# Notes on Eliashberg's 1992 paper, "Contact 3-manifolds twenty years since J. Martinet's work

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March 6, 2007

## Contents

# 1 Introduction

These are the most relevant points (to me!) arising in this paper. I have written them informally and without proper proof. It's no substitute for reading the real paper! I just like to convey some of the gist, without doing any real work.

# 2 Significance of the paper

It's one of the seminal papers spurring the resurgence of contact geometry, along with Eliashberg's classification of overtwisted contact structures and Giroux's work on convex surfaces.

Until this paper, the definition of overtwisted was "contains a standard overtwisted disk": that is, a disk with characterstic foliation which has precisely one elliptic point, in the interior, and precisely one limit cycle on the boundary. After this paper, an equivalent definition was "contains a surface with a limit cycle in its characteristic foliation", or eqivalently, "contains a surface with a closed loop without singular points in its singular foliation". Eliashberg proved they are equivalent, by manipulating characteristic foliations. One direction is easy; the other direction is his theorem.

**Theorem 2.1** *Let $(M, \xi)$ be a closed oriented contact 3-manifold. Suppose there is an embedded disc $D$ in $M$ with a limit cycle in its characteristic foliation. Then $M$ contains a standard overtwisted disc.*

Clearly, this is a result about simplifying the characteristic foliation: if we have a limit cycle, then we have a simple-looking limit cycle. The next result is also about simplifying the characteristic foliation: it turns out to amount to removing elliptic points wherever possible. The result is (ostensibly!) about the Euler class $e(\xi)$ of the contact structure on a closed oriented surface $S$ in $M$.

**Theorem 2.2** *Let $(M, \xi)$ be a tight contact manifold. If $S = S^2$ then $e(\xi)[S] = 0$. Otherwise*

$$\chi(S) \leq e(\xi)[F] \leq -\chi(S).$$

This is clearly a finiteness result about the possible homology classes of contact structures. In fact we can say more (relying also on later results).

**Theorem 2.3** *Let $M$ be a closed manifold. Only finitely many homotopy classes of plane fields can contain tight contact structures.*

This is in stark contrast to the result for overtwisted contact structures — there is precisely one (up to isotopy) in each homotopy class of plane fields.

The other major theorem is the classification of tight contact structures on $S^3$. (Since the classification of overtwisted contact structures was already known, and he just clarified the distinction between tight and overtwisted, this is a complete classification of *all* contact structures on $S^3$.)

**Theorem 2.4** *A tight contact structure on $S^3$ is isotopic to the standard contact structure $\xi_0$.*

In fact we can prove a stronger result. (To see it's stronger, consider $S^3$ as a union of two balls.)

**Theorem 2.5** *Two tight contact structures on $B^3$ which coincide at the boundary are isotopic relative to $\partial B^3$.*

The corollaries are many and varied. They are not all complete pushovers, though.

**Theorem 2.6 (Cerf's theorem)** *Any diffeomorphism of $S^3$ extends to $B^4$.*

**Theorem 2.7** *All tight contact structures on $\mathbb{R}^3$ are isomorphic.*

# 3 Clearing up overtwisted disks

The first task is to clear up all these previous difficulties with overtwisted discs, and prove theorem 2.1. The idea is to take a disc with a limit cycle; and then manipulate the foliation, perturbing the disc, until it contains a standard overtwisted disc.

Eliashberg in this paper gives new techniques for manipulating the foliation; they are now all completely standard.

In general, if you have a surface in a contact manifold, you have a characteristic foliation on it. If you move the surface around, you alter the foliation. And if you want to alter the foliation in certain ways, you can sometimes move the surface around to achieve it. The detail of how this works is crucial and is in general not too difficult.

## 3.1 $e_\pm, h_\pm, d_\pm, \chi, c$

We begin by defining some variables. We have a contact 3-manifold $(M, \xi)$ and a surface in it $S$. The surface $S$ may or may not have boundary. As usual $\chi(S)$ denotes the Euler characteristic of the surface. We let:

(i) $e_\pm$ denote the number of positive/negative interior elliptic singular points of $S_\xi$;

(ii) $h_\pm$ denote the number of positive/negative interior hyperbolic singular points of $S_\xi$;

(iii) $d_\pm = e_\pm - h_\pm$.

