# Construction of geometric cone-manifold structures with prescribed holonomy

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## Geometric structures and holonomy

Recall a geometric $(X, \mathrm{Isom}\, X)$ structure on a manifold $M$ is a metric on $M$ locally isometric to $X$.

A geometric $(X, \mathrm{Isom}\, X)$ structure on a manifold $M$ gives a local isometry with $X$ which extends to a *developing map* $\mathcal{D} : \tilde{M} \longrightarrow X$.

For each $\alpha \in \pi_1(M)$, the developing map gives a path in $X$. The start and end points are related by an isometry $\rho(\alpha)$.

The map $\rho : \pi_1(M) \longrightarrow \mathrm{Isom}\, X$ so obtained is a homomorphism and is called the *holonomy*.

A geometric *cone manifold* structure determines a holonomy map $\rho : \pi_1(M) \longrightarrow \mathrm{Isom}\, X$ if all interior cone angles are multiples of $2\pi$.

## The problem

Given a homomorphism $\rho : \pi_1(M) \longrightarrow \text{Isom}\, X$, is $\rho$ the holonomy of some structure on $M$?

- geometric structure on $M$?

- geometric cone-manifold structure with interior cone angles multiples of $2\pi$?

- if $M$ has boundary, with totally geodesic boundary? With corners? Allowing too many corners trivialises the problem.

**Some known answers.**

(i) Leleu 2000: 3-dimensional hyperbolic, Euclidean, spherical manifolds with nonempty (non-geodesic) boundary.

$\rho : \pi_1(M) \longrightarrow \text{Isom}\, X$ is the holonomy of a geometric structure if and only if $\rho$ lifts to $\tilde{\rho} : \pi_1(M) \longrightarrow \widetilde{\text{Isom}}\, X$.

No control on boundary.

(ii) Gallo, Kapovich, Marden 2000: closed surfaces with complex projective structures.

Any nonelementary $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{C}$ is the holonomy of a complex projective cone manifold structure. No cone points are required iff $\rho$ lifts to $SL_2\mathbb{C}$. Otherwise a single $4\pi$ cone point is required.

## 2-dimensional hyperbolic geometry

**Theorem 1 (Goldman 1988)** *For a closed surface $S$, $\chi(S) < 0$, the Euler class $\mathcal{E}(\rho)$ classifies the topological components of the algebraic variety of representations $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$.*

The Euler class $\mathcal{E}(\rho) \in H^2(S)$ and

$$\chi(S) \leq \mathcal{E}(\rho)[S] \leq -\chi(S).$$

**Theorem 2 (Goldman 1980)** *For a closed surface $S$, $\chi(S) < 0$, the representations with $\mathcal{E}(\rho)[S] = \pm\chi(S)$ are precisely the holonomy representations of (non-singular) hyperbolic structures on $S$.*

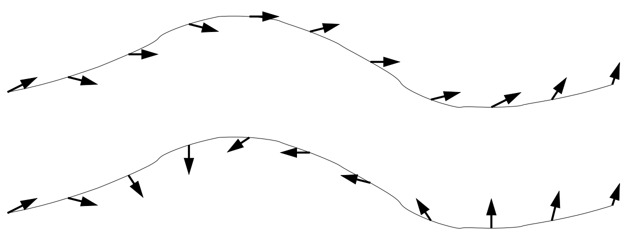
I.e. the holonomy representations are those "least liftable" to $\widetilde{PSL_2\mathbb{R}}$.

## The Euler class of a representation

Most useful description in terms of
$\widetilde{PSL_2\mathbb{R}} \cong \widetilde{UT\mathbb{H}^2} \cong \widetilde{\mathbb{H}^2 \times S^1} \cong \mathbb{H}^2 \times \mathbb{R}$.

Elements of $\widetilde{PSL_2\mathbb{R}}$ are homotopy classes of paths in $UT\mathbb{H}^2$.

