lambda_1^{-n}\mu_2^{-1}\lambda_1^n\mu_2\lambda_1^{-n}\right\rangle.$$

If we let $a = \mu_2, b = \lambda_1^n\mu_2\lambda_1^{-n}$ then we may rewrite in terms of these generators

$$\begin{aligned}\pi_1(K_{2n}) &= \left\langle a, b, \mu_2, \lambda_1 \mid \lambda_1 = ab^{-1}a^{-1}b,\ a = \mu_2,\ b = \lambda_1^n\mu_2\lambda_1^{-n}\right\rangle \\ &= \left\langle a, b \mid a\left(b^{-1}aba^{-1}\right)^n = \left(b^{-1}aba^{-1}\right)^n b\right\rangle.\end{aligned}$$

We have obtained the same presentation for $\pi_1(K_{2n})$ in a slightly different form: here $a = g, b = h$. Finally, from the Schubert normal form, since the sum of exponents in w is 0, we have the longitude λ given by

$$\begin{aligned}\lambda &= w\tilde{w} = \left(hg^{-1}h^{-1}g\right)^n\left(gh^{-1}g^{-1}h\right)^n \\ &= hg^{-1}\left(h^{-1}ghg^{-1}\right)^n gh^{-1}gh^{-1}\left(g^{-1}hgh^{-1}\right)^n hg^{-1}.\end{aligned}$$

We will make use of these calculations in 8

Chapter 3

Hyperbolic Geometry and Manifolds

3.1 The Upper Half Space Model

Throughout, we will consider $\mathbb{H}^3$, hyperbolic 3-space, through the upper half space model. The sphere at infinity is identified with $\mathbb{C} \cup \infty$, and points $(x, y, z) \in \mathbb{H}^3$ can be thought of as quaternions $x + yi + zj = \alpha + \beta j$ with $\alpha = x + yi$ complex and $\beta = z > 0$ real.

The geodesics in $\mathbb{H}^3$ are Euclidean circles orthogonal to the sphere at infinity. The "Euclidean circle" meeting ∞ and another point at infinity orthogonally is a Euclidean vertical line. We refer to all of these as *hyperbolic lines.* Any geodesic has two *endpoints at infinity* which are distinct elements of $\mathbb{C} \cup \infty$. A Euclidean sphere orthogonal to the sphere at infinity is a geodesic surface. A "sphere" passing through ∞ is a vertical Euclidean plane. We refer to these as *hyperbolic planes.*

The metric $ds^2 = \dfrac{dx^2 + dy^2 + dz^2}{z}$ gives a logarithmic form to vertical distances: an easy integration shows the distance between (x, y, z_1) and (x, y, z_2) is $\log(z_2/z_1)$. If we consider a horizontal plane $z = c$ with c a constant, the metric reduces to the Euclidean metric. The region $z \geq c$ is a closed ball around ∞ and is called a (closed) *horoball.* Its boundary is called a *horosphere* and has Euclidean geometry. A horosphere around any other point $\omega \in \mathbb{C}$ at infinity is a Euclidean sphere tangent to $\mathbb{C}$ at ω.

3.1.1 Isometries

An isometry is a transformation $\mathbb{H}^3 \longrightarrow \mathbb{H}^3$ which preserves hyperbolic distances, hence geodesics. Since two geodesics with a common endpoint at infinity eventually become arbitrarily close, and otherwise do not, an isometry extends to a well-defined and continuous action on the sphere at infinity. The group of orientation-preserving isometries of $\mathbb{H}^3$ is denoted $\text{Isom}^+ \mathbb{H}^3$.

Because horosphere geometry is Euclidean, $\text{Isom}^+ \mathbb{H}^3$ includes Euclidean isometries preserving horizontal planes; in particular, translations $z \mapsto z + c, c \in \mathbb{C}$ (this action applies to all points in $\mathbb{H}^3$, considered as quaternions). Euclidean dilations of $\mathbb{R}^3$ centred at a point on the plane at infinity are also hyperbolic isometries — this is clear from the factor of $\frac{1}{z}$ in the expression for ds. For example the Euclidean dilation $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$, $\lambda \in \mathbb{R}$, is a hyperbolic translation of $\log \lambda$ along the line $(0, 0, z)$. In fact, taking $\lambda \in \mathbb{C}$, $\lambda = re^{i\theta}$ we obtain a Euclidean spiral symmetry of $\mathbb{R}^3$ which is a hyperbolic screw motion — translation by $\log r$ and rotation by θ along this line. Then $\log r + i\theta$ is called the *complex translation distance* of the isometry.

It is not too difficult to check that inversion in a Euclidean sphere S orthogonal to the plane at infinity is an orientation-reversing isometry. This is a hyperbolic reflection in the plane S. Composing with a reflection in the vertical plane above the real axis, we obtain an orientation-preserving isometry which acts as $z \mapsto \frac{1}{z}$.

It can be proved that the isometries described above generate all the isometries of $\mathbb{H}^3$: see e.g. [44]. The Euclidean translations $z \mapsto z + c$, the hyperbolic translations $z \mapsto \lambda z$, and the reflected inversions $z \mapsto \frac{1}{z}$ generate the group of *Möbius transformations*: transformations of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C},\ ad - bc \neq 0$. Hence the orientation-preserving isometries of $\mathbb{H}^3$ are the Möbius transformations.

We can describe a Möbius transformation as above by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then Möbius transformations compose exactly as matrices multiply. However a, b, c, d and $\lambda a, \lambda b, \lambda c, \lambda d$ represent the same Möbius transformation. So we can identify Möbius transformations with matrices considered equivalent if they differ by a nonzero scalar multiple. If we choose a, b, c, d with $ad - bc = 1$, this leaves only an ambiguity of ± 1. Thus we have

$$\text{Isom}^+(\mathbb{H}^3) \cong \frac{SL_2(\mathbb{C})}{\pm I} = PSL_2(\mathbb{C}).$$

Hence we speak of the eigenvalues and trace of an isometry, which are well-defined up to sign. We can notate isometries either by Möbius transformations or matrices with determinant 1 up to sign.

3.1.2 Types of isometries

We now briefly categorise isometries of $\mathbb{H}^3$ and outline some of their properties. Given a non-trivial isometry

$$f : z \mapsto \frac{az + b}{cz + d} \quad \text{represented by} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{C}),$$

we can conjugate by an isometry g to put f in Jordan normal form. Conjugation of an isometry can be thought of as a change of perspective. The entries on the diagonal of this normal form are the eigenvalues, which must multiply to 1 (the determinant). Thus gfg^{-1} has matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \qquad \text{or} \qquad \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The first case is the isometry $z \mapsto \lambda^2 z$, which is a Euclidean rotation, dilation or spiral symmetry from the origin, respectively as $|\lambda| = 1$, $\lambda^2 \in \mathbb{R}$, or otherwise.

If $|\lambda| = 1$, $\lambda^2 = e^{i\theta}$ then gfg^{-1} rotates around the vertical line above the origin by angle θ, so f rotates around some geodesic l by angle θ. Then f fixes every point on l and preserves hyperbolic planes orthogonal to l. In this case f is called *elliptic*. Elliptic isometries are characterised by the presence of fixed points, and real trace lying in the interval $(-2, 2)$.

If $|\lambda| \neq 1$, $\lambda^2 = re^{i\theta}$ then gfg^{-1} acts as a hyperbolic screw motion around the vertical line above origin with complex translation distance $\log r + i\theta$. So f is a screw motion with the complex translation distance $\log r + i\theta$ about some geodesic l. If we consider a collection of planes orthogonal to l, filling $\mathbb{H}^3$, then f preserves the collection, translating by $|\log r|$ and rotating each by θ. It follows that every point is translated by at least $|\log r| > 0$ by f; the set of translation distances of f has positive infimum. In this case f is called *loxodromic*. A loxodromic isometry can have any trace in $\mathbb{C} - [-2, 2]$.

In the second case gfg^{-1} acts as the Euclidean translation $z \mapsto z + 1$ with no fixed points in $\mathbb{H}^3$ but one fixed point at infinity (∞). This transformation fixes horospheres around ∞, on which it acts as a Euclidean translation. Points are translated arbitrarily small distances as we approach ∞. Thus f also has no fixed points but fixes one point at infinity ω, around which f preserves horospheres. Such an f is called *parabolic*. A parabolic isometry has trace ± 2.

Proposition 3.1.1 *Two hyperbolic isometries f, g commute if and only if they have the same fixed point(s) at infinity, or are $180°$ rotations about orthogonal axes.*

Proof. Clearly two parabolics with the same endpoint, or two loxodromics or elliptics with the same axis, commute. Suppose $fg = gf$ and let $\omega \in \mathbb{C} \cup \infty$ be a fixed point of f. Then $gfg^{-1}(g(\omega)) = g(\omega)$, so $g(\omega)$ is a fixed point of $gfg^{-1} = f$. Similarly $g^{-1}fg$ fixes $g^{-1}(\omega)$ is also. So ω is a fixed point of f if and only if $g(\omega)$ is a fixed point of f. If f, g are parabolic we are done. Otherwise conjugate f, g so that f is represented by a diagonal matrix. Since $fg = gf$, a straightforward computation gives g is diagonal or $g = \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. In the first case, after conjugation both f, g fix $(0, \infty)$; hence they fix the same points at infinity. In the second case, after conjugation f, g are $180°$ rotations about orthogonal axes; hence they were before conjugation, also. □

We now run through a number of transitivity results which will be useful.

Proposition 3.1.2

1. *For any distinct* $\alpha, \beta, \gamma \in \mathbb{C} \cup \infty$*, there is a parabolic transformation* f *fixing* α *and mapping* $\beta \mapsto \gamma$*.*

2. *Isom*$^+ \mathbb{H}^3$ *acts transitively on pairs of points on the sphere at infinity. That is, for any two pairs of distinct elements of* $\mathbb{C} \cup \infty$ (α_1, β_1) *and* (α_2, β_2)*, there exists* $f \in$ *Isom*$^+ \mathbb{H}^3$ *with* $f(\alpha_1) = \alpha_2, f(\beta_1) = \beta_2$*.*

3. *For any distinct* $\alpha, \beta, \gamma, \delta \in \mathbb{C} \cup \infty$*, there is a loxodromic or elliptic transformation fixing* α, β *and mapping* $\gamma \mapsto \delta$*.*

4. *Isom*$^+ \mathbb{H}^3$ *acts transitively on triples of points on the sphere at infinity. That is, for any two triples of distinct elements of* $\mathbb{C} \cup \infty$ $(\alpha_1, \beta_1, \gamma_1)$ *and* $(\alpha_2, \beta_2, \gamma_2)$*, there exists* $f \in$ *Isom*$^+ \mathbb{H}^3$ *with* $f(\alpha_1) = \alpha_2, f(\beta_1) = \beta_2, f(\gamma_1) = \gamma_2$*.*

Proof.

1. Since parabolic isometries fixing ∞, $z \mapsto z + c$ can be chosen to map any $\beta \mapsto \gamma$ for $\beta, \gamma \neq \infty$, the same must be true of a conjugate fixing α.

2. First we take a parabolic transformation f_1 (fixing a random point) which takes α_1 to α_2. Then we compose with a parabolic transformation fixing α_2 and taking $f_1(\beta_1)$ to β_2.

3. An isometry $z \mapsto (\delta/\gamma)\, z$ (with $\gamma, \delta \neq 0$) fixes $(0, \infty)$ and maps $\gamma \mapsto \delta$. So isometries fixing $(0, \infty)$ act transitively on all other $\gamma, \delta \in \mathbb{C}$. Applying a conjugation taking $(0, \infty) \mapsto (\alpha, \beta)$ (which exists by the previous part), isometries fixing (α, β) act transitively on all $\gamma, \delta (\neq \alpha\beta) \in \mathbb{C} \cup \infty$.

4. Using the second part we take f_1 sending $\alpha_1 \mapsto \alpha_2$ and $\beta_1 \mapsto \beta_2$. Then we use the previous part to find f_2 fixing α_2, β_2 and sending $f_1(\gamma_1)$ to γ_2 and compose.

□

3.1.3 Hyperbolic triangles and tetrahedra

Unlike Euclidean geometry, triangles in a hyperbolic plane are determined up to congruence by their angles. The angles always sum to less than π, and in fact the deficiency from π is the area. An *ideal triangle* is one with all vertices at infinity, hence with all angles 0 and area π. From proposition 3.1.2(4), there is an isometry taking any ideal triangle to any other, so all ideal triangles in $\mathbb{H}^3$ are congruent. An *ideal tetrahedron* is one with all vertices at infinity. Topologically, an ideal tetrahedron is one with its vertices removed.

Ideal tetrahedra can be classified nicely up to congruence. If one vertex is at ∞, three edges of our tetrahedron are vertical Euclidean lines. Considering a horosphere cross-section of the tetrahedron, we see that the dihedral angles

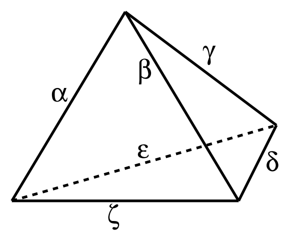


Figure 3.1: an ideal tetrahedron with dihedral angles labelled

are the angles of a Euclidean triangle, which sum to π. Applying this at each vertex and labelling angles as in figure 3.1 yields

$$\alpha + \beta + \gamma = \alpha + \epsilon + \zeta = \beta + \delta + \zeta = \gamma + \delta + \epsilon = \pi.$$

It follows that opposite dihedral angles are equal: $\alpha = \delta,\ \beta = \epsilon,\ \gamma = \zeta$.

Associated to each edge of an ideal tetrahedron is an *edge parameter*. Take a tetrahedron T with dihedral angles α, β, γ as above. Take an edge e_0 with dihedral angle α. Consider one of the endpoints at infinity of e_0 and a horosphere cross-section about this endpoint. We obtain a Euclidean triangle on which 3 edges e_0, e_1, e_2 of T are projected to vertices of a Euclidean triangle with angles α, β, γ, as shown in figure 3.2 (viewed from above).

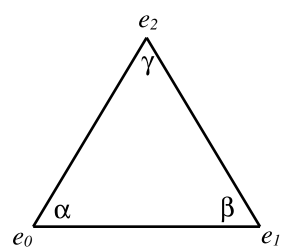


Figure 3.2: horosphere cross-section of an ideal tetrahedron

Consider the horosphere as $\mathbb{C}$ so that the three edges have associated complex numbers z_0, z_1, z_2. The edge parameter associated to e_0 is

$$z(e_0) = \frac{z_2 - z_0}{z_1 - z_0}.$$

$z(e_0)$ tells us what complex factor we multiply the (horospherical) edge e_0e_1 by to send it to e_0e_2. Thus $\arg z(e_0) = \alpha$. Similarly

$$z(e_1) = \frac{z_0 - z_1}{z_2 - z_1}, \qquad z(e_2) = \frac{z_1 - z_2}{z_0 - z_2}.$$

The parameters associated to two similar Euclidean triangles are identical. Thus the tetrahedral edge parameter is well-defined and opposite edges have the same parameter. Further

$$z(e_1) = \frac{1}{1 - z(e_0)}, \qquad z(e_2) = \frac{z(e_0) - 1}{z(e_0)},$$

so one edge parameter determines the other two.

By 3.1.2(4), we can apply an orientation-preserving isometry to place the first 3 vertices of T at $\infty, 0, 1$. Let the fourth vertex then go to ω. By choosing vertices in the appropriate order we assume $\operatorname{Im} \omega > 0$. If e_0 is the edge mapped to $(0, \infty)$ then $z(e_0) = \frac{\omega - 0}{1 - 0} = \omega$. So the tetrahedral edge parameters are $\omega, \frac{1}{1-\omega}, \frac{\omega - 1}{\omega}$. It is often convenient think of edge parameters from placing ideal tetrahedra in this "standard position".

From this, we can see that two ideal tetrahedra are related by an orientation-preserving isometry if and only if they have the same edge parameters. One edge parameter determines the other two, so we can parametrise ideal (non-degenerate) tetrahedra by a single variable which ranges over all complex numbers with positive imaginary part, up to a 3–1 ambiguity.

An ideal tetrahedron has a well-defined finite hyperbolic volume, which is a function of its edge parameters, in fact its dihedral angles α, β, γ. The volume is

$$L(\alpha) + L(\beta) + L(\gamma)$$

where $L(x)$ is the Lobachevsky function

$$L(x) = -\int_0^x \log |2 \sin t| \, dt.$$

3.2 Hyperbolic manifolds

In this section we deal with a 3-manifold M as a geometric object. Starting from a purely topological viewpoint, we would like to put a metric on M, consistent with a well-known 'model' geometry — here we are concerned with hyperbolic geometry.

3.2.1 Charts and Atlases

The philosophy is "act hyperbolic locally, think hyperbolic globally". We start with a "map" (in the cartographic and functional sense) of the local neighbourhood in our manifold M: a *coordinate chart* (U, ϕ), where U is the local open neighbourhood and $\phi : U \longrightarrow \mathbb{H}^3$ is a homeomorphism onto its image. Then ϕ "maps" the local neighbourhood into well-known $\mathbb{H}^3$ where there are well-defined coordinates. So as we walk around our local neighbourhood we can keep track of our journey by the coordinates in $\mathbb{H}^n$ given by ϕ.

If the local neighbourhoods U cover the whole manifold M, then we always have at least one coordinate chart to keep track of ourselves. We then have a global *atlas* of the whole manifold.

However, we need our local maps to make sense. If we need to turn to a different page of the atlas, we would like to know how the two maps fit together. We need to know how ϕ_i and ϕ_j differ on $U_i \cap U_j$. So we define the *transition map* or *coordinate change* $\gamma_{ij} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ to take us from the ϕ_j chart map, to ϕ_i:

$$\gamma_{ij}(x) = \phi_i \left(\phi_j^{-1}(x)\right).$$

Each γ_{ij} is a homeomorphism between open sets of $\mathbb{H}^n$. If we consider M identified with open chunks of $\mathbb{H}^n$ through the charts, then we can imagine γ_{ij} as the map gluing together the chunks to give M. The nature of the coordinate changes γ_{ij} determines the sort of structure we obtain on M.

There is not much point in using $\mathbb{H}^n$ as our "reference space" unless the coordinate changes preserve the geometry of hyperbolic space — they should be hyperbolic isometries. In this case we say M has a *hyperbolic structure*. A manifold for which there exists a hyperbolic structure is called a *hyperbolic manifold*. If there exists a hyperbolic structure on M where all the coordinate changes are orientation-preserving isometries, then M is *orientable*. We will only be concerned with orientation-preserving isometries and orientable manifolds throughout.

Thus, in a hyperbolic manifold, we can navigate by plotting our position in hyperbolic space. Our journeys have the full geometric notions of distance, angle and area inherited from hyperbolic geometry. Every local ball is a hyperbolic ball. Whenever we turn to different pages of our hyperbolic atlas, we shift our plot by a given isometry.

Of course, there is nothing special about using $\mathbb{H}^n$ as our "reference space" here. Euclidean and spherical manifolds are analogously defined. In general we can use any manifold, and constrain coordinate changes to lie in some pseudogroup to obtain a structure: see generally [82].

3.2.2 Geometric structures on knot and link complements

We are mostly concerned with knot complements, hence proceed directly to the problem of placing geometric structures on them. As we will see, a powerful technique in knot theory is to find knot invariants through geometric structures on their complements: "better knot theory through geometry". We will see "most" knot complements have a hyperbolic structure.

The strategy to find a geometric structure on a knot K is as follows. First, we decompose $S^3 - K$ into a set of ideal tetrahedra or polyhedra with faces glued in pairs. Second, we try to place these in $\mathbb{H}^3$ (the chart maps) so that their faces are glued by isometries (the coordinate changes).

3.2.3 From link complement to ideal triangulation

We will present a decomposition of the Whitehead link complement, following [82], since it is important subsequently. But there is a much more general method due to Thurston (see [82]; [13]; [52]; [65]) which can be used to obtain such a decomposition directly from a picture of the knot.

We arrange the Whitehead link as shown in figure 3.3, and add three "props" joining parts of the link. Then we glue four discs in, with their boundaries mapping to the knot and/or props as shown (they are very twisted!). Three discs are adjacent to prop 1; similarly prop 3. Prop 2 has all four discs adjacent, which is difficult to visualise, but is important to understand the decomposition.

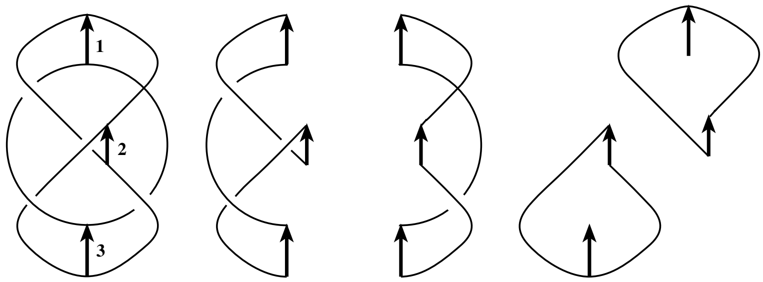


Figure 3.3: Whitehead link and boundaries of twisted discs

The link, props and discs form a 2-complex which cut the link complement into a connected 3-manifold with 8 boundary faces — it is adjacent to each of the four discs on both sides. This is an ideal octahedron, with faces glued as shown in figure 3.4. So we have a decomposition of the link complement into an ideal polyhedron.

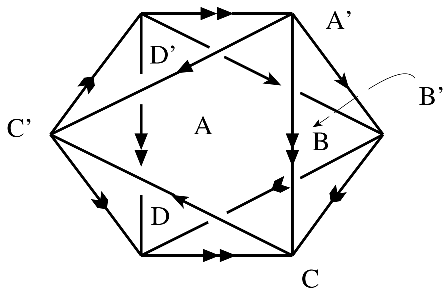


Figure 3.4: Whitehead link complement as an ideal octahedron

Following [56] we can obtain an ideal triangulation by cutting the octahedron into four ideal tetrahedra.

3.2.4 From ideal triangulation to hyperbolic structure

We now use the geometry of hyperbolic ideal tetrahedra to obtain a hyperbolic structure on the Whitehead link complement.

First we clarify the task: see generally [82]. We ask generally when a knot complement (or any orientable 3-manifold) obtained by gluing together ideal

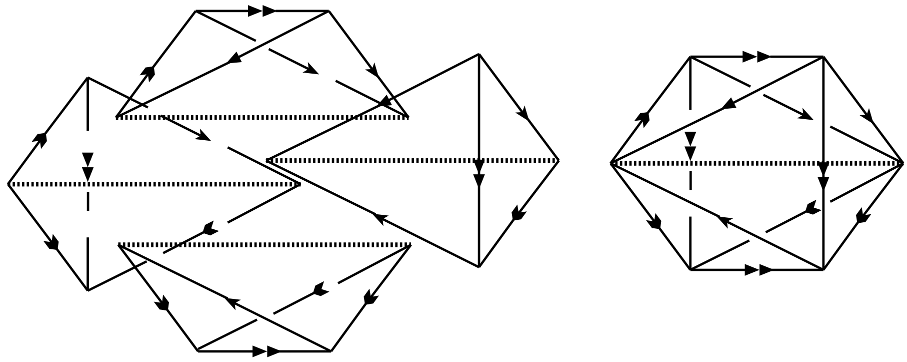


Figure 3.5: ideal triangulation of Whitehead link complement

hyperbolic tetrahedra by orientation-preserving isometries is hyperbolic. Navigation in the manifold must look like navigation in $\mathbb{H}^3$. Every point must have a neighbourhood which is isometric to a ball in hyperbolic space. This is a necessary and sufficient criterion for a hyperbolic structure.

This criterion is automatically satisfied for points interior to ideal tetrahedra, however placed in $\mathbb{H}^3$. Provided faces are glued via isometries, the criterion is easily satisfied for interior points of faces. Since we only have ideal tetrahedra there are no vertices to consider. Therefore, we obtain a hyperbolic structure if and only if the dihedral angles around each edge in the triangulation sum to 2π, and the edges are glued without translations.

Returning to the Whitehead link complement, we have 4 ideal tetrahedra with faces glued in pairs, hence 24 edges which are glued in two groups of 8 and two groups of 4. The 16 ideal vertices are glued together in two groups of 8, corresponding to the two components of the link.

We now follow [56]. We cut off a neighbourhood of each ideal vertex, and obtain a triangulation of the boundary of these neighbourhoods (the *link* of each ideal vertex: for a more precise definition see [82, p. 120]), shown in figure 3.6. In a hyperbolic structure, these can be considered as horosphere cross-sections, which have Euclidean geometry. In these cross-sections, the dihedral angles around each edge become the Euclidean angles around a vertex. The tetrahedral parameters in the first tetrahedron are labelled w, w', w''; those in the second x, x', x''; and so on.

We obtain a hyperbolic structure if and only if the Euclidean triangles in these cross-sections piece together nicely. This is equivalent to the condition that the tetrahedral edge parameters $z_1, \ldots, z_k$ around each edge multiply to 1. To avoid wrapping around a vertex more than once, we also require that $\arg z_1, \ldots, \arg z_k$ sum to 2π. This is equivalent to

$$\log z_1 + \log z_2 + \ldots + \log z_k = 2\pi i$$

where we take the branch of the natural logarithm on the complex plane split

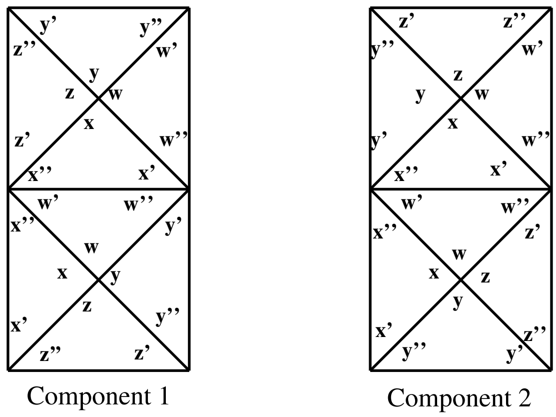


Figure 3.6: triangulations on links of ideal vertices

along $(-\infty, 0]$.

With tetrahedral edge parameters labelled

$$w, \frac{w-1}{w}, \frac{1}{1-w}, x, \frac{x-1}{x}, \frac{1}{1-x}, y, \frac{y-1}{y}, \frac{1}{1-y}, z, \frac{z-1}{z}, \frac{1}{1-z}$$

as shown, we obtain

$$\begin{aligned}
2\pi i &= \log\frac{1}{1-w} + \log\frac{z-1}{z} + \log\frac{1}{1-x} + \log\frac{w-1}{w} + \log\frac{1}{1-x} \\
&\quad + \log\frac{y-1}{y} + \log\frac{1}{1-w} + \log\frac{x-1}{x} \\
2\pi i &= \log w + \log x + \log y + \log z \\
2\pi i &= \log\frac{1}{1-z} + \log\frac{w-1}{w} + \log\frac{1}{1-y} + \log\frac{z-1}{z} + \log\frac{1}{1-y} \\
&\quad + \log\frac{x-1}{x} + \log\frac{1}{1-z} + \log\frac{y-1}{y} \\
2\pi i &= \log w + \log x + \log y + \log z
\end{aligned}$$

which simplifies to two *consistency equations*:

$$\log w + \log x + \log y + \log z = 2\pi i \tag{3.1}$$

$$\log(1-w) + \log(1-x) - \log(1-y) - \log(1-z) = 0 \tag{3.2}$$

Thus any solution w, x, y, z to these equations (with all imaginary parts greater than 0) gives a set of hyperbolic tetrahedra which piece together appropriately to give a hyperbolic structure.

3.2.5 Developing map and holonomy

We now return to hyperbolic structures in general, following [82], [41]. A hyperbolic structure on a manifold M is a way to navigate around M by mapping the local neighbourhood into hyperbolic space, plotting our journey there, and knowing that turning pages in the atlas corresponds to applying hyperbolic isometries. We want to produce a global (cartographic) map of M.

As a start, we note that two local maps ϕ_i, ϕ_j which map two local neighbourhoods U_i, U_j ($U_i \cap U_j \neq \emptyset$), into $\mathbb{H}^3$, can be combined. Provided $U_i \cap U_j$ is connected, replacing ϕ_j with $\phi'_j = \gamma_{ij} \circ \phi_j$, the two local maps agree on $U_i \cap U_j$ and we obtain another 'more global' map. M still has a hyperbolic structure, since any two coordinate charts still differ by an isometry.

There is a problem of course. We can't expect to obtain a homeomorphism $M \longrightarrow \mathbb{H}^3$. We need to distinguish between different possible *paths* to a point in M. Starting at a basepoint $x_0 \in M$ and chart ϕ_0 applying there, we can give coordinates in $\mathbb{H}^3$ to the endpoint of each path from x_0 in M, by *analytic continuation* of our original chart ϕ_0 in the manner described above. Recall that the space of all homotopy classes of paths in M starting from x_0 is the universal cover $\tilde{M}$. So we can obtain the *developing map* $D : \tilde{M} \longrightarrow \mathbb{H}^3$ in this way. It's not too difficult to check that D is well defined, once we choose our x_0 and ϕ_0: homotopic paths to the same point and different choices of chart maps along the way lead to the same result. The developing map now lets us walk around M, and if we lift our path to $\tilde{M}$, it explicitly plots our course in $\mathbb{H}^3$.

We now consider what happens if we walk around a loop α in M. If our walk is trivial in $\pi_1(M)$, then it is trivial in $\tilde{M}$ hence we end up at the base point $\phi_0(x_0) \in \mathbb{H}^3$. But if α is non-trivial loop, something more interesting happens. Our walk passes through several charts $U_0, U_1, \ldots, U_n = U_0$ with coordinate changes $\gamma_{0,1}, \ldots, \gamma_{n-1,n}$ which are hyperbolic isometries. So we end up at

$$\gamma_{n,n-1} \circ \gamma_{n-1,n-2} \circ \cdots \circ \gamma_{1,0} \phi_0(x_0),$$

The map $H : \pi_1(M) \longrightarrow \text{Isom}^+ \mathbb{H}^3$ taking α to the isometry $\gamma_{n,n-1} \circ \cdots \circ \gamma_{1,0}$ is called the *holonomy map*. It's easy to see that H is a homomorphism of groups.

Recall that in a covering space $p : \tilde{M} \longrightarrow M$, an element $\alpha \in \pi_1(M)$ determines a unique *deck transformation* $T_\alpha : \tilde{M} \longrightarrow \tilde{M}$. Representing elements of $\tilde{M}$ by equivalence classes $[x]$ of paths x in M, T_α maps $[x]$ to $[\alpha][x]$, i.e. the equivalence class of paths represented by following the loop α, before journeying down x. A deck transformation T satisfies $p \circ T = p$. The deck transformation T_α of $\tilde{M}$ induces the isometry $H(\alpha)$ by the following commutative diagram.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{T_\alpha} & \tilde{M} \\ D \downarrow & & \downarrow D \\ \mathbb{H}^3 & \xrightarrow{H(\alpha)} & \mathbb{H}^3 \end{array}$$

3.2.6 Complete and incomplete structures

A knot complement M can have many hyperbolic structures, as is clearly true for the Whitehead link complement. Once it has a hyperbolic structure, any path in M gives a well-defined path in $\mathbb{H}^3$ via the developing map. Thus M inherits a metric from $\mathbb{H}^3$. We ask whether M is complete as a metric space.