We also let $c(S)$ denote the "Euler class of $\xi$ on $S$". This makes sense if $S$ is a closed surface, and denotes the obstruction to finding a vector field tangent to $S$ and $\xi$. If $S$ has boundary, then we take a vector field tangent to $S$ and $\xi$ along $\partial S$, and consider the obstruction to extending this to a vector field tangent to $S$ and $\xi$ all over $S$.

From Poincare-Hopf, the Euler characteristic is given by the sums of indices of singular points in a vector field on $S$, so

$$\chi(S) = e_+ + e_- - h_+ - h_- = d_+ + d_-.$$

On the other hand, the obstruction to extending a vector field along $S_\xi$ takes into account the signs of singularities. A positive elliptic point is a $+1$ obstruction, but a negative elliptic point is a $-1$ obstruction. (It's easy to see they can cancel each other out.) Similarly for hyperbolic singularities. We obtain

$$c(S) = e_+ - e_- - h_+ + h_- = d_+ - d_-.$$

Thus we find the following proposition, which Eliashberg in this 1992 paper attributed to Harlamov–Eliashberg in 1982 or Eliashberg 1991.

**Proposition 3.1**

$$d_\pm = \frac{1}{2}\left(\chi(S) \pm c(S)\right).$$

## 3.2 Manipulating Legendrian boundaries

This proposition is stated as "easy to see": maybe for some!

**Proposition 3.2** *Let $S$ be a surface bounded by a piecewise smooth legendrian curve $\Gamma$. Then $S$ can be deformed to an embedded surface $\tilde{S}$ bounded by $\partial\tilde{S} = \tilde{\Gamma}$ that preserves the Thurston-Bennequin invariant, and such that the characteristic foliations $S_\xi, \tilde{S}_\xi$ are homeomorphic. Hyperbolic corners of $\Gamma$ disappear, elliptic corners become smooth elliptic points, and all interior singular points remain the same.*

Maybe it's easier to see this after reading Giroux's paper or something...

In any case, the Thurston–Bennequin invariant can be found simply by looking at the characteristic foliation on the Legendrian boundary of a surface. If the singular points are all the same sign, then the contact structure never gets to turn around very far, so must turn zero in total.

**Proposition 3.3** *Let $S$ be a surface in a contact manifold $(M, \xi)$ bounded by a Legendrian curve $\gamma$. If all singular points on $\gamma$ are of the same sign then $tb(\gamma|S) = 0$.*

## 3.3 The elimination lemma

It turns out that it's possible to kill some singular points in pairs. Namely, if there exist an elliptic point and an hyperbolic point of the same sign, connected by a trajectory $\Gamma$ of the characteristic foliation, then we can cancel them both. The change only occurs in a neighbourhood of the single trajectory. It's achieved by a $C^0$ small perturbation of our surface $S$ in $M$, but the perturbation is fixed on the trajectory $\Gamma$ itself. This works equally well, whether the points are in the interior or boundary of $S$. In fact, if the trajectory is along the boundary, we can say a little more: the perturbation can be taken to fix the boundary (not just the trajectory from one singular point to the other, but little chunks on either side in the support of the perturbation, also).

Eliashberg attributes this proposition to Fuchs, improving a slightly weaker result of Giroux.

**Lemma 3.4 (Elimination Lemma)** *Let $M, \xi, S$ be as above. Let $\Gamma$ be a trajectory of $S_\xi$ between an elliptic point $p$ and a hyperbolic point $q$ of the same sign. Let $U$ be a neighbourhood of $\Gamma$ in $M$ which contains no other singular points of $S_\xi$.*

*Then there exists an isotopy of $S$ in $M$ such that:*

*(i) it is $C^0$ small;*

*(ii) it is supported in $U$;*

*(iii) it is fixed at $\Gamma$;*

*(iv) the new surface $S'$ has no singular points in its characteristic foliation inside $U$.*

*(v) If $p, q$ are both on the (Legendrian) boundary of $S$ then we can keep $\partial S$ fixed.*

We can also reverse the process (a *creation* lemma!) and create elliptic and hyperbolic singular points along a trajectory — having the same, but pre-specified sign, determined by the relative orientation of $\xi$ and $S$ along a trajectory.

