Contact forms on 3-dimensional manifolds "Formes de contact sur les variétés de dimension 3" Proceedings of Liverpool Singularities Symposium II

J. Martinet
Translated by Daniel Mathews

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Contents

Translator's note

My French is not great, but with a dictionary hopefully most of this is accurate. Any doubts are written in footnotes. All footnotes are mine. Figures and references can be found in the original.

Introduction

The aim of this paper is to prove that

Theorem 0.1 *Let M be a compact orientable 3-dimensional manifold. There exists on M a contact form, that is a differential form ω such that $\omega \wedge d\omega \neq 0$ at every point of M.*

This theorem answers a question of S.S.Chern [1].

The problem of the existence of a contact form on an *open* odd-dimensional manifold has recently (almost) been resolved by Gromov [8]; it is equivalent in this case to the existence of an "almost-contact structure", that is a pair (α, β) of a differential 1-form α and a 2-form β such that $\alpha \wedge \beta^p \neq 0$ at every point of M (where β^p denotes the p'th exterior power of β).

By contrast, almost nothing is known in the case of compact manifolds (see however Gray [4], who gives the first obstructions to the existence of an almost contact structure). The theorem announced here thus resolves the first problem that arises in this direction.

Let us mention also a remarkable theorem of R. Lutz [5], indicating in particular that, on the sphere S^3, there exists a contact form in every homotopy class of forms without zeroes; the generalisation of this result to compact orientable 3-dimensional manifolds seems easy.

The proof of the main theorem is fundamentally based on the *stability* property of contact structures; this property has been established by Gray [4], in the framework of the theory of Kodaira–Spencer deformations; I will give an elementary and geometric proof here, which lends itself particularly well to the applications.

The two other keys to the proof are Lickorish's theorem on the structure of compact orientable 3-dimensional manifolds [6], and the existence of a large number of contact forms on the solid torus $S^1 \times D^2$ (cf. section 4).

1 Preliminaries

All objects considered are of class C^∞.

1.1

Let M be a manifold of dimension n. We denote by $T^*M \longrightarrow M$ the *cotangent* bundle of M, and by $P \longrightarrow M$ the projective bundle associated to $T * M$; the fibre P_x of P at $x \in M$ is the projective space of lines of the fibre $T^*_x M$ of T^*M at x.

Any section σ (C^∞) of P will be called a *Pfaffian equation on* M ([2]); it is therefore a sub-bundle of lines of T^*M; by passing to the orthogonal, σ defines a sub-bundle $\bar\sigma$ of codimension one of the tangent bundle TM.

Let us denote by T^*_0 the complement of the zero section in T^*M, and by $\pi : T^*_0 \longrightarrow P$ the canonical projection.

Let σ be a Pfaffian equation on M, and U an open set of M; a differential 1-form ω, defined and nonzero at each point of U, is called a *lift of* σ *on* U if $\sigma = \pi \circ \omega$ on U; in other words, ω is a section of σ on U without zeroes; again, the equation $\omega = 0$ defines a bundle $\bar\sigma$ on U. If ω and ω' are two lifts of σ on U, then $\omega' = f\,\omega$ where f is a nonzero function on U.

The existence of a *global* lift of a Pfaffian equation σ is equivalent to the triviality of the line bundle σ, that is to the fact that the sub-bundle $\bar\sigma$ of TM is *tranversally orientable.*

1.2

Let $V \longrightarrow P$ be the vector bundle of tangent vectors to P, *vertical* with respect to the projection $P \longrightarrow M$; if $p \in P$, the fibre V_p of V is the tangent space at p to the projective space of $T_x M$, where x is the projection of p on M.

Let us denote by Σ the set of sections (C^∞) of P.

For each $\sigma \in \Sigma$, we write σ^*V for the vector bundle on M which is the pullback of V by $\sigma : M \longrightarrow P$; a section of σ^*V attaches to each point x of M a vector vertically tangent to P at $\sigma(x)$: it is an "infinitesimal deformation" of σ. The space of sections (C^∞) of σ^*V will naturally be denoted $T_\sigma\Sigma$ (tangent space to Σ at σ).