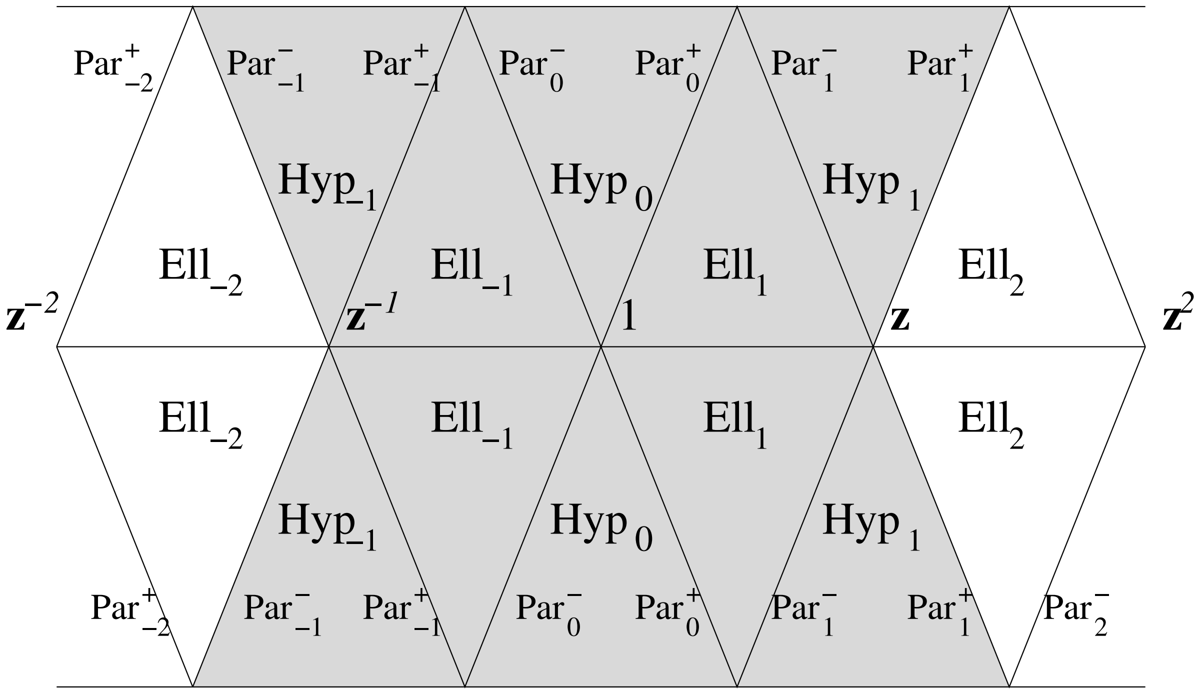
Any $\alpha \in PSL_2\mathbb{R}$ has infinitely many lifts to $\widetilde{PSL_2\mathbb{R}}$. These are paths in $UT\mathbb{H}^2$ with the same start and end tangent vectors, but differ according to the number of times that the tangent vectors "spin".



The fibre over $1 \in PSL_2\mathbb{R}$ is an infinite cyclic group $\{\mathbf{z}^n\}$.

The lifts of $\alpha \in PSL_2\mathbb{R}$ are precisely $\mathbf{z}^n\tilde{\alpha}$.

## Schematic diagram of $\widetilde{PSL_2\mathbb{R}}$




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Tw | $-4\pi$ | $-3\pi$ | $-2\pi$ | $-\pi$ | 0 | $\pi$ | $2\pi$ | $3\pi$ | $4\pi$ |
| θ | $-2\pi$ | | $-\pi$ | | 0 | | $\pi$ | | $2\pi$ |
| Tr | 2 | | $-2$ | | 2 | | $-2$ | | 2 |

A hyperbolic/parabolic element $\alpha \in PSL_2\mathbb{R}$ has a *simplest* lift to $\widetilde{PSL_2\mathbb{R}}$ ("drive straight along the axis"). These form $\text{Hyp}_0/\text{Par}_0$. Their *twist* is small.

## Commutators

Commutators are well-defined: if $\alpha, \beta \in PSL_2\mathbb{R}$ then $[\alpha, \beta] \in \widetilde{PSL_2}\mathbb{R}$ makes sense.

Commutators are "not too twisty": lie in the shaded region.

Take presentation for $\pi_1(S)$:
generators $G_1, H_1, \ldots, G_k, H_k, C_1, \ldots, C_n$
relator $[G_1, H_1] \cdots [G_k, H_k] C_1 \cdots C_n$.

A representation into $PSL_2\mathbb{R}$ is equivalent to choice of $g_i, h_i, c_i \in PSL_2\mathbb{R}$ with

$$[g_1, h_1] \cdots [g_k, h_k] c_1 \cdots c_n = 1.$$

For closed surfaces, this relator is well-defined in $\widetilde{PSL_2}\mathbb{R}$. For surfaces with boundary, we may take lifts $\tilde{c}_i$ with small twist.