**Lemma 3.5 (Creation lemma!)** *Let $M, \xi, S$ be as before. Near a nonsingular point of $S_\xi$, one can create a pair of singular points, one elliptic and one hyperbolic, having the same (pre-specified) sign.*

Why are these true? Well at least the creation of singular points is believable, simply by drawing a picture. We can take a standard neighbourhood of a legendrian curve in the standard contact $\mathbb{R}^3$, say along the $x$-axis. Perturbing a little horizontal strip by bending it a little can give us the singularities we want. And by a Darboux-type argument, the elimination of such singularities doesn't sound too far fetched either.

## 3.4 Manipulating the characteristic foliation I: simplifying inside a limit cycle

Now we can give the proof of theorem 2.1. Recall we have a limit cycle in a disc $D$: we have to find a limit cycle in a disc with only one elliptic point inside. The idea is to use the tricks we have just discussed to simplify the characteristic foliation inside a limit cycle.

WLOG we can assume the limit cycle is innermost on $D$, and we can assume it is attractive. We also assume the characteristic foliation is *generic*: so all singularities are standard and there are no connections between hyperbolic points.

What are the possibilities for the singularities inside $D$? By Poincare–Hopf, we know that $e - h = \chi(D) = 1$. So if there is only 1 elliptic point, there are no hyperbolic points, and we are done. If there are more elliptic points, then there are more hyperbolics, also; in particular, there exists a hyperbolic point $q$. We will proceed by removing hyperbolic points, so that the situation is eventually reduced to the case of a single elliptic point, which is a standard overtwisted disc.

Where can the separatrices of $q$ go? The incoming ones can only come from positive elliptic points. (Not other hyperbolic points, since $D_\xi$ is generic; and not from the attracting limit cycle either!) So if $q$ is positive, we can apply the elimination lemma and cancel $q$, and simplify the situation; by induction then we are done.

We may assume then that $q$ is (indeed all hyperbolic points are) negative. Now where do the outgoing separatrices form $q$ go? They can go to negative elliptic points or to the limit cycle. (Not to other hyperbolic points, by genericity!) If anything goes to a negative elliptic point, we can apply the elimination lemma and cancel $q$. So we may assume the outgoing separatrices go to the limit cycle.

But in this case there is a sneaky trick: we use the creation lemma to create a pair of singularities, so that the separatrices from $q$ go to the hyperbolic singularity. (This is possible, if you think about it.) We can even do it so that the two separatrices now form a smooth closed loop $\gamma$. And if we take the created singularities to be negative, then all singularities along $\gamma$ are negative; so its Thurston-Bennequin number is 0. Now we can use the Legendrian perturbation lemma to smooth it into a limit cycle. This gives us a limit cycle with fewer hyperbolic points inside, and we are done.

# 4 Manipulating the characteristic foliation II: removing elliptic points

We will shortly see that, by use of the tricks previously discussed, we can almost remove any elliptic singularity from a characteristic foliation. In particular we can prove:

**Proposition 4.1** *Any elliptic point of $S$ can be destroyed via a perturbation of $S$, cancelling it with a hyperbolic point, unless $S$ is a sphere and $e_+ = e_- = 1$, $h_+ = h_- = 0$.*

Theorem 2.2 follows immediately. For if $S$ is a sphere then we can cancel until we are left with $e_+ = e_- = 1$, $h_+ = h_- = 0$, so that $d_+ = d_- = 1$ and $c(S) = e(\xi)[S] = d_+ - d_- = 0$. If $S$ is not a sphere then we can cancel until

$e_+ = e_- = 0$ so that $d_+, d_- \leq 0$. Then $c(S) = d_+ - d_-$ and we have bounds:

$$\chi(S) = d_+ + d_- \leq c(S) = d_+ - d_- \leq -d_+ - d_- = -\chi(S)$$

so we are done. Theorem 2.2 is really just about cancelling elliptic points.

### 4.1 Basins and Legendrian polygons

To remove elliptic points, we are going to look at their *basins*. Given a subset $V$ of a surface $S$ in $M$, the *basin* $B(V)$ is the set of points of $S$ which can be reached from $V$ along (oriented!) trajectories of $S_\xi$.