It can be proved (see [82]) that the following conditions are equivalent:

1. M is complete as a metric space
2. $D : \tilde{M} \longrightarrow \mathbb{H}^3$ is a homeomorphism
3. For some $\epsilon > 0$, every closed ϵ-ball in M is compact.
4. For every $a > 0$, every closed a-ball in M is compact.
5. There is a family of compact subsets S_t of M for $t \in (0, \infty)$ with the following properties:
   - $\displaystyle\bigcup_{t \in (0,\infty)} S_t = M$, and
   - the family S_t increases "at a constant rate". That is, S_{t+a} contains a neighbourhood of radius a about S_t.

In this situation we say the hyperbolic structure on M is *complete*. Because of the second criterion, we can think of $\mathbb{H}^3$ *as* the universal cover of M, and directly lift paths in M to paths in $\mathbb{H}^3$. Then the commutative diagram above collapses and the holonomy $H(\alpha)$ actually *is* the deck transformation T_α. Then by standard properties of covering spaces M is homeomorphic to $\frac{\mathbb{H}^3}{H(\pi_1(M))}$. We let $\Gamma = H(\pi_1(M)) \subset \text{Isom}^+ \mathbb{H}^3$ be this holonomy group. (Since $H(\pi_1(M)) \cong \pi_1(M)$, H is injective.)

It is not too hard to prove that every closed hyperbolic manifold is automatically complete (see [82] 3.4.10). Since we are interested in knot complements, built out of ideal tetrahedra, we must examine ideal vertices.

Consider an ideal vertex v, a neighbourhood of v, and its boundary (the link of v). If v is placed at ∞ in the half space model, then the link of v is composed of triangular horosphere segments, which fit together in a Euclidean plane. Consider a loop in the link of v and its holonomy. This must be a hyperbolic isometry fixing v, hence acts as a Euclidean similarity on the horosphere segments. Thus our triangular horosphere segments are glued by Euclidean similarities. If each holonomy is actually a Euclidean isometry, then every loop in the link of v does not bring us closer to v, so there is a well-defined horosphere neighbourhood of v in M. Then removing smaller neighbourhoods N_t gives a family S_t which demonstrates completeness. On the other hand, if there is a loop in the link of v whose holonomy is not a Euclidean isometry, then following the loop takes us closer to v or further from v. Repeatedly following the loop in the appropriate direction we spiral into v. Taking corresponding points on each loop, we obtain a non-convergent Cauchy sequence. So

Proposition 3.2.1 *A hyperbolic manifold M is complete if and only if the holonomy of the link of every ideal vertex consists of Euclidean isometries.*

We now find all complete structures on the Whitehead link complement. We have already obtained triangulations on the links of the two ideal vertices of the Whitehead link complement, which are both tori (as we would expect, since each ideal vertex corresponds to a component of the link, with a torus neighbourhood). In fact, with some consideration of the knot and its triangulation, we can add in standard meridians μ_1, μ_2 and longitudes λ_1, λ_2, as shown in figure 3.7.

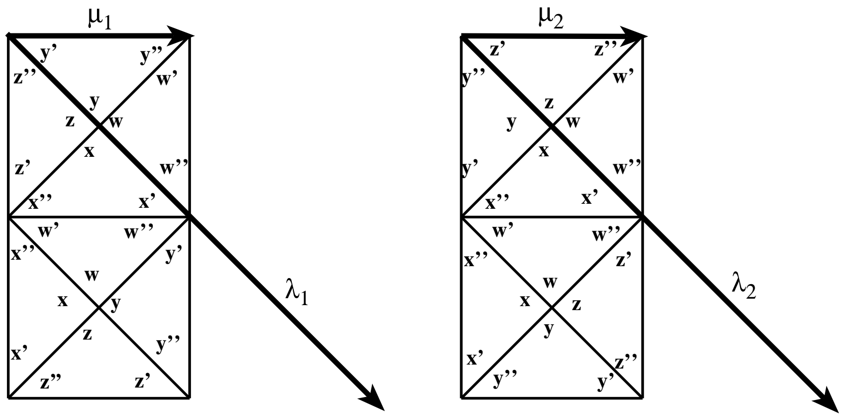


Figure 3.7: Links of ideal vertices, with longitudes and meridians

We obtain a complete structure if and only if $H(\mu_i), H(\lambda_i)$ are Euclidean isometries. These holonomies can be obtained by chasing around edge parameters to relate corresponding sides under the action of $H(\mu_i), H(\lambda_i)$. We thus obtain complex numbers u_i', v_i' respectively which describe how one side should be multiplied to be equal and parallel to the other. Expressing this logarithmically, with $u_i = \log u_i', v_i = v_i'$, we obtain

$$\begin{aligned}
u_1 &= \log\frac{1}{1-z} + \log\frac{y-1}{y} + \log\frac{1}{1-y} + \log\frac{w-1}{w} - \pi i \\
v_1 &= \log\frac{1}{1-z} + \log z + \log x + \log\frac{x-1}{x} + \log\frac{1}{1-w} + \log\frac{y-1}{y} \\
&\quad + \log\frac{1}{1-x} + \log x + \log z + \log\frac{z-1}{z} + \log\frac{1}{1-y} + \log\frac{w-1}{w} - 4\pi i \\
u_2 &= \log\frac{1}{1-y} + \log\frac{z-1}{z} + \log\frac{1}{1-z} + \log\frac{w-1}{w} - i\pi \\
v_2 &= \log\frac{1}{1-y} + \log y + \log x + \log\frac{x-1}{x} + \log\frac{1}{1-w} + \log\frac{z-1}{z} \\
&\quad + \log\frac{1}{1-x} + \log x + \log y + \log\frac{y-1}{y} + \log\frac{1}{1-z} + \log\frac{w-1}{w} - 4\pi i
\end{aligned}$$

which simplifies (from the consistency relations 3.1) to

$$\begin{aligned} u_1 &= \log(w-1) + \log x + \log z - \log(z-1) - \pi i \\ v_1 &= 2\log x + 2\log z - 2\pi i \\ u_2 &= \log(w-1) + \log x + \log y - \log(y-1) - \pi i \\ v_2 &= 2\log x + 2\log y - 2\pi i \end{aligned}$$

We have a complete structure if and only if $u_1 = u_2 = v_1 = v_2 = 0$, which is equivalent to $x = y = z = w = i$. In this case our original octahedron was an ideal regular octahedron.

It's a standard result (see e.g. [82] 3.5.7) that if a group Γ acts on a Hausdorff manifold X, then the quotient map $p : X \longrightarrow X/\Gamma$ is a covering space with X/Γ a manifold if and only if Γ acts *freely* (i.e. no $1 \neq \gamma \in \Gamma$ has any fixed points) and *properly discontinuously* (i.e. for every compact $K \subseteq X$ the set $\{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\}$ is finite). But for subgroups of $\mathrm{Isom}^+\,\mathbb{H}^3 \cong PSL_2(\mathbb{C})$, freeness is equivalent to Γ containing no elliptics, and proper discontinuity is equivalent to *discreteness.* So with $X = \mathbb{H}^3$, a complete hyperbolic structure has a discrete holonomy group with no elliptic elements. This is not generally true in an incomplete structure. Conversely, any discrete subgroup Γ of $\mathrm{Isom}^+\,\mathbb{H}^3$ acting freely gives a complete manifold quotient $\mathbb{H}^3/\Gamma$. Thus we have:

Proposition 3.2.2 *Let M be a (orientable differentiable) 3-manifold. Complete hyperbolic structures on M are in one-to-one correspondence with conjugacy classes of discrete subgroups of Isom$^+\,\mathbb{H}^3$ that are isomorphic to $\pi_1(M)$ and act freely on $\mathbb{H}^3$ with quotient M.*

It is a powerful and deep result that, in dimension 3 and above, complete structures with quotients of finite hyperbolic volume are essentially unique. 'Essentially', because conjugate holonomy groups clearly give the same manifold.

Theorem 3.2.3 (Mostow rigidity theorem [54] [81]) *If Γ_1 and Γ_2 are discrete isomorphic subgroups of Isom$^+\,\mathbb{H}^3$, with $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ finite volume, then Γ_1 and Γ_2 are conjugate in Isom $\mathbb{H}^3$.*

In general, there is a family of hyperbolic structures of which the complete structures are (conjugate) isolated points.

3.3 Hyperbolic knot theory

We now consider some possible knot invariants derived from a hyperbolic structure on knot complements. Since the complete structure on a hyperbolic knot is essentially unique, any algebraic data from this structure preserved under conjugacy is a knot invariant.

One simple such invariant is *volume.* Being obtained by gluing finitely many ideal tetrahedra, the complete structure has a well-defined volume: the sum of the volumes of the tetrahedra. By the Mostow rigidity theorem, a knot complement has at most one complete hyperbolic structure up to conjugacy by isometries, so the volume of this structure is a topological invariant of the knot.

We say a knot K is *hyperbolic* if $S^3 - K$ admits a complete hyperbolic structure (equivalently, $S^3 - K$ admits a hyperbolic structure of finite volume: see [13]). We now ask which knots are hyperbolic.

Proposition 3.3.1 *Satellite knots are not hyperbolic.*

Proof. Suppose K is a hyperbolic satellite knot. By definition $S^3 - \nu(K)$ contains a boundary torus T_1 and a second torus T_2 which is incompressible and not boundary parallel. Consider $\pi_1(T_1) \cong \pi_1(T_2) \cong \mathbb{Z} \oplus \mathbb{Z}$, which inject into $\pi_1(K)$, but which are not conjugate in $\pi_1(K)$. Let λ_i, μ_i denote standard longitudes and meridians. We consider the image of $\pi_1(T_i)$ under the holonomy map of the complete hyperbolic structure. $H(\lambda_i)$ and $H(\mu_i)$ must commute, hence are either parabolics with common fixed point at infinity, or loxodromics with common axis. (Elliptics cannot occur in the holonomy group of the complete structure.) It's not too difficult to see that two linearly independent loxodromics generate a non-discrete group of isometries. So $\pi_1(T_1), \pi_1(T_2)$ are sent to subgroups of commuting parabolics, non-conjugate in $H(\pi_1(K))$. But a quotient by two non-conjugate subgroups of commuting parabolics will produce an extra cusp: more precisely, all parabolics occurring in the holonomy group must be peripheral. This follows from the Margulis lemma and the thick-thin decomposition: see [81], [82]. This is a contradiction. □

An analysis of the fundamental group of a torus knot, which has nontrivial centre, also shows that

Proposition 3.3.2 *Torus knots are not hyperbolic.*

It is a remarkable and deep theorem of Thurston that all other knots are hyperbolic ([83]).

Theorem 3.3.3 *A non-trivial knot is hyperbolic if and only if it is not a torus or satellite knot.*

The existence of such hyperbolic structures makes hyperbolic geometry a powerful tool in knot theory.

Now take a hyperbolic knot K with a complete structure with holonomy H and $S^3 - \nu(K)$ having boundary T. Using the idea in the proof of 3.3.1 above we see that $H(\pi_1(T))$ consists of parabolic isometries of $\mathbb{H}^3$. But the Wirtinger presentation shows that knot groups are generated by meridians $g_1, \ldots, g_n$, all of which are conjugate. So in fact

Proposition 3.3.4 *The complete structure has holonomy H which takes all meridians, including $g_1, \ldots, g_n$, and longitudes, to parabolic transformations.*

By appropriately conjugating we can assume $H(\pi_1(T))$ fixes ∞ — then $H(\pi_1(T))$ consists of Euclidean translations. So T is seen as the quotient of a horosphere (Euclidean plane) in the universal cover $\mathbb{H}^3$ by two independent Euclidean translations. The *cusp* $\nu(K) - K \cong T \times (0,1)$ is seen as the quotient of a corresponding horoball. If we conjugate so that $H(\mu)$ is $z \mapsto z+1$ then $H(\lambda)$ is $z \mapsto z+c$, and c here is called the *cusp shape* or *cusp parameter.* Since μ, λ can be chosen canonically, up to conjugacy, the cusp parameter is a knot invariant.

Finally, in hyperbolic structures we can generalise the notion of Dehn surgery: see generally [81]. We generalise the idea that (p,q)-Dehn surgery makes $p\mu+q\lambda$ trivial, where μ, λ are a standard meridian and longitude on T. In any hyperbolic structure, by conjugation if necessary we may assume $H(\pi_1(T))$ fixes ∞; it acts as a Euclidean similarity on a horosphere neighbourhood of ∞, which we identify with $\mathbb{C}$. Since $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ is commutative, either $H(\pi_1(T))$ consists of commuting parabolics, giving a complete structure, or $H(\pi_1(T))$ consists of commuting loxodromics or elliptics about a vertical axis which we may assume is $(0,\infty)$. We consider only this latter case, in which $(0,\infty)$ is not part of $D(\tilde{M})$, since the tetrahedra we are gluing together "spiral" around the axis, glued by various spiral symmetries.

Given $x \in \pi_1(T)$, as for the example of the Whitehead link complement, we can define a complex number $H'(x)$ which denotes the complex translation distance of $H(x)$ upwards along $(0,\infty)$. The effect of $H(x)$ on the horosphere $\mathbb{C}$ is to multiply by $e^{H'(x)}$. If we have a hyperbolic structure such that

$$pH'(\mu) + qH'(\lambda) = \pm 2\pi i,$$

then we have a structure in which the path $p\mu + q\lambda$ is mapped by D to a loop around $(0,\infty)$, so $H(p\mu + q\lambda)$ is trivial. Note that $p\mu + q\lambda$ only has meaning for $p, q \in \mathbb{Z}$, but the notion here applies for any $p, q \in \mathbb{R}$.

Metrically completing M at this cusp corresponds to adding $(0,\infty)$ to $\mathbb{H}^3$, which in M corresponds to adding in a circle (if $\{|H'(x)| : x \in \pi_1(T)\} \subset \mathbb{R}^+$ is discrete) or a point (if $\{|H'(x)| : x \in \pi_1(T)\} \subset \mathbb{R}$ is dense). This in fact gives a complete structure on a manifold where $p\mu + q\lambda$ has become trivial, in an analogous way to (p,q)-Dehn surgery. If p, q are relatively prime integers, then our completed hyperbolic structure is actually a complete structure for the (p,q)-Dehn-filled manifold. But p, q can now be any real numbers.

We return to our example of the Whitehead link complement. We will obtain complete hyperbolic structures on the twist knots by performing hyperbolic Dehn surgery. Recall that

$$\begin{aligned}
H'(\mu_1) = u_1 &= \log(w-1) + \log x + \log z - \log(z-1) - \pi i \\
H'(\lambda_1) = v_1 &= 2\log x + 2\log z - 2\pi i \\
H'(\mu_2) = u_2 &= \log(w-1) + \log x + \log y - \log(y-1) - \pi i \\
H'(\lambda_2) = v_2 &= 2\log x + 2\log y - 2\pi i.
\end{aligned}$$

We only perform surgery on the first component, so we need the complete structure at the second component: this requires $u_2 = v_2 = 0$. The requirement $v_1 = 0$ implies

$$\log x + \log y = \pi i.$$

so $x = -y^{-1}$. From 3.1 we then have

$$\log w + \log z = \pi i$$

so $z = -w^{-1}$. The second requirement $u_1 = 0$ then yields

$$\begin{aligned} \log(w-1) + \log\left(-y^{-1}\right) + \log y - \log(y-1) &= \pi i \\ \log(w-1) - \log(1-y) &= \pi i \\ w - 1 = y - 1 & \end{aligned}$$

hence $w = y$ and $(x, y, z, w) = (x, -x^{-1}, x, -x^{-1})$. All such solutions satisfy the consistency relations 3.1 and $u_2 = v_2 = 0$, hence we have obtained all hyperbolic structures which are complete on the first cusp.

Since we are interested in the first cusp, drop the "1" subscripts and u, v simplify to

$$u = \log\left(\frac{(x+1)x}{x-1}\right), \qquad v = 4\log x - 2\pi i.$$

Thus the complete hyperbolic structure on the twist knot K_{2n} arises from solutions to $u + nv = \pm 2\pi i$, that is

$$\pm 2\pi i = 4n \log x - 2n\pi i + \log\left(\frac{x(x+1)}{x-1}\right)$$

or exponentiating,

$$x^{4n}\frac{x(x+1)}{x-1} = 1. \tag{3.3}$$

Note that while the second equation gives us a polynomial, the first exactly locates which root of the polynomial must be taken for x: see [43] or chapter 8 later. Thus we have obtained the complete structure on K_{2n} — gluing together an octahedron, or equivalently 4 tetrahedra, with parameters as shown.

In [13] it is shown that hyperbolic (p, q) Dehn surgery on the second component of the Whitehead link complement can be done for (p, q) outside the closed parallelogram with vertices at $(-4, 1)$, $(0, 1)$, $(0, -1)$, $(4, -1)$ in $\mathbb{R}^2$. This is called *hyperbolic Dehn surgery space* and the twist knots K_{2n} correspond to the points $(1, n)$.

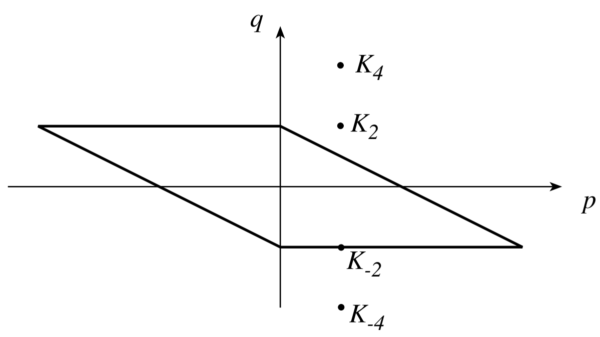


Figure 3.8: hyperbolic Dehn surgery space

Chapter 4

Algebraic Geometry and Matrix Fun

In this chapter we define some basic concepts of algebraic geometry. The definition of the A-polynomial relies heavily on these concepts. Our basic references are [76] and [37]. Our primary interest is in matrix representations of groups.

4.1 Basic Principles

An *affine algebraic set* $\mathbb{V}\left(\{F_i\}_{i\in I}\right) \subseteq \mathbb{C}^n$ is the common zero set of a collection of complex polynomials F_i in n variables. For example, the whole space $\mathbb{C}^n = \mathbb{V}(\emptyset)$, the null set $\mathbb{V}(6)$ and the single point $(a_1, a_2, \ldots, a_n) = \mathbb{V}(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$ are all affine algebraic sets.

The union of two affine algebraic sets and the intersection of arbitrarily many affine algebraic sets are both affine algebraic sets, since

$$\begin{aligned} \mathbb{V}\left(\{F_i\}_{i\in I}\right) \cup \mathbb{V}\left(\{F_j\}_{j\in J}\right) &= \mathbb{V}\left(\{F_iF_j\}_{(i,j)\in I\times J}\right) \\ \bigcap_{j\in J} \mathbb{V}\left(\{F_i\}_{i\in I_j}\right) &= \mathbb{V}\left(\{F_i\}_{i\in\bigcup_{j\in J} I_j}\right). \end{aligned}$$

Thus we can form the *Zariski topology* on $\mathbb{C}^n$, whose closed sets are affine algebraic sets.

This topology is quite counterintuitive. For example, the only Zariski-closed sets in $\mathbb{C}^1$ are the finite sets and the whole space. Thus the topology on $\mathbb{C}^1$ is not Hausdorff, as two nonempty open sets intersect almost everywhere, and the Zariski closure of any infinite subset of $\mathbb{C}^1$ (such as an interval) is $\mathbb{C}^1$ itself.

Every Zariski-closed set is closed in the standard Euclidean topology. For the zero set of a polynomial is a Euclidean-closed subset of $\mathbb{C}^n$, and an affine algebraic set is an intersection of such sets. But not every Euclidean-closed set is Zariski-closed.

Given a subset V of $\mathbb{C}^n$, we let $I(V)$ be the ideal of polynomials in n variables $x_1, \dots, x_n$ vanishing on V. This ideal is finitely generated, since $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian. It is also radical, since $F^n \in I(V)$ implies $F^n(x) = 0$, hence $F(x) = 0$ for all $x \in V$, so $F(x) \in I(V)$. The *coordinate ring* of V is defined to be

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{I(V)}.$$

The coordinate ring is a ring of polynomials, but two polynomials differing by polynomials vanishing on V are equal — so the ring represents 'polynomials on V'.

Conversely, given an ideal I of $\mathbb{C}[x_1, x_2, \dots, x_n]$, we can associate the affine algebraic set $Z(I) \subseteq \mathbb{C}^n$ on which all polynomials in I vanish. It turns out that the maps I and Z give a one-to-one inclusion-reversing correspondence between affine algebraic sets in $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[x_1, \dots, x_n]$. For a generic ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ we have $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, the radical of $\mathfrak{a}$. This is Hilbert's *Nullstellensatz*. And conversely, for a generic set $V \subseteq \mathbb{C}^n$, $Z(I(V)) = \overline{V}$, the closure of V in the Zariski topology.

A nonempty affine algebraic set V is *reducible* if it can be expressed as a union of nonempty affine algebraic proper subsets. For example the 'skewered plane' $\{x, y, 0\} \cup \{0, 0, z\} = \mathbb{V}(xz, yz)$ can be expressed as a union $\mathbb{V}(z) \cup \mathbb{V}(x, y)$. A general nonempty subset Y of $\mathbb{C}^n$ is *reducible* if it can be expressed as the union of two proper subsets Y_1, Y_2, where Y_1, Y_2 are closed in Y. (That is, each is the intersection of a Zariski-closed subset of $\mathbb{C}^n$ with Y.) A set which is not reducible is *irreducible*. The empty set is not considered irreducible. Irreducible affine algebraic sets are called *affine algebraic varieties* (or just *affine varieties*). Every affine algebraic set can be uniquely expressed as a finite union of affine varieties, none of which contains another. If V is irreducible then $\mathbb{C}(V)$ is an integral domain, so we can define its field of fractions to be the *function field* $\mathbb{C}(V)$ of V.

Affine algebraic sets have a notion of dimension, which equates with the intuitive concept. Where an affine algebraic set consists of pieces of (intuitively) different dimensions, (like the 2-dimensional plane skewered by a 1-dimensional line) we take the maximal dimension. The *dimension* of an affine algebraic set V is the length d of the longest properly ascending chain of (irreducible) affine varieties

$$V_0 \subset V_1 \subset \dots \subset V_d \subseteq V.$$

There is a natural type of map between affine varieties, which is essentially a polynomial map. A *morphism* between affine varieties $V \subseteq \mathbb{C}^m$, $W \subseteq \mathbb{C}^n$ is a map $V \longrightarrow W$ which is the restriction of a polynomial map $\mathbb{C}^m \longrightarrow \mathbb{C}^n$. Of course, morphisms need not be injective or surjective. Even if a morphism is bijective, its inverse need not be a morphism. A bijective morphism for which the inverse function is a morphism is called an *isomorphism*. In this case we say V, W are *isomorphic*.

4.2 Matrix representations of knot groups

We are interested in *representations* of a group π into $SL_2(\mathbb{C})$, that is, homomorphisms $\pi \longrightarrow SL_2(\mathbb{C})$, particularly where π is the fundamental group of a knot complement. Such representations are important for 2 reasons. First, a group like $SL_2(\mathbb{C})$ is much better understood than a knot group $\pi_1(K)$, enabling us to study $\pi_1(K)$ more easily, but still retains sufficient complexity to produce interesting results. Second, there is a geometric connection. Any hyperbolic structure on $S^3 - K$ has a holonomy representation

$$H : \pi_1(K) \longrightarrow \text{Isom}^+ \mathbb{H}^3 \cong PSL_2(\mathbb{C}) \cong \frac{SL_2(\mathbb{C})}{\pm I}$$

which is obviously closely related to representations into $SL_2(\mathbb{C})$. In fact *any* representation into $PSL_2(\mathbb{C})$ lifts into $SL_2(\mathbb{C})$.

Proposition 4.2.1 *Any representation $\rho : \pi_1(K) \longrightarrow PSL_2(\mathbb{C})$ lifts to two representations $\tilde{\rho}, \tilde{\rho}' : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$.*

Proof. Let $\pi_1(K) =< g_1, \ldots, g_n | r_1, \ldots, r_m >$ be a Wirtinger presentation for K and $\rho : \pi_1(K) \longrightarrow PSL_2(\mathbb{C})$ a representation. There are two possibilities for $\tilde{\rho}(g_1)$: say $A, -A$. Choose one. Then walk around the knot. At the end of the maximal arc corresponding to g_1, there is an intersection with relator $g_1 = g_i^{\pm 1} g_k g_i^{\mp 1}$.

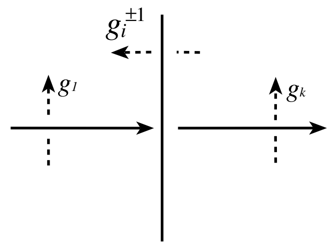


Figure 4.1: Given $\tilde{\rho}(g_1)$, there is only one choice for $\tilde{\rho}(g_k)$

Thus, though there are (a priori) two possibilities for $\tilde{\rho}(g_i)$, there is a unique choice for $\tilde{\rho}(g_k)$. This is the next maximal arc on our journey. Continuing our walk around the knot we obtain a unique $\tilde{\rho}(g_i)$ for $i = 1, \ldots, n$ at each stage until the last. The last relator is a consequence of the others, so is automatically satisfied. All relators being satisfied, we have a representation $\tilde{\rho} : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$. Depending on our initial choice $\pm A$ we obtain two different lifts. □

These two lifts $\tilde{\rho}, \tilde{\rho}'$ into $SL_2(\mathbb{C})$ satisfy $\tilde{\rho}(g_i) = -\tilde{\rho}'(g_i)$ for all generators g_i. Hence for $x \in \pi_1(K)$, $\tilde{\rho}(x) = (-1)^{w(x)} \tilde{\rho}'(g_i)$, where w is the "winding number homomorphism" discussed in 2.1.

There are in fact much more general results, due to Thurston.

Theorem 4.2.2 ([25]) *Let M be a hyperbolic 3-manifold. The holonomy representation of the complete structure $H : \pi_1(M) \longrightarrow PSL_2(\mathbb{C})$ lifts to a representation $\tilde{H} : \pi_1(M) \longrightarrow SL_2(\mathbb{C})$.*

Theorem 4.2.3 ([81]) *Let M be a hyperbolic 3-manifold with k cusps. Then there is a k-parameter family of hyperbolic structures on M obtained from deforming the complete structure on M at each cusp.*

Since those representations $\pi_1(K) \longrightarrow SL_2(\mathbb{C})$ which are lifts of holonomies of hyperbolic structures have an obvious geometric interpretation, one interesting question is whether there is a geometric interpretation for other representations into $SL_2(\mathbb{C})$, hence into $PSL_2(\mathbb{C}) \cong \mathrm{Isom}^+ \mathbb{H}^3$.

Our first task is to give the set of representations $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ the structure of an affine algebraic set.

4.3 Representation varieties

A matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$$

can be identified with the point $(a, b, c, d) \in \mathbb{C}^4$. The group $SL_2(\mathbb{C})$ is then identified with the affine algebraic set

$$\left\{(a, b, c, d) \in \mathbb{C}^4 |\ ad - bc = 1\right\} = \mathbb{V}(ad - bc) \subset \mathbb{C}^4.$$

More generally, any $m \times n$ matrix with complex entries can be thought of as a point in $\mathbb{C}^{mn}$.

Using this idea we can identify the representations of $\pi_1(K)$ with an affine algebraic set. (The technique here applies to any finitely generated group.) Let $R(\pi_1(K))$ denote the set of all representations $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$. We give $R(\pi_1(K))$ the structure of an affine algebraic set. Here and for the rest of this chapter, let $\pi_1(K)$ have a presentation

$$\pi_1(K) = \langle g_1, g_2, \ldots, g_n \mid r_1, r_2, \ldots, r_m \rangle$$

and for a representation $\rho \in R(\pi_1(K))$, let

$$\rho(g_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Since $g_1, \ldots, g_n$ generate $\pi_1(K)$, we can identify ρ uniquely with $\{a_i, b_i, c_i, d_i\}_{i=1}^n$, a point in $\mathbb{C}^{4n}$. Thus we identify $R(\pi_1(K))$ with a subset of $\mathbb{C}^{4n}$.

A relator r_i is a word in the g_j and their inverses which must be equal to the identity, so gives a product of the matrices $\rho(g_j)$ and their inverses which must be the identity matrix. Each such matrix equation gives four polynomial equations $p_{i,1} = 0, p_{i,2} = 0, p_{i,3} = 0, p_{i,4} = 0$ in the $4n$ variables $\{a_i, b_i, c_i, d_i\}_{i=1}^n$. Hence a representation in $R(\pi_1(K))$ corresponds exactly to a point $\{a_i, b_i, c_i, d_i\}_{i=1}^n \in$

$\mathbb{C}^{4n}$ satisfying these $4m$ equations. So $R(\pi_1(K))$ is identified with the affine algebraic set

$$\left\{(a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n) \in \mathbb{C}^{4n} \mid p_{1,1} = 0, p_{1,2} = 0, \ldots, p_{m,4} = 0\right\}.$$

For this reason $R(\pi_1(K))$ is referred to as the *representation space* or *representation variety* of $\pi_1(K)$ (the language is confusing as $R(\pi_1(K))$ need not be irreducible, hence need not be an affine variety). We will henceforth identify $R(\pi_1(K))$ with this affine algebraic set. However, $\pi_1(K)$ may have many different presentations, giving different representation varieties. We show that all such varieties are isomorphic:

Proposition 4.3.1 *Suppose the group π has two presentations*

$$\pi = \langle g_1, g_2, \ldots, g_n \mid r_1, r_2, \ldots, r_m \rangle = \langle h_1, h_2, \ldots, h_l \mid s_1, s_2, \ldots, s_k \rangle,$$

then the representation varieties associated with these two presentations

$$\begin{aligned} R_1(\pi) &= \left\{(a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n) \in \mathbb{C}^{4n} \mid p_{1,1} = \cdots = p_{m,4} = 0\right\} \\ R_2(\pi) &= \left\{(a_1, b_1, c_1, d_1, \ldots, a_l, b_l, c_l, d_l) \in \mathbb{C}^{4n} \mid q_{1,1} = \cdots = q_{k,4} = 0\right\} \end{aligned}$$

are isomorphic as affine algebraic sets.

Proof. Given that $g_1, \ldots, g_n$ is a generating set for π, each element of π is a word in the g_i and their inverses. In particular, each h_j is. Let $h_j = w_j(g_1, g_2, \ldots, g_n)$ where w_j denotes such a word. Define a morphism $\phi : R_1(\pi) \longrightarrow R_2(\pi)$ as follows. Take $\rho \in R_1(\pi)$, which is uniquely defined by a_i, b_i, c_i, d_i, $i = 1, \ldots, n$, where

$$\rho(g_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

We let $\phi(\rho) \in R_2(\pi)$ be defined on the generators $h_1, \ldots, h_l$ by $\phi(\rho)(h_j) = \rho(h_j)$, i.e. the "same" representation according to different generators. Then $\phi(\rho)$ is identified with a point of $\mathbb{C}^{4l}$ by the entries of $\phi(\rho)(h_i)$, $i = 1, \ldots, l$. We show the change of coordinates is a morphism.

$$\begin{aligned} (\phi(\rho))(h_j) = \rho(h_j) &= w_j\left(\rho(g_1), \rho(g_2), \ldots, \rho(g_n)\right) \\ &= w_j\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right) \end{aligned}$$

The coordinates of $\mathbb{C}^{4l}$, i.e. the entries of $\phi(\rho(h_i))$ for $i = 1, \ldots, l$, are clearly polynomials in the coordinates of $\mathbb{C}^{4n}$ $\{a_i, b_i, c_i, d_i\}_{i=1}^n$, Since ρ is a representation, the relators $\{s_j\}_{j=1}^k$ are automatically satisfied. So $\phi(\rho)$ is a representation in $R_2(\pi)$ and ϕ is a morphism.