**Representations of surface groups: the Milnor-Wood inequality**

We have

$$\left[\tilde{g}_1, \tilde{h}_1\right] \cdots \left[\tilde{g}_k, \tilde{h}_k\right] \tilde{c}_1 \cdots \tilde{c}_n = \mathbf{z}^m$$

for some $m$. Since "twist" is approximately additive $m$ is bounded:

$$\chi(S) \leq m \leq -\chi(S).$$

For a closed surface, $m$ depends only on $\rho$ and gives the Euler class of $\rho$. $\mathcal{E}(\rho)[S] = m$.

For a surface with boundary we may define a *relative Euler class* if we require $\rho$ to take boundary curves to non-elliptics.

If $\rho$ is the holonomy of a (non-singular) hyperbolic structure then $\mathcal{E}(\rho)[S] = \pm\chi(S)$.

For a closed surface, every step towards 0 requires an extra $2\pi$ worth of cone angle.

**Punctured torus theorem:**
Let $S$ be a punctured torus. A representation $\rho$ is the holonomy of a hyperbolic cone-manifold structure with at most one corner point and no interior cone points if and only if $\rho$ is not virtually abelian.

**Genus 2 surface theorem:**
Let $S$ be a genus 2 closed surface. Let $\rho$ be a representation with

(i) $\mathcal{E}(\rho)[S] = \pm 1$ (i.e. one off extremal).

(ii) for some separating curve $C$, $\rho(C)$ is not hyperbolic.

Then $\rho$ is the holonomy of a hyperbolic cone-manifold structure with one $4\pi$-angle cone point.

**"Almost every" theorem:**
Let $S$ be a closed orientable surface, $\chi(S) < 0$. Almost every representation $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ such that

(i) $\mathcal{E}(\rho)[S] = \pm(\chi(S) + 1)$ (i.e. "one off maximal"), and

(ii) $\rho$ sends some non-separating simple closed curve to an elliptic,

is the holonomy of a hyperbolic cone-manifold structure on $S$ with a single cone point with cone angle $4\pi$.

## The relevant measure

Goldman (1984) constructed a measure on the character variety $X(S)$ of representations $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$. ("Representations up to conjugacy.")

The tangent space of $X(S)$ at $[\rho_0]$ is

$$H^1(\pi_1(S); \mathfrak{sl}_2\mathbb{R}_{\mathrm{Ad}\rho_0})$$

($\mathfrak{sl}_2\mathbb{R}$ as a $\pi_1(S)$-module via $\rho_0$ and the adjoint representation).

The cup product and the Killing form allow us to define a pairing

$$\omega_{\rho_0} : H^1(\pi_1(S); \mathfrak{sl}_2\mathbb{R}) \times H^1(\pi_1(S); \mathfrak{sl}_2\mathbb{R}) \longrightarrow \mathbb{R}$$

which gives a closed non-degenerate 2-form $\omega$ on $X(S)$ (singular at abelian representations), i.e. a symplectic structure.

This defines a measure on $X(S)$ by integration.

**Questions:**

The hypothesis of an elliptic s.c.c. is ungainly.

**Question:** For a general closed surface with $\chi(S) < 0$, is almost every representation with Euler class $\pm(\chi(S) + 1)$ a holonomy representation?

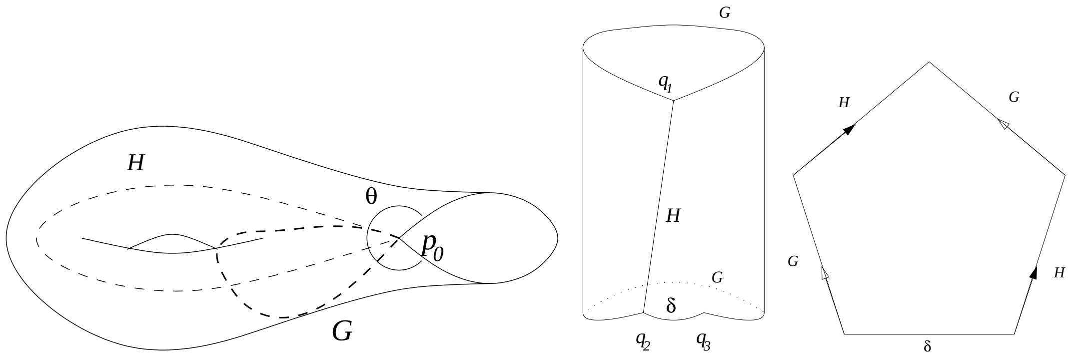
Can the "almost" be dropped?