What does the basin of a set look like? Well, if you start at a point and follow the trajectory, you can end up at a negative fixed point, or a hyperbolic fixed point, or a limit cycle. Put limit cycles to one side for a minute. Then the basin looks like a "Legendrian polygon", with boundary consisting of Legendrian curves.

We can formalise the idea of "Legendrian polygons".

Such a polygon will arise where you have a surface (not necessarily a polygon! The "polygonal" part of the polygon is solely its boundary!) with a piecewise smooth boundary; and you immerse it into a surface $S$ in a contact manifold $(M, \xi)$ so that the boundary runs along the characteristic foliation, with vertices mapping to singularities.

Note that this is a bit ambiguous: a smooth edge of $S$ could run through a singularity of $S_\xi$. Do we still count it as a vertex? In a strange convention, we *do* if the singularity is *elliptic*. We can have hyperbolic singularities along smooth edges, and they may not count as vertices; if so, they are called *pseudovertices*.

A Legendrian polygon is required to be "almost" injective: it can overlap on its vertices or edges, but not in its interior. So, for instance, a Legendrian polygon which is a disc with two vertices and two edges could fold up into a ball. (The two edges glued together.)

Basins (still putting limit cycles to one side) have this structure as Legendrian polygons; vertices can overlap; singularities can be approached from different directions; trajectories may join up along a edge; but the basin cannot overlap with itself in its interior.

It turns out, even if there are limit cycles present, then the polygon can still be defined. Suppose there are trajectories from $V$ approaching a limit cycle $C$. Note that if we were happy to say "a limit cycle is a vertex", we could define an extended version of Legendrian polygon. But we won't do that; we'll perturb to avoid the entire situation! Suppose two adjacent boundary edges of our proposed polygon approach the limit cycle $C$. We perturb $S$ near $C$ to create a hyperbolic and elliptic fixed point; so that the edges approach the elliptic point. (The hyperbolic and elliptic point we create are negative.) In this way, by a series of small perturbations of $S$, we can assume that $B(V)$ is in fact a Legendrian polygon.

### 4.2 Killing elliptic points

So, to the proof of theorem 2.2. Let $E$ be an elliptic point on $S$; we want to kill it without creating any more elliptic points. WLOG we can assume $E$ is positive. And WLOG we may also perturb $S$ so that it is generic; in particular, we can asssume there are no separatrix connections between hyperbolic points.

If $\overline{B(E)}$ is a sphere, then certainly $S$ must be a sphere! The polygon must fold up so that its boundary maps to one point; and there are precisely two singularities. One is positive elliptic; the other is negative elliptic. So $e_+ = e_- = 1$, $h_+ = h_- = 0$ as desired.

We now assume $\overline{B(E)}$ is not a sphere. Well then, what does its boundary look like? It may involve sinks, hyperbolic points and limit cycles; it may be complicated. But we can say immediately that if there is any positive hyperbolic point involved, we can cancel it by the elimination lemma, and reduce the number of elliptic points; by induction, we are done.

So we may assume that $\overline{B(E)}$ involves no positive hyperbolic points. But it may contain limit cycles. So we perturb $S$ near limit cycles, if necessary, so that the basin $B(E)$ is nice and its closure $\overline{B(E)}$ is a Legendrian polygon in $S$. As discussed above, this involves creating negative elliptic and negative hyperbolic. (No matter that we have created extra elliptic points, since we will now find a contradiction!)

The boundary of $\overline{B(E)}$ then consists of piecewise Legendrian curves, with singularities on them, negative elliptic (sinks!) and negative hyperbolic. By genericity, there are no separatrices connecting hyperbolics. This only makes sense if the elliptic and hyperbolic points alternate along the boundary. So the picture is very standard. And by the elimination lemma, we can cancel all these singular points in pairs! This then reduces $\overline{B(E)}$ to something bounded by a limit cycle; and with only one singularity inside; so it's a standard overtwisted disc, contradicting tightness.

# 5 Extending a characteristic foliation over a 3-ball

We now enter upon the proof of theorem 2.5, that a tight contact structure on $B^3$ is determined up to isotopy by its restriction to the boundary $S$: that is, determined by the characteristic foliation $S_\xi$.