By symmetry, we can construct a morphism $\varphi : R_2(\pi) \longrightarrow R_1(\pi)$. It's clear that ϕ and φ are mutual inverses. So we have an isomorphism. □

We say that two representations $\rho, \sigma \in R(\pi_1(K))$ are *conjugate* if there exists $g \in SL_2(\mathbb{C})$ such that for all $x \in \pi_1(K)$, $\rho(x) = g\sigma(x)g^{-1}$. Many conjugate representations exist for a given representation.

It is customary to introduce the notion of the *character* of a representation. We define it here for the sake of completeness; however we later define the A-polynomial in a way which avoids characters. Given $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$, the character of ρ is the map $t\rho : \pi_1(K) \longrightarrow \mathbb{C}$ given by $t\rho(g) = \operatorname{tr}(\rho(g))$. Since conjugate matrices have equal trace, the character can be thought of (in a vague sense) as the representation "modulo conjugacy". A character can be uniquely identified with a point in $\mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}}$, since the trace of a product of 4 matrices can be expressed in terms of traces of products of 3 or fewer of them. The set of all such points in $\mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}}$, as ρ ranges over all representations in $R(\pi_1(K))$, is called the *character variety* $X(\pi_1(K))$ of $\pi_1(K)$. It is in fact an affine algebraic set: see [25].

4.4 Parabolic representations

Returning to the geometric point of view, proposition 3.3.4 tells us that the holonomy H of the complete structure of a hyperbolic knot K sends all generators $g_1, \ldots, g_n$, indeed all meridians, to parabolic isometries. Similarly, a *parabolic representation* $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ is one where each $\rho(g_i)$ (hence the image of every meridian under ρ) has eigenvalues ± 1. So a lift $\tilde{H}$ of H to $SL_2(\mathbb{C})$ is parabolic.

There are in general parabolic representations other than $\tilde{H}$, i.e. not conjugate to $\tilde{H}$. In particular, if $\tilde{H} \in R(\pi_1(K)) \subseteq \mathbb{C}^{4n}$ represents a lift of H, then any Galois conjugate of $\tilde{H}$ satisfies the same polynomial equations, hence also gives a parabolic representation. But there exist knot groups with parabolic representations which are not Galois conjugates of $\tilde{H}$, whenever the polynomials involved are reducible.

We will investigate further in subsequent chapters.

4.5 Abelian representations

An *abelian representation* is a $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ which has abelian image. Abelian representations are in some sense "boring".

By proposition 2.1.1, a knot group has abelianization $\mathbb{Z}$, and it follows from the proof of the proposition that an abelian representation has $\rho(g_1) = \rho(g_2) = \cdots = \rho(g_n) = A$ for some (constant) matrix $A \in SL_2(\mathbb{C})$. If w is the winding number homomorphism of 2.1, then $\rho(x) = A^{w(x)}$. Since a longitude is nullhomologous, we have $\rho(\lambda) = I$ for any abelian representation.

Part II

Mahler Measure

Chapter 5

Mahler Measure in One Variable

We now consider a type of measure on a polynomial known as Mahler measure. On a one-variable polynomial the measure has a distinct form with an interesting and perhaps serendipitous usage in classical number theory. The relevance of Mahler measure to the remainder of this thesis however depends on a more general form discussed in the next chapter. The reader interested only in topological and geometrical results may therefore skip this chapter. But the following results are included, for their independent value, to indicate the breadth of application of Mahler measure, and because there are tantalizing hints of an application to hyperbolic geometry (see section 5.3).

Throughout, let $F(x)$ be a polynomial (in $\mathbb{C}[x]$ or $\mathbb{Z}[x]$)

$$F(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_0 = a_r\left(x-\alpha_1\right)\left(x-\alpha_2\right)\cdots\left(x-\alpha_r\right).$$

5.1 A Rather Odd Measure

The *Mahler measure* of $F(x) \in \mathbb{C}[x]$ is

$$M(F) = |a_r| \prod_{i=1}^{r} \max\{1, |\alpha_i|\} = |a_r| \prod_{|a_i|\geq 1} |\alpha_i|.$$

Mahler studied this quirky measure for the purpose of comparing it with "more natural" measures of polynomials (see [29], [50], [51]). It measures how far those roots outside the unit circle deviate from the unit circle, and ignores those inside the unit circle.

Although it is a seemingly strange type of measure, it is commensurate with some other, more intuitive, types of measure. For instance the *height* and *length* of F are respectively

$$H(F) = \max \{|a_i|\}_{i=0}^{r}, \qquad L(F) = \sum_{i=0}^{r} |a_i|.$$

$L(F)$ and $M(F)$ are commensurate in the following sense:

Proposition 5.1.1 *If r is the degree of F,*

$$2^{-r} L(F) \leq M(F) \leq L(F).$$

Proof. Since, when we divide $F(x)$ by $|a_r|$, all measures are decreased by a factor of $|a_r|$, we may assume $|a_r| = 1$. For the lower bound note that the largest modulus of products $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_j}$, as $\{i_1, i_2, \ldots, i_j\}$ range over all subsets of $\{1, 2, \ldots, r\}$, is the product found in the Mahler measure. Since every a_i is a symmetric sum of $\binom{r}{n-i} = \binom{r}{i}$ such products, we have

$$|a_i| \leq \binom{r}{i} M(F)$$

for $i = 0, 1, \ldots, r$. Hence

$$L(F) = \sum_{i=0}^{r} |a_i| \leq M(F) \sum_{i=0}^{r} \binom{r}{i} = 2^r M(F)$$

as required. We postpone the proof that $M(F) \leq L(F)$ until the next chapter, since it uses the form of $M(F)$ described there. □

Proposition 5.1.2 *$H(F)$ and $M(F)$ are also commensurate in a similar sense. In particular*

$$2^{-r} H(F) \leq M(F) \leq \sqrt{r+1} H(F)$$

Proof. For the lower bound, again we have each $|a_i| \leq \binom{r}{i} M(F) \leq 2^r M(F)$ so $H(F) = \max |a_i| \leq 2^r M(F)$. For the upper bound, see [51]: it uses the form of $M(F)$ described in the next chapter and the Hardy-Littlewood-Polya inequality. □

We now present Kronecker's beautiful proof describing when the Mahler measure of an integer polynomial takes the trivial value 1.

Theorem 5.1.3 (Kronecker) *Let $F(x) \in \mathbb{Z}[x]$ be monic. Then $M(F) = 1$ if and only if every root of F is zero or a root of unity.*

Proof. As $M(F) = 1$, all roots have modulus ≤ 1. Let $F(x) = F_1(x) = (x - \alpha_1) \cdots (x - \alpha_r)$. Now let

$$F_n(x) = \prod_{i=1}^{r} (x - \alpha_i^n) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0.$$

Consider a coefficient a_k of $F_n(x)$.

$$a_k = \pm \sum_{i_1 < \ldots < i_{r-k}} \alpha_{i_1}^n \alpha_{i_2}^n \cdots \alpha_{i_{r-k}}^n$$

This is a symmetric integral expression in the algebraic integers α_i. Hence it is an algebraic integer, because the algebraic integers form a ring. It is also a rational number, since it is invariant under any $\mathbb{Q}$-monomorphism of the splitting field of F_n. So $a_k \in \mathbb{Z}$ and $F_n(x) \in \mathbb{Z}[x]$. Further

$$|a_k| \leq \sum_{i_1 < \ldots < i_{r-k}} \left| \alpha_{i_1}^n \alpha_{i_2}^n \cdots \alpha_{i_{r-k}}^n \right| \leq \sum_{i_1 < \ldots < i_{r-k}} 1 \leq \binom{r}{k}.$$

Hence, no matter what n is, there are only finitely many possibilities for a_i. By the pigeon hole principle, $F_s = F_t$ for some $s \neq t$. So

$$\{\alpha_1^s, \alpha_2^s, \ldots, \alpha_r^s\} = \{\alpha_1^t, \alpha_2^t, \ldots, \alpha_r^t\}.$$

Let σ be a permutation of $\{1, \ldots, r\}$ such that $\alpha_i^s = \alpha_{\sigma(i)}^t$. Then

$$\alpha_i^{s^2} = (\alpha_i^s)^s = \left(\alpha_{\sigma(i)}^t\right)^s = \left(\alpha_{\sigma(i)}^s\right)^t = \left(\alpha_{\sigma^2(i)}^t\right)^t = \alpha_{\sigma^2(i)}^{t^2}.$$

We proceed by induction to obtain $\alpha_i^{s^k} = \alpha_{\sigma^k(i)}^{t^k}$. Since the permutation group on r elements is finite, σ has finite order T and

$$\alpha_i^{s^T} = \alpha_{\sigma^T(i)}^{t^T} = \alpha_i^{t^T}.$$

We conclude either $\alpha_i = 0$ or $\alpha_i^{s^T - t^T} = 1$, i.e. α_i is a root of unity. □

5.2 Cyclotomic Functions and Large primes

Lehmer in [47] used the Mahler measure in a method for manufacturing large primes. His method is aided by finding integer monic polynomials of small Mahler measure. The ideas are interesting and we review them here.

Given $F(x) \in \mathbb{Z}[x]$ monic with splitting field K, let

$$\Delta_n(F) = \prod_{i=1}^{r} (\alpha_i^n - 1).$$

Thus $\Delta_n(F)$ is a symmetric integral expression in the algebraic integers α_i, so by the argument used earlier, $\Delta_n(F) \in \mathbb{Z}$.

Lemma 5.2.1 $\Delta_n(F)$ *is a divisibility sequence, i.e., if* $n|m$ *then* $\Delta_n(F)|\Delta_m(F)$.

Proof. Let $m = nk$. Then we have the factorisation

$$\alpha^m - 1 = (\alpha^n)^k - 1^k = (\alpha^n - 1)\left(\alpha^{n(k-1)} + \alpha^{n(k-2)} + \cdots + 1\right)$$

so taking a product over all α_i,

$$\Delta_m(F) = \Delta_n(F)\left(\prod_{i=1}^{d} \alpha_i^{n(k-1)} + \alpha_i^{n(k-2)} + \cdots + 1\right).$$

The product above is a symmetric integral expression in algebraic integers, so by a similar argument to above, lies in $\mathbb{Z}$. The result follows. □

In fact, we can do better and partially factorise $\Delta_n(F)$ (as an integer) directly. We can factorise $x^n - 1$ into a product of cyclotomics, i.e.

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where Φ_d is the d'th *cyclotomic polynomial* (i.e. the monic polynomial with roots the primitive d'th roots of unity). Let

$$Q_d = \prod_{i=1}^{r} \Phi_d(\alpha_i)$$

so Q_d is a symmetric integral expression in algebraic integers, hence in $\mathbb{Z}$. Then

$$\Delta_n(F) = \prod_{i=1}^{r} (\alpha_i^n - 1) = \prod_{i=1}^{r} \prod_{d|n} \Phi_d(\alpha_i) = \prod_{d|n} Q_d.$$

To obtain a prime factorisation of $\Delta_n(F)$, therefore, it is necessary only to factor Q_d for $d \,|\, n$. But for proper factors d, $Q_d \,|\, \Delta_d$. So we take an inductive approach to factorising $\Delta_n(F)$, starting from 1 and building up our calculations of $\Delta_n(F)$ from its smallest factors. If we assume that we have factorised $\Delta_d(F)$ for all proper factors d, then we are left only with Q_n to factorise. For this reason Q_n is called the *essential factor* of $\Delta_n(F)$. A prime factor of Q_n, i.e. a prime factor of $\Delta_n(F)$ which is not a factor of $\Delta_d(F)$ for any proper factor d of n, is called a *characteristic prime factor* of $\Delta_n(F)$.

Our search is aided by two powerful results proved by Lehmer.

Theorem 5.2.2 *A characteristic prime factor p of $\Delta_n(F)$ is not a factor of n.*

Because of this theorem, p is coprime to n, so $p^{\phi(n)} \equiv 1 \bmod n$ (where ϕ is the Euler ϕ function). Let ω be the least positive integer with $p^\omega \equiv 1 \bmod n$, i.e. the *exponent* of p mod n, so $\omega \,|\, \phi(n)$.

Theorem 5.2.3 *Let p be a characteristic prime factor of $\Delta_n(F)$, which is a prime factor of multiplicity e. Suppose F is irreducible over $\mathbb{Z}$. If ω is the exponent of p mod n, then $\omega \le r$ and $\omega \,|\, e$.*

The proofs are elegant. We need a lemma from elementary number theory.

Lemma 5.2.4 *For any $x_1, \ldots, x_n \in \mathbb{Z}$ and prime p,*

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \quad \textit{mod } p.$$

Proof. For $j = 1, 2, \ldots, p-1$, the binomial coefficient

$$\binom{p}{j} = \frac{p!}{j!(p-j)!}$$

is divisible by p, as there is a factor of p in $p!$ which cannot occur in $j!$ or $(p-j)!$. Hence

$$(x+y)^p = \sum_{j=0}^{p} \binom{p}{j} x^j y^{p-j} \equiv x^p + y^p \mod p,$$

and by repeated application the result follows. □

Proof. [Of theorem 5.2.2] Suppose $p \mid n$. Let $n = pk$. We expand the expression for $\Delta_n(F)$ as follows.

$$\begin{aligned} \Delta_n(F) &= \prod_{i=1}^{r} \left(\alpha_i^{pk} - 1\right) = \left(\alpha_1^{pk} - 1\right)\left(\alpha_2^{pk} - 1\right)\left(\alpha_3^{pk} - 1\right) \cdots \left(\alpha_r^{pk} - 1\right) \\ &= 1 - \sum_{i_1} \alpha_{i_1}^{pk} + \sum_{i_1, i_2} (\alpha_{i_1}\alpha_{i_2})^{pk} + \cdots \pm (\alpha_1\alpha_2 \cdots \alpha_r)^{pk} \end{aligned}$$

Each sum in the above expression is an algebraic integer symmetric in the α_i, hence in $\mathbb{Z}$. Taking both sides of our expression for $\Delta_n(F)$ modulo p, and remembering $p \mid \Delta_n(F)$ and $a^p \equiv a \bmod p$ yields

$$\begin{aligned} 0 &\equiv \left(1 - \sum_{i_1} \alpha_{i_1}^{k} + \sum_{i_1, i_2} (\alpha_{i_1}\alpha_{i_2})^{k} + \cdots \pm (\alpha_1\alpha_2 \cdots \alpha_r)^{k}\right)^p \\ &= (\Delta_k(F))^p \equiv \Delta_k(F) \mod p. \end{aligned}$$

So $p \mid \Delta_k(F)$. But this contradicts the definition of a characteristic prime factor. □

Proof. [Of theorem 5.2.3] Let α be a root of F. As F is irreducible, $K = \mathbb{Q}(\alpha)$ is a degree r extension of $\mathbb{Q}$. Let $\mathcal{O}_K$ be the ring of integers of K. Note that $\Delta_n(F) = N(\alpha^n - 1)$, where N denotes the norm. We analyse the ideal $(\alpha^n - 1)\mathcal{O}_k$ and its factorisation.

First, $p\mathcal{O}_K$ and $(\alpha^n - 1)\mathcal{O}_K$ have a factor in common. Otherwise there exist $x, y \in \mathcal{O}_K$ such that $x(\alpha^n - 1) + yp = 1$, or equivalently,

$$x(\alpha^n - 1) = 1 - yp.$$

Upon taking norms we have

$$N(x)\Delta_n(F) = N(1 - yp) \equiv 1 \bmod p$$

which is a contradiction, as p is a prime factor of $\Delta_n(F)$.

Let the factorisation of $p\mathcal{O}_K$ be

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_u^{e_u}$$

where each $\mathfrak{p}_i$ is a prime ideal and e_i is the ramification index. If the inertia degree $\left[\frac{\mathcal{O}_K}{\mathfrak{p}_i} : \frac{\mathbb{Z}}{p\mathbb{Z}}\right]$ of $\mathfrak{p}_i$ is f_i then taking norms gives $p^r = p^{e_1 f_1}p^{e_2 f_2}\cdots p^{e_u f_u}$ so $r = e_1 f_1 + \cdots + e_u f_u$. In particular each $f_i \leq r$.

So let $p\mathcal{O}_K$ and $(\alpha^n - 1)\mathcal{O}_K$ have highest common factor

$$\mathfrak{p}_1^{g_1}\mathfrak{p}_2^{g_2}\cdots\mathfrak{p}_t^{g_t}$$

where $t \leq u$ and $g_i \leq e_i$. Then we have

$$(\alpha^n - 1)\mathcal{O}_K = \mathfrak{k}\mathfrak{p}_1^{g_1}\mathfrak{p}_2^{g_2}\cdots\mathfrak{p}_t^{g_t} \tag{5.1}$$

where $\mathfrak{k}$ is an ideal relatively prime to $p\mathcal{O}_k$.

Now $\alpha^n \equiv 1 \bmod \mathfrak{p}_i$ for each i, so let ϵ_i be the exponent of α mod $\mathfrak{p}_i$. Clearly $\epsilon_i \mid n$. Then we have $(\alpha^{\epsilon_i} - 1)\,\mathcal{O}_K = \mathfrak{c}\mathfrak{p}_i$, and taking norms gives

$$\Delta_{\epsilon_i}(F) = N(\mathfrak{c})p^{f_i} \equiv 0 \bmod p.$$

Since p is a characteristic prime factor of $\Delta_n(F)$, p is not a factor of any smaller $\Delta_m(F)$, so $\epsilon_i = n$.

Now $\frac{\mathcal{O}_K}{\mathfrak{p}_i}$ is a finite field, which is a degree f_i extension of the finite field $\frac{Z}{p\mathbb{Z}}$ with p elements. Thus $\frac{\mathcal{O}_K}{\mathfrak{p}_i}$ is a field with p^{f_i} elements, hence has a multiplicative group of size $p^{f_i} - 1$. By Lagrange's theorem then we have

$$\alpha^{p^{f_i}-1} \equiv 1 \bmod \mathfrak{p}_i$$

but the exponent of α mod $\mathfrak{p}_i$ is n. Hence $n \mid p^{f_i} - 1$, that is,

$$p^{f_i} \equiv 1 \bmod n.$$

Since the exponent of p mod n is ω, we have $\omega \mid f_i$ for each i. As $f_i \leq r$ also $\omega \leq r$. Let $f_i = \omega\sigma_i$.

Taking norms of 5.1 above then gives

$$\Delta_n(F) = N(\mathfrak{k})p^{f_1 g_1 + \cdots + f_t g_t} = N(\mathfrak{k})p^{\omega(f_1\sigma_1 + \cdots + f_t\sigma_t)}.$$

As $N(\mathfrak{k})$ is relatively prime to p, the highest factor e of p dividing $\Delta_n(F)$ is that shown, which is divisible by ω. □

The effect of the above theorems is to reduce dramatically the number of possible characteristic prime factors of $\Delta_n(F)$. Once we check through these, and take out appropriate prime powers, by induction we have factorised $\Delta_n(F)$.

It was mentioned earlier that Lehmer had a technique for producing large prime numbers. The above appear to be geared for factorising $\Delta_n(F)$ — precisely what we don't want to happen! In fact the divisibility sequence result guarantees us that $\Delta_n(F)$ can only be prime when n is also. Nevertheless, for n prime, we can check the limited number of possibilities for characteristic prime factors, to test whether $\Delta_n(F)$ is prime.

We give an example of the method, based on these results, to prove primality.

Theorem 5.2.5 *Let* $F(x) = x^3 - x - 1$.

$$\Delta_{113}(F) = 63,088,004,325,217$$

is a prime number.

Proof. Suppose $\Delta_{113}(F)$ is not prime. Then as 113 is prime, any prime factor of $\Delta_{113}(F)$ is a characteristic prime. It's easy to check F is irreducible. From above, a characteristic prime p is not a factor of 113 and has exponent ω modulo 113 where $\omega \le 3$. Since $p^{\phi(113)} = p^{112} \equiv 1 \bmod 113$ we have $\omega | 112$ also, so $\omega = 1$ or 2. A prime p with exponent 1 mod 113 is $\equiv 1 \bmod 113$, and a prime p with exponent 2 mod 113 is $\equiv -1 \bmod 113$.

If there is a characteristic prime p with exponent $\omega = 2$, then since $\omega \mid e$ we have $e \ge 2$, where p^e is the largest power of p dividing $\Delta_{113}(F)$. It's easily verified $\Delta_{113}(F)$ is not square, so $\Delta_{113}(F)$ has at least 3 prime factors (counted with multiplicity). One of these, p, is $\le \sqrt[3]{\Delta_{113}(F)} < 39810$, and $p \equiv \pm 1 \bmod 113$. We verify there is no such p. First let $p = 113x + 1$. If x is odd or $x \equiv 1 \bmod 3$, then p is divisible by 2 or 3, not prime, so $x \equiv 0, 2 \bmod 6$. Similarly, if $p = 113x - 1$, then we only need check $x \equiv 0, 4 \bmod 6$. There are 117 numbers to check in each case.

We may now assume all characteristic primes p belongs to exponent 1, so all prime factors are $\equiv 1 \bmod 113$. Since $\Delta_{113}(F) \equiv 1 \bmod 113$, we have $\frac{\Delta_{113}(F)}{p} \equiv 1 \bmod 113$ also. So let $p = 113x + 1$, $\frac{\Delta_{113}(F)}{p} = 113y + 1$, $2a = 113x + 113y + 2$, $2b = 113x - 113y$ and we obtain

$$\Delta_{113}(F) = (113x + 1)(113y + 1) = (a - b)(a + b).$$

We can verify that all such p up to 100,000 are not prime factors. Again $x \equiv 0, 2 \bmod 6$ and we have $x \le 884$. There are 294 numbers to check.

So we may assume $p = 113x + 1$ and $\frac{\Delta_{113}(F)}{p} = 113y + 1$ are both $> 100,000$ and hence both less than

$$\frac{\Delta_{113}(F)}{100,000} < 630,880,043.$$

Hence $a < 630,880,043$. But we have

$$\Delta_{113}(F) \equiv 10171 \equiv 1 + 113 \cdot 90 \bmod 113^2$$

hence

$$1 + 113 \cdot 90 \equiv \Delta_{113}(F) = (113x + 1)(113y + 1) \equiv 113(x + y) + 1 \bmod 113^2$$

so $x + y \equiv 90 \bmod 113$. Then $a = \frac{113(x+y)+2}{2} \equiv 5086 \bmod 113^2 = 12769$. But further, $\Delta_{113}(F) \equiv 1 \bmod 16$ so $(113x + 1)(113y + 1) \equiv 1 \bmod 16$. Thus $\{113x + 1, 113y + 1\} \equiv \{1, 1\}, \{3, -5\}, \{5, -3\}$ {7,7}, {-7, -7} or $\{-1, -1\}$ mod 16 and hence $2a = 113x + 113y + 2 \equiv \pm 2 \bmod 16$ so $a \equiv \pm 1 \bmod 8$. Similarly, $\Delta_{113}(F) \equiv 4 \bmod 9$ so $a \equiv \pm 2 \bmod 9$; $\Delta_{113}(F) \equiv 2 \bmod 5$ so $a \equiv \pm 1 \bmod 5$; and $\Delta_{113}(F) \equiv 4 \bmod 7$ so $a \equiv \pm 1, \pm 2 \bmod 7$. Combining all this with the Chinese Remainder Theorem gives 32 possible congruence classes for a modulo 32177880. There are 627 numbers to check. We find that none is a prime factor of $\Delta_{113}(F)$. □

Lehmer also proved that

$$\Delta_{127}(F) = 3,233,514,251,032,733$$

is a prime.

It is worth noting that for $F(x) = x - 2$ we obtain Mersenne primes, i.e. those of the form $2^n - 1$.

We can see that, as we attempt to find large primes by this method, we are assisted if n is large relative to $\Delta_n(F)$. In our example above with $n = 113$, there were many modulo 113 calculations. So it is preferable that $\Delta_n(F)$ grow slowly, that is, if

$$\left| \frac{\Delta_{n+1}(F)}{\Delta_n(F)} \right|$$

is small. This quantity usually converges to the Mahler measure of F.

Proposition 5.2.6 *Provided no root of F has modulus 1,*

$$\lim_{n\to\infty} \left| \frac{\Delta_{n+1}(F)}{\Delta_n(F)} \right| = \prod_{|\alpha_i|>1} |\alpha_i| = M(F).$$

Proof.

$$\lim_{n\to\infty} \left| \frac{\alpha_i^{n+1} - 1}{\alpha_i^n - 1} \right| = \begin{cases} |\alpha_i|, & \text{if } |\alpha_i| > 1 \\ 1, & \text{if } |\alpha_i| < 1. \end{cases}$$

Taking a product over i yields the result. □

Finally, the quantity $\Delta_n(F)$ also turns up in knot theory as the order of the first homology group of an n-sheeted cyclic cover of S^3 branched over a knot with Alexander polynomial F: [59], [69]

5.3 Lehmer's Problem and Salem numbers

Consider the set of all Mahler measures of integer polynomials. It is clear that the minimum possible Mahler measure is 1, and that this is achieved. But it is an open problem whether there are integer polynomials with Mahler measures arbitrarily close to 1. This is *Lehmer's problem.* Surprisingly, it seems that this is not the case.

Conjecture 5.3.1 *The minimum Mahler measure greater than 1 is the measure of the polynomial (*Lehmer's Polynomial*)*

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

which has measure approximately 1.17628.

$L(x)$ has 8 roots on the unit circle and two real roots: one equal to $M(L) \sim 1.17628$, and one equal to its reciprocal. This is a very difficult problem. Some other low-degree polynomials with small Mahler measure are:

$$\begin{aligned} M(x^8 - x^5 - x^4 - x^3 + 1) &= 1.280\ldots \\ M(x^3 - x - 1) &= 1.324\ldots \\ M(x^5 - x^3 - 1) &= 1.362\ldots \\ M(x^6 - x - 1) &= 1.370\ldots \\ M(x^7 - x^3 - 1) &= 1.379\ldots \\ M(x^4 - x - 1) &= 1.380\ldots \end{aligned}$$

Note that there are many polynomials with Mahler measure in between these; for instance, there are 71 primitive, irreducible, non-cyclotomic, integer polynomials of degree 38 with Mahler measure less than 1.3: see [60] or [59]. In these papers, Mossinghoff performs a computer search and demonstrates that no smaller Mahler measures exist up to polynomials of degree 40. The best bound towards the conjecture is quite tight, but is in terms of the degree r of F.

Theorem 5.3.2 (Dobrowolski [26])

$$\log M(F) > \frac{1}{1200}\left(\frac{\log\log r}{\log r}\right)^3.$$

One may note that Lehmer's polynomial is palindromic. It is known that any better polynomial must also be palindromic. For polynomials in 1 variable palindromic is equivalent to *reciprocal*: $F(x) = 0$ if and only if $F(x^{-1}) = 0$.

Theorem 5.3.3 (Smyth [78]) *If $F(x) \in \mathbb{Z}[x]$ is non-reciprocal and neither of $F(0)$ or $F(1)$ is 0 then*

$$M(F) \geq M(x^3 - x - 1) = 0.281$$

Lehmer's polynomial $L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ turns up suspiciously often in a number of distinct areas of mathematics. For a good summary see [34]. We mention a few appearances here.

The number $M(L)$ is also a *Salem number*: a real algebraic integer greater than 1 with all conjugates on or within the unit circle, and at least one conjugate on the unit circle. Related to Lehmer's problem is the question whether there exists a smallest Salem number. It is conjectured the answer is $M(L)$. Any smaller Salem number would have to be the root of a polynomial of degree above 40 ([5], [4], [60]).

Salem numbers are closely related to arithmetic hyperbolic surfaces. In fact, if l is the length of a closed geodesic on the quotient of $\mathbb{H}^2$ by an arithmetic Fuchsian group, then $e^{\frac{l}{2}}$ is a Salem number; and every Salem number is of this form: see [34]. The existence of a closed geodesic of minimal length amongst all arithmetic hyperbolic surfaces is equivalent to the existence of a minimal Salem number.

Salem numbers arise in the growth series of hyperbolic planar reflection groups: [31]. Given a group G and a set of generators S, the *growth series* for G with respect to S is the formal power series

$$f(x) = \sum_{n=1}^{\infty} N_S(n) x^n$$

where $N_S(n)$ is the number of elements of G that can be expressed as a word of length n in the generators of S, but no shorter. The *Coxeter reflection group* $G_{p_1,\ldots,p_d}$ is the group generated by reflections in the sides of a polygon with interior angles $\frac{\pi}{p_i}$. If S is chosen suitably "geometrically" to generate $G_{p_1,\ldots,p_d}$ then usually $f(x) = \pm f(\frac{1}{x})$ ([31]). In this case, it can be shown that $f(x)$ is a rational function of x ([31], [79])) and the denominator $\Delta_{p_1,\ldots,p_d}(x)$ is a product of cyclotomic polynomials and at most one Salem polynomial: [31], [14], [64]. Suspiciously, $\Delta_{2,3,7}(x) = L(x)$, and the $(2, 3, 7)$-hyperbolic triangle has the smallest volume among such polygons. It has been shown that $M(L)$ is the smallest number arising as a root of any $\Delta_{p_1,\ldots,p_d}(x)$: [39].

In fact, $\Delta_{p_1,\ldots,p_d}(-x)$ is the Alexander polynomial of the $(p_1, \ldots, p_d, -1)$-pretzel link, with respect to a suitable orientation (see [39]). Thus $L(-x)$ is the Alexander polynomial of the (-2,3,7)-pretzel knot ([39], [34]). So the minimal Mahler measure of Alexander polynomials of suitably oriented $(p_1, \ldots, p_d, -1)$-pretzel links is attained by $M(L)$.

Salem numbers and Mahler measures also appear in special values of L-functions. Some of this is related to explicit evaluations in the next chapter: see generally [34].

Finally, the Mahler measure (although in a form closer related to the logarithmic version of the next chapter) also arises in dynamical systems: see, e.g., [29]. It is also related to exceptional units in number fields ([30], [74], [75]), polylogarithm relations ([15]). For a good summary see [59].

Chapter 6

Mahler Measure in Two Variables

We now end our number-theoretical frolic and examine a generalised form of the Mahler measure. We will see that the original definition of $M(F)$ is rather peculiar to one-variable polynomials.

6.1 Logarithmic Mahler Measure

The *logarithmic Mahler measure* of a nonzero polynomial $F(x_1, x_2, \ldots, x_n) \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is defined as

$$\mathcal{M}(F) = \int_0^1 \cdots \int_0^1 \log \left| F\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots, e^{2\pi i\theta_n}\right) \right| d\theta_1 d\theta_2 \cdots d\theta_n.$$

We will not consider Mahler measure in more than two variables. We first assure the reader that this entirely different-looking definition is the (logarithm of the) measure of the previous chapter.