Let $\omega \in T^*_0$, and $p = \pi(\omega) \in P$; let x be the basepoint of the tangent covector ω in M. It is clear that the derivative of π at ω defines a *surjective* linear function

$$T_\omega\pi \ : \ T^*_x M \longrightarrow V_p.$$

The kernel of $T_\omega\pi$ is the line defined by the vector $\omega \in T^*_x M$.

Now let ω be a *section* of T^*_0, and $\sigma = \pi \circ \omega$ the Pfaffian equation defined by ω; we will further denote

$$T_\omega\pi \ : \ T^*M \longrightarrow \sigma^*V$$

the morphism defined at each point as above.

1.3

The group Diff(M) of diffeomorphisms of M to itself acts in a natural way on Σ. In effect, let ϕ be an automorphism of M, we have the diagram:

$$\begin{array}{ccc} T^*M & \stackrel{T^*\phi}{\longleftarrow} & T^*M \\ \downarrow & & \downarrow (\uparrow \omega) \\ M & \stackrel{\phi}{\longrightarrow} & M \end{array}$$

where $T^*\phi$ denotes the automorphism of T^*M induced from ϕ by derivation; $T^*\phi$ leaves invariant the open set T^*_0, and commutes with scalar multiplication in T^*M, therefore passes to a quotient by π, and defines an automorphism of P, again denoted $T^*\phi$, such that the following diagram commutes:

$$\begin{array}{ccc} P & \stackrel{T^*\phi}{\longleftarrow} & P \\ \downarrow & & \downarrow (\uparrow \sigma) \\ M & \stackrel{\phi}{\longrightarrow} & M. \end{array}$$

If ω (resp. σ) is a section of T^*M (resp. P), we set $\phi^*\omega = T^*\phi \circ \omega \circ \phi$ (resp. $\phi^*\sigma = T^*\phi \circ \sigma \circ \phi$); and, if ω is a lift of σ, we have:

$$\phi^*\sigma = \pi \circ \phi^*\omega. \tag{1}$$

If now X is a vector field on M and σ is a Pfaffian equation, the *Lie derivative* $\theta(X)\sigma$ of σ with respect to X is a section of σ^*V defined in the following manner: let ϕ_t be the (local) 1-parameter group obtained by integration of X; in a neighbourhood of each point of M, we have a 1-parameter family $\sigma_t = \phi_t^*\sigma$ of Pfaffian equations; we then put

$$\theta(X)\sigma = \frac{\partial \sigma_t}{\partial t}|_{t=0}.$$

It is clear that if ω is a (local) lift of σ, and X is any vector field, we have, by 1.2 and formula (1):

$$\theta(X)\sigma = T_\omega \pi \circ \theta(X)\omega$$

at every point of the open set of definition of ω, where $T_\omega\pi$ is the morphism defined in 1.2.

Lemma 1.1 *Let σ_t ($t \in [0,1]$) be a 1-parameter family of Pfaffian equations on M. Let ϕ_t be a 1-parameter family of diffeomorphisms of M. We set $\dot\sigma_t = \frac{\partial \sigma_t}{\partial t}$, and let X_t be the 1-parameter family of vector fields $X_t = \frac{\partial \sigma_t}{\partial t} \circ \phi_t^{-1}$. Then, the following conditions are equivalent:*

(a) $\phi_t^*\sigma_0 = \sigma_t$ for all $t \in [0,1]$

(b) $\theta(X_t)\sigma_t = \dot\sigma_t$ for all $t \in [0,1]$.

Proof Let us set $\mu_t = \left(\phi_t^{-1}\right)^* \sigma_t$; a standard and easy calculation shows that:

$$\frac{\partial \mu_t}{\partial t} = \left(\phi_t^{-1}\right)^* [\dot{\sigma}_t - \theta(X_t)\sigma_t]$$

where, this time, $\left(\phi_t^{-1}\right)^* \sigma_t$ represents the action of ϕ_t^{-1} on the bundle V. The equivalence between (a) and (b) results immediately. ∎

1.4

Henceforth, the manifold M will always be supposed to be of odd dimension $2p+1$.