**Question:** Is *every* representation of a general closed surface, $\chi(S) < 0$, with Euler class $\pm(\chi(S) + 1)$ a holonomy representation?

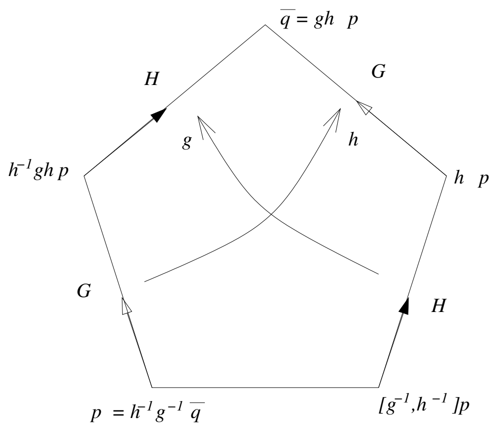
For two-off-extremal Euler class representations, we know not all are holonomy representations.

**Question:** For a given integer $m \neq 0$, $|m| \leq -\chi(S) - 1$, and representations with $\mathcal{E}(\rho)[S] = m$, are holonomy representations dense/conull?

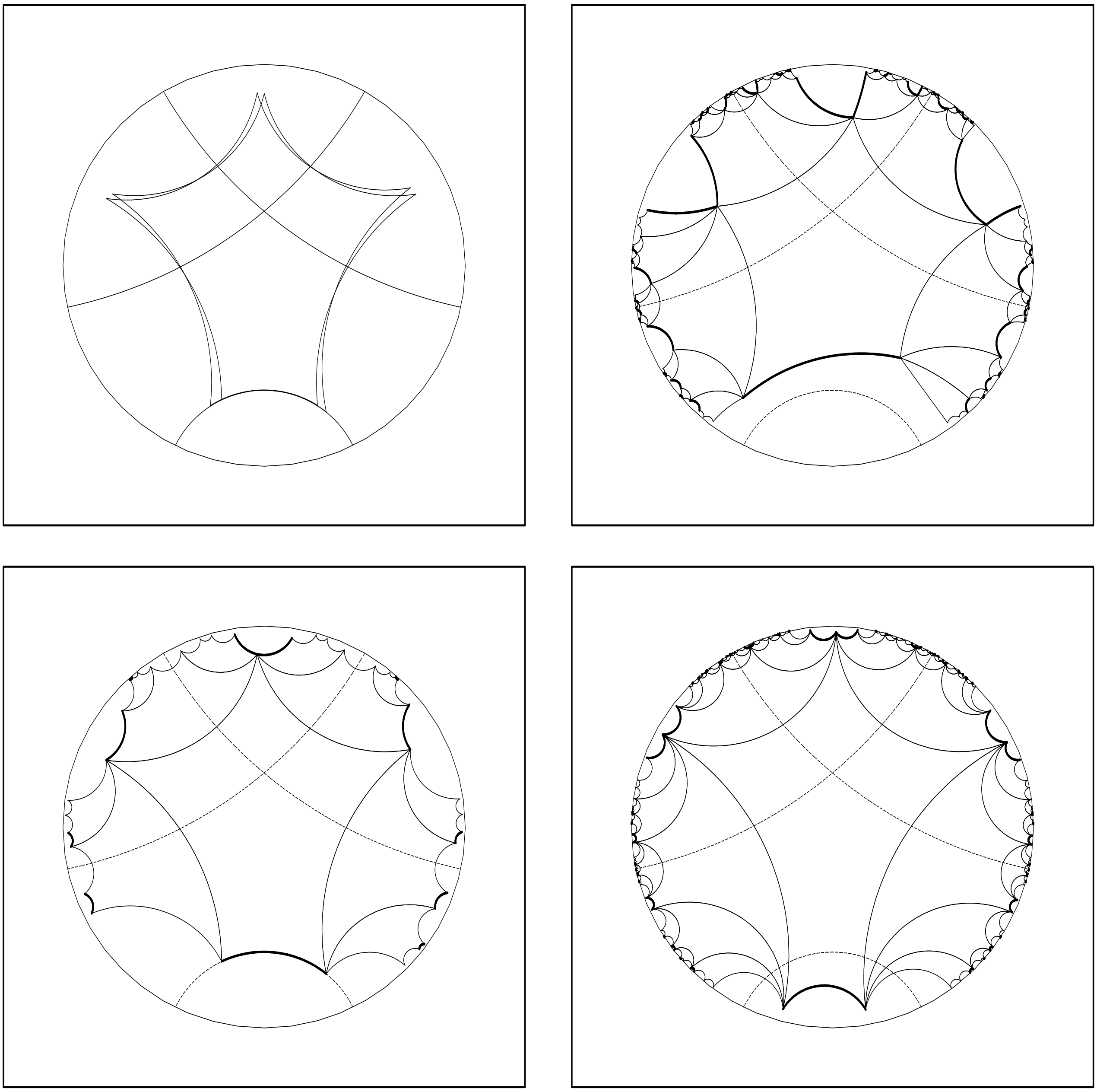
**Ideas in proof of punctured torus theorem**
Construct a pentagonal fundamental domain.