Proposition 6.1.1 *For a one-variable polynomial*

$$F(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + x_0 = a_r\left(x - \alpha_1\right)\left(x - \alpha_2\right)\cdots\left(x - \alpha_r\right),$$

$\mathcal{M}(F)$ *is the logarithm of* $M(F)$*; that is,*

$$\int_0^1 \log \left| F(e^{2\pi i\theta}) \right| d\theta = \mathcal{M}(F) = \log M(F) = \log |a_r| + \sum_{|\alpha_i| \geq 1} \log |\alpha_i|.$$

Lemma 6.1.2 (Jensen's Formula) *For any* $\alpha \in \mathbb{C}$*,*

$$\int_0^1 \log \left| \alpha - e^{2\pi i\theta} \right| d\theta = \log \max\{1, |\alpha|\} = \begin{cases} 0, & \text{if } |\alpha| \leq 1 \\ \log |\alpha|, & \text{if } |\alpha| \geq 1. \end{cases}$$

Proof. If $\alpha = 0$ then the integral is clearly 0. We prove the result for $|\alpha| \neq 1$ and refer to [29] for the more difficult case $|\alpha| = 1$. First consider $|\alpha| < 1$. We then have

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \int_0^1 \log\left|1 - e^{-2\pi i\theta}\alpha\right| d\theta = \int_0^1 \log\left|1 - e^{2\pi i\theta}\alpha\right| d\theta$$

(the last equality is obtained by substituting $-\theta$ for θ). Then $\log|z| = \mathrm{Re}\log z$ and $\left|e^{2\pi i\theta}\alpha\right| < 1$ so we can expand a Taylor series and the integral is

$$\begin{aligned}\mathrm{Re}\int_0^1 \log\left(1 - \alpha e^{2\pi i\theta}\right) d\theta &= -\mathrm{Re}\int_0^1 \sum_{n=1}^{\infty} \frac{\left(\alpha e^{2\pi i\theta}\right)^n}{n}\, d\theta \\ &= -\mathrm{Re}\sum_{n=1}^{\infty} \frac{\alpha^n}{n}\int_0^1 e^{2\pi i\theta n} d\theta = 0.\end{aligned}$$

If $|\alpha| > 1$ then

$$\int_0^1 \log\left|\alpha - e^{2\pi i\theta}\right| d\theta = \log|\alpha| + \int_0^1 \log\left|1 - \frac{1}{\alpha}e^{2\pi i\theta}\right| d\theta.$$

Since $\left|\frac{1}{\alpha}\right| < 1$ this reduces to the first case and the integral is 0, so the original integral is equal to $\log|\alpha|$ as required. □

Proof. (of proposition 6.1.1). Let $F(x) = a_r(x - \alpha_1)\cdots(x - \alpha_r)$. Then

$$\log\left|F(e^{2\pi i\theta})\right| = \log|a_r| + \log\left|e^{2\pi i\theta} - \alpha_1\right| + \log\left|e^{2\pi i\theta} - \alpha_2\right| + \cdots + \log\left|e^{2\pi i\theta} - \alpha_r\right|$$

so from Jensen's formula

$$\mathcal{M}(F) = \int_0^1 \log\left|F\left(e^{2\pi i\theta}\right)\right| d\theta = \log|a_r| + \sum_{i=1}^{r} \log\max\{1, |\alpha_i|\} = \log M(F).$$

□

We can now prove $M(F) \leq L(F)$ for F in one variable, as promised in proposition 5.1.1 of the previous chapter. In fact the proof generalises immediately: see [51].

Proof. We bound $\left|F(e^{2\pi i\theta})\right|$, then $M(F)$.

$$\begin{aligned}\left|F(e^{2\pi i\theta})\right| &= \left|a_r e^{2\pi i\theta r} + a_{r-1}e^{2\pi i\theta(r-1)} + \cdots + a_0\right| \\ &\leq \left|a_r e^{2\pi i\theta r}\right| + \left|a_{r-1}e^{2\pi i\theta(r-1)}\right| + \cdots + |a_0| \\ &= |a_r| + |a_{r-1}| + \cdots + |a_0| = L(F) \\ M(F) &= \exp\int_0^1 \log\left|F(e^{2\pi i\theta})\right| d\theta \\ &\leq \exp\int_0^1 \log L(F)\, d\theta = L(F).\end{aligned}$$

□

We will henceforth refer to the logarithmic Mahler measure simply as *the Mahler measure*.

In two variables, the Mahler measure is an integral taken over the torus $S^1 \times S^1$:

$$\mathcal{M}(F) = \int_0^1 \int_0^1 \log \left| F\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2.$$

It turns out that $\mathcal{M}(F)$ defined this way exists for all $F \neq 0$, and is nonnegative for all $f \in \mathbb{Z}[x, y]$: see [29].

Calculating exact values of the Mahler measure is in general exceedingly difficult. To get some idea of the sort of complexity involved, observe some of the simplest cases, from [29]:

$$\begin{aligned} \mathcal{M}(2 + x_1 + x_2) &= \log 2 \\ \mathcal{M}(1 + x_1 + x_2) &= \frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{n}{3}\right)}{n^2} \\ \mathcal{M}(1 + x_1 + x_2 + x_3) &= \frac{7}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3}. \end{aligned}$$

(here $\left(\frac{n}{3}\right)$ denotes the Legendre symbol, equal to 1, -1, 0 respectively as $n \equiv 1$, 2, 3 mod 3.)

6.2 Basic Properties

We establish some simple properties of the Mahler measure. We will express a 2-variable polynomial $F(x, y) \in \mathbb{C}[x, y]$ in the form

$$F(x, y) = \sum_{m,n} a_{m,n} x^m y^n.$$

The *degree in* x of F is the highest m for which there exists n such that $a_{m,n} \neq 0$. Similarly, the degree in y. The *total degree* of $F(x, y)$ is the largest $m + n$ for which $a_{m,n} \neq 0$. The *coefficient matrix* of $F(x, y)$ is the matrix of $a_{m,n}$. The *support* of $F(x, y)$ is the set $\{(m, n) : a_{mn} \neq 0\} \subset \mathbb{Z}^2$, a finite set of lattice points. The convex hull of the support of F is the *Newton polygon* of F, denoted $\mathcal{C}(F)$.

By multiplying out terms in the multiplication of two polynomials, it's quite easy to see that $\mathcal{C}(FG) = \mathcal{C}(F) + \mathcal{C}(G)$, where the addition is a pointwise addition of subsets of $\mathbb{R}^2$ (see [29] for a proof). We can obtain the Newton polygon of a product of two polynomials by 'sliding' the polygon of one polynomial around the other.

Proposition 6.2.1 $\mathcal{M}(FG) = \mathcal{M}(F) + \mathcal{M}(G)$

Proof.

$$\begin{aligned}
\mathcal{M}(FG) &= \int_0^1\int_0^1 \log\left|F\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)G\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2 \\
&= \int_0^1\int_0^1 \log\left|F\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right|\cdot\left|G\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2 \\
&= \int_0^1\int_0^1 \log\left|F\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2 \\
&\quad + \int_0^1\int_0^1 \log\left|G\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2 \\
&= \mathcal{M}(F)+\mathcal{M}(G)
\end{aligned}$$

□

We omit the proof of the following basic fact about Mahler measure, since the proof is long and off track: it may be found in [29] or [77].

Proposition 6.2.2 *If F is an integer Laurent polynomial in n variables whose coefficients are integers with no non-trivial common factor, then $\mathcal{M}(F)=0$ if and only if F is a monomial times a product of cyclotomic polynomials evaluated on monomials.*

The integral is over the torus and hence other parametrisations of the torus will give the same result.

Proposition 6.2.3 *Let $G(x,y)=F(x^ay^b,x^cy^d)=F(u,v)$ where*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}).$$

Then $\mathcal{M}(F)=\mathcal{M}(G)$.

Proof.

$$\begin{aligned}
\mathcal{M}(G) &= \int_0^1\int_0^1 \log\left|G\left(e^{2\pi i\theta_1},e^{2\pi i\theta_2}\right)\right| d\theta_1 d\theta_2 \\
&= \int_0^1\int_0^1 \log\left|F\left(e^{2\pi ia\theta_1+2\pi ib\theta_2},e^{2\pi ic\theta_1+2\pi id\theta_2}\right)\right| d\theta_1 d\theta_2
\end{aligned}$$

Let $\phi_1=a\theta_1+b\theta_2, \phi_2=c\theta_1+d\theta_2$, so

$$\frac{\partial(\phi_1,\phi_2)}{\partial(\theta_1,\theta_2)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with determinant 1. Then

$$\begin{aligned}
\mathcal{M}(G) &= \int_0^1\int_0^1 \log\left|F\left(e^{2\pi i\phi_1},e^{2\pi i\phi_2}\right)\right| \left|\det\frac{\partial(\phi_1,\phi_2)}{\partial(\theta_1,\theta_2)}\right| d\theta_1 d\theta_2 \\
&= \int\int_R \log\left|F\left(e^{2\pi i\phi_1},e^{2\pi i\phi_2}\right)\right| d\phi_1 d\phi_2
\end{aligned}$$

where the integral is taken over the region R, which is the image of the square $[0,1]^2$ under the area-preserving linear transformation $(\theta_1, \theta_2) \mapsto (a\theta_1 + b\theta_2, c\theta_1 + d\theta_1)$. The region R is another fundamental domain for the torus $\mathbb{R}^2/\mathbb{Z}^2$, so the integral gives the same result. Hence $\mathcal{M}(F) = \mathcal{M}(G)$. □

The matrices of coefficients of F and G above are related by the linear transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since this transformation is area-preserving, the area of the Newton polygon is preserved, along with the Mahler measure.

6.3 Calculation

I created a crude algorithm, using MAPLE, to calculate (very) approximate Mahler measures of two-variable polynomials. The idea is very simple. Consider $F(x,y) \in \mathbb{C}[x,y]$ with degree d_x in x and degree d_y in y. Substituting a particular value y_0 for y gives a polynomial $F(x, y_0)$ in x only, which has degree $\leq d_x$. It will have degree strictly less than d_x iff the terms of highest degree in x vanish at $y = y_0$. Letting the terms of highest degree in x be $P(y)x^{d_x}$, where $P \in \mathbb{C}[y]$, then $F(x, y_0)$ has degree strictly less than d_x if and only if y_0 is a root of $P(y)$. In all of the polynomials considered, $P(y)$ was a monomial, so problems could only arise at $y_0 = 0$.

Assuming that y_0 is not a root of $P(y)$, we let the roots of $F(x, y_0)$ be $x_1(y_0), x_2(y_0), \ldots, x_{d_x}(y_0)$ (counted with multiplicity; for now assume these roots are distinct). For nearby values of y_0 (avoiding roots of $P(y)$), we obtain d_x nearby values of x. In this way each x_i can be considered as a continuous function of y_0 — or simply y. In fact the functions $x_i(y)$ so defined are *algebraic functions* or *algebraic branches* of $F(x,y) = 0$. Thus we have a factorisation of $F(x,y)$

$$F(x,y) = a_0(y)\,(x - x_1(y))\,(x - x_2(y)) \cdots (x - x_{d_x}(y))\,.$$

From proposition 6.2.1 it's clear that $\mathcal{M}(F) = \mathcal{M}(a_0(y)) + \mathcal{M}(F_1) + \cdots + \mathcal{M}(F_{d_x})$, where $F_i(x,y) = x - x_i(y)$. Consider the integral for $\mathcal{M}(F_i)$.

$$\int_0^1 \int_0^1 \log\left|e^{2\pi i\theta_1} - x_i(e^{2\pi i\theta_2})\right| d\theta_1 d\theta_2,$$

Jensen's formula 6.1.2 above yields

$$\int_0^1 \log\left|e^{2\pi i\theta_1} - x_i(e^{2\pi i\theta_2})\right| d\theta_1 = \log^+\left|x_i(e^{2\pi i\theta_2})\right|$$

where $\log^+(x) = \log\max\{1, |x|\}$. Thus

$$\mathcal{M}(F_i) = \int_0^1 \log^+\left|x_i(e^{2\pi i\theta_2})\right| d\theta_2$$

and our computations are reduced to a sum of single integrals.

We can simplify a little. To remain a little more geometric, we replace $2\pi\theta_2$ with θ, so that

$$\mathcal{M}(F_i) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |x_i(e^{i\theta})| d\theta. \tag{6.1}$$

We will perform more simplifications later, using the symmetries of the A-polynomial.

We can thus regard y (or θ) as an independent variable, solve for $x_i(y) = x_i(e^{i\theta})$ for many values of θ (the more values, the greater the 'refinement'), take $\log^+$ and easily obtain a Riemann sum for the integral. In fact, we can easily obtain the separate integrals $\mathcal{M}(F_i)$ also, provided that as we solve for x_i at each stage, we ensure that consecutive solutions to $F(x, e^{i\theta_2})$ are allocated to the correct x_i. One obvious way to do this would be to take $\theta = 2\pi\frac{k}{N}$ for a large 'refinement' N and $k = 1, 2, \ldots, N$, and choose $x_i(e^{\frac{i(k+1)}{N}})$ to be the closest of the roots of $F(x, e^{\frac{2\pi i(k+1)}{N}})$ to $x_i(e^{\frac{2\pi ik}{N}})$. This is quite crude however, and I opted for a slightly more accurate option: predicting the location of $x_i(e^{\frac{2\pi i(k+1)}{N}})$ based on a linear interpolation from the past two values $x_i(e^{\frac{2\pi ik}{N}})$ and $x_i(e^{\frac{2\pi i(k-1)}{N}})$.

As it turns out, when we take consider the A-polynomial of a knot, each of the separate integrals $\mathcal{M}(F_i)$ has a separate interpretation, so our calculations are important.

It is possible that for some values y_0 of y, $F(x, y_0)$ has repeated roots. This poses problems for our calculations of separate $\mathcal{M}(F_i)$, but not for calculations of $\mathcal{M}(F)$. Given two polynomials $f(x), g(x)$, the *resultant* gives us a polynomial whose roots are the common roots of $f(x)$ and $g(x)$. The resultant of $f(x)$ and $f'(x)$ gives us the common roots of $f(x)$ and $f'(x)$ — that is, the repeated roots of $f(x)$. So the resultant of $F(x, y)$ and $\frac{\partial F}{\partial x}$ over x is a polynomial in y which tells us the values of y for which $F(x, y)$ has multiple roots in x. In our calculations, often branches did meet, but on the unit circle, where integrals became zero.

Part III

Group representations

Chapter 7

The A-Polynomial

7.1 Philosophy

The A-polynomial is a polynomial associated to a 3-manifold with boundary a torus T; we consider only knot exteriors $S^3 - \nu(K)$. We first give some motivation for the use of this polynomial.

The starting point is the result that the fundamental group and peripheral subgroup form a complete knot invariant. The problem is that fundamental groups are difficult to work with; all the problems of combinatorial group theory arise. Our philosophy, then, is to simplify this data in some way, preserving as much structure as possible, while finishing with an easily manipulated object, such as a polynomial. Both the A-polynomial and Alexander polynomial are results of this philosophy, but unlike the Alexander polynomial, no combinatorial procedure is known for computing the A-polynomial from a knot diagram.

As discussed in section 4.2, there are good reasons to investigate representations into $SL_2(\mathbb{C})$. The representation variety $R(\pi_1(M))$ and character variety $X(\pi_1(M))$ are affine algebraic sets, but are still difficult to work with, so we simplify further. We do not consider the image of the entire group under a representation ρ, but restrict to the peripheral subgroup. The image of $\pi_1(T)$ is determined by $\rho(\lambda)$ and $\rho(\mu)$, where λ and μ are a standard longitude and meridian respectively, so we consider only these.

We strip down our data to a manageable minimum, and avoid the redundancy of conjugate representations, by taking only the eigenvalues of $\rho(\lambda)$ and $\rho(\mu)$. Since the eigenvalues of a matrix in $SL_2(\mathbb{C})$ multiply to 1, we take only one eigenvalue.

Therefore, we take those $(l, m) \in \mathbb{C}^2$ for which there exists $\rho \in R(\pi_1(M))$ with $\rho(\lambda), \rho(\mu)$ having (ordered) eigenvalues $\left(l, l^{-1}\right)$ and $\left(m, m^{-1}\right)$ respectively. This set turns out to be (almost) the zero set of a polynomial $A_K(x, y)$, which is the A-polynomial, and (after some normalisation) is a knot invariant. It is not a complete knot invariant, and some mutants (perhaps all) have the same A-polynomial: [19, section 7].

The A-polynomial is built entirely from algebraic data — the fundamental group and peripheral subgroup. In this sense the A-polynomial has *nothing* to do with the geometry of the manifold. We will see soon enough however, that it has *plenty* to tell us about the geometry of the manifold!

7.2 Definition

We make the above definition precise. In the exposition here we avoid the use of character varieties of [16] and instead follow [19] (also [18]). Let

$$\pi_1(K) =< g_1, \ldots, g_n | r_1, \ldots, r_m >$$

so, as in section 4.3,

$$R(\pi_1(K)) = \left\{ (a_1, b_1, c_1, d_1, \ldots, d_n) \in \mathbb{C}^{4n} \mid p_{1,1} = p_{1,2} = \cdots, p_{m,4} = 0 \right\}.$$

First we restrict to those representations ρ with $\rho(\lambda)$ and $\rho(\mu)$ upper triangular. Now λ, μ can be expressed as words in $g_1, \ldots, g_n$ and their inverses so $\rho(\lambda)$ and $\rho(\mu)$ are matrices with entries which are polynomials in $\{a_i, b_i, c_i, d_i\}_{i=1}^n$. Let q_λ, q_μ be the polynomials in the lower left entries of $\rho(\lambda), \rho(\mu)$ respectively. Consider the affine algebraic set $R_U(\pi_1(K)) \subseteq R(\pi_1(K))$ obtained by adjoining $q_\lambda = 0, q_\mu = 0$ (we abbreviate to R_U and R when the context is clear).

Given any $\rho \in R$, since λ and μ commute, we can conjugate ρ such that $\rho(\lambda)$ and $\rho(\mu)$ are both diagonal, or are both of the form $\pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (representing commuting parabolics). In particular, any representation $\pi_1(K) \longrightarrow SL_2(\mathbb{C})$ is conjugate to one which is upper triangular on $\pi_1(T)$. So restricting to R_U loses no information.

Next we define two projection functions $\xi_\lambda, \xi_\mu : R_U \to \mathbb{C}$. For $\rho \in R_U$, let $\xi_\lambda(\rho), \xi_\mu(\rho)$ be the upper left entries of $\rho(\lambda), \rho(\mu)$ respectively. Since these entries are polynomials in $\{a_i, b_i, c_i, d_i\}_{i=1}^n$, both ξ_λ, ξ_μ are polynomial maps, hence morphisms. We then define

$$\xi : R_U(\pi_1(M)) \to \mathbb{C}^2, \qquad \xi(\rho) = (\xi_\lambda(\rho), \xi_\mu(\rho)) = (l, m).$$

Note ξ is not necessarily injective: there is nothing a priori to stop non-conjugate representations ρ_1, ρ_2 having $\rho_1(\lambda) = \rho_2(\lambda)$ and $\rho_1(\mu) = \rho_2(\mu)$. But $\xi(R_U) \subseteq \mathbb{C}^2$ is the image of the affine algebraic set R_U under a morphism. It may consist of different components of different dimensions and need not be Zariski closed. Thus we take a component C of R_U and consider the Zariski closure $\overline{\xi(C)}$. By definition, $\overline{\xi(C)}$ is the zero set of a family of polynomials, but there could be more than one. If $\overline{\xi(C)}$ is defined by a single polynomial (i.e. it is a *curve*), then we take this polynomial F_C. In this case $\overline{\xi(C)}$ consists of only finitely more points than $\xi(C)$. (This follows from some standard arguments about projective varieties: see, e.g., [55, ch. 2].) The A-polynomial $A_K(l, m)$ is provisionally defined to be the product of such F_C over all the components C

of R_U. Curves here (like [19] but unlike [16]) are counted with multiplicity, so that $A_K(x,y)$ may have repeated factors.

Our provisional A-polynomial is defined up to a scalar multiple. In [16] it is shown that a scalar can be chosen so that all coefficients are integers. We require the coefficients to have no common factor, so A_K is well defined up to sign.

There is one further convention. Abelian representations are not interesting, since they are the same for all knots: we found all abelian representations in section 4.5. We can construct an abelian representation ρ with $\rho(\mu)$ arbitrary, but we always have $\rho(\lambda) = I$. Thus $\xi_\mu(\rho)$ is arbitrary but $\xi_\lambda(\rho) = 1$. Hence $A_K(l,m)$ is divisible by $l-1$ for any knot K in S^3; upon dividing through by $l-1$, we have our A-polynomial $A_K(l,m)$.

7.3 Basic Properties

We run through some of the most fundamental properties of the A-polynomial of a knot.

Proposition 7.3.1 *If K is the unknot, $A_K(l,m) = \pm 1$.*

Proof. Since $\pi_1(K) = \mathbb{Z}$ we have only abelian representations, so $\xi_\mu(\rho)$ is arbitrary but $\xi_\lambda(\rho) = 1$. This gives a polynomial $\pm(l-1)$; upon division by $l-1$ we have ± 1. □

Proposition 7.3.2 ([16]) *The A-polynomial of a hyperbolic knot K is non-trivial (i.e. $\neq \pm 1$).*

Proof. The knot group $\pi_1(K)$ is non-abelian. By theorem 4.2.3 (see more generally [81]), there is a 1-parameter family of hyperbolic structures obtained from deforming the complete structure, each with holonomy representation ρ into $PSL_2(\mathbb{C})$, none of which have $\rho(\lambda)$ (hence nor $\rho(\mu)$) parabolic. Upon lifting to $\tilde{\rho}$ into $SL_2(\mathbb{C})$ and conjugating so $\tilde{\rho}(\lambda)$ is upper triangular, we see $\xi_\lambda(\tilde{\rho}) \neq 1$. So there exist l, m with $l \neq 1$ and $A_K(l,m) = 0$. Upon dividing by $l-1$ we still have a nontrivial polynomial. □

Proposition 7.3.3 $A_K(l,m) = \pm A(l^{-1}, m^{-1})$*, after multiplying by appropriate powers of l and m.*

Proof. Let $\rho \in R_U$ have $\xi(\rho) = (l,m)$. Then $\rho(\lambda), \rho(\mu)$ are commuting upper triangular matrices in $SL_2(\mathbb{C})$. Conjugating in $SL_2(\mathbb{C})$ if necessary, we may assume

$$\rho(\lambda) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \qquad \rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

Conjugating by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we obtain a representation ρ' such that

$$\rho'(\lambda) = \begin{pmatrix} l^{-1} & 0 \\ 0 & l \end{pmatrix}, \qquad \rho(\mu) = \begin{pmatrix} m^{-1} & 0 \\ 0 & m \end{pmatrix}.$$

Clearly $\xi(\rho') = (l^{-1}, m^{-1})$. (The conjugation can be visualised as rotating the fixed geodesic $(0, \infty)$ of $\rho(\lambda), \rho(\mu)$, considered in $PSL_2(\mathbb{C})$, through 180° back onto itself in the opposite direction.)

Taking closures of the components of R_U under ξ which are curves, which only adds finitely many points to those arising in R_U, we must have $A_K(l, m) = 0$ if and only if $A_K(l^{-1}, m^{-1}) = 0$. So after multiplying by appropriate powers of l and m, these polynomials are identical up to sign. □

Corollary 7.3.4 *Reversing the orientation on* K *(i.e. reversing* λ, μ*) does not change* A_K*.*

Proposition 7.3.5 $A_K(l, m)$ *is even in* m*.*

Proof. Let $\tilde{\rho} \in R_U$, with $\xi(\rho) = (l, m)$, be a lift of $\rho : \pi_1(K) \longrightarrow PSL_2(\mathbb{C})$. Let $\tilde{\rho}'$ be the other lift, as guaranteed by proposition 4.2.1. Then $\tilde{\rho}'(g_i) = -\tilde{\rho}(g_i)$ for g_i any generator in the Wirtinger presentation. So $\tilde{\rho}'(\alpha) = (-1)^{w(\alpha)}\rho(\alpha)$, where w is a winding number (abelianization) homomorphism $\pi_1(K) \longrightarrow \mathbb{Z}$ discussed in 4.5. Since μ is a meridian and λ a (nullhomologous) longitude, $w(\mu) = 1$ and $w(\lambda) = 0$. So $\tilde{\rho}'(\mu) = -\tilde{\rho}(\mu)$ but $\tilde{\rho}'(\lambda) = \tilde{\rho}(\lambda)$, giving $\xi(\rho') = (l, -m)$. Upon taking closures of components of R_U under ξ which are curves, which only adds finitely many points to those in R_U, we have $A_K(l, m) = 0$ if and only if $A_K(l, -m) = 0$. □

Proposition 7.3.6 *If* $A_K(l, \pm 1) = 0$ *then* $l = \pm 1$*.*

Proof. If we have a representation $\tilde{\rho} : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ with $\tilde{\rho}(\mu)$ having eigenvalue ± 1 then $\tilde{\rho}(\mu)$ is $\pm I$, or a lift of $\rho : \pi_1(K) \longrightarrow PSL_2(\mathbb{C})$ with $\rho(\mu)$ parabolic. In the first case, since $\pi_1(K)$ is generated by meridians conjugate to μ, ρ is the identity representation, so $\tilde{\rho}$ is abelian and $l = 1$. In the second case $\rho(\lambda)$ is parabolic, so $\tilde{\rho}(\lambda)$ has eigenvalues ± 1 and $l = \pm 1$. Taking closures of curve components of $\xi(R_U)$ adds only finitely many points, so $A_K(l, \pm 1) = 0$ implies $l = \pm 1$. □

7.4 Calculation

In general the calculation of the A-polynomial is a grungy exercise in elimination of variables (see also [16]). Given a presentation $\pi_1(K) =< g_i \mid r_i >$, we set each $\rho(g_i)$ to some general form involving pronumerals $x_{i1}, \ldots, x_{ij}$. Then using the relators, we find polynomial equations in the x_{ij} which define R_U. Then, obtain $m = \xi_\mu(\rho)$ (best done by letting μ be a generator, so m is some x_{ij}) and $l = \xi_\lambda(\rho)$, and eliminate all other variables from these equations, leaving a polynomial in l, m. Provided that all components of $\xi(R_U)$ are curves, this calculation will give us the A-polynomial, up to scalar multiple. I do not know of any knot where this is not the case.

For two-bridge knots however, we shall see that these computations are quite manageable, since $\pi_1(K)$ has a nice form and μ, λ are readily computable.

Let K be a 2-bridge knot. Recall from theorem 2.4.4 and subsequent discussion that we can let

$$\pi_1(K) =< g, h \mid gw = wh >, \qquad \mu = g, \qquad \lambda = g^{-2\sigma} w\tilde{w}.$$

We find a more agreeable normal form (in R_U) for a non-abelian representation $\tilde{\rho} : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$. Since $\tilde{\rho}(g)$ and $\tilde{\rho}(h)$ both have determinant 1, and have the same trace (being conjugate by w), they have identical eigenvalues m, m^{-1}. Descend to $PSL_2(\mathbb{C})$ to obtain a representation ρ giving isometries of $\mathbb{H}^3$. Let $\rho(g)$ have fixed points at infinity α, β; possibly $\alpha = \beta$ if $\rho(g)$ is parabolic, i.e. $m = \pm 1$. Since we have a non-abelian representation, $\rho(h)$ has at least one fixed point γ different from α, β. Take a hyperbolic isometry f mapping $(\alpha, \beta, \gamma) \mapsto (\infty, \frac{m}{1-m^2}, 0)$ (guaranteed by proposition 3.1.2). Then $\rho'(g) = f\rho(g)f^{-1}$ fixes ∞ and $\frac{m}{1-m^2}$ (which are equal when $m = \pm 1$) and $\rho'(h) = f\rho(h)f^{-1}$ fixes 0. So $\rho'(g)$ is $z \mapsto az + b$ for some $a, b \in \mathbb{C}$, but considering a lift to $SL_2(\mathbb{C})$ we see that $a = m^2$.

If $\rho'(g)$ is not parabolic, then since $\rho'(g)$ fixes $\frac{m}{1-m^2}$ we then have $b = m$, so $\rho'(g)$ is $z \mapsto m^2 z + m$. Then taking the appropriate lift $\tilde{\rho}'$ back to $SL_2(\mathbb{C})$ (the one with $\tilde{\rho}'(g)$ having trace $m + m^{-1}$), we have a representation $\tilde{\rho}'$ conjugate to $\tilde{\rho}$. Then $\tilde{\rho}'(g)$ is upper triangular, and $\tilde{\rho}'(h)$ is lower triangular, both with eigenvalues m, m^{-1}. If the eigenvalues are not in the same order, repeat the argument, but choose f mapping $(\alpha, \beta, \gamma) \mapsto (\frac{m}{1-m^2}, \infty, 0)$.

If $\rho'(g)$ is parabolic, then conjugate by a Euclidean spiral symmetry around a vertical axis so that $\rho'(g)$ is $z \mapsto z+1$, and take the lift $\tilde{\rho}'$ with $\operatorname{tr} \tilde{\rho}'(g) = m + m^{-1}$ (which is ± 2). Then $\tilde{\rho}'(h)$ is lower triangular, with the same equal eigenvalues as $\tilde{\rho}'(h)$.

So, in all cases, there is a conjugate $\tilde{\rho}'$ of $\tilde{\rho}$ such that

$$\tilde{\rho}'(g) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \qquad \tilde{\rho}'(h) = \begin{pmatrix} m & 0 \\ -t & m^{-1} \end{pmatrix}$$

for some $t \in \mathbb{C}$. $\tilde{\rho}'(g)$, the image of our standard meridian, is upper triangular, hence so is $\tilde{\rho}(\lambda)$ (which must commute with it), hence $\tilde{\rho}' \in R_U$. So we take this as our normal form for a non-abelian representation in R_U, and lose no information. Call our representation simply ρ.

Next we find polynomial equations defining such representations by considering the relator $gw = wh$. We evaluate $\rho(gw) - \rho(wh)$ (which involves lower degree expressions than $\rho(gwh^{-1}w^{-1})$) and obtain four Laurent polynomial equations stating that this is the zero matrix. We shall see (chapter 9) that one suffices. After multiplying by an appropriate power of m this gives a defining polynomial $\Phi(m, t) = 0$ for non-abelian representations in R_U; in particular, every component of R_U is a curve. We then calculate the upper triangular $\rho(\lambda)$, setting $l = \xi_\lambda(\rho)$, giving a Laurent polynomial equation, which after multiplying by an appropriate power of m is the polynomial $\Xi(l, m, t) = 0$.

All that remains is to eliminate t from Φ, Ξ: we use the resultant over t. The set of (l, m) thus obtained clearly gives $\bigcup_C \overline{\xi(C)}$ over non-abelian components C of R_U (with multiplicity), and all are curves. So we obtain the A-polynomial.

We will investigate the special case of twist knots further in chapter 8. For these knots, we can explicitly perform some of the above calculations in generality.

7.5 Boundary slopes

The most startling results about the A-polynomial are geometric ones. Despite the purely algebraic pretensions of the A-polynomial, it has a remarkable relationship with the geometry of the knot (or cusped manifold) under examination.

In fact, there is a relationship between the geometry of the knot complement and the *geometry of the polynomial*: its Newton polygon.

Theorem 7.5.1 *Suppose the Newton polygon $\mathcal{C}(A_K(l,m))$ has a side of slope q. Then there is an incompressible surface S in $S^3 - \nu(K)$ with boundary which is a curve on the torus ∂M of slope q.*

We will only indicate the flavour of the proof. There are a number of steps. We follow [16] and [25] generally.