A Pfaffian form ω defined on an open set U of M is called a *contact form* on U if $\omega \wedge d\omega^p \neq 0$ at each point of U (where $d\omega^p$ denotes the p'th exterior power of the exterior differential $d\omega$ of ω).

A Pfaffian equation σ for M is called a *contact structure* on M if, for every open set U of M and every lift ω of σ on U, ω is a contact form on U.

We remark that if ω is a differential form and f is a numerically-valued function, we have:

$$\omega' \wedge d\omega'^p = f^{p+1}\, \omega \wedge d\omega^p \quad \text{where} \quad \omega' = f\omega.$$

We infer that a Pfaffian equation σ is a contact structure if and only if there exists a family $(U_i, \omega_i)_{i \in I}$ where the U_i constitute an open cover of M, and ω_i is a contact form lifting σ over U_i for each i.

A contact structure is therefore a Pfaffian equation of maximal class at each point (cf. [3]), *that is without singularities in the sense of* [2].

Based on *Darboux's theorem* (see for example [2] or [3]), the giving of a contact structure on M is equivalent to the giving of an atlas of M such that the transition functions belong to the *pseudo-group of contact transformations* (cf. [4]).

More geometrically, if σ is a contact structure, the sub-bundle $\bar{\sigma}$ of TM (see 1.1), considered as a field of contact elements of codimension 1, is as distant as possible from complete integrability (which is equivalent to the Frobenius condition $\omega \wedge d\omega = 0$, where ω is any local lift of σ); this will be made precise below (remark 2.1).

The condition, for a Pfaffian equation, to be a contact structure is, evidently, an *open* condition on the 1-jet of σ at each point; and, if we consider the space Σ in the C^1-topology of Whitney ([2]), the set of contact structures is C^1-open in Σ.

We remark further that a contact structure σ is defined by a global contact form ω if and only if $\bar{\sigma}$ is transversally orientable in TM.

Recall finally that if the manifold M is of dimension $2p+1$, *p odd*, any contact structure on M canonically defines an orientation on M (Gray [4], Prop. 2.2.1).

Thus, in dimension 3, the existence problem for a contact structure only arises in the case of orientable manifolds.

2 Stability of contact structures

2.1

Always let M be a manifold of odd dimension $2p+1$. Let σ be a Pfaffian equation on M. We propose to study the equation:

$$\theta(X)\sigma = \tau$$

where τ is a given section of σ^*V, and where we solve for the vector field X.

We remark first that $X \mapsto \theta(X)\sigma$ is a differential operator of order 1 from the space of vector fields on M to the space $T_\sigma\Sigma$ of sections of σ^*V.

We will consider the restriction of this operator to the subspace of vector fields which are sections of the bundle $\bar{\sigma} \subset TM$.

Proposition 2.1 *The operator $X \mapsto \theta(X)\sigma$, restricted to the space of sections of $\bar{\sigma}$, is of order 0, that is, it is defined by a morphism $u : \bar{\sigma} \longrightarrow \sigma^*V$. Further, u is an isomorphism from $\bar{\sigma}$ to σ^*V if and only if σ is a contact structure.*

PROOF The proposition is of a local nature. We can therefore use the local lift ω of σ. We then have, by 1.3, for each vector field X:

$$\theta(X)\sigma = T_\omega\pi \circ \theta(X)\omega$$
$$\theta(X)\omega = d(X \lrcorner \omega) + X \lrcorner d\omega$$

where $\lrcorner$ represents the interior product.

But, if X is a section of $\bar{\sigma}$, we have:

$$X \lrcorner \omega = 0$$

by definition of ω. Thus

$$\theta(X)\sigma = T_\omega\pi \circ (X \lrcorner d\omega)$$

and the first part of the proposition is shown, the morphism u being defined by:

$$u(\xi) = T_\omega\pi(\xi \lrcorner d\omega).$$

Now, we remark that the bundles $\bar{\sigma}$ and σ^*V have the same dimension $2p$.