### 5.1 Strategy of the proof.

Given a contact structure $\xi$ on $B$, with characteristic foliation $S_\xi$ on the boundary, we want to find a standard contact structure $\zeta$, which only depends on the foliation $\mathcal{F} = S_\xi$, such that $\xi$ is isotopic to $\zeta$.

How are we going to find $\zeta$, starting only from a foliation $\mathcal{F}$ on $S$?

We're going to use all manner of tricks about foliations and pseudoconvex embeddings. As it turns out, given an embedding $\alpha : B \longrightarrow \mathbb{R}^3$ with certain properties and a map $\gamma : \alpha(B) \longrightarrow \mathbb{R}$ (also with certain properties!), we can obtain a contact structure on $B$! How do we get this? We consider our $\mathbb{R}^3$ as lying inside $\mathbb{C}^2$, where the coordinates on $\mathbb{C}^2$ are $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ and the coordinates on $\mathbb{R}^3$ are $(x_1, y_1, x_2)$. Then from $\alpha$ (the *embedding*) and $\gamma$ (the *function*) we obtain the *graph*

$$\Gamma_\gamma = \{y_2 = \gamma(u) : u \in \alpha(B)\} \subset \mathbb{C}^2.$$

If $\alpha$ and $\gamma$ satisfy certain requirements, it turns out that the complex tangencies on $\Gamma_\gamma$, a 3-manifold in $\mathbb{C}^2$, give a contact structure on $\alpha(B)$, and hence on $B$. For instance, one of these requirements is that $\gamma = 0$ on $S$ and is positive on the interior of $B$; so that in $\Gamma_\gamma$, $S$ correpsonds to $y_2 = 0$.

It will turn out that from a foliation $\mathcal{F}$ on $S$, we will find a function $\varphi : S \longrightarrow \mathbb{R}$ which is said to *tame* $\mathcal{F}$. There are many choices of such $\varphi$, but it turns out they are homotopic. The pair $(\varphi, \mathcal{F})$ form a *function–foliation pair* and will lie in a space called $FF$. From $(\varphi, \mathcal{F})$ we can obtain an embedding $\alpha : B \longrightarrow \mathbb{R}^3$ satisfying the required properties mentioned above, in fact so that $\varphi$ is the $y_1$-coordinate of the restriction of $\alpha$ to $S$. The pair $(\alpha, \mathcal{F})$ is an *embedding–foliation pair* and lies in a space called $EF$. The choice of $\alpha$ is unique up to homotopy; it turns out $EF \longrightarrow FF$ is a Serre fibration with contractible fibre. From $(\alpha, \mathcal{F})$ we want to obtain a function $\gamma : \alpha(B) \longrightarrow \mathbb{R}$ which has the properties desired of $\gamma$ above. The foliation of $\mathcal{F}$ determines the 1-jet of $\gamma$ on the boundary $\alpha(S)$, and so we have a pair $(\alpha, \gamma_1^\partial)$ where $\alpha$ is an embedding and $\gamma_1^\partial$ is a 1-jet of a function $\alpha(S) \longrightarrow \mathbb{R}$; such a pair lies in the space $\text{Conv}_1^\partial$, which is homeomorphic to $EF$. From $(\alpha, \gamma_1^\partial)$ the 1-jet gives us a germ $\gamma^\partial$ of a function $\alpha(S) \longrightarrow \mathbb{R}$; the pair $(\alpha, \gamma^\partial)$ lies in a space $\text{Conv}^\partial$. The choice of $\gamma_1$ is unique up to homotopy, and indeed the map $\text{Conv}^\partial \longrightarrow \text{Conv}_1^\partial$ is a Serre fibration with contractible fibre. Then from $\gamma_1$ we obtain a full function $\gamma : \alpha(B) \longrightarrow \mathbb{R}$ of the desired type. The pair $(\alpha, \gamma)$ lies in a space Conv and the space of such $\gamma$ is unique up to homotopy; again we have a Serre fibration $\text{Conv} \longrightarrow \text{Conv}^\partial$ which has contractible fibre. Our pair $(\alpha, \gamma)$ now determines a graph $\Gamma_\gamma$ and a contact structure $\zeta$ on $B$, which is unique up to homotopy; hence only depends on the original foliation $\mathcal{F}$.