1. *Boundary slopes of the Newton polygon give valuations on the function field of* $\mathbb{V}(A_K)$. Given $(l, m) \in \mathbb{V}(A_K(l,m))$, a parametrisation known as a *Puiseux parametrisation* can give rise to a valuation. Denote the function field by $\mathbb{C}(A)$. Given $p, q \in \mathbb{Z}$, we define a Puiseux parametrisation of $(l, m) \in \mathbb{V}(A(l,m))$ by

$$l(t) = t^p, \qquad m(t) = t^q \sum_{k=0}^{\infty} b_k t^k.$$

Puiseux parametrisations can only be given to points on the A-polynomial variety in situations corresponding to boundary slopes of the Newton Polygon. The equation $A(l(t), m(t)) = 0$ gives a power series in t which must be identically zero. If $A(l,m) = \sum a_{ij} l^i m^j$ then the series is

$$\sum_{i,j} a_{ij} t^{ip+jq} \left(\sum_{k=0}^{\infty} b_n t^n \right)^j.$$

Let the lowest degree of t occurring in this series be d, so d is the minimum value of $ip + jq$, as i, j range over the nonzero terms $a_{ij} l^i m^j$ occurring in $A(l, m)$. The terms of lowest degree in t give

$$\sum_{ip+jq=d} a_{ij} b_0^j t^d = 0.$$

A solution can only exist if there is more than one term in this sum — i.e. there are at least two a_{ij} satisfying $ip + jq = d$. Since d is chosen to be minimal, this is equivalent to saying that there is a lower side of the Newton polygon of slope $-\frac{p}{q}$.

The valuation v on $\mathbb{C}(A)$ is defined as follows. Take a rational function $R(l, m) \in \mathbb{C}(A)$. Substitute the above parametrisation and obtain $R(l(t), m(t)) = t^k E(t)$ where E is a power series with nonzero constant term. We define $v(R(l, m)) = k$. It can be shown [46] that two rational functions representing the same element of $\mathbb{C}(A)$ have the same valuation, so that v is well-defined. Note that $v(l) = p, v(m) = q$.

Valuations can also be obtained for upper sides of the Newton polygon, applying birational equivalences: see [16].

2. *A valuation on the function field gives an action of $\pi_1(K)$ on a certain tree.* Given the valuation v above, let $\mathcal{O}$ be the valuation ring (i.e. $\mathcal{O} = \{x \in \mathbb{C}(A) \mid v(x) \geq 0\}$), and π π a uniformizing element (i.e. $v(\pi)$ positive and minimal). We consider $W = \mathbb{C}(A) \oplus \mathbb{C}(A)$, a 2-dimensional vector space over the field $\mathbb{C}(A)$. A *lattice* in W is an $\mathcal{O}$-submodule of W which generates W over $\mathbb{C}(A)$. Given two lattices L, L' in W we see that $\frac{L}{L'} \cong \pi^a \mathcal{O} \oplus \pi^b \mathcal{O}$ for some integers a, b. We define $|a - b|$ to be the *distance* $d(L, L')$ between L and L'. Since $\frac{xL}{L'} \cong \pi^{a-c} \mathcal{O} \oplus \pi^{b-c} \mathcal{O}$ for some integer c, we see that distance is not affected by scalar multiplication. Thus we define lattices L, L' to be *equivalent* if $L = xL'$ for some $x \in \mathbb{C}(A)$. Two equivalence classes Λ, Λ' of lattices have a well-defined distance $d(\Lambda, \Lambda')$ from the above.

   Our tree then has for vertices the equivalences classes of lattices in W. An edge joins two equivalence classes Λ, Λ' iff $d(\Lambda, \Lambda') = 1$. That this is in fact a tree is proven in [73].

   The idea of the action of $\pi_1(K)$ on the tree is that for a representation ρ and $g \in \pi_1(K)$, $\rho(g) \in SL_2(\mathbb{C})$ is a 2×2 matrix, with entries in $\mathbb{C} \subseteq \mathbb{C}(A)$, which we can consider as a linear transformation on W, hence on lattices, hence an action on the tree. Actually there are significant technicalities involved: see [16].

   Alternatively, an action of $\pi_1(M)$ on a simplicial tree can be obtained by regarding elements of $\rho\pi_1(M)$ as isometries of $\mathbb{H}^3$ which degenerate as ρ tends towards an ideal point given by a boundary slope of the Newton polygon: see [20]

3. *From this action a non-trivial tree exists which gives a non-trivial splitting of $\pi_1(K)$.* According to a theorem of Serre, if $\pi_1(K)$ acts on a tree $\mathcal{T}$ without inversions (i.e. without reversing an edge) then $\pi_1(K)$ is isomorphic to the fundamental group of the quotient graph of groups $\mathcal{T}/\pi_1(K)$. This is a splitting of $\pi_1(K)$. It must be checked that the splitting is non-trivial [16].

4. *From a non-trivial splitting of* $\pi_1(K)$ *we can construct an incompressible surface in* $S^3 - \nu(K)$. The idea is now to construct a space with fundamental group equal to the 'fundamental group of the graph of groups' above. One constructs each vertex group from loops and glued-on discs, and does the same for each edge group. The monomorphisms from edge groups to vertex groups can be realised by gluing the 'edge space' X_e onto 'vertex spaces' X_v — in particular, forming $X_e \times [0,1]$ and gluing $X_e \times \{0\}$ and $X_e \times \{1\}$ onto vertex spaces. We thus obtain a space L with fundamental group corresponding to the groups of the graph. There is then a map $f : S^3 - \nu(K) \to L$ inducing the isomorphism. Taking the inverse image of midpoints of edges in K, and homotoping sufficiently (see [25]), and taking only components which are not boundary-parallel, gives incompressible surfaces.

7.6 Representations and triangulations

We now bring together the notions of A-polynomial, representations of a knot group into $SL_2(\mathbb{C})$, and hyperbolic structures.

Given a hyperbolic knot K and a decomposition of $S^3 - K$ into ideal tetrahedra, we can obtain hyperbolic structures by solving the consistency equations, considered purely algebraically. Such solutions include the complete structure and incomplete structures obtained from deforming the complete structure; Galois conjugates of these solutions are also solutions. There could be more solutions in addition to these, if the polynomial we solve is reducible. Such solutions could give tetrahedral shape parameters which have negative imaginary part (negatively oriented tetrahedra with negative volume), or are real (flat tetrahedra with zero volume; or degenerate if a parameter is 0, 1 or ∞). Together these are called *pseudotriangulations* of $S^3 - K$, and may in general involve nasty overlap or folding.

However a pseudotriangulation gives tetrahedra with the same combinatorial relations as a hyperbolic structure, still piecing together in $\mathbb{H}^3$. So there is still a *pseudo-developing map* $\widetilde{S^3 - K} \longrightarrow \mathbb{H}^3$. And a loop α at a given basepoint in $S^3 - K$, which corresponds to an element of $\pi_1(K)$, can be associated a *pseudo-holonomy* $H(\alpha) \in PSL_2(\mathbb{C})$ obtained from the isometries gluing together corresponding faces, which lifts (see 4.2.1) to a representation $\tilde{H} : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$. This pseudotriangulation also has a well-defined *pseudovolume* V, given by the algebraic sum of the volumes of the ideal tetrahedra (some of which may be negative).

Conversely, given a representation $\tilde{\rho} : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ (which descends to $\rho : \pi_1(K) \longrightarrow PSL_2(\mathbb{C})$), we can construct a pseudotriangulation of $S^3 - K$ with pseudo-holonomy ρ. We will first define a pseudo-developing map D_ρ : $\widetilde{S^3 - K} \longrightarrow \mathbb{H}^3$. Let $\overline{S^3 - K}$ be a one-point compactification of $S^3 - K$, collapsing the knot to an ideal vertex, and let $\overline{\mathbb{H}^3}$ denote $\mathbb{H}^3$ adjoined with its sphere at infinity. In fact we will define $\overline{D_\rho} : \widetilde{\overline{S^3 - K}} \longrightarrow \overline{\mathbb{H}^3}$.

Take an ideal hyperbolic tetrahedron in our decomposition of $S^3 - K$ and an ideal vertex v. Now loops "near" v (i.e. loops from our basepoint, via some path γ to a point near v, running around near v, and then back along γ^{-1}) form a conjugate of the peripheral subgroup $\pi_1(T)$ in $\pi_1(K)$; in fact these form the *stabilizer* of $\tilde{v}$, denoted $\mathrm{Stab}\,\tilde{v}$, for some lift $\tilde{v}$ of v in $\widetilde{S^3 - K}$. $\mathrm{Stab}\,\tilde{v}$ is an abelian subgroup of $\pi_1(K)$, so $\rho(\mathrm{Stab}\,\tilde{v})$ consists of commuting isometries of $\mathbb{H}^3$, with a fixed point w at infinity (and possibly another). Let $D_\rho(\tilde{v}) = w$. We define D_ρ on every other ideal vertex by setting, for all $\alpha \in \pi_1(K)$, $D_\rho(\alpha(\tilde{v})) = \rho(\alpha)(\tilde{v})$ — there is no other choice, if D_ρ is to have ρ as pseudo-holonomy. Now that we have defined where D_ρ takes the ideal vertices, the rest is easy. We map each ideal triangular face in our triangulation to the convex hull of its ideal vertices, mapping paired faces in the same way; this is fine since all ideal triangles are congruent. (It is possible that tetrahedra could be mapped to flat or degenerate hyperbolic tetrahedra.) Then we map the interior of each tetrahedron into the convex hull of its vertices in the obvious way. Having constructed D_ρ in this way, the pseudo-holonomy is clearly ρ, and the induced pseudotriangulation on $S^3 - K$ will satisfy the consistency equations. See [16], [33] for more details. In particular we have:

Proposition 7.6.1 ([3]) *Every representation $\rho : \pi_1(K) \longrightarrow SL_2(\mathbb{C})$ has a well-defined pseudovolume.*

7.7 Parabolic representations and cusp shapes

A_K tells us more about hyperbolic structures on $S^3 - K$. It contains information about the cusp shape, which can be defined purely algebraically and generalised. Here and subsequently we assume all components of $\xi(R_U)$ are curves, so that all non-abelian representations are encoded in the A-polynomial: see section 7.4. Suppose ρ_0 is a parabolic representation. After conjugation, we may assume

$$\rho_0(\mu) = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and then $\rho_0(\lambda)$, which must commute, has the form

$$\rho_0(\lambda) = \pm \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

We let c be the *pseudo-cusp shape*. Equivalently, this is the ratio of the translations of the parabolic isometries of $\mathbb{H}^3$ induced by projection to $PSL_2(\mathbb{C})$. Note c depends on our choice of basis λ, μ for $\pi_1(T)$. Other choices shift c by an integral Möbius transformation. If we choose λ, μ canonically, then c is a knot invariant.

This is clearly a generalisation of the previously defined geometric notion of cusp shape. If c_0 is the cusp shape of the complete structure, then it also a pseudo-cusp shape. The field $\mathbb{Q}(c_0)$, called the *cusp field* of K, is a knot invariant *independent* of our choice of basis λ, μ (c_0', the result of applying an integral Möbius transformation to c_0, satisfies $\mathbb{Q}(c_0') = \mathbb{Q}(c_0)$). Galois conjugates of c_0 satisfy the same algebraic relations as c_0, so are all pseudo-cusp shapes. But there may be others, if the polynomials involved are reducible.

At parabolic representations we clearly have $(l, m) = (\pm 1, \pm 1)$. Since the two lifts of a representation to $PSL_2(\mathbb{C})$ have identical geometric meaning, we may assume $m = 1$. It has been proved that we must have $l = -1$ at the complete structure ([48]). By considering values $(l, m) \in \mathbb{V}(A_K)$ close to $(-1, \pm 1)$, we can determine the pseudo-cusp shapes associated to parabolic representations.

For a non-parabolic representation $\rho \in R(\pi_1(K))$ near ρ_0, corresponding to $(l, m) \in \mathbb{V}(A_K)$, we may examine the complex translation distances associated to $\rho(\lambda), \rho(\mu)$. Since $\rho(\lambda)$ has eigenvalues l, l^{-1}, it induces an isometry conjugate to $z \mapsto l^2 z$, hence has complex translation distance $t_\lambda = 2 \log |l| + 2i \arg l = 2 \log l$ (taking appropriate branches of the logarithm). Similarly $\rho(\mu)$ induces an isometry with complex translation distance $t_\mu = 2 \log |m| + 2i \arg m = 2 \log m$. As $\rho \longrightarrow \rho_0$, $\rho(\mu)$ and $\rho(\lambda)$ tend to commuting parabolics, and the ratio of their translations is the pseudo-cusp shape c. So $\dfrac{t_\lambda}{t_\mu} = \dfrac{\log l}{\log m} \longrightarrow c$ as $m \longrightarrow \pm 1$: see, e.g. [19, lemma 6.1].

By L'Hopital's rule, we can simplify this limit. Taking the case $m = -1$ (the other possibility $m = 1$ is analogous), the limit is equal to

$$\frac{d \log l}{d \log m} = \frac{d \log l}{dl} \frac{dl}{dm} \frac{dm}{d \log m} = \frac{m}{l} \frac{dl}{dm} = \frac{dl}{dm}.$$

To calculate the values of $\dfrac{dl}{dm}$ at $m = -1$, we can set $m = -1 + t$, $l = -1 + st$ in A_K. Then we obtain a polynomial in s, t. Let d_l be the degree of $A_k(l, m)$ in l. Since $A_K(-1, 1) = 0$ (there are parabolic representations there!), this polynomial has no constant terms, and is divisible by t. For t small the lowest degree terms in t dominate. So we collect the terms of lowest degree in t. This gives us a polynomial in s of degree d_l, the roots of which are all possible pseudo-cusp shapes of parabolic representations.

Clearly all Galois conjugates of the complete structure's cusp shape c_0 will be found by this procedure; and possibly others. Hence we have

Proposition 7.7.1

$$d_l \geq \deg \mathbb{Q}(c_0)$$

That is, the degree of the A-polynomial in l is at least the degree of the cusp field of K.

Note here we have really investigated the algebraic branches A_j of $A_K(l, m)$. If we factorise A_K as

$$A_K(l, m) = a_0(m) \prod_{j=1}^{d_l} A_j(l, m) = a_0(m) \prod_{j=1}^{d_l} (l - l_j(m)), \tag{7.1}$$

then we have calculated $\dfrac{dl_j}{dm}$ at $(-1,\ pm1)$, for each $j = 1, \ldots, d_l$.

7.8 Mahler measure and hyperbolic volume

Having seen the connection between A-polynomial and hyperbolic structures, we now add Mahler measure and hyperbolic volume. Good references are [3], [16], [33].

We first consider changes in the pseudovolumes of representations associated to points $(l, m) \in \mathbb{V}(A_K)$. It is known that (see [40], [16], [27], [3]) for (l, m) varying in $\mathbb{V}(A_K)$, the volume changes as

$$dV = -\frac{1}{2}\left(\log|l|\; d(\arg m) - \log|m|\; d(\arg l)\right).$$

If we consider a path $m = e^{i\theta}$, as we do in evaluating Mahler measure, then we have $\log|m| = 0,\ \arg m = \theta,\ d(\arg m) = d\theta$. So

$$dV = -\frac{1}{2}\log|l| d\theta$$

which is almost exactly the integrand of the Mahler measure in equation 6.1.

Next we consider now the Mahler measure of A_K. Factorising A_K as above (equation 7.1), we have

$$\mathcal{M}(A_K) = \mathcal{M}(a_0) + \sum_{j=1}^{d_l} \mathcal{M}(A_j); \qquad \mathcal{M}(A_j) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |l_j(e^{i\theta})| d\theta.$$

In [16, prop. 5.10], [19, thm. 11.1], [21] it is proved that every side of the Newton polygon of A_K corresponds to a cyclotomic polynomial, so $\mathcal{M}(a_0) = 0$. There is also much symmetry in these integrals. As $A_K(l, m)$ is even in m, if $A_K(l, e^{i\theta}) = 0$ then $A_K(l, -e^{i\theta}) = A_K(l, e^{i(\theta+\pi)})$ also. As $A_K(l, m) \in \mathbb{Z}[l, m]$, if $A_K(l, e^{i\theta}) = 0$ then $A_K(\bar{l}, e^{-i\theta}) = A_K(\bar{l}, e^{i(2\pi-\theta)}) = 0$ also. And as $A_K(l, m)$ is reciprocal, if $A_K(l, e^{i\theta}) = 0$ then $A_K(l^{-1}, e^{-i\theta}) = A_K(l^{-1}, e^{i(2\pi-\theta)}) = 0$ also. The graph of $|l_j(e^{i\theta})|$ for $0 \le \theta \le 2\pi$ thus has symmetry as shown in the example of figure 7.1.

So it suffices to evaluate the above integrals for $\mathcal{M}(A_j)$ on smaller intervals:

$$\mathcal{M}(A_K) = \frac{1}{\pi}\sum_{j=1}^{d_l}\int_0^{\pi} \log^+ |l_j(e^{i\theta})|\; d\theta = \frac{2}{\pi}\sum_{j=1}^{d_l}\int_0^{\frac{\pi}{2}} \log^+ |l_j(e^{i\theta})|\; d\theta$$

To simplify further, we let $A_K(l, m) = B_K(l, m^2)$, as in [3], and let B_K have branches $B_j = l - l_j^B(m)$. Then $l_j(m) = l_j^B(m^2)$ so that

$$\mathcal{M}(A_K) = \frac{1}{\pi}\sum_{j=1}^{d_l}\int_0^{\pi} \log^+ |l_j^B(e^{2i\theta})|\; d\theta = \frac{1}{\pi}\sum_{j=1}^{d_l}\int_0^{\pi} \log^+ |l_j^B(e^{i\phi})|\; d\phi = \frac{1}{\pi}\sum_{j=1}^{d_l}\mathcal{M}_j,$$

where $\mathcal{M}_j$ denotes the integral involving l_j^B. The centre equality follows from the symmetry of $B_K(l, m^2)$, which is inherited from A_K. A graph of $|l_j^B(e^{i\theta})|$ for $0 \le \theta \le 2\pi$ in general has symmetry as shown in figure 7.2.

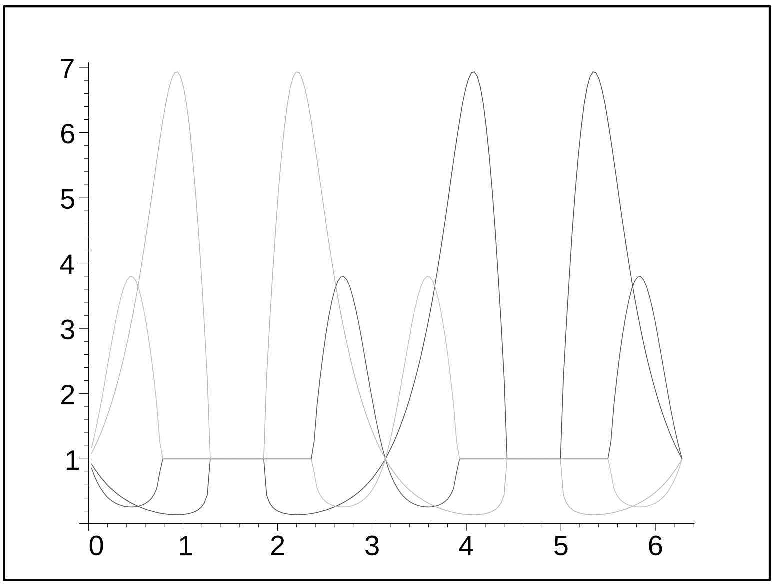


Figure 7.1: Graph of $|l_j(e^{i\theta})|$ for the branches of A_{6_1}

In fact, the integral for B_K is more closely related to the volume than A_K. For here the angle $\phi = 2\theta$ means

$$dV = -\log |l| \; d\phi.$$

So, if $|l_j^B(e^{i\theta})| \geq 1$ for all $0 \leq \phi \leq \pi$, then the integral $\mathcal{M}_j$ evaluates the change in volume of the representations corresponding to $(l_j^B(e^{i\phi}), e^{i\phi}) \in \mathbb{V}(A_K)$.

For example, the figure-8 knot has $d_l = 2$, with $|l_1^B(e^{i\phi})| \geq 1$ and $|l_2^B(e^{i\phi})| \leq 1$ for all $0 \leq \phi \leq \pi$. So $\mathcal{M}_2 = 0$ and $\pi\mathcal{M}(A_{4_1}) = \mathcal{M}_1$. We find that

$$\mathcal{M}_1 = \pi\mathcal{M}(A_{4_1}) = \text{vol } 4_1 \sim 2.029883.$$

As discussed earlier, for any hyperbolic knot K there is a 1-parameter family of representations obtained from deforming the complete hyperbolic structure on $S^3 - K$, corresponding to hyperbolic Dehn surgery. One of the branches $A_j(l, m)$ of A_K will correspond to these representations; the point $(l_j(e^{i\theta}), e^{i\theta}) \in \mathbb{V}(A_K)$ or $(l_j^B(e^{2i\theta}), e^{2i\theta}) \in \mathbb{V}(B_K)$ corresponds to a representation ρ where $\rho(\mu)$ has eigenvalues $e^{i\theta}, e^{-i\theta}$, so $\rho(\mu)$ is conjugate to the Möbius transformation $z \mapsto e^{2i\theta} z$, i.e. elliptic with rotation angle 2θ. This is the structure obtained from $(\frac{\pi}{\theta}, 0)$-surgery on the knot complement. The complete structure corresponds to $\theta = 0$. If the volume decreases or collapses to 0 as θ increases to $\pi/2$ (or ϕ increases to π), and does not go below zero, and $|l_j| \geq 1$ throughout, then the appropriate $\mathcal{M}_j$ is exactly the volume! Though these conditions might seem restrictive, they are often satisfied — for instance, we know this to be the case

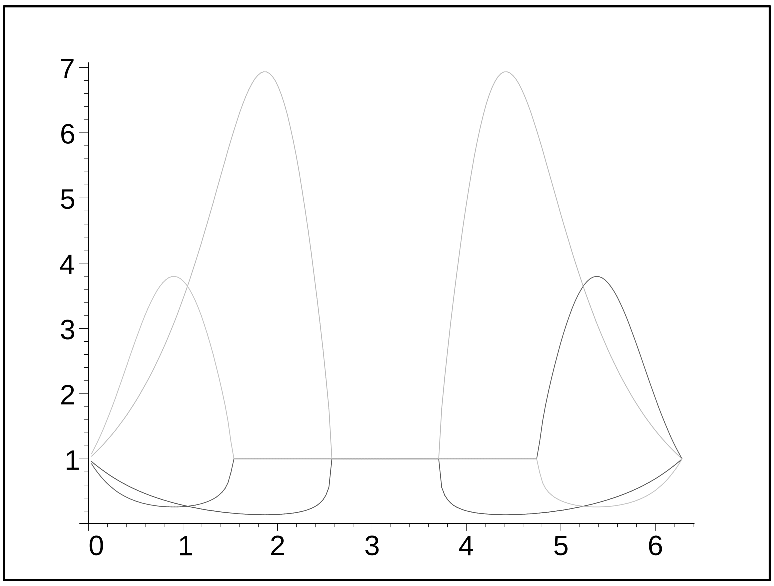


Figure 7.2: Graph of $|l_j^B(e^{i\theta})|$ for the branches of B_{6_1}

for 2-bridge knots, following from ideas involved in the proof of the orbifold theorem [17].

The same principle applies to other branches of the A-polynomial corresponding to pseudotriangulations and pseudovolumes. An interesting question is to give other representations and changes in pseudovolume a geometric interpretation. This can be done for twist knots; it can be done in many cases (perhaps all) for two-bridge knots; perhaps it is doubtful though, in general.

Chapter 8

Twist knots

We now investigate the representations of the fundamental group of the complement of twist knots into $SL_2(\mathbb{C})$ and give them a geometric interpretation. We vary and expand upon the analysis in [43].

One representation of a twist knot into $SL_2(\mathbb{C})$ which has a geometric interpretation is the holonomy representation of the complete hyperbolic structure on the knot complement. But other representations have geometric interpretations also.

The holonomy representation given by the complete hyperbolic structure is parabolic — all meridians must have parabolic image. We will thus first consider parabolic representations, which have an interesting geometric interpretation, before moving on to more general representations.

Twist knots have a particularly tractable structure — though by no means too simple. Recall from section 2.5 that there are two similar presentations for the fundamental group of the twist knot K_{2n}. The first comes from performing Dehn surgery on the Whitehead link complement, the second from considering the Schubert normal form:

$$\begin{aligned}\pi_1(K_{2n}) &= \left\langle g, h \mid g\left(h^{-1}ghg^{-1}\right)^n = \left(h^{-1}ghg^{-1}\right)^n h \right\rangle \\ &= \langle g, h \mid gw = wh \rangle = \left\langle g, h \mid g\left(hg^{-1}h^{-1}g\right)^n = \left(hg^{-1}h^{-1}g\right)^n h \right\rangle.\end{aligned}$$

The group elements g, h are the same in both presentations, and the longitude λ is given by

$$\lambda = hg^{-1}\left(h^{-1}ghg^{-1}\right)^n gh^{-1}gh^{-1}\left(g^{-1}hgh^{-1}\right)^n hg^{-1}$$

Henceforth we stick to the first presentation, following [43]. Throughout, since $K_m = K_{-1-m}$, we only consider K_{2n}.

8.1 Parabolic representations

This section is an outline of results from [43]. We present them here, before continuing further.

First take a normal form for a parabolic representation $\rho : \pi_1(K_{2n}) \longrightarrow SL_2(\mathbb{C})$. Since $\rho(g), \rho(h)$ are conjugate in $SL_2(\mathbb{C})$ and parabolic, the four eigenvalues of $\rho(g)$, $\rho(h)$ are all 1 or all -1. But note that, as the relator has even length, $\rho' : \pi_1(K_{2n}) \longrightarrow SL_2(\mathbb{C})$ defined by $\rho'(g) = -\rho(g)$, $\rho'(h) = -\rho(h)$ is also a representation. These representations descend to the same representation in $PSL_2(\mathbb{C})$ and are related as those described in proposition 4.2.1; we will not consider representations with eigenvalues -1 further until considering general representations in section 8.4. So we may assume all the eigenvalues are 1. Then by the discussion of section 7.4, by conjugating ρ we have

$$\rho(g) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \rho(h) = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$$

for some $t \in \mathbb{C}$.

We have thus parametrised all possible parabolic representations in terms of the single complex variable t. The same technique is available whenever dealing with a group with 2 conjugate generators. Of course, only some values of t (finitely many) will give a representation. This depends on the relator, and so we must compute the value of ρ on this.

An easy computation gives

$$\rho\left(h^{-1}ghg^{-1}\right) = \begin{bmatrix} 1-t & t \\ -t^2 & t^2+t+1 \end{bmatrix} = \Omega.$$

To compute Ω^n we use the fact that Ω satisfies its own characteristic polynomial. The characteristic polynomial for a matrix in $SL_2(\mathbb{C})$ is simple to calculate; we only need find its trace.

$$\Omega^2 - (\operatorname{tr}\Omega)\,\Omega + I = \Omega^2 - (t^2+2)\Omega + I = 0$$

Multiplying by Ω^{n-2} gives a recurrence for Ω^n.

$$\Omega^n - (t^2+2)\Omega^{n-1} + \Omega^{n-2} = 0. \tag{8.1}$$

This is a linear equation in the matrices — so the same equation holds for the entries of powers of Ω. It holds for all $n \in \mathbb{Z}$. We let Ω^n_{ij} denote the i,j element of Ω^n, which is a polynomial in t. From the recurrence some relationships between the Ω^n_{ij} can be established by easy induction.

$$\Omega^n_{21} = -t\Omega^n_{12}, \qquad \Omega^n_{11} + (t+2)\Omega^n_{12} - \Omega^n_{22} = 0. \tag{8.2}$$

We obtain a parabolic representation if and only if the relator is satisfied:

$$\rho\left(g\left(h^{-1}ghg^{-1}\right)^n\right) = \rho\left(\left(h^{-1}ghg^{-1}\right)^n h\right),$$

which is equivalent to the matrix equation

$$\rho(g)\Omega^n = \Omega^n\rho(h).$$

Substituting $\Omega^n = [\Omega^n_{ij}]$ gives

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Omega^n_{11} & \Omega^n_{12} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} = \begin{bmatrix} \Omega^n_{11} & \Omega^n_{12} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$$
$$\begin{bmatrix} \Omega^n_{11} - t\Omega^n_{12} & \Omega^n_{12} + \Omega^n_{22} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} = \begin{bmatrix} \Omega^n_{11} - t\Omega^n_{12} & \Omega^n_{12} \\ -t\Omega^n_{12} - t\Omega^n_{22} & \Omega^n_{22} \end{bmatrix}$$

which is equivalent to $\Omega^n_{22} = 0$. Now Ω^n_{22} is a polynomial in t which we denote $\Phi_n(t)$. The first few polynomials $\Phi_n(t)$ (with $n \geq 0$) are:

$$\begin{aligned} \Phi_0(t) &= 1 \\ \Phi_1(t) &= t^2 + t + 1 \\ \Phi_2(t) &= t^4 + t^3 + 3t^2 + 2t + 1 \\ \Phi_3(t) &= t^6 + t^5 + 5t^4 + 4t^3 + 6t^2 + 3t + 1 \end{aligned}$$

The degree of $\Phi_n(t)$ is $2n$ for $n \geq 0$ and $-2n-1$ for $n < 0$. The coefficients of these polynomials are actually binomial coefficients, which follow an interesting meandering pattern on Pascal's triangle. Both are established by easy induction: see [43]. Explicitly,

$$\Phi_n(t) = \begin{cases} \displaystyle\sum_{i=0}^{2n} \binom{n + \lfloor \frac{i}{2} \rfloor}{i} t^i & \text{if } n \geq 0 \\ \displaystyle\sum_{i=0}^{-2n-1} \binom{-n + \lfloor \frac{i-1}{2} \rfloor}{i} t^i & \text{if } n < 0 \end{cases}.$$

In [43] it is proved that $\Phi_n(t)$ is irreducible and has distinct roots. Thus parabolic representations of K_{2n} are in one-to-one correspondence with the roots of Φ_n.

Substituting $t = x - x^{-1}$ simplifies matters greatly. We let $w_n(x)$ be the polynomial obtained from $\Phi_n(x - x^{-1})$ after clearing denominators by multiplying by the appropriate power of x. Thus $w_n(x) = x^{\deg \Phi_n} \Phi_n(x - x^{-1})$. Then we have

$$\begin{aligned} w_0(x) &= 1 \\ w_1(x) &= x^4 + x^3 - x^2 - x + 1 = \frac{x^6 + x^5 - x + 1}{x^2 + 1} \\ w_2(x) &= x^8 + x^7 - x^6 - x^5 + x^4 + x^3 - x^2 - x + 1 \\ &= \frac{x^{10} + x^9 - x + 1}{x^2 + 1} \end{aligned}$$

and the recurrence becomes (for $n \geq 2$)

$$w_n(x) - (x^4 + 1)w_{n-1}(x) + x^4 w_{n-2}(x).$$

From this it is established by induction that

$$w_n(x) = \begin{cases} \dfrac{x^{4n+2} + x^{4n+1} - x + 1}{x^2 + 1} & \text{if } n \geq 0, \\ \dfrac{-x^{-4n} + x^{-4n-1} + x + 1}{x^2 + 1} & \text{if } n < 0. \end{cases}$$

and, for all $n \in \mathbb{Z}$, $w_n(x) = 0$ if and only if

$$x^{4n} \left(\frac{x(x+1)}{x-1} \right) = 1 \tag{8.3}$$

and $x \neq \pm i$. This is exactly equation 3.3. In fact the x here and in section 3.3 are identical: [43].