For $g, h \in PSL_2\mathbb{R}$ and $p \in \mathbb{H}^2$, we have a *pentagon generated by* $g, h$ *at* $p$, $\mathcal{P}(g, h, p)$.




We have a holonomy representation iff we can find a basis $G, H$ and a point $p \in \mathbb{H}^2$ giving a nice pentagon (non-degenerate bounding an immersed disc).



## Algebra of punctured torus representations

**Fricke, Klein 1897:** For irreducible $\rho$, the triple $(\mathrm{Tr}\, g, \mathrm{Tr}\, h, \mathrm{Tr}\, gh)$ determines $\rho$ up to conjugacy.
Thus the character variety $\Omega \subset \mathbb{R}^3$.

**Nielsen 1918:** An automorphism of $\langle G, H\rangle$ takes $[G, H]$ to a conjugate of itself or its inverse.

If $(\mathrm{Tr}\, g, \mathrm{Tr}\, h, \mathrm{Tr}\, gh) = (x, y, z)$ then

$$\mathrm{Tr}[g, h] = x^2 + y^2 + z^2 - xyz - 2 = \kappa(x, y, z).$$

Consider action of $\mathrm{Out}\, \pi_1(S) \cong GL_2\mathbb{Z}$ on $\Omega$, which preserves each $\kappa^{-1}(t)$.

Two characters are equivalent if they are related by the following moves:

$$(x, y, z) \mapsto \begin{cases} \text{permutations of coords} \\ (x, y, xy - z) \text{ (“Markoff”)} \\ (-x, -y, z) \end{cases}$$

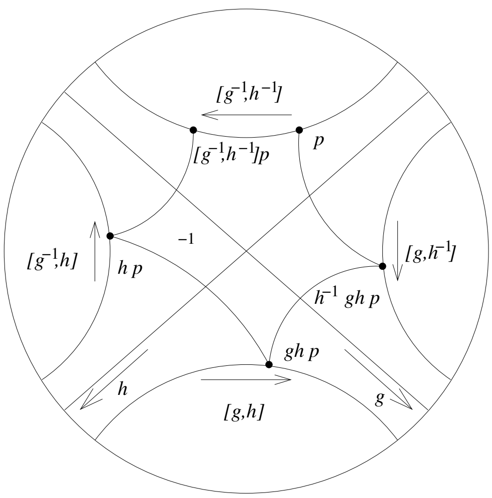
**Proof by picture.**

Key lemma: for $g, h \in PSL_2\mathbb{R}$.