That's quite a lot of construction! We have a whole lot of maps, which are all Serre fibrations with contractible fibres (or better).

$$\begin{array}{cccccc} \text{Conv} \longrightarrow & \text{Conv}^\partial \longrightarrow & \text{Conv}_1^\partial \longrightarrow & EF \longrightarrow & FF \longrightarrow & \text{Fol} \\ (\alpha, \gamma) & (\alpha, \gamma^\partial) & (\alpha, \gamma_1^\partial) & (\alpha, \mathcal{F}) & (\varphi, \mathcal{F}) & \mathcal{F} \end{array}$$

Having obtained our $\zeta$, we then need to show it's isotopic to the original contact structure $\xi$. We do this by taking a family of spheres $S_t$, $t \in [0, 1]$, which fill $B$ (okay, $S_0$ is a point). We can apply the procedure above to obtain a $\zeta_t$ for each $t$, and since $\zeta_t$ has the same characteristic foliation on $S_t$ as $\xi$, we can replace the foliation $\xi$ with $\zeta_t$ on expanding balls. (Okay, we can't do this at $t = 0$, but by Darboux both $\xi$ and $\zeta$ can be assumed standard there.) As long as $\zeta_t$ varies smoothly, we will have our isotopy. The upshot is: provided we can do the whole intricate construction of the previous paragraph not only once, but *in families*, we are done. It's lucky that we mentioned everything is a Serre fibration, because that means homotopies lift and this is all possible!

And that is the gist of the proof.

(I won't go into further detail on all the Serre fibrations, since I'm not that interested, I don't know much of the background about pseudoconvex embeddings, and Eliashberg doesn't go into that background.)

Where is the tightness used? It's used in constructing the taming function. We couldn't just start with any foliation $\mathcal{F}$: it had to be a characeristic foliation of a tight contact structure. Thus the taming function is crucial. I *will* go into this part! (Well, not homotopies of families of them, but why they exist.)

## 5.2 Taming functions

We start with a two-sphere $S$ in $M$ and a foliation $\mathcal{F}$ on $S$. Since $\mathcal{F}$ is eventually going to lie in a family, it might not be quite so generic. We allow simple singularities or birth–death singularities. Let $X$ be a vector field generating $\mathcal{F}$. We will define a function $\varphi : S \longrightarrow \mathbb{R}$ to *tame* $\mathcal{F}$ if:

(i) $X$ is gradient-like for $\varphi$ (recall this means that there exists a Riemannian metric so that it behaves like a gradient, well not quite, with an inequality, more or less, critical points are the same);

(ii) the positive/negative elliptic points of $\mathcal{F}$ are the local minima/maxima of $\varphi$ (so that with the increasing flow of $\varphi$, positive elliptic points are indeed sources and vice versa);

(iii) if we pass through a critical value with a negative/positive hyperbolic singularity in the increasing direction of $\varphi$, the number of components of the level set increases/decreases.

For a sphere $S$ embedded in a tight $(M, \xi)$, it turns out there always exists a function taming $S_\xi$; and any two such taming functions are homotopic through taming functions; and given a nicely-behaved family of contact structures (with only simple singularities or birth-death singularities on the characteristic foliation of $S$), we can obtain a smooth family of taming functions; and so any two such smooth families of taming functions are homotopic through families of taming functions.

Let's see why.

## 5.3 Tricks with Legendrian boundaries

Let's recall some tricks with Legendrian boundaries of surfaces in contact manifolds.

First, suppose we have a surface $S$ bounded by a piecewise smooth legendrian curve $\Gamma$. If there are no singularities on $\Gamma$, we have a limit cycle. If not, then there may be singularities of any type: elliptic or hyperbolic, positive or negative. Generically recall there are no connections between hyperbolic singularities. We can often perturb $S$, sometimes even fixing $\Gamma$, to simplify the situation.

(i) If the Thurston-Bennequin invariant is 0, then we can $C^0$ perturb $S$ near $\Gamma$ (but hold $\Gamma$ fixed!) to remove the singularities. Thus we obtain a limit cycle.