On the other hand, the kernel of the morphism

$$T_\omega\pi \ : \ T^*M \longrightarrow \sigma^*V$$

is, at each point $x \in M$, the line defined by $\omega(x) \in T^*_xM$, by 1.2.

Finally, for each $x \in M$, the subsapce $S_x = \{\xi \lrcorner d\omega \ : \ \xi \in \bar{\sigma}(x)\}$ lies in the *support* of $d\omega$ at x ([2], I.1.3).

Thus, if u is surjective, S_x is transverse to $\omega(x)$ for all $x \in M$. As a result, $d\omega$ is of rank $2p$ at each point, and its support is exactly S_x; therefore ([2], Prop. I.4.1) $d\omega^p \neq 0$ at every point, and the support of $d\omega^p$ at x is S_x; by ([2], Prop. I.1.4.5), $\omega \wedge d\omega^p \neq 0$ at every point, and σ is a contact stucture.

The converse is immediate. ∎

By the preceding proposition, if σ is a contact structure, and if τ is any section of σ^*V, there exists a unique vector field X, a section of $\bar{\sigma}$, such that $\theta(X)\sigma = \tau$.

This explains the "infinitesimal stability" of σ. It is remarkable that this can be established simply by the intermediary of a linear operator.

Remark. The preceding proposition also has the following interpretation, which represents a characteristic geometric property of contact structures: let σ be a contact structure and X a *nowhere vanishing* vector field on a neighbourhood of a point of M, lying in the field of contact elements $\bar{\sigma}$; then $\theta(X)\sigma$ is nonzero. Let S be a hypersurface element transverse to X, and let π be the projection on S whose fibres are the integral curves of X; for every x, the projection of $\bar{\sigma}(x)$ under π is a hyperplane tangent to S at $\pi(x)$; the fact that $\theta(X)\sigma \neq 0$ means that, when x moves along a trajectory of X, the hyperplane projection of $\bar{\sigma}(x)$ *"pivots"* around $\pi(x)$ (which is fixed).

2.2 Theorem

Theorem 2.2 *([4], Th.5.2.1) Let M be a compact manifold, and σ_t, $t \in [0,1]$, be a 1-parameter family of contact structures on M. Then, there exists an isotopy ϕ_t of M such that:*

$$\phi_t^* \sigma_0 = \sigma_t \quad \textit{for every} \quad t \in [0,1].$$

PROOF We set

$$\dot{\sigma}_t = \frac{\partial \sigma_t}{\partial t}$$

For each t, by proposition 2.1, there exists a unique vector field X_t, which is a section of $\bar{\sigma}_t$, such that

$$\theta(X_t)\sigma_t = \dot{\sigma}_t.$$

It is clear that X_t depends differentiably on t; integration of the differential equation

$$\frac{dx}{dt} = X_t(x)$$

gives the desired isotopy, by lemma 1.3. ∎

2.3 Corollary

Corollary 2.3 *(Stability of contact structures). Let M be a compact manifold; let $\sigma \in \Sigma$ be a contact structure on M. Then there exists a neighbourhood U of σ in Σ, in the C^1-topology, such that, for all $\sigma' \in U$, σ' is a contact structure and there exists a diffeomorphism ϕ of M such that $\phi^*\sigma = \sigma'$ (i.e. σ' is isomorphic to σ).*

In effect, by 1.4, the contact structures form a C^1-open set of Σ; it then suffices to choose a neighbourhood U of σ such that, for all $\sigma' \in U$, there exists a 1-parameter family σ_t of contact structures, with $\sigma_0 = \sigma$ and $\sigma_1 = \sigma'$, and to apply the preceding theorem.

2.4 Remarks

(1) The proof of theorem 2.2, unlike that of Gray ([4]), does not suppose that the contact structures are representable by global contact forms and, in particular, does not suppose that the manifold M is orientable.