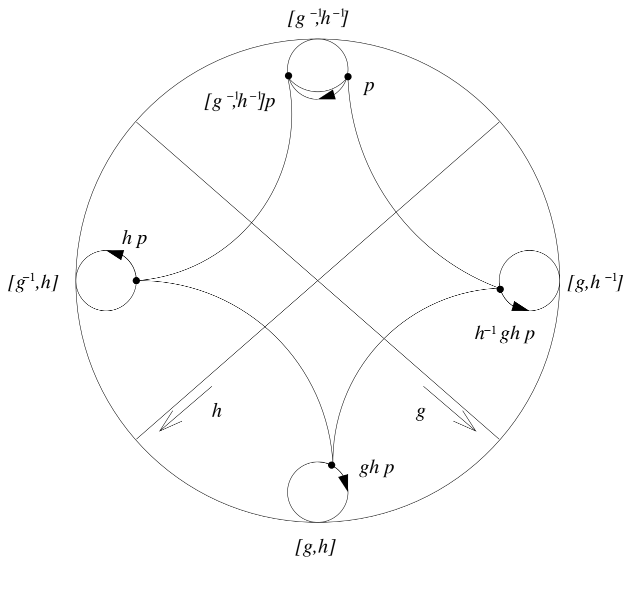
$\mathrm{Tr}[g, h] < 2 \Leftrightarrow \{g, h$ hyperbolic and axes cross$\}$

Take cases according to value of $\mathrm{Tr}[g, h]$.

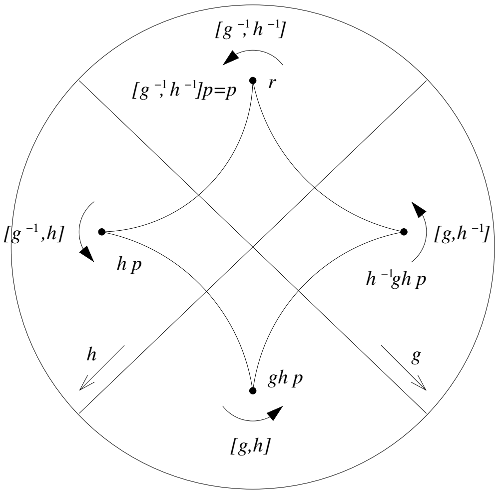
Case (i)
$\mathrm{Tr}[g, h] < -2$



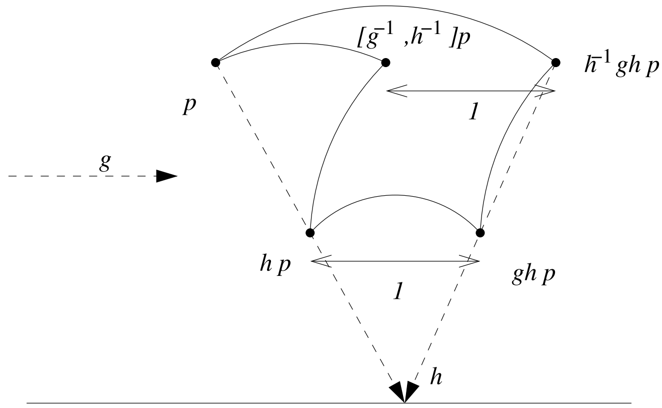

Case (ii)
$\mathrm{Tr}[g, h] = -2$



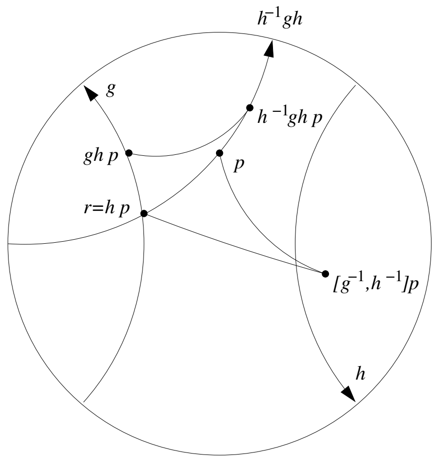
Case (iii)
Tr$[g,h] \in (-2,2)$
(Perturb carefully)




Case (iv)
Tr$[g,h] = 2$
(Reducible reps)




Case (v): algorithm to change basis until character $(x,y,z) \in (2,\infty)^3$. Deduce arrangements of axes using results of Gilman–Maskit. Then construct.



## Ergodicity

Recall Aut $\pi_1(S)$ acts on $\Omega$, describing how the character changes when we change basis of $\pi_1(S)$.

Goldman (1984) showed that there is a measure on $\Omega$ (in fact, a symplectic structure on each level set $\kappa^{-1}(t)$) which is invariant under the action of $\pi_1(S)$. In certain regions of $\Omega$ (occurring in the present case) the action is ergodic.

Recall *ergodic* means that the only invariant sets under the action are null or conull.

Now by the mixing properties of ergodicity, by changing basis we can move from almost any such $\rho$ to arbitrarily close to a specific $\rho^*$, which has the desired properties.