Using some complex analysis, it can be shown that the roots of $w_n(x)$ lie approximately evenly spaced close to the circle $|x| = 1$, with $|n|$ roots in each quadrant. When $n > 0$, there are n roots in each quadrant, and of the n roots in the first quadrant, there is one with argument in each of the intervals

$$\left(0, \frac{\pi}{4n}\right), \left(\frac{2\pi}{4n}, \frac{3\pi}{4n}\right), \left(\frac{4\pi}{4n}, \frac{5\pi}{4n}\right), \ldots, \left(\frac{2(n-1)\pi}{4n}, \frac{(2n-1)\pi}{4n}\right).$$

When $n < 0$, there are two real roots, and $-n-1$ non-real roots in each quadrant. Those in the first quadrant have arguments lying one each in the intervals

$$\left(\frac{\pi}{4(-n)}, \frac{2\pi}{4(-n)}\right), \left(\frac{3\pi}{4(-n)}, \frac{4\pi}{4(-n)}\right), \left(\frac{5\pi}{4(-n)}, \frac{6\pi}{4(-n)}\right), \ldots, \left(\frac{(2(-n)-3)\pi}{4(-n)}, \frac{(2(-n)-2)\pi}{4(-n)}\right).$$

Whenever x is a root, $\bar{x}, -x^{-1}, -\bar{x}^{-1}$ are also roots, so all $4|n|$ roots have similar spacings. The roots also lie on a curve with equation in polar coordinates

$$\cos\theta = \left(\frac{r + r^{-1}}{2}\right) \left(\frac{r^{-4n-1} - r^{4n+1}}{r^{-4n-1} + r^{4n+1}}\right),$$

where $z \in \mathbb{C}$ is taken as $z = re^{i\theta}$. Thus we have a good idea where the roots of $w_n(x)$ are, and hence the roots of $\Phi_n(x)$. For $n = 10$ the 40 roots are as shown in figure 8.1.

8.2 Geometry of parabolic representations

We now find a geometric interpretation for the parabolic representations, expanding upon the analysis in [43].

One parabolic representation must be a lift into $SL_2(\mathbb{C})$ of the (unique) holonomy representation of the complete hyperbolic structure on K_{2n}. We locate the root of $w_n(x)$, hence $\Phi_n(x)$, corresponding to the complete hyperbolic

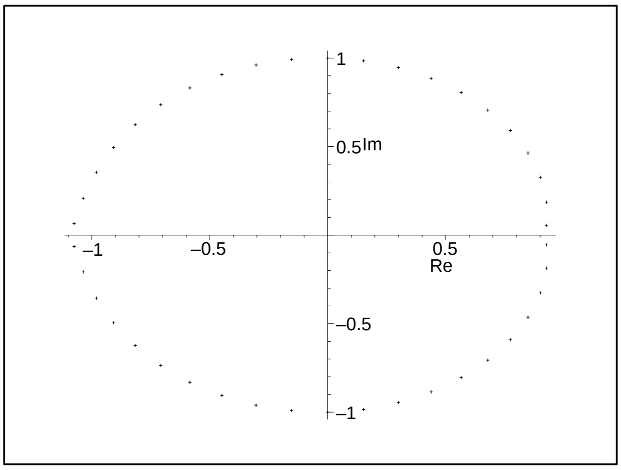


Figure 8.1: roots of $w_{10}(x)$, for the knot K_{20}

structure on K_{2n}. We deal with $n > 0$ in detail and summarise the $n < 0$ case, which is similar. As discussed in section 3.3 and [43], the particular root of $w_n(x)$ corresponding to the complete structure is the one that satisfies

$$\pm 2\pi i = 4n \log x - 2n\pi i + \log\left(\frac{x(x+1)}{x-1}\right).$$

Simplifying with equation 8.3 and taking the imaginary part gives

$$\pm 2\pi = 4n \arg x - 2n\pi + \arg x^{-4n}.$$

Each root of $w_n(x)$ is parametrised by $k \in \mathbb{Z}$ where $-2n \leq k \leq 2n-1$ (or in fact by any set of $4n$ consecutive integers) and

$$\frac{2k\pi}{4n} < \arg x < \frac{(2k+1)\pi}{4n},$$

or equivalently

$$(2k-2n)\pi < 4n \arg x - 2n\pi < (2k-2n+1)\pi.$$

Now $4n \arg x - 2n\pi + \arg x^{-4n}$ must be an integer multiple of 2π, and $\arg x^{-4n} \in (-\pi, \pi)$, so

$$4n \arg x - 2n\pi + \arg x^{-4n} = (2k-2n)\pi. \qquad (8.4)$$

Thus the complete structure corresponds to the roots x_1, x_2 with $2k - 2n = -2, 2$, that is, $k = n \pm 1$. We then have $x_2 = -x_1^{-1}$, which gives the same value of $t = x - x^{-1}$ and the same root of Φ_n. This root of $\Phi_n(t)$ for $n > 0$ lies in the region bounded by the imaginary axis, the circle $|z| = 2$, and the two hyperbolas $|z+2i|-|z-2i| = 4\sin\frac{(2n-2)\pi}{4n}$ and $|z+2i|-|z-2i| = 4\sin\frac{(2n-1)\pi}{4n}$. For $n < 0$ the root lies in a similar region but the bounding hyperbolas are $|z+2i|-|z-2i| = 4\sin\frac{(2n+3)\pi}{4n}$ and $|z+2i|-|z-2i| = r\sin\frac{(2n+2)\pi}{4n}$. These regions arise as the images of regions containing x, (bounded by circles and lines) under the transformation $x \mapsto x - x^{-1}$. Thus as $n \longrightarrow \pm\infty$, the root of $\Phi_n(t)$ corresponding to the complete structure tends to $2i$. The 20 roots for $n = 10$ are shown in figure 8.2; the complete structure corresponds to the point $\sim -.00114 + 1.97660i$ closest to $2i$.

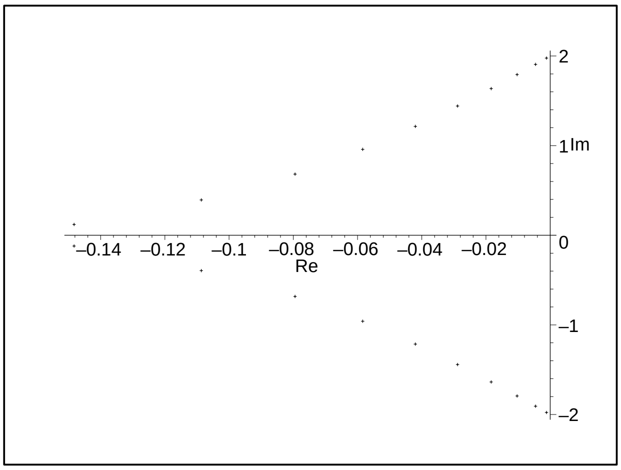


Figure 8.2: roots of $\Phi_{10}(t)$, giving parabolic representations of K_{20}

A similar analysis applies to the other parabolic representations (which does not appear in [43]). Recall the notation of section 3.3, where

$$H'(\mu_1) = u = \log\left(\frac{(x+1)x}{x-1}\right), \qquad H'(\lambda_1) = v = 4\log x - 2\pi i$$

are the complex translation distances of the holonomy of the meridian and longitude on the first component of the Whitehead link at an incomplete structure, which is complete at the second component. The other roots of $w_n(x)$ correspond to structures on the Whitehead link complement which are complete at the second cusp (since 8.3 is satisfied) but incomplete at the first. Equations 8.4 and 8.3 above give us that, at the root x of $w_n(x)$ with parameter k,

$$u + nv = 2(k-n)\pi i$$

or equivalently

$$\frac{1}{k-n}H'(\mu_2) + \frac{n}{k-n}H'(\lambda_2) = 2\pi i$$

so that the completion of this space is the complete structure on the Whitehead link with $\left(\frac{1}{k-n}, \frac{n}{k-n}\right)$-filled first component. This is equivalent to the twist knot K_{2n} (obtained by $(1, n)$-filling), with $\left(\frac{1}{k-n}, 0\right)$ surgery on a loop a around the twist as shown in figure 8.3. This can be considered as a complete cone-manifold structure on $S^3 - K_{2n}$ with angle $2\pi\,|k-n|$ around a.

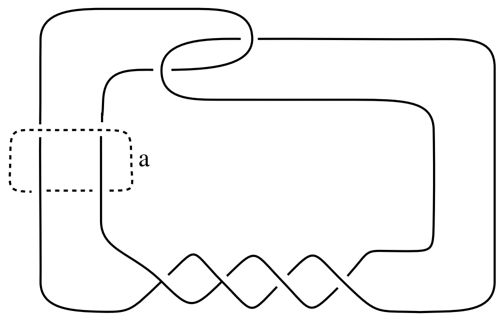


Figure 8.3: geometric interpretation of parabolic representations of K_{2n}

We obtain a hyperbolic structure if and only if $\left(\frac{1}{k-n}, \frac{n}{k-n}\right)$ lies in the hyperbolic Dehn surgery space described in section 3.3. Recall this is the complement of a closed parallelogram, with vertices $(-4, 1), (0, 1), (0, -1), (4, -1)$. So, considering figure 8.4, the root $w_n(x)$ gives a hyperbolic structure if and only if

$$\frac{n}{k-n} > \frac{-1}{2(k-n)} + 1 \text{ and } k > n, \qquad \text{or} \qquad \frac{n}{k-n} > \frac{-1}{2(k-n)} - 1 \text{ and } k < n.$$

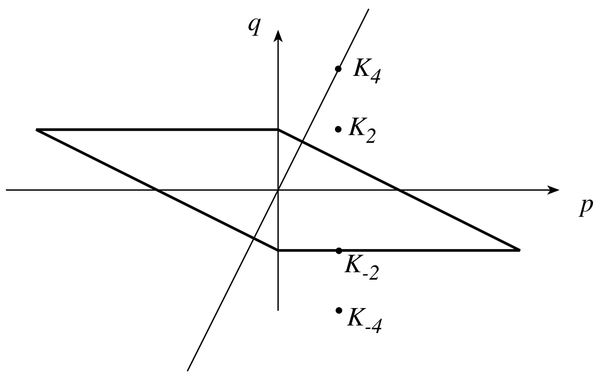


Figure 8.4: parabolic representations lie on a line in Dehn surgery space

These are equivalent to $\frac{1}{2} < k < \frac{4n+1}{2}$, or, as k is an integer,

$$1 \leq k \leq 2n.$$

So the admissible cone angles around a are $2\pi, 4\pi, \ldots, 2n\pi$.

For $n < 0$ we similarly parametrise roots of $w_n(x)$ by

$$\frac{(2k-1)\pi}{-4n} < \arg x < \frac{2k\pi}{-4n}$$

and obtain, at the root parametrised by k,

$$\begin{aligned} 4n \arg x - 2n\pi + \arg x^{-4n} &= -\pi(2k+2) \\ \frac{1}{k+n} H'(\mu_2) + \frac{n}{k+n} H'(\lambda_2) &= -2\pi. \end{aligned}$$

Then $\left(\frac{1}{k+n}, \frac{n}{k+n}\right)$-filling gives a hyperbolic structure if and only if

$$\frac{n}{k+n} < -1 \text{ and } k+n > 0, \qquad \text{or} \qquad \frac{n}{k+n} > 1 \text{ and } k+n < 0,$$

which simplifies to $1 \leq k \leq -2n-1$, so the admissible cone angles around a are $2\pi, 4\pi, \ldots, 2(-n-1)\pi$.

If we calculate the Mahler measures $\mathcal{M}_j/\pi$ of the separate branches of $A_{K_{2n}}$, we obtain the volumes of these cone manifolds as $\mathcal{M}_j$, to within the accuracy of our crude calculations. In fact, we will see that every $\mathcal{M}_j$ represents the volume of some such cone manifold.

8.3 Longitude and cusp shape

The remainder of this chapter goes beyond [43], and as far as I know is new: perhaps few have dared to brave such dense algebra, but some explicit calculations are possible and quite neat.

We consider the image of the longitude under a parabolic representation, which determines its "cusp shape". Recall that

$$\begin{aligned} \lambda &= hg^{-1}\left(h^{-1}ghg^{-1}\right)^n gh^{-1}gh^{-1}\left(g^{-1}hgh^{-1}\right)^n hg^{-1} \\ &= j\left(h^{-1}ghg^{-1}\right)^n j^{-2}\left(g^{-1}hgh^{-1}\right)^n j, \end{aligned}$$

where $j = hg^{-1}$ and hence

$$\rho(j) = \begin{bmatrix} 1 & -1 \\ -t & t+1 \end{bmatrix}, \qquad \rho\left(j^{-1}\right) = \begin{bmatrix} t+1 & 1 \\ t & 1 \end{bmatrix}$$

We compute powers of $\rho\left(g^{-1}hgh^{-1}\right) = \Upsilon$, using no new methods.

$$\Upsilon = \rho\left(g^{-1}hgh^{-1}\right) = \begin{bmatrix} t^2+t+1 & t \\ -t^2 & 1-t \end{bmatrix}$$

The matrix Υ contains the same elements as Ω, but flipped in a diagonal. It has the same trace and determinant, hence obeys an identical recurrence. Thus

$$\Upsilon^n = \begin{bmatrix} \Upsilon_{11}^n & \Upsilon_{12}^n \\ \Upsilon_{21}^n & \Upsilon_{22}^n \end{bmatrix} = \begin{bmatrix} \Omega_{22}^n & \Omega_{12}^n \\ \Omega_{21}^n & \Omega_{11}^n \end{bmatrix}.$$

We can compute $\rho(\lambda)$. We use the relationship $\Omega_{21}^n = -t\Omega_{12}^n$ and the fact that any representation satisfies $\Omega_{22}^n = \Phi_n(t) = 0$.

$$\begin{aligned}\rho(\lambda) &= \rho(j)\rho\left(h^{-1}ghg^{-1}\right)^n \rho(j)^{-2}\rho\left(g^{-1}hgh^{-1}\right)^n \rho(j) \\ &= \begin{bmatrix} 1 & -1 \\ -t & t+1 \end{bmatrix} \begin{bmatrix} \Omega_{11}^n & \Omega_{12}^n \\ -t\Omega_{12}^n & 0 \end{bmatrix} \begin{bmatrix} t+1 & 1 \\ t & 1 \end{bmatrix}^2 \\ & \quad \begin{bmatrix} 0 & \Omega_{12}^n \\ -t\Omega_{12}^n & \Omega_{11}^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -t & t+1 \end{bmatrix}\end{aligned}$$

Now setting $\Omega_{11}^n = -(t+2)\,\Omega_{12}^n$ (from equation 8.2) gives

$$\rho(\lambda) = \begin{bmatrix} -t\left(\Omega_{12}^n\right)^2 & (2t+4)\left(\Omega_{12}^n\right)^2 \\ 0 & -t\left(\Omega_{12}^n\right)^2 \end{bmatrix}$$

The determinant of this matrix is 1 so $t\left(\Omega_{12}^n\right)^2 = \pm 1$. Equivalently this must be so if $\rho(\lambda)$ is to be parabolic, to commute with $\rho(g)$. The Möbius transformation of $\rho(\lambda)$ is

$$z \mapsto \frac{-t\left(\Omega_{12}^n\right)^2 z + (2t+4)\left(\Omega_{12}^n\right)^2}{-t\left(\Omega_{12}^n\right)^2} = z - \frac{2t+4}{t}$$

so the cusp shape is

$$-\frac{2t+4}{t}.$$

The transformation $t \mapsto -\frac{2t+4}{t}$ is an elliptic with fixed axis the geodesic between $(-1-\sqrt{3}i, -1+\sqrt{3}i)$ and an angle of $\frac{2\pi}{3}$. Thus the roots of $\Omega_{22}^n = \Phi_n(t)$ are all shifted "to the left" to give the corresponding pseudo-cusp shapes. They are shown in figure 8.4 for $n = 10$.

Since the value of t corresponding to the complete structure converges to $2i$ as $n \longrightarrow \pm\infty$, the cusp shape converges to $-2+2i$.

Proposition 8.3.1 *The degree of the cusp field of K_{2n} is $2n$ for $n > 0$, and $-2n-1$ for $n < 0$.*

Proof. The cusp shape $c = -\frac{2t+4}{t}$ where $\Phi_n(t) = 0$. Then $\mathbb{Q}(c) = \mathbb{Q}(t)$; clearly $c = -\frac{2t+4}{t} \in \mathbb{Q}(t)$ so $\mathbb{Q}(c) \subseteq \mathbb{Q}(t)$, and applying the inverse Möbius transformation shows $\mathbb{Q}(t) \subseteq \mathbb{Q}(c)$. Since $\Phi_n(t)$ is irreducible and has degree $2n$ or $-2n-1$ respectively as n is positive or negative, the result follows. □

8.4 General representations

We can extend the technique of [43] to general, not necessarily parabolic, representations. The computations become horrendously more difficult and so a computer was used to perform much of the algebra involved.

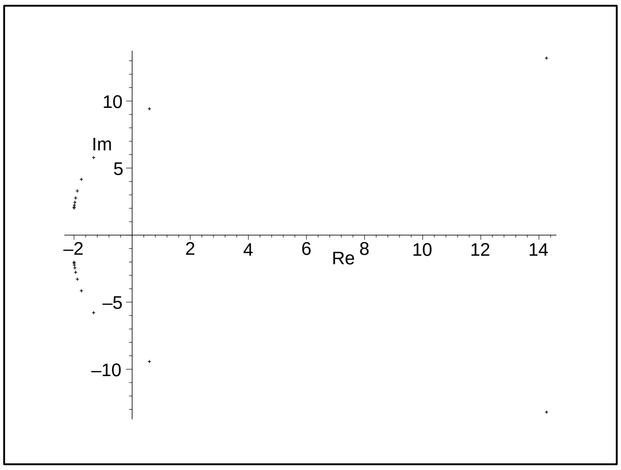


Figure 8.5: pseudo-cusp shapes of K_{20}

We take the normal form for our representation $\rho : \pi_1(K_{2n}) \longrightarrow SL_2(\mathbb{C})$ as discussed in section 7.4:

$$\rho(g) = \begin{bmatrix} m & 1 \\ 0 & m^{-1} \end{bmatrix}, \qquad \rho(h) = \begin{bmatrix} m & 0 \\ -t & m^{-1} \end{bmatrix}.$$

As before, to find representations we compute and equate $\rho\left(g\left(h^{-1}ghg^{-1}\right)^n\right)$ and $\rho\left(\left(h^{-1}ghg^{-1}\right)^n h\right)$. We first obtain

$$\rho\left(h^{-1}ghg^{-1}\right) = \begin{bmatrix} -tm^{-2}+1 & tm^{-1}-m+m^{-1} \\ -t^2m^{-1}+tm-tm^{-1} & t^2-tm^2+2t+1 \end{bmatrix}$$

which we again call Ω. The recurrence for Ω is

$$\Omega^n - \left(t^2 - tm^2 + 2t - tm^{-2} + 2\right)\Omega^{n-1} + \Omega^{n-2} = 0. \tag{8.5}$$

We again let Ω^n_{ij} denote the i, j entry of Ω^n, which is a Laurent polynomial involving negative powers of m (but not t). For convenience we denote the trace $t^2 - tm^2 + 2t - tm^{-2} + 2$ by $T(m, t)$.

Lemma 8.4.1 *For all $n \geq 1$,*

$$\Omega^n_{21} = -t\Omega^n_{12},$$

Proof. An easy induction. □

Lemma 8.4.2 *The highest and lowest powers of m and t occurring in Ω^n_{11}, Ω^n_{12}, Ω^n_{21}, Ω^n_{22} for $n \geq 1$ are as follows.*

Entry	*Lowest power of m*	*Highest power of m*	*Lowest power of t*	*Highest power of t*
Ω^n_{11}	$-2n$	$2n-2$	0	$2n-1$
Ω^n_{12}	$-2n+1$	$2n-1$	0	$2n-1$
Ω^n_{21}	$-2n+1$	$2n-1$	1	$2n$
Ω^n_{22}	$-2n+2$	$2n$	0	$2n$

Proof. We will present this proof slightly more carefully as it is important for the degree of the A-polynomial. It is still routine.

First we consider Ω^n_{11}. We have

$$\Omega^1_{11} = -tm^{-2}+1, \qquad \Omega^2_{11} = \left(t^2 - tm^2 + 2t - tm^{-2} + 2\right)\left(-tm^{-2}+1\right) - 1$$

establishing the required degrees for $n = 1, 2$. Assume the result for all $n < k$. Consider the recurrence

$$\Omega^k_{11} = \left(t^2 - tm^2 + 2t - tm^{-2} + 2\right)\Omega^{k-1}_{11} - \Omega^{k-2}_{11}.$$

The product $\left(t^2 - tm^2 + 2t - tm^{-2} + 2\right)\Omega^{k-1}_{11}$ contains terms of degree t^{2k-1} which do not sum to zero, terms of degree m^{2k-2} which do not sum to zero, and terms of degree m^{-2k} which do not sum to zero. The second term contains no powers of t greater than or equal to $2k-1$ or powers of m outside $[-2k+4, 2k-6]$, establishing maximal and minimal degrees in m and maximal degree in t. Consider now the terms of minimal degree in t. Clearly there are never any negative powers of t. The term of degree 0 in t is 1, for both Ω^1_{11} and Ω^2_{11}. By induction, the term of degree 0 in t is 1 in all Ω^n_{11}.

The proof for Ω^n_{22} is similar.

The proof for Ω^n_{12} differs only in considering the terms of minimal degree in t. The term of degree 0 in t is $m^{-1}-m$ for Ω^1_{12} and $2(m^{-1}-m)$ for Ω^2_{12}. By induction, the term of degree 0 in t is $n(m^{-1}-m)$ for all Ω^n_{12}.

The result for Ω^n_{21} follows easily since $\Omega^n_{21} = -t\Omega^n_{12}$. □

For negative n we have a similar result.

Lemma 8.4.3 *The highest and lowest powers of m and t occurring in Ω^n_{11}, Ω^n_{12}, Ω^n_{21}, Ω^n_{22} for $n \le -1$ are as follows.*

Entry	*Lowest power of m*	*Highest power of m*	*Lowest power of t*	*Highest power of t*
Ω^n_{11}	$2n+2$	$-2n$	0	$-2n$
Ω^n_{12}	$2n+1$	$-2n-1$	0	$-2n-1$
Ω^n_{21}	$2n+1$	$-2n-1$	1	$-2n$
Ω^n_{22}	$2n$	$-2n-2$	0	$-2n-1$

Next we try to find a linear relationship between entries of Ω^n in analogy to the second equation of 8.2. There is hope, since there is a relatively simple analogous relation when $n = 1$:

$$m^2\Omega^1_{11} + (tm^{-1} + m + m^{-1})\Omega^1_{12} - m^{-2}\Omega^1_{22} = 0.$$

This expression does not equal zero for $n \ge 2$, but we find it always to be divisible by $m^2 - m^{-2}$.

Lemma 8.4.4 *For all $n \in \mathbb{Z}$,*

$$m^2\Omega_{11}^n + \left(tm^{-1} + m + m^{-1}\right)\Omega_{12}^n - m^{-2}\Omega_{22}^n + \left(m^2 - m^{-2}\right)Q_n(m,t) = 0$$

where the sequence of polynomials $Q_n(m,t)$ is defined for all $n \in \mathbb{Z}$ by $Q_1(m,t) = 0$, $Q_2(m,t) = 1$ and

$$Q_{n+2}(m,t) = Q_{n+1}(m,t)T(m,t) - Q_n(m,t).$$

Proof. Induction on n. Computations for $n = 1, 2$ are routine. If

$$m^2\Omega_{11}^n + \left(tm^{-1} + m + m^{-1}\right)\Omega_{12}^n - m^{-2}\Omega_{22}^n + \left(m^2 - m^{-2}\right)Q_n(m,t) = 0,$$

$$m^2\Omega_{11}^{n+1} + \left(tm^{-1} + m + m^{-1}\right)\Omega_{12}^{n+1} - m^{-2}\Omega_{22}^{n+1} + \left(m^2 - m^{-2}\right)Q_{n+1}(m,t) = 0$$

then multiplying the second equation by $T(m,t)$, subtracting the first and using the recurrence for Ω^n gives

$$\begin{aligned} m^2\Omega_{11}^{n+2} &+ \left(tm^{-1} + m + m^{-1}\right)\Omega_{12}^{n+2} - m^{-2}\Omega_{22}^{n+2} \\ &+ \left(m^2 - m^{-2}\right)\left\{Q_{n+1}(m,t)T(m,t) - Q_n(m,t)\right\} = 0 \end{aligned}$$

establishing the required equation for $n + 2$. We may similarly establish the required equation for $n - 1$, proving the result for all positive and negative n. □

The polynomials $Q_n(m,t)$ can be as considered polynomials in $T(m,t)$. Their coefficients also form a pattern on Pascal's triangle.

$$\begin{aligned} Q_1(T) &= 0 \\ Q_2(T) &= 1 \\ Q_3(T) &= T \\ Q_4(T) &= T^2 - 1 \\ Q_5(T) &= T^3 - 2T \\ Q_6(T) &= T^4 - 3T^2 + 1 \\ Q_7(T) &= T^5 - 4T^3 + 3T \\ Q_8(T) &= T^6 - 5T^4 + 6T^2 - 1 \end{aligned}$$

We have an explicit form for the polynomials $Q_n(t)$.

Lemma 8.4.5 *For all positive integers n, expressed as $2k$ or $2k+1$ for an integer k,*

$$\begin{aligned} Q_{2k}(m,t) &= \sum_{i=0}^{k-1} T(m,t)^{2i}(-1)^{k-i-1}\binom{k+i-1}{2i} \\ Q_{2k+1}(m,t) &= \sum_{i=0}^{k-1} T(m,t)^{2i+1}(-1)^{k-i-1}\binom{k+i}{2i+1} \end{aligned}$$

For $n \leq 0$ we have $Q_n = -Q_{1-n}$

Proof. Again a simple induction. Small cases are verified above. Note that, if we interpret binomial coefficients as 0 wherever they are not defined, the all sums are over all $i \in \mathbb{Z}$, so we drop terminals from our sums. Assuming Q_{2k} and Q_{2k+1} are as above, the recurrence for Q gives us

$$\begin{aligned}
Q_{2k+2} &= Q_{2k+1}T - Q_{2k} \\
&= \sum \left[T^{2i+2}(-1)^{k-i-1}\binom{k+i}{2i+1} - T^{2i}(-1)^{k-i-1}\binom{k+i-1}{2i} \right] \\
&= \sum T^{2i} \left[(-1)^{k-i}\binom{k+i-1}{2i-1} - (-1)^{k-i-1}\binom{k+i-1}{2i} \right] \\
&= \sum T^{2i}(-1)^{k-i} \left[\binom{k+i-1}{2i-1} + \binom{k+i-1}{2i} \right] \\
&= \sum T^{2i}(-1)^{k-i}\binom{k+i}{2i}
\end{aligned}$$

as required. Similarly we can check Q_{2k+3}. To see why $Q_n = -Q_{1-n}$, we compute $Q_0 = -1 = -Q_2$. The recurrence for Q_n is symmetric in the sense that $Q_{n+2} = Q_{n+1}T - Q_n$ and $Q_n = Q_{n+1}T - Q_{n+2}$. So the sequence $Q_1, Q_0, Q_{-1}, \ldots$ follows the same recurrence as $Q_0, Q_1, Q_2, \ldots$. Since the first two terms are negatives of each other, so are all remaining terms. $\square$

Finally, we investigate under what conditions for m, t we obtain a representation. Again we check the relator, which gives us the matrix equation

$$\rho(g)\Omega^n = \Omega^n \rho(h).$$

Substituting $\Omega^n = [\Omega^n_{ij}]$ and using lemma 8.4.1 we obtain

$$\begin{aligned}
\begin{bmatrix} m & 1 \\ 0 & m^{-1} \end{bmatrix} \begin{bmatrix} \Omega^n_{11} & \Omega^n_{12} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} &= \begin{bmatrix} \Omega^n_{11} & \Omega^n_{12} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} \begin{bmatrix} m & 0 \\ -t & m^{-1} \end{bmatrix} \\
\begin{bmatrix} m\Omega^n_{11} - t\Omega^n_{12} & m\Omega^n_{12} + \Omega^n_{22} \\ -tm^{-1}\Omega^n_{12} & m^{-1}\Omega^n_{22} \end{bmatrix} &= \begin{bmatrix} m\Omega^n_{11} - t\Omega^n_{12} & m^{-1}\Omega^n_{12} \\ -tm\Omega^n_{12} - t\Omega^n_{22} & m^{-1}\Omega^n_{22} \end{bmatrix}.
\end{aligned}$$

Thus we have a representation if and only if

$$\left(m - m^{-1}\right)\Omega^n_{12} + \Omega^n_{22} = 0. \tag{8.6}$$

We call this polynomial $\Phi_n(m,t)$. These polynomials generalise the polynomials of the same name obtained earlier for parabolic representations, which we now refer to as $\Phi^p_n(t)$. Thus $\Phi_n(1,t) = \Phi^p_n(t)$. For $n \geq 0$, the polynomial $\Phi_n(m,t)$ has degree $2n$ in t and is a Laurent polynomial in m with maximum degree $2n$ and minimum degree $-2n$.