(ii) If all the singularities on $\Gamma$ have the same sign, then the Thurston-Bennequin invariant is 0, so by the same reasoning we can perturb $S$, fixing $\Gamma$, to obtain a limit cycle.

(iii) Any hyperbolic singularities can be made to disappear by a perturbation of $S$ (moving $\Gamma$ along with it). So if there are only hyperbolic singularities, we can remove them all. But on the other hand, connections between hyperbolic singularities are extremely rare and non-generic, so this shouldn't happen in the first place.

## 5.4 Manipulating the characteristic foliation III: removing hyperbolic points, Legendrian polygons

Recall, given a characteristic foliation $\mathcal{F} = S_\xi$ on a sphere in a tight contact manifold, we want to construct a taming function. How are we going to do it? We are going to start from the bottom and work our way up! The minima will be the sources in $\mathcal{F}$, and we will then construct $\varphi$ so that as $C$ increases, the subsets $\varphi \leq C$ engulf one other singularity at a time. How will we make sure we can do this? We will look at the basins of whatever we have already engulfed! These, as we know, are Legendrian polygons. So we will now perform some more tricks in this regard.

**Lemma 5.1** *Let $D$ be an embedded disc in a tight contact manifold $(M, \xi)$, such that $\partial D$ is transverse to $\xi$ and the characteristic foliation exits through $\partial D$. Then by a $C^0$ small isotopy of $D$, fixed near $\partial D$, we can kill all positive hyperbolic points of $D_\xi$.*

Just think of the incoming separatrices to a positive hyperbolic point: where do they come from? Not a limit cycle, because it's tight; not a hyperbolic point if $D$ is generic; hence from a positive elliptic point. Then we can cancel with the elimination lemma.

**Lemma 5.2** *Suppose we have a Legendrian polygon $Q$ in a surface $S$ in a tight $(M, \xi)$, where $Q$ is an immersed disc, possibly with vertices identified (but not edges). Then $\partial Q$ contains at least one positive and at least one negative singular point.*

Suppose otherwise. If there are no singularities on the boundary we have a limit cycle and are done. If there are singularities, we can perturb $S$ to make the boundary smooth; and if all the singularities have the same sign, we can perform the legendrian boundary trick described above and perturb $Q$, fixed on the boundary, to obtain a limit cycle bounding a disc, a contradiction.

**Lemma 5.3** *Suppose we have a Legendrian polygon which is a (closure of a) basin $\overline{B(V)}$ in a sphere $S$ in a tight $(M, \xi)$, and which is injective except possibly for identified vertices. Then the boundary $\Gamma$ has at least one positive hyperbolic point.*

The polygon must be an immersed disc, and by the previous lemma it must contain at least one positive singular point; it can't be a positive elliptic (source!) since it's a basin, so must be positive hyperbolic.

**Lemma 5.4** *Suppose we have a Legendrian polygon which is a (closure of a) basin $\overline{B(V)}$ in a sphere in a tight $(M, \xi)$, where $d_+(V) = 1$. Then any identified pseudovertices (recall these must be hyperbolic) are negative.*

The region $V$ is an embedded disc as in the first lemma above; so we can cancel positive hyperbolic points in $V$. Hence we may assume $V$ has $e_+ = 1$, $h_+ = 0$. So the basin $\overline{B(V)}$ has a simple form. Any identified hyperbolic pseudovertices will form a smooth loop when we join them to our positive hyperbolic point in $V$ (after we perturb the Legendrian curve). This loop only has positive singularities, so we can perturb to get a limit cycle, which bounds a disc since we're on a sphere.

In particular, in our case of a sphere $S$ in a tight $(M, \xi)$: if we are looking at basins $B(V)$ then what can happen?

(i) The boundary $\Gamma$ of the basin could be null; then we have the whole of $S^2$; then extending $\varphi$ is easy.

(ii) The polygon $\overline{B(V)}$ could be injective except possibly for identified vertices. Then from above we have at least one positive hyperbolic point on $\Gamma$. We'll engulf this for the next extension of $\Gamma$ and add it to $V$. Note this joins together two components of $\varphi \leq C$, so if both these regions had $d_+ = 1$, the new combined region has $d_+ = 1 + 1 - 1 = 1$ also.