(2) If ω_t denotes a 1-parameter family of contact forms, we obtain, applying theorem 2.2 to the family σ_t defined by ω_t, an isotopy of M such that

$$\phi_t^*\omega_0 = f_t\omega_t$$

where f_t denotes for every t a nonzero function on M. It is in general impossible to find ϕ_t such that $f_t = 1$, that is *a contact form is not stable.*

(3) The argument employed in theorem 2.2 furnishes a particularly simple and natural proof of *Darboux's theorem*:

If ω is a germ of a contact form at the origin of $\mathbb{R}^{2p+1}$, there exist local coordinates $z, x_1, \ldots, x_p, y_1, \ldots, y_p$ such that:

$$\omega = f\left[dz + \sum_{i=1}^{p} x_i\, dy_i\right], \quad f \neq 0.$$

In effect, we write the Taylor series of order 1 at the origin:

$$\omega = \omega^0 + \omega^1 + \omega'$$

where ω^0 is a linear form on $\mathbb{R}^{2p+1}$, ω^1 a differential form with linear coefficients, and ω' a form with coefficients of order greater than or equal to 2.

We set $\omega^0+\omega^1 = \omega_0$; ω_0 is a form with affine coefficients, which is identified with 1-jets of ω at the origin; ω_0 is evidently contact and, by ([2], Prop. I.4.3.2), there exist local coordinates for which

$$\omega_0 = dz + \sum_{i=1}^{p} x_i\, dy_i.$$

We then set

$$\omega_t = \omega_0 + t\omega', \quad t \in [0, 1].$$

Applying the argument of 2.2 and the remark 2 above, we obtain a 1-parameter family ϕ_t of germs of difeomorphisms *preserving the origin*, such that

$$\phi_t^*\omega_0 = f_t\omega_t.$$

Therefore

$$\phi_1^*\omega_0 = f_1\omega \quad \text{and} \quad f_1 \neq 0.$$

And, via ϕ_1, ω is written in the required form.

3 A model for the contact structures transverse to a closed curve

3.1

We consider the manifold $S^1 \times \mathbb{R}^{2p}$; we set $\Gamma = S^1 \times \{0\}$; let the differential form

$$\omega_0 = d\theta + \sum_{i=1}^{p} (x_i \, dy_i - y_i \, dx_i)$$

where $d\theta$ represents the fundamental form on S^1, and $(x_1, \ldots, x_p, y_1, \ldots, y_p)$ the natural coordinates on $\mathbb{R}^{2p}$.

We have:

$$\omega_0 \wedge d\omega_0^p = p! \; 2^p \; d\theta \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_p \wedge dy_p;$$

thus ω_0 is a contact form.

We denote by σ_0 the contact structure defined by ω_0. The expression for ω_0 at any point $m \in \Gamma$ ($x_i = 0$, $y_i = 0$) is $\omega_0(m) = d\theta$; at this point, the hyperplane $\bar{\sigma}_0(m)$, defined by the equation $d\theta = 0$, is therefore *transverse to* Γ.

In general, we will say that a Pfaffian equation σ, defined on a neighbourhood of Γ in $S^1 \times \mathbb{R}^{2p}$, is *transverse to* Γ if $\bar{\sigma}(m)$ is transverse to Γ at every point $m \in \Gamma$.

3.2 Proposition

Proposition 3.1 *Let σ be a contact structure defined on a neighbourhood of Γ in $S^1 \times \mathbb{R}^{2p}$, and transverse to Γ. Then there exists a diffeomorphism ϕ from a neighbourhood of Γ to a neighbourhood of Γ, leaving fixed each point of Γ, such that*

$$\phi^* \sigma = \sigma_0.$$

In other words, the Pfaffian equation

$$\omega_0 = d\theta + \sum_{i=1}^{p} (x_i \, dy_i - y_i \, dx_i) = 0$$

represents a *model* for the *germs* of contact