Using the recursive formula 8.5 for Ω^n, $\Phi_n(m,t)$ can be equally well defined by

$$\begin{aligned}
\Phi_0(m,t) &= 1,\\
\Phi_1(m,t) &= t^2 - tm^2 + 3t - tm^{-2} + t - m^2 + 3 - m^{-2}\\
&= t^2 - t\left(m - m^{-1}\right)^2 + t - \left(m - m^{-1}\right)^2 + 1,\\
\Phi_n(m,t) &= \left(t^2 - tm^2 + 2t - tm^{-2} + 2\right)\Phi_{n-1}(m,t) - \Phi_{n-2}(m,t)\\
&= \left(t^2 - t\left(m - m^{-1}\right)^2 + 2\right)\Phi_{n-1}(m,t) - \Phi_{n-2}(m,t).
\end{aligned}$$

When written in this way there is an obvious substitution $u = -(m - m^{-1})^2$ and we obtain polynomials (not Laurent polynomials) in t, u.

$$\begin{aligned}
\Phi_0(t,u) &= 1\\
\Phi_1(t,u) &= t^2 + tu + t + u + 1\\
\Phi_n(t,u) &= \left(t^2 + tu + 2\right)\Phi_{n-1}(m,t) - \Phi_{n-2}(m,t)
\end{aligned}$$

Listing the first few polynomials provides a fun pattern-spotting puzzle to find a rule for the coefficients.

$$\Phi_1(t,u) = \begin{array}{ccccccc} & 1 & + & t & + & t^2 & \\ + & u & + & tu & & & \end{array}$$

$$\Phi_2(t,u) = \begin{array}{ccccccccccc} & 1 & + & 2t & + & 3t^2 & + & t^3 & + & t^4 \\ + & 2u & + & 3tu & + & 2t^2u & & & & \\ & & + & tu^2 & + & t^2u^2 & & & & \end{array}$$

To remain printable we retain only coefficients in a similar arrangement.

$$\Phi_3(t,u) = \begin{array}{ccccccc} 1 & 3 & 6 & 4 & 5 & 1 & 1 \\ 3 & 6 & 8 & 10 & 3 & 3 & \\ & 4 & 5 & 3 & 3 & & \\ & & 1 & 1 & & & \end{array}$$

$$\Phi_4(t,u) = \begin{array}{ccccccccc} 1 & 4 & 10 & 10 & 15 & 6 & 7 & 1 & 1 \\ 4 & 10 & 20 & 30 & 18 & 21 & 4 & 4 & \\ & 10 & 15 & 18 & 21 & 6 & 6 & & \\ & & 6 & 7 & 4 & 4 & & & \\ & & & 1 & 1 & & & & \end{array}$$

Lemma 8.4.6 *For $n \geq 0$ we have*

$$\Phi_n(t,u) = \sum_{i=0}^{2n}\sum_{j=0}^{n} \binom{\left\lfloor \frac{i+j+1}{2} \right\rfloor}{j} \binom{n + \left\lfloor \frac{i+j}{2} \right\rfloor}{i+j} t^i u^j.$$

In terms of the original variables m, t,

$$\Phi_n(m,t) = \sum_{i=0}^{2n}\sum_{j=0}^{n} \binom{\left\lfloor \frac{i+j+1}{2} \right\rfloor}{j} \binom{n + \left\lfloor \frac{i+j}{2} \right\rfloor}{i+j} (-1)^j t^i \left(m - m^{-1}\right)^{2j}.$$

Proof. Proof by induction. We have demonstrated several small cases. So we set $\Phi_n(t,u)$ as above into the recurrence for Φ_n. Equating coefficients of $t^i u^j$, we must show that

$$\begin{aligned}
\binom{\lfloor \frac{i+j+1}{2} \rfloor}{j}\binom{n+1+\lfloor \frac{i+j}{2} \rfloor}{i+j} &= \binom{\lfloor \frac{i+j-1}{2} \rfloor}{j}\binom{n+\lfloor \frac{i+j-2}{2} \rfloor}{i+j-2} \\
&+ \binom{\lfloor \frac{i+j-1}{2} \rfloor}{j-1}\binom{n+\lfloor \frac{i+j-2}{2} \rfloor}{i+j-2} \\
&+ 2\binom{\lfloor \frac{i+j+1}{2} \rfloor}{j}\binom{n+\lfloor \frac{i+j}{2} \rfloor}{i+j} \\
&- \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j}\binom{n-1+\lfloor \frac{i+j}{2} \rfloor}{i+j},
\end{aligned}$$

where binomial coefficients $\binom{a}{b}$ are interpreted as 0 if $b > a$ or $a < 0$. We make repeated use of the formula $\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}$. Then we have

$$\begin{aligned}
\binom{\lfloor \frac{i+j-1}{2} \rfloor}{j} + \binom{\lfloor \frac{i+j-1}{2} \rfloor}{j-1} &= \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j} \\
\binom{n+\lfloor \frac{i+j}{2} \rfloor}{i+j} - \binom{n-1+\lfloor \frac{i+j}{2} \rfloor}{i+j} &= \binom{n-1+\lfloor \frac{i+j}{2} \rfloor}{i+j-1}
\end{aligned}$$

so the right hand side simplifies to

$$\begin{aligned}
&\binom{\lfloor \frac{i+j+1}{2} \rfloor}{j} \left[\binom{n+\lfloor \frac{i+j-2}{2} \rfloor}{i+j-2} + \binom{n+\lfloor \frac{i+j}{2} \rfloor}{i+j} + \binom{n-1+\lfloor \frac{i+j}{2} \rfloor}{i+j-1}\right] \\
&= \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j} \left[\binom{n+\lfloor \frac{i+j}{2} \rfloor}{i+j-1} + \binom{n+\lfloor \frac{i+j}{2} \rfloor}{i+j}\right] \\
&= \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j} \binom{n+\lfloor \frac{i+j}{2} \rfloor + 1}{i+j}
\end{aligned}$$

as required. □

The corresponding result for negative n is

Lemma 8.4.7 *For $n < 0$ we have*

$$\Phi_n(t,u) = \sum_{i=0}^{-2n-1} \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j}\binom{-n+\lfloor \frac{i+j-1}{2} \rfloor}{i+j} t^i u^j.$$

Lemma 8.4.8 *If we collect terms by powers of t so that*

$$\Phi_n(m,t) = \sum_{i=0}^{2n} \phi_i(m)t^i \quad (n > 0) \qquad \textit{or} \qquad \sum_{i=0}^{-2n-1} \phi_i(m)t^i \quad (n < 0)$$

where $\phi_i(m)$ is a Laurent polynomial involving only m, then maximum and minimum degrees of m occurring in $\phi_i(m)$ are

$$2\min\{i+1, 2n-i\}, \qquad -2\min\{i+1, 2n-i\}$$

respectively for $n > 0$, and for $n < 0$ the same degrees are

$$2\min\{i+1, -2n-1-i\}, \qquad -2\min\{i+1, -2n-1-i\}.$$

Proof. From 8.4.6 above we have

$$\phi_i(m) = \sum_{j=0}^{n} \binom{\lfloor \frac{i+j+1}{2} \rfloor}{j} \binom{n + \lfloor \frac{i+j}{2} \rfloor}{i+j} (-1)^j \left(m - m^{-1}\right)^{2j}.$$

The term indexed by j, $0 \le j \le n$ is nonzero if and only if

$$j \le \left\lfloor \frac{i+j+1}{2} \right\rfloor \text{ and } i+j \le n + \left\lfloor \frac{i+j}{2} \right\rfloor.$$

We now simplify using the fact that for integers a and b, $a \le \lfloor b/2 \rfloor$ if and only if $2a \le b$. We obtain that the term indexed by j is nonzero if and only if

$$j \le \min\{i+1, 2n-i\}.$$

Obviously the highest and lowest powers of m occurring in $\phi_i(m)$ will both arise only in the summand for $j = \min\{i+1, 2n-i\}$. The proof for $n < 0$ is similar. □

8.5 Longitude and A-polynomial

Having found explicitly the polynomial defining representations into $SL_2(\mathbb{C})$, we turn to the longitude. We will use a similar analysis to section 8.3, but the complexity of calculations is horrendous.

Again we let $j = hg^{-1}$ and $\Upsilon = \rho\left(g^{-1}hgh^{-1}\right)$ and denote the entries of Υ^n by Υ^n_{ij}, which will be Laurent polynomials in m, t. We obtain

$$\rho(j) = \begin{bmatrix} 1 & -m \\ -tm^{-1} & t+1 \end{bmatrix}, \qquad \rho(j)^{-1} = \begin{bmatrix} t+1 & m \\ tm^{-1} & 1 \end{bmatrix}$$

and

$$\Upsilon = \begin{bmatrix} t^2+2t-tm^{-2}+1 & tm+m-m^{-1} \\ -t^2m-tm+tm^{-1} & -tm^2+1 \end{bmatrix}.$$

Note that Υ has identical determinant and trace as Ω, hence obeys an identical recurrence. In fact, we note that Υ is obtained from Ω by flipping the entries in a diagonal (as in the parabolic case), and then replacing m with m^{-1}. We will use $\widetilde{\Omega^n_{ij}}$ to denote the expression Ω^n_{ij} after replacing m with m^{-1}. The recurrence relation makes it clear that there is a similar relationship between the entries of Ω^n and Υ^n:

$$\Upsilon^n = \begin{bmatrix} \widetilde{\Omega^n_{22}} & \widetilde{\Omega^n_{12}} \\ \widetilde{\Omega^n_{21}} & \widetilde{\Omega^n_{11}} \end{bmatrix} = \begin{bmatrix} \widetilde{\Omega^n_{22}} & \widetilde{\Omega^n_{12}} \\ -t\widetilde{\Omega^n_{12}} & \widetilde{\Omega^n_{11}} \end{bmatrix}.$$

We can now calculate $\rho(\lambda)$.

$$\begin{aligned} \rho(\lambda) &= \rho(j)\Omega^n\rho(j)^{-2}\Upsilon^n\rho(j) \\ &= \begin{bmatrix} 1 & -m \\ -tm^{-1} & t+1 \end{bmatrix} \begin{bmatrix} \Omega^n_{11} & \Omega^n_{12} \\ -t\Omega^n_{12} & \Omega^n_{22} \end{bmatrix} \begin{bmatrix} t+1 & m \\ tm^{-1} & 1 \end{bmatrix}^2 \\ &\qquad \begin{bmatrix} \widetilde{\Omega^n_{22}} & \widetilde{\Omega^n_{12}} \\ -t\widetilde{\Omega^n_{12}} & \widetilde{\Omega^n_{11}} \end{bmatrix} \begin{bmatrix} 1 & -m \\ -tm^{-1} & t+1 \end{bmatrix} \end{aligned}$$

The ensuing algebraic mayhem does not deserve reproduction here. The result, we know, must commute with $\rho(g)$ and hence is upper triangular. Let its eigenvalues be $l = \Psi_n(m,t)$ (in the top left) and l^{-1} (lower right). Then $\Psi_n(m,t)$ is again a polynomial in t and Laurent polynomial in m.

As it is we have 6 "unknowns" $\Omega^n_{11}, \Omega^n_{12}, \Omega^n_{22}, \widetilde{\Omega^n_{11}}, \widetilde{\Omega^n_{12}}, \widetilde{\Omega^n_{22}}$. There are 5 relations which can be used to eliminate them:

1. The relation 8.6: $\Phi_n(m,t) = \left(m - m^{-1}\right)\Omega^n_{12} + \Omega^n_{22} = 0$.

2. The corresponding relation under $m \mapsto m^{-1}$: $\left(m^{-1} - m\right)\widetilde{\Omega^n_{12}} + \widetilde{\Omega^n_{22}} = 0$

3. The relation in lemma 8.4.4:

$$m^2\Omega^n_{11} + \left(tm^{-1} + m + m^{-1}\right)\Omega^n_{12} - m^{-2}\Omega^n_{22} + \left(m^2 - m^{-2}\right)Q_n(m,t) = 0$$

4. The corresponding relation under $m \mapsto m^{-1}$:

$$m^{-2}\widetilde{\Omega^n_{11}} + \left(tm + m^{-1} + m\right)\widetilde{\Omega^n_{12}} - m^2\widetilde{\Omega^n_{22}} + \left(m^{-2} - m^2\right)Q_n(m^{-1},t) = 0$$

5. The expression found for the lower left entry of $\rho(\lambda)$, which must be 0.

Only the final relation is difficult to use, since it is an unwieldy expression. There is some hope that a simpler form might be found for $l = \Psi(m,t)$, but I have been unable to find it. In any case, by direct inspection of the expanded top left term, since we know the degrees of Ω^n_{ij}, we obtain the following result.

Lemma 8.5.1 *For $n > 0$, the degree in t of $\Psi_n(m,t)$ is $4n+2$. For $n < 0$, the degree is $-4n+2$.* □

We let $\Psi_n(m,t) = \sum_{i=0}^{4n+2}\psi_i(m)t^i$. As explained in chapter 7, we can obtain the A-polynomial by taking the resultant of $l - \Psi_n(m,t)$ and $\Phi_n(m,t)$, eliminating t. The more explicit our closed form for Ψ_n, the more explicit our knowledge of the A-polynomial. The resultant is the determinant of the following $(6n+2) \times (6n+2)$ matrix. The first $4n+2$ rows consist of ϕ's and the last $2n$ rows consist of ψ's.

$$\begin{pmatrix}
\phi_{2n} & \phi_{2n-1} & \cdots & \phi_0 & & & & & & \\
 & \phi_{2n} & \cdots & \phi_1 & \phi_0 & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & \phi_{2n} & \phi_{2n-1} & \cdots & \phi_0 \\
\psi_{4n+2} & \psi_{4n+1} & \cdots & \psi_{2n+2} & \psi_{2n+1} & \cdots & \psi_1 & \psi_0 - l & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & \psi_{4n+2} & \psi_{4n+1} & \cdots & \cdots & \psi_{2n} & \psi_{2n-1} & \cdots & \psi_0 - l
\end{pmatrix}$$

We now have the following result, which appears to be new.

Theorem 8.5.2 *The degree in l of $A_{K_{2n}}(l,m)$ is $2n$ for $n > 0$, and $-2n-1$ for $n < 0$.*

Proof. Every $(l,m) \in \mathbb{V}(A_K)$ satisfies the resultant polynomial, so A_K is a factor of the resultant above. The resultant clearly has degree $\leq 2n$ in l. So the degree of A_K in l is $\leq 2n$. On the other hand, degree of A_K in l is greater than or equal to the degree of the cusp field of K, which is $2n$ by 8.3.1. The same argument applies for negative n. □

Corollary 8.5.3 *The A-polynomial $A_{K_{2n}}(l,m)$ does not factorise as the product of two polynomials $F(l,m)\ G(l,m)$ with positive degree in l. That is, the branches $l_j(m)$ of $A_{K_{2n}}(l,m)$ are all Galois conjugates.*

Proof. We consider $n > 0$; the case $n < 0$ is analogous. Suppose $A_{K_{2n}}(l,m) = F(l,m)\ G(l,m)$ where $F, G \in \mathbb{Z}[l,m]$. Following the idea of 8.3.1, we can obtain the cusp shape c_0 of the complete structure by setting $l = -1 + st, m = \pm 1 + t$, taking the terms of lowest degree in t and solving for s. Let $F_1(s)$, $G_1(s)$ be the result of this procedure on $F(l,m), G(l,m)$ respectively. Then c_0 is a root of F_1 or G_1. By proposition 8.3.1, the minimal polynomial of c_0 has degree $2n$, so one of F_1, G_1 has degree $2n$ in l. □

For $n > 0$, it follows from the symmetry properties discussed in section 7.8, that n of the branches have $|l_j^B(e^{i\phi})| \geq 1$ and the other n have $|l_j^B(e^{i\phi})| \leq 1$. In section 8.2 we found that there were n hyperbolic cone-manifold structures on $S^3 - K_{2n}$, with cone angles $2\pi, 4\pi, \ldots, 2n\pi$ around a, corresponding to Galois conjugates of the (lifted) holonomy representation $\pi_1(K_{2n}) \longrightarrow SL_2(\mathbb{C})$. Deformations of these cone-manifold hyperbolic structures, which are Galois conjugate to deformations of the complete hyperbolic structure on $S^3 - K_{2n}$, are represented by n branches of $A_{K_{2n}}$. For these hyperbolic structures, the integrals $\mathcal{M}_j$ of section 7.8) give changes in pseudovolume. For $\mathcal{M}_j$ to be nonzero, we require $|l_j^B(e^{i\phi})| > 1$. So the n branches with $|l_j^B(e^{i\phi})| > 1$ for small $\phi > 0$ correspond precisely to deformations of these n cone-manifold. Similar properties occur when $n < 0$.

Chapter 9

Two-bridge knots

This final chapter is a brief discussion of representations and A-polynomials of two-bridge knots. Being a much broader class of knots than the twist knots (though still decidedly narrow when compared to the set of all knots), explicit calculations in the nature of the previous chapter are much more difficult. However we can perform some computations in general, and from empirical investigation we make some observations and pose some questions.

9.1 Parabolic representations

We recall the fundamental group of a two-bridge knot, and a standard meridian and longitude, from section 2.4.

$$\pi_1(S(\alpha,\beta)) = < g, h \mid gw = wh >,$$

where

$$w = h^{\epsilon_1} g^{\epsilon_2} \cdots g^{\epsilon_{\alpha-1}}, \qquad \epsilon_j = (-1)^{\lfloor \frac{j\beta}{\alpha} \rfloor}, \qquad j = 1, \ldots, \alpha - 1.$$

As in the previous chapter (but slightly varying notation for Ω), parabolic representations are found by setting

$$\rho(g) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho(h) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}, \qquad \rho(w) = \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

and solving the matrix equation given by the relator

$$\rho(g)\Omega = \Omega\rho(h). \tag{9.1}$$

We will also make use of an alternative, symmetric normal form. Let $t = u^2$ and create a new representation τ by conjugating by

$$U = \begin{pmatrix} u^{\frac{1}{2}} & 0 \\ 0 & u^{-\frac{1}{2}} \end{pmatrix}, \qquad \tau(x) = U\rho(x)U^{-1}.$$

We obtain

$$\tau(g) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad \tau(h) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}, \qquad \tau(w) = \begin{pmatrix} \Omega_{11} & u\Omega_{12} \\ u^{-1}\Omega_{21} & \Omega_{22} \end{pmatrix}$$

The entries Ω_{ij} are polynomials in t and we perform a little devious algebra. Firstly, as each Ω_{ij} is a polynomial in $t = u^2$, all these polynomials are even in u.

Lemma 9.1.1 *The polynomials* Ω_{21}, Ω_{12} *satisfy*

$$\Omega_{21} = -t\Omega_{12}, \qquad \textit{or equivalently,} \qquad \Omega_{21} = -u^2\Omega_{12}.$$

Proof. This proof is contained in the proof of theorem 2 of [71]; he also refers to [49]. Define four matrices as shown:

$$I_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad I_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so $I_{ij}I_{kl} = \delta_{jk}I_{il}$ (where δ_{jk} is the Kronecker delta) and

$$\begin{aligned} \tau(w) &= \tau(h)^{\epsilon_1}\tau(g)^{\epsilon_2}\cdots\tau(g)^{\epsilon_{\alpha-1}} \\ &= (I - uI_{21})^{\epsilon_1}(I + uI_{12})^{\epsilon_2}\cdots(I + uI_{12})^{\epsilon_{\alpha-1}} \end{aligned}$$

We transpose $\tau(w)$.

$$\tau(w)^T = (I + uI_{21})^{\epsilon_{\alpha-1}}\cdots(I - uI_{12})^{\epsilon_1}$$

Now we use the fact that $\epsilon_j = \epsilon_{\alpha-j}$. To see this, recall from section 2.4 that β is odd and α, β are coprime, so none of $\left\lfloor \frac{j\beta}{\alpha} \right\rfloor$ are integers for $j = 1, \ldots, \alpha - 1$. Then

$$\left\lfloor \frac{j\beta}{\alpha} \right\rfloor + \left\lfloor \frac{(\alpha - j)\beta}{\alpha} \right\rfloor = \frac{j\beta}{\alpha} + \frac{(\alpha - j)\beta}{\alpha} - 1 = \beta - 1$$

so $\left\lfloor \frac{j\beta}{\alpha} \right\rfloor$ and $\left\lfloor \frac{(\alpha-j)\beta}{\alpha} \right\rfloor$ have the same parity, hence $\epsilon_j = \epsilon_{\alpha-j}$. Thus

$$\tau(w)^T = (I + uI_{21})^{\epsilon_1}(I - uI_{12})^{\epsilon_2}\cdots(I - uI_{12})^{\epsilon_{\alpha-1}} = \tau(w)(-u),$$

i.e. $\tau(w)$ with all u's replaced with $-u$'s. This gives in particular

$$u\Omega_{12}(u) = -u^{-1}\Omega_{21}(-u)$$

and since Ω_{21} is even in u we have

$$u\Omega_{12}(u) = -u^{-1}\Omega_{21}(u)$$

from which the result follows. □

We now return to our matrix equation 9.1 and obtain

$$\begin{aligned}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ -t\Omega_{12} & \Omega_{22}\end{pmatrix} &= \begin{pmatrix}\Omega_{11} & \Omega_{12}\\ -t\Omega_{12} & \Omega_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ -t & 1\end{pmatrix}\\ \begin{pmatrix}\Omega_{11}-t\Omega_{12} & \Omega_{12}+\Omega_{22}\\ -t\Omega_{12} & \Omega_{22}\end{pmatrix} &= \begin{pmatrix}\Omega_{11}-t\Omega_{12} & \Omega_{12}\\ -t\Omega_{12}-t\Omega_{22} & \Omega_{22}\end{pmatrix}.\end{aligned}$$

which is satisfied if and only if $\Omega_{22} = 0$. Following notation of the previous chapter we call this polynomial $\Phi_{(\alpha,\beta)}(t)$. Unlike twist knots, not all polynomials $\Phi_{(\alpha,\beta)}(t)$ are irreducible: see section 9.4.

We next turn to the longitude and cusp shape. There is an analogous result to that for twist knots. Recall that $\lambda = g^{-2\sigma}w\tilde{w}$. We use the symmetric normal form to calculate $\tau(\tilde{w})$.

$$\begin{aligned}\tau(\tilde{w}) &= (I+uI_{12})^{\epsilon_{\alpha-1}}\cdots(I-uI_{21})^{\epsilon_1}\\ \tau(\tilde{w})^{-1} &= (I+uI_{21})^{\epsilon_1}\cdots(I-uI_{12})^{\epsilon_{\alpha-1}}\\ &= \tau(w)(-u)\end{aligned}$$

Thus $\tau(\tilde{w})$ is the matrix for $\tau(w)^{-1}$, with $-u$ substituted for u. Using the fact that all polynomials Ω_{ij} are even in u, we obtain explicitly

$$\tau(w) = \begin{pmatrix}\Omega_{11} & u\Omega_{12}\\ -u\Omega_{12} & \Omega_{22}\end{pmatrix}, \qquad \tau(\tilde{w}) = \begin{pmatrix}\Omega_{22} & u\Omega_{12}\\ -u\Omega_{12} & \Omega_{11}\end{pmatrix}.$$

Thus we compute $\tau(\lambda)$, using the fact that at a parabolic representation, Ω_{22} must equal 0:

$$\begin{aligned}\tau(\lambda) &= \tau(g)^{-2\sigma}\tau(w)\tau(\tilde{w})\\ &= \begin{pmatrix}1 & -2\sigma u\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Omega_{11} & u\Omega_{12}\\ -u\Omega_{12} & 0\end{pmatrix}\begin{pmatrix}0 & u\Omega_{12}\\ -u\Omega_{12} & \Omega_{11}\end{pmatrix}\\ &= \begin{pmatrix}1 & -2\sigma u\\ 0 & 1\end{pmatrix}\begin{pmatrix}-u^2\Omega_{12}^2 & 2u\Omega_{11}\Omega_{12}\\ 0 & -u^2\Omega_{12}^2\end{pmatrix}\\ &= \begin{pmatrix}-u^2\Omega_{12}^2 & 2u\Omega_{11}\Omega_{12}+2\sigma u^3\Omega_{12}^2\\ 0 & -u^2\Omega_{12}^2\end{pmatrix}\end{aligned}$$

Conjugating back by U we obtain

$$\rho(\lambda) = \begin{pmatrix}-t\Omega_{12}^2 & 2\Omega_{11}\Omega_{12}+2\sigma t\Omega_{12}^2\\ 0 & -t\Omega_{12}^2\end{pmatrix}$$

so the parabolic isometries of $\mathbb{H}^3$ given to μ and λ respectively are therefore

$$z \mapsto z+1, \qquad z \mapsto z - \frac{2}{t}\frac{\Omega_{11}}{\Omega_{12}} - 2\sigma.$$

The pseudo-cusp shape is then $-\frac{2}{t}\frac{\Omega_{11}}{\Omega_{12}} - 2\sigma$, analogously to twist knots.

Thus, the calculation of parabolic representations , hence cusp shapes, is no more difficult than for twist knots, in the individual case.

9.2 General representations

We can follow the procedure of 7.4 to calculate the A-polynomial, extending the technique of the previous chapter and [71]. We use the more general forms

$$\rho(g) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \qquad \rho(h) = \begin{pmatrix} m & 0 \\ -t & m^{-1} \end{pmatrix}, \qquad \rho(w) = \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix},$$

and the analogous symmetric form $\tau = U\rho U^{-1}$, with $t = u^2$ and U as in the previous section:

$$\tau(g) = \begin{pmatrix} m & u \\ 0 & m^{-1} \end{pmatrix}, \qquad \tau(h) = \begin{pmatrix} m & 0 \\ -u & m^{-1} \end{pmatrix}, \qquad \tau(w) = \begin{pmatrix} \Omega_{11} & u\Omega_{12} \\ u^{-1}\Omega_{21} & \Omega_{22} \end{pmatrix}$$

Again we have all the polynomials Ω_{ij} even in u, and we have an analogous relation to lemma 9.1.1:

Lemma 9.2.1 *The polynomials* Ω_{12}, Ω_{21} *satisfy* $\Omega_{21} = -t\Omega_{12} = -u^2\Omega_{12}$.

Proof. This proof is analogous to that of lemma 9.1.1. Instead of I, here we use the matrix

$$M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}.$$

We use the fact that $I_{ij}M$ and MI_{ij} are both of the form $m^{\pm 1}I_{ij}$, and prove that $\tau(w)^T(m,u) = \tau(w)(m,-u)$, i.e. the transpose of $\tau(w)$ is equal to $\tau(w)$ with u's replaced with $-u$'s. The result follows. □

To find representations we find solutions to the matrix equation given by the relator,

$$\rho(g)\Omega = \Omega\rho(h).$$

Substituting $\Omega = [\Omega_{ij}]$ and using 9.2.1 above gives

$$\begin{aligned} \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ -t\Omega_{12} & \Omega_{22} \end{pmatrix} &= \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ -t\Omega_{12} & \Omega_{22} \end{pmatrix} \begin{pmatrix} m & 0 \\ -t & m^{-1} \end{pmatrix} \\ \begin{pmatrix} m\Omega_{11} - t\Omega_{12} & m\Omega_{12} + \Omega_{22} \\ -m^{-1}t\Omega_{12} & m^{-1}\Omega_{22} \end{pmatrix} &= \begin{pmatrix} m\Omega_{11} - t\Omega_{12} & m^{-1}\Omega_{12} \\ -mt\Omega_{12}\Omega_{22} - t\Omega_{22} & m^{-1}\Omega_{22} \end{pmatrix} \end{aligned}$$

which is satisfied if and only if

$$\left(m - m^{-1}\right)\Omega_{12} + \Omega_{22} = 0, \tag{9.2}$$

identically to 8.6. So one equation suffices to relate m and t, as for twist knots.

As is becoming quite familiar by now, we next consider the longitude $\lambda = g^{-2\sigma}w\tilde{w}$. Using the symmetric normal form τ, we have

$$\tau(g) = M + uI_{12}, \qquad \tau(h) = M - uI_{21}, \qquad M = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

so that $\tau(g)^{-1} = M^{-1} - uI_{12}$ and $\tau(h)^{-1} = M^{-1} + uI_{21}$. Then we can compute $\tau(w)$ and $\tau(\tilde{w})$:

$$\begin{aligned}
\tau(w) &= (M - uI_{21})^{\epsilon_1} \cdots (M + uI_{12})^{\epsilon_{\alpha-1}} \\
\tau(\tilde{w}) &= (M + uI_{12})^{\epsilon_{\alpha-1}} \cdots (M - uI_{21})^{\epsilon_1} \\
\tau(\tilde{w})^{-1} &= \left(M^{-1} + uI_{21}\right)^{\epsilon_1} \cdots \left(M^{-1} - uI_{12}\right)^{\epsilon_{\alpha-1}} \\
&= \tau(w)(m^{-1}, -u).
\end{aligned}$$

That is, $\tau(\tilde{w})$ is the matrix $\tau(w)^{-1}$ with m replaced with m^{-1} and u replaced with $-u$. Using this and lemma 9.2.1, we obtain explicitly

$$\tau(w) = \begin{pmatrix} \Omega_{11} & u\Omega_{12} \\ -u\Omega_{12} & \Omega_{22} \end{pmatrix}, \qquad \tau(\tilde{w}) = \begin{pmatrix} \widetilde{\Omega_{22}} & u\widetilde{\Omega_{12}} \\ -u\widetilde{\Omega_{12}} & \widetilde{\Omega_{11}} \end{pmatrix}$$

where, as in the previous chapter, $\widetilde{\Omega_{ij}}$ denotes the polynomial Ω_{ij} after replacing m with m^{-1}.

We now calculate $\tau(g)^{-2\sigma}$ (recall $\sigma = \sum_{i=1}^{\alpha-1} \epsilon_i$). The recurrence for $\tau(g)^n$, obtained from the characteristic polynomial as in the previous chapter, is

$$\tau(g)^n - \left(m + m^{-1}\right)\tau(g)^{n-1} + \tau(g)^{n-2} = 0$$

for all $n \in \mathbb{Z}$. Then we obtain

$$\tau(g)^n = \begin{pmatrix} m^n & R_n(m,u) \\ 0 & m^{-n} \end{pmatrix}$$

where the polynomials $R_n(m,u)$ are defined by $R_0(m,u) = 0$, $R_1(m,u) = u$ and, for all $n \in \mathbb{Z}$,

$$R_n(m,u) = \left(m + m^{-1}\right) R_{n-1}(m,u) - R_{n-2}(m,u).$$

We can now calculate $\tau(\lambda)$.

$$\begin{aligned}
\tau(\lambda) &= \tau(g)^{-2\sigma}\tau(w)\tau(\tilde{w}) \\
&= \begin{pmatrix} m^{-2\sigma} & R_{-2\sigma} \\ 0 & m^{2\sigma} \end{pmatrix} \begin{pmatrix} \Omega_{11} & u\Omega_{12} \\ -u\Omega_{12} & \Omega_{22} \end{pmatrix} \begin{pmatrix} \widetilde{\Omega_{22}} & u\widetilde{\Omega_{12}} \\ -u\widetilde{\Omega_{12}} & \widetilde{\Omega_{11}} \end{pmatrix} \\
&= \begin{pmatrix} m^{-2\sigma} & R_{-2\sigma} \\ 0 & m^{2\sigma} \end{pmatrix} \begin{pmatrix} \Omega_{11}\widetilde{\Omega_{22}} - u^2\Omega_{12}\widetilde{\Omega_{12}} & u\Omega_{11}\widetilde{\Omega_{12}} + u\Omega_{12}\widetilde{\Omega_{11}} \\ -u\Omega_{12}\widetilde{\Omega_{22}} - u\Omega_{22}\widetilde{\Omega_{12}} & -u^2\Omega_{12}\widetilde{\Omega_{12}} + \Omega_{22}\widetilde{\Omega_{11}} \end{pmatrix}
\end{aligned}$$

Since $\tau(\lambda)$ commutes with $\tau(g)$, it will be upper triangular, so the lower-left entry is zero:

$$m^{2\sigma}\left(-u\Omega_{12}\widetilde{\Omega_{22}} - u\Omega_{22}\widetilde{\Omega_{12}}\right) = 0.$$

Since $m \neq 0$, the expression in brackets (i.e. the lower-left entry of $\tau(w)\tau(\tilde{w})$ is zero. We now obtain l as the upper-left entry of upper triangular matrix $\tau(g)$

$$l = m^{-2\sigma}\left(\Omega_{11}\widetilde{\Omega_{22}} - u^2\Omega_{12}\widetilde{\Omega_{12}}\right). \tag{9.3}$$

Thus, the A-polynomial can be calculated by eliminating $t = u^2$ from 9.2 and 9.3. So, in any individual case, the procedure is no different from that for twist knots.

In the style of the last chapter, we can obtain a bound on the degree of the A-polynomial in l.