(iii) The polygon $\overline{B(V)}$ could have identified pseudovertices. But we can assume $d_+(V) = 1$, and so by the above lemma the identified pseudovertices are negative hyperbolic. We engulf this to extend $\varphi$, and $d_+ = 1$ still.

This is how we construct $\varphi$; and that is all I am going to say about the proof of theorem 2.5 and its corollary theorem 2.4.

# 6 Finiteness result

So we have proved theorems 2.2 and 2.5. Now theorem 2.3 is not too difficult.

Given $M$, we want to show that there are only finitely many homotopy classes of plane fields which can occur as tight contact structures. For any such plane field $\xi$ and any 2-dimensional homology class $S$, we have $e(\xi)[S] = 0$ (if $S$ is a sphere) or $e(\xi)[S] \leq -\chi(S)$ (otherwise). How many 2-dimensional cohomology classes on $M$ are there with satisfying these conditions? Only finitely many.

On the other hand, consider homotopy classes of 2-plane fields on $M$. Consider in the method of obstruction theory, skeleton by skeleton. Since $S^2$ is connected and $\pi_1(S^2) = 1$, we can extend over 1-skeleta in only one way. To extend over 2-skeleta is possible since $\pi_1(S^2) = 1$ but we have many choices since $\pi_2(S^2) \cong \mathbb{Z}$. For the plane field to be extendable over $M$ we require that in each 3-cell the boundary obstructions sum (with signs) to 0; so the homotopy classes of plane fields on the 2-skeleton of $M$ which extend over $M$ is in bijective correspondence with $H^2(M)$. From theorem 2.2, there are only finitely many such cohomology classes available for $e(\xi)$.

Now suppose we are given $\xi$ on the 2-skeleton. Can $\xi$ be extended over $M$ as a tight contact structure? By theorem 2.5, and in at most one way, up to isotopy, on each 3-cell. Hence, being given the homotopy class of $\xi$ on the 2-skeleton determines the homotopy class of $\xi$ as a 2-plane field. So there are only finitely many possibilities.

# 7 Cerf

The idea of proving Cerf's theorem is now easy. If we have an orientation preserving diffeomorphism $f : S^3 \longrightarrow S^3$, we want to extend it to $B^4$. The idea is that $f$ is isotopic to a contactomorphism — this is easy to show, since the space of tight contact structures on $S^3$ is connected.

Then one needs to show that any contactomorhism of $S^3$ with the standard tight contact structure extends to $B^4$. This requires some holomorphic curve methods which refer back to Gromov.

# 8 Tight contact structures on $\mathbb{R}^3$

We can prove theorem 2.7, that all tight contact structures on $\mathbb{R}^3$ are isomorphic. The idea is to exhaust $\mathbb{R}^3$ by spheres.

Consider a sphere $S$ in $\mathbb{R}^3$. For a tight contact structure $\xi$ we know now that $e(\xi)[S] = 0$, and clearly $\chi(S) = 2$; so $d_+ = d_- = 1$. Using the elimination lemma, then, we can perturb our sphere a little to have $e_+ = e_- = 1$, $h_+ = h_- = 0$. And then the characteristic foliation is standard, just like the unit sphere in the standard $\mathbb{R}^3$! If we perturb a little more, the characteristic foliation is *diffeomorphic* to the standard foliation.

So, we can exhaust $\mathbb{R}^3$ by a nested sequence of such spheres. Given a tight contact structure $\xi$ on $\mathbb{R}^3$, we will show it isomorphic to the standard $\xi_0$. So take nested spheres $\mathbb{R}^3 = \cup^\infty V_k$ for $\xi$ and $\mathbb{R}^3 = \cup^\infty B_k$ for $\xi_0$. For each $k$, $(B_k, \xi_0)$ and $(V_k, \xi)$ are contactomorphic to the standard unit ball, hence to each other, by a map $h_k$. Since the space of embeddings of the standard ball to itself is connected, we can take each $h_k$ to agree with $h_{k-1}$ on $B_{k-1} \cong V_{k-1}$. Then taking their union gives us a contactomorphism of $\mathbb{R}^3$.

## References

[1] Yakov Eliashberg, Contact 3-manifolds twenty years since J. Martinet's work, Annales de l'institut Fourier, 42, 1–2 (1992), 165–192.