Proposition 9.2.2 *The degree of the A-polynomial of $S(\alpha, \beta)$ in l is at most*

$$\frac{\alpha-1}{2}.$$

Proof. From above we have

$$\begin{aligned}\tau(w) &= \begin{pmatrix}\Omega_{11} & u\Omega_{12} \\ -u\Omega_{12} & \Omega_{22}\end{pmatrix} = (M - uI_{21})^{\epsilon_1} \cdots (M + uI_{12})^{\epsilon_{\alpha-1}} \\ &= (M^{\epsilon_1} - \epsilon_1 uI_{21}) \cdots (M^{\epsilon_{\alpha-1}} + \epsilon_{\alpha-1} uI_{12})\end{aligned}$$

There is only one way to obtain a $u^{\alpha-1}$ term in the product above, and that is of the form $\pm u^{\alpha-1} I_{22}$. Clearly no higher powers of u are possible. So the degree of Ω_{22} in u is $\alpha - 1$. There are only two ways to obtain a nonzero $u^{\alpha-2}$ term, since $I_{jk}I_{lm} = \delta_{kl}I_{jm}$ and $MI_{jk} = m^{\pm 1}I_{jk}$, $I_{jk}M = m^{\pm 1}I_{jk}$. These give $\pm u^{\alpha-2}MI_{12} = \pm u^{\alpha-2}m^{\pm 1}I_{12}$ and $\pm u^{\alpha-2}I_{21}M = \pm u^{\alpha-2}m^{\pm 1}I_{21}$ respectively. So the degree of $u\Omega_{12}$ in u is $\alpha - 2$, and the degree of Ω_{12} is $\alpha - 3$. Since we have accounted for all terms of higher degree, the degree of Ω_{11} in u is $\leq \alpha - 3$. Setting $t = u^2$ we obtain the degrees of Ω_{22}, Ω_{12} in t as $\frac{\alpha-1}{2}, \frac{\alpha-3}{2}$ respectively, and the degree of Ω_{11} is at most $\frac{\alpha-3}{2}$..

The two equations relating l, m, t are

$$\begin{aligned}\left(m - m^{-1}\right)\Omega_{12} + \Omega_{22} = 0 \\ l - m^{-2\sigma}\left(\Omega_{11}\widetilde{\Omega_{22}} - t\Omega_{12}\widetilde{\Omega_{12}}\right) = 0.\end{aligned}$$

The first expression is the sum of two polynomials of degree $\frac{\alpha-1}{2}$ and $\frac{\alpha-3}{2}$ respectively in t, so has degree $\frac{\alpha-1}{2}$. The second is the sum of two terms of degree $\leq \alpha - 2$, so has degree $\leq \alpha - 2$.

We follow the argument about the resultant in theorem 8.5.2. The number of times that l occurs in the matrix for the resultant over t is equal to the degree of $\left(m - m^{-1}\right)\Omega_{12} + \Omega_{22}$ in t, which is $\frac{\alpha-1}{2}$. So the resultant has degree $\leq \frac{\alpha-1}{2}$ in l. The A-polynomial is a factor of this resultant, so has degree $\leq \frac{\alpha-1}{2}$. $\square$

9.3 Cone manifolds and geodesics

As discussed at the end of the previous chapter, twist knots have the property that parabolic representations and A-polynomial branches of positive Mahler measure correspond to cone manifold structures on $S^3 - K_{2n}$, placing a cone angle of $2k\pi$ around a loop a.

This property raises some interesting questions: how general is it?

Question 9.3.1 *Given a knot K and a branch $l_j(m)$ of $A_K(l,m)$, does l_j represent a family of cone manifold structures on $S^3 - K$, with singular locus $a_1 \cup \ldots \cup a_n$, for some loops $a_1, \ldots, a_n$, with cone angles $2k_1\pi, \ldots, 2k_n\pi$?*

Question 9.3.2 *For which knots K do all of the branches of $A_K(l,m)$ of positive Mahler measure have such an interpretation?*

I conjecture that there is such an interpretation for all two-bridge knots. For non-two-bridge knots, we may be more dubious, as the branches $l_j(e^{i\theta})$ may have modulus which is greater than 1 in some ranges of θ, and less than 1 in other ranges. Boyd in [3] gives the example of the knot 10_{125}. But we can still ask whether the integral

$$\int_0^\pi \log |l_j^B(e^{i\phi})| \, d\phi$$

(which is not a Mahler measure) represents a cone-manifold.

One simple way to test this question experimentally is to find loops a in $S^3 - K$ such that $S^3 - K - a$, with the appropriate cone angle, has a volume equal to a "Galois conjugate" of the volume of the complete structure. The volumes of such cone manifolds can be easily computed from a knot diagram by SnapPea ([87]). But we need to be more precise about a "Galois conjugate" of a hyperbolic volume, which need not be an algebraic number.

We take a triangulation for $S^3 - K$ and find the tetrahedral parameters $z_1, \ldots, z_k$. Then we take the image of $(z_1, \ldots, z_k)$ under each complex embedding of $\mathbb{Q}(z_1, \ldots, z_k)$, and find the pseudovolume associated with this solution: this set of pseudovolumes is the *Borel regulator* of $S^3 - K$ ([3], [58]).

We then have the following related question:

Question 9.3.3 *Given a knot K, is every pseudovolume in the Borel regulator of K equal in magnitude to the volume of the cone-manifold $S^3 - K$ with singular locus $a_1 \cup \ldots \cup a_n$, for some loops $a_1, \ldots, a_n$ in $S^3 - K$, with cone angles $2k_1\pi, \ldots, 2k_n\pi$?*

I have verified that this is the case for every two-bridge knot K through 8 crossings, and more. I conjecture it is true for all two-bridge knots. Note however that, for given K, there may be many different sets of loops $\{a_i\}$ and angles $\{k_i\}$ giving the same volume. For instance, in the knot 8_{13}, the cone-manifolds with cone angles 6π and 4π about b and c respectively, as shown in figure 9.2, both have volume approximately 0.470548, which is an element of the Borel regulator.

The appendix contains a list of all two-bridge knots through 7 crossings, including Borel regulators and loops which give cone-manifold structures with these pseudovolumes.

9.4 Factorisation Phenomena

For twist knots, we know that the polynomial defining parabolic representations is irreducible, so all parabolic representations are Galois conjugates of each other. We also know that the all the branches of the A-polynomial are Galois conjugates of each other. But such irreducibility is not true in general, and there are several intriguing examples where the polynomials involved factorise.

For instance, consider the knot 7_4, which is $S(15,11)$ and the knot formed from the basic horizontal tangle $H(3,1,3)$.

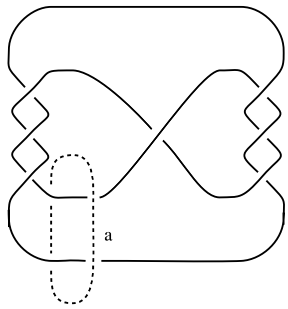


Figure 9.1: knot $7_4 = S(15,11)$, with loop a

We find that parabolic representations are given by the roots of the polynomial

$$\begin{aligned}\Phi_{(15,11)}(t) &= t^7+7t^6+18t^5+19t^4+6t^3+2t^2+4t-1\\ &= \left(t^3+4t^2+4t-1\right)\left(t^4+3t^3+2t^2+1\right)\end{aligned}$$

and the A-polynomial factorises as

$$\begin{aligned}A_{S(15,11)}(l,m) &= \big(lm^{14}-2l^2m^{14}+l^3m^{14}-2lm^{12}+6l^2m^{12}+3lm^{10}\\ &\quad +2l^2m^{10}+2lm^8-7l^2m^8-7lm^6+2l^2m^6+2lm^4\\ &\quad +3l^2m^4+6lm^2-2l^2m^2+l^2-2l+1\big)\\ &\quad \big(1-l+lm^2+2lm^4+lm^6+l^2m^8-lm^8\big)^2\\ &= A_1(l,m)\,A_2(l,m)^2\end{aligned}$$

where the Mahler measures of these two components, decomposed into branches, are

$$\pi\mathcal{M}(A_1)\sim 5.136+0+0, \qquad \pi\mathcal{M}(A_2)\sim 2.033+0.$$

The branches of the A-polynomial are graphed in figure 9.2; one curve corresponds to the nonzero term in $\mathcal{M}(A_1)$, the other curve corresponds to the nonzero term in $\mathcal{M}(A_2)$.

The volume of 7_4 is $\sim 5.137941(\sim 5.136)$, and the Borel regulator consists only of this volume. And note that

$$A_2(l,m)=m^{-4}A_{4_1}(lm^4,m)$$

so A_2 is almost the A-polynomial of the figure-8 knot! By proposition 6.2.3, these two polynomials have the same Mahler measure, for which $\pi\mathcal{M}(A_{4_1})=$

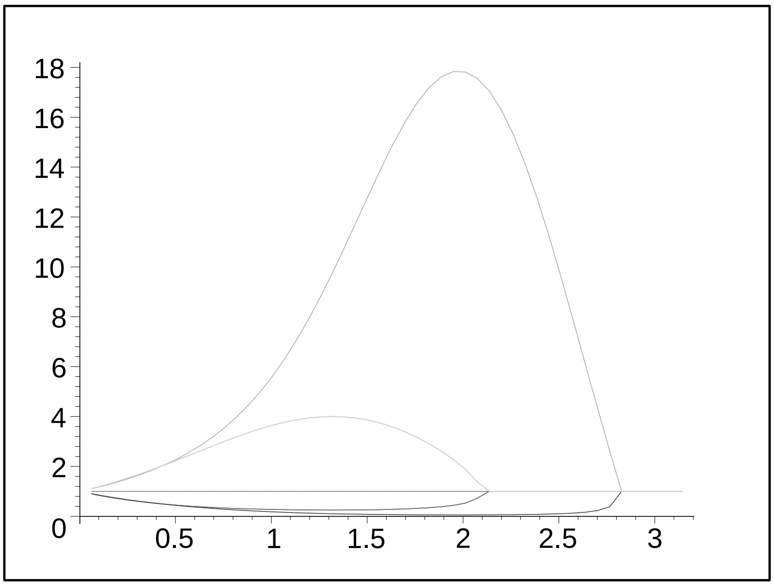


Figure 9.2: Graphs of $|l_j^B(e^{i\phi})|$ for A_{7_4}

vol 4_1. Indeed, placing around a cone angle of 4π around a gives a volume of 2.029883, the volume of the figure-8 knot.

These fascinating occurrences are not unique to the 7_4 and the figure-8 knot. Indeed, the knot $7_7 = S(21, 13)$ has a similar relationship to the tweeny knot $5_2 = S(7, 3)$, as does $9_{10} = S(33, 23)$ to $6_2 = S(11, 3)$. In a vague sense, all the "factor" knots are about "half as complicated" as the "larger" knots.

What is common to these examples? We find that all these knots are associated to rational tangles whose *continued fractions are palindromic*, which implies $\beta^2 \equiv \pm 1 \mod \alpha$ (see further proposition 2.4.3).

Conjecture 9.4.1 *The polynomial $\Phi_{(\alpha,\beta)}(t)$ defining parabolic representations, and $A_{S(\alpha,\beta)}(l, m)$ both factorise when α/β has a palindromic continued fraction.*

This phenomenon seems related to other special properties of such (α, β). In [63], Ohtsuki claims a proof that the representation variety $R(\pi_1(S(\alpha, \beta)))$ is reducible when $\beta^2 \equiv 1 \mod \alpha$. His investigations involve the tree theory of [73], as mentioned in section 7.5. Further, $\beta^2 \equiv 1 \mod \alpha$ implies that the Lens space $L(\alpha, \beta)$ has extra orientation-preserving involutions (see [42]). And the 2-fold cover of S^3 branched along $S(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$ (see, e.g., [10, thm. 12.3]). But neither supplies a full explanation of which smaller, "factor" knots are involved.

Perhaps the simplest explanation would be a homomorphism, for example, from $\pi_1(7_4) \longrightarrow \pi_1(4_1)$. However this does not seem to be the case.

Proposition 9.4.2 *There is no surjective homomorphism* $\pi_1(7_4) \longrightarrow \pi_1(4_1)$.

Proof. Suppose there were such a homomorphism α. Now there exists a homomorphism of $\pi_1(4_1)$ onto the permutation group $S5$, taking all meridians to permutations of the form $(abc)(de)$; this can easily be found, for instance, using the program Knotscape [80]. But composition with α would then give a homomorphism of $\pi_1(7_4)$ onto $S5$. It is easily checked that such a homomorphism does not exist. □

I have not been able to rule out a homomorphism onto, for instance, an order-2 subgroup. But obvious homomorphisms do not appear to work.

Question 9.4.3 *Find an adequate explanation for these phenomena.*

Perusal of the data in the appendix may also lead to other conjectures: for instance, the loops giving cone-manifolds predominantly seem to run around some twists. Is there a way to find where such loops lie, short of educated trial and error? Is there an explanation in terms of Dehn surgery on a link, analogous to the twist knot example? There are many avenues along which to proceed from here.

Appendix A

A-polynomial data and geometric interpretation

This appendix lists all the two-bridge knots through 7 crossings, and two selected trickier examples with 8 crossings. For each knot, the following is given:

1. Names for the knot, including the standard notation (e.g. 4_1), Schubert normal form (e.g. $S(5,3)$), the name of the knot complement manifold in the SnapPea census (e.g. $m004$), twist knot notation K_2, and notation in terms of the minimal number of ideal tetrahedra in a triangulation for the knot complement (e.g. $k3_1$ = first knot with complement consisting of 3 ideal tetrahedra).

2. A knot diagram

3. A graph of the branches $|l_j^B(e^{i\phi})|$, for $0 \leq \phi \leq \pi$, in the notation of section 7.8.

4. The volume of the knot

5. The degree of the A-polynomial: actually the degree of $B(l,m)$ where $B(l,m^2) = A(l,m)$, which is more closely related to hyperbolic volume. If $B(l,m)$ factorises, then the degree is expressed as a sum of the degrees of the factors.

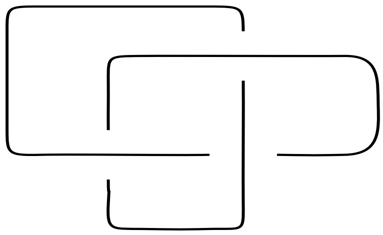
6. The Mahler measures $\pi\mathcal{M}_j$, and total Mahler measure.

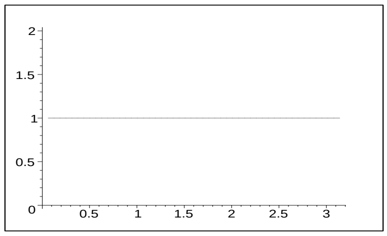
7. The Borel regulator of the knot complement.

8. A description of the cusp field (which is identical to the invariant trace field, shape field, group coefficient field and trace field in all these cases: see [56]), in the form [minimal polynomial, [real embeddings, complex embeddings], discriminant, root of polynomial generating field].

9. A list of loops and angles giving cone-manifold structures with the volumes in the Borel regulator and Mahler measures. With each, I have included whether the solution is geometric (G) or involves negative tetrahedra (NT).

Knot $3_1, S(3,1)$

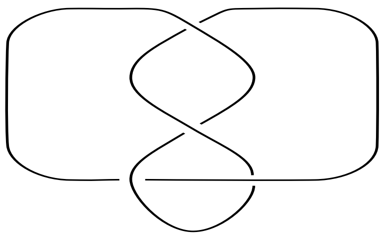


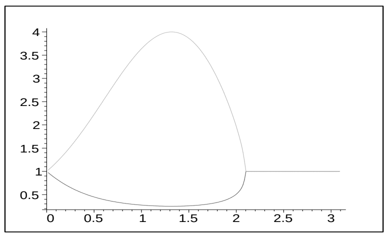


Volume $= 0$ (torus knot)
$\mathcal{M} = 0 = 0$

Deg B= (1,3)
Ireducible

Knot $4_1, S(5,3), m004, K_2, k3_1$

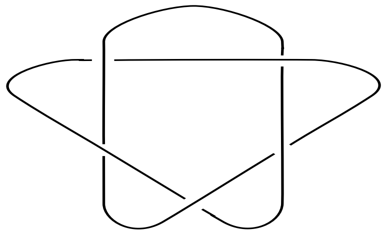


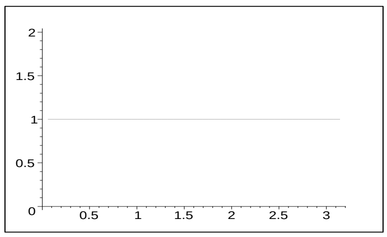


Volume $= 2.029883$
B.R.$=$ [2.029883]
$\pi\mathcal{M} = 2.033 + 0 = 2.033$

Field: $x^2 - x + 1$,
$[0,1], -3, \frac{1}{2} + \frac{\sqrt{3}}{2}i$
Deg B= (2,4)
Irreducible

Knot $5_1, S(5,1)$

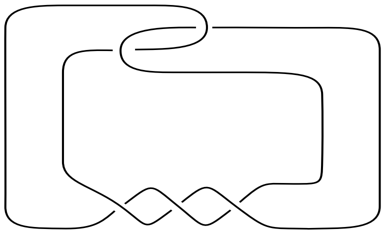


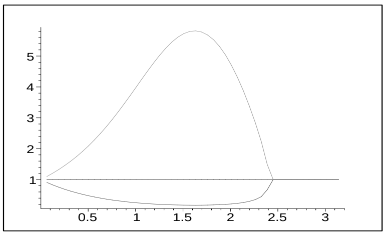


Volume $= 0$ (torus knot)
$\pi\mathcal{M} = 0 = 0$

Deg B $=$ (2,10)
Perfect square $= B_1^2$

Knot $5_2, S(7,3), m015, K_3, k3_2$

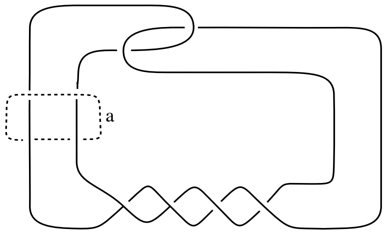


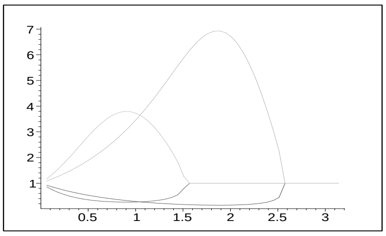


Volume = 2.828122
B.R.= [-2.828122]
$\pi\mathcal{M} = 2.831 + 0 + 0 = 2.831$

Field : $x^3 - x^2 + 1$,
$[1,1], -23, 0.877 - 0.745i$
Deg B = (3,14)
Irreducible

Knot $6_1, S(9,5), m032, K_4, k4_1$

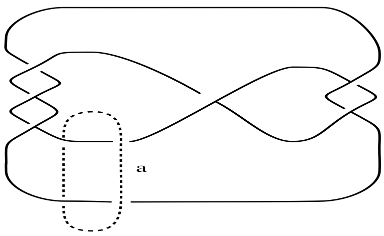


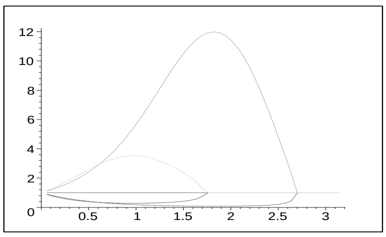


Volume = 3.163963
B.R.= [1.415105, -3.163963]
$\pi\mathcal{M} = 3.154 + 1.417 = 4.571$
a angle 4π vol = 1.415105 (G)

Field: $x^4 + x^2 - x + 1$
$[0,2], 257, 0.547 - 0.86i$
Deg B = (4,8)
Irreducible

Knot $6_2, S(11,3), m289, K_3$

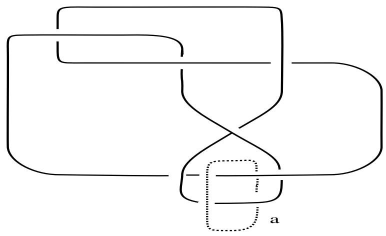


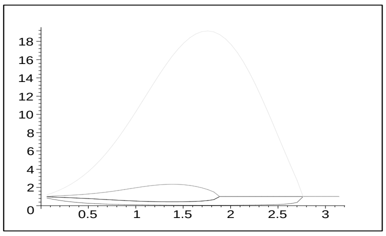


Volume = 4.400833
B.R.= [-1.530580, 4.400833]
$\pi\mathcal{M} = 4.402 + 1.530 + 0 \times 3 = 5.932$
a angle 4π vol= 1.503580 (NT)

Field: $x^5 - x^4 + x^3 - 2x^2 + x - 1$
$[1,2], 177, 0.277 + 0.728i$
Deg B = (5,14)
Irreducible

$6_3, S(13,5), s912$

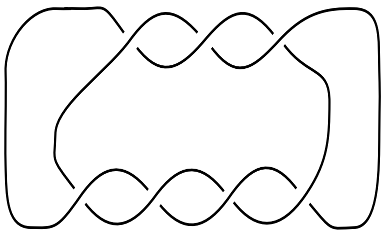


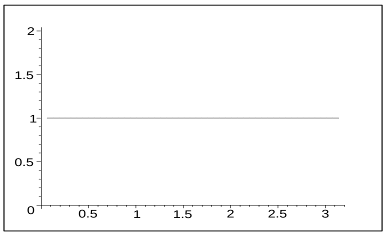


Volume = 5.693021
B.R.= [-0.924305, -0.924305, 5.693021]
$\pi\mathcal{M} = 5.683 + 0.920 \times 2 + 0 \times 3 = 7.523$
(2 pairs of branches equal in modulus)
a angle 4π vol = 0.924305 (NT)

Field: $x^6 - x^5 - x^4 + 2x^3 - x + 1,$
$[0,3], -10571, 1.074 + 0.559i$
Deg B = (6,14)
Irreducible

Knot $7_1, S(7,1)$

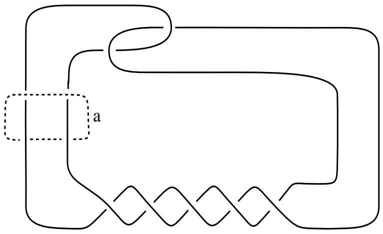


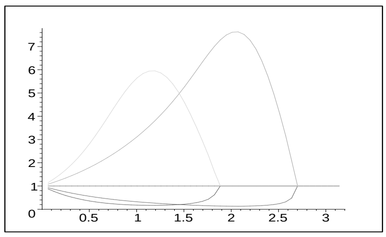


Volume = 0 (torus knot)
$\pi\mathcal{M} = 0 \times 3 = 0$

Deg B = (3,21)
Perfect cube = B_1^3

Knot $7_2, S(11,5), m053, K_5, k4_2$

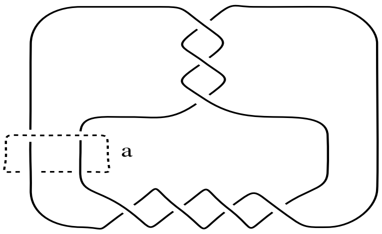


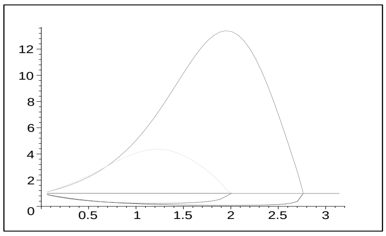


Volume = 3.331744
B.R.= [2.213969, -3.331744]
$\pi\mathcal{M} = 3.332 + 2.216 + 0 \times 3 = 5.548$
a angle 4π vol = 2.213969 (G)

Field: $x^5 - x^4 + x^2 + x - 1,$
$[1,2], 4409, 0.936 - 0.904i$
Deg B = (5, 11)
Irreducible

Knot $7_3, S(13,3), m340$

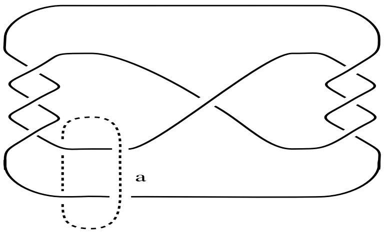


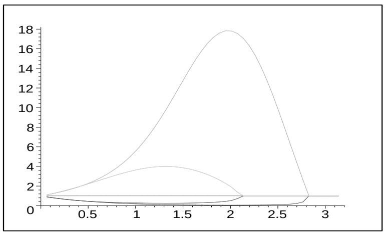


Volume = 4.592126
B.R.= [-1.972412, 4.592126]
$\pi\mathcal{M} = 4.584 + 1.975 + 0 \times 4 = 6.559$
a angle 4π vol=1.972412 (NT)

Field: $x^6 - x^5 + 3x^4 - 2x^3 + 2x^2$
$-x - 1, [2,2], 78301, 0.409 + 1.276i$
Deg B = (6,26)
Irreducible

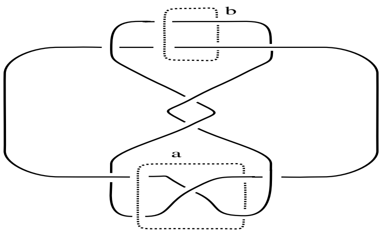
Knot $7_4, S(15,11), s648$

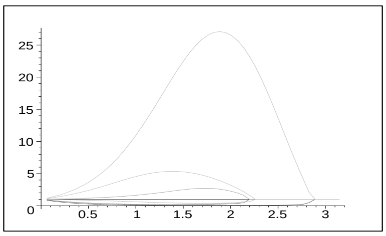




Volume = 5.137941
Field: $x^3 + 2x - 1, [1,1], -59, -0.227 - 1.468i$
B.R.= [-5.137941]
$\pi\mathcal{M} = 5.136 + 2.033 \times 2 + 0 \times 4 = 9.201$
Deg B $= (7,15) = (3,7) + (2,4) \times 2$, $B = B_1 B_2^2$, $B_2(l,m) = m^{-2} B_{4_1}(lm^2, m)$
a angle 4π vol=2.029883 (NT)

Knot $7_5, S(17,5), v3310$

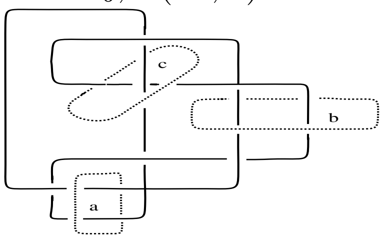


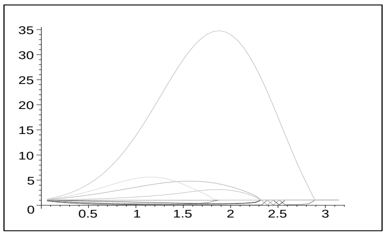


Volume = 6.443537
B.R.= [-2.578494, -1.131234, 6.443537]
$m = 6.450 + 2.580 + 1.127 + 0 \times 5$
$= 10.157$
a angle 4π vol = 2.578494 (NT)
b angle 4π vol = 1.131234 (NT)

Field: $x^8 - x^7 - x^6 + 2x^5 + x^4$
$-2x^3 + 2x - 1, [2,3],$
$-4690927, 1.032 + 0.655i$
Deg B = (8,34)
Irreducible

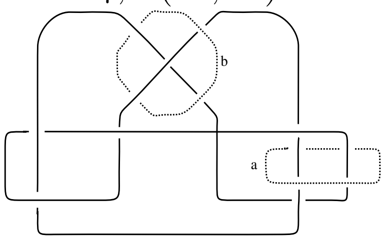
Knot $7_6, S(19,7)$

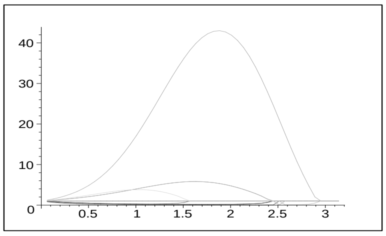




Volume = 7.084926
Field: $x^9 - x^8 + 2x^7 - x^6 + 3x^5 - x^4 + 2x^3 + x + 1$, $[1,4], 90320393, 0.729 - 0.986i$
B.R.= [2.454417, 2.093374, -7.084926, 1.336171]
$\pi\mathcal{M} = 7.076 + 2.447 + 2.094 + 1.335 = 12.952$
Deg B = (9,27), Irreducible
a angle 4π vol = 2.454417 (NT)
b angle 4π vol = 2.093374 (NT)
c angle 4π vol = 1.336171 (NT)

Knot $7_7, S(21,13)$



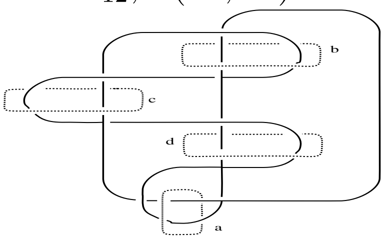


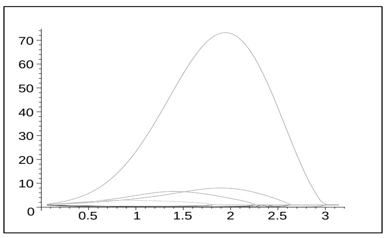
Volume = 7.643375
B.R.= [-7.643375, 1.397088]
$-0.547 - 1.121i$
$\pi\mathcal{M} = 7.650 + 2.832 \times 2 + 1.400$
$+0 \times 6 = 14.714$
a angle 4π vol = 2.828122 (NT)
b angle 4π vol = 1.397088 (NT)

Field: $x^4 + x^2 - x + 1$
$[0,2], 257,$

Deg B = $(10,24) = (4,14) + (3,5) \times 2$
$= B_1 B_2^2$, note $B_2(l,m) = m^{-2} B_{5_2}(lm^2, m)$

Knot $8_{12}, S(29,17)$

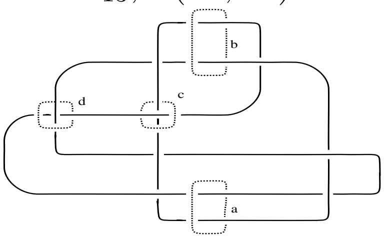


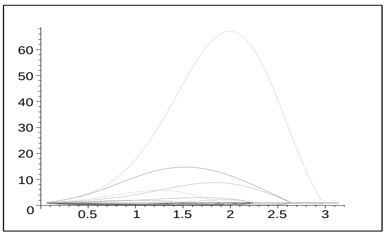


Volume = 8.935856
Field: $x^{14} - 2x^{13} + 3x^{12} - 4x^{11} + 4x^{10} - 5x^9 + 7x^8 - 7x^7 + 7x^6 - 5x^5 + 4x^4 - 4x^3 + 3x^2 - 2x + 1, [0,7], -15441795725579, 0.385 + 0.293i$
B.R.= [-2.767472, -2.767472, 1.377701, 1.377701, 8.935857, -3.412711, -3.412711]

$\pi\mathcal{M} = 8.945 + 1.376 \times 2 + 2.761 \times 2 + 3.413 \times 2 + 0 \times 7 = 24.044$
Deg B $= (14, 36)$, Irreducible (note branches equal in pairs)
a angle 4π vol $= 3.412711$ (NT)
b angle 4π vol $= 3.412711$ (NT)
c angle 4π vol $= 2.767472$ (NT)
d angle 4π vol $= 2.767472$ (NT)
a and d angle 4π vol $= 1.377701$ (NT)
b and c angle 4π vol $= 1.377701$ (NT)

Knot $8_{13}, S(29, 11)$





Volume $= 8.531232$
Field: $x^{14} - x^{13} - 3x^{12} + 4x^{11} + 4x^{10} - 7x^9 - x^8 + 6x^7 - 2x^6 - 2x^5 + 2x^4 - x + 1$,
$[0, 7]$, -759929100364387, $1.143 - 0.547i$
B.R.= [-0.470548, 5.071847, -0.628586, 3.628787, -2.191283, -1.404837, -8.531232]
$\pi\mathcal{M} = 5.068 + 8.540 + 2.193 + 3.628 + 0.624 + 1.408 + 0.524 = 21.986$
Deg B $= (14, 36)$, Irreducible
a angle 4π vol $= 5.071847$ (NT)
a angle 6π vol $= 1.404837$ (NT)
b angle 4π vol $= 3.628787$ (NT)
b angle 6π vol $= 0.470548$ (NT)
c angle 4π vol $= 0.470548$ (NT)
d angle 4π vol $= 2.191283$ (NT)
a and b angle 4π vol $= 0.628586$ (NT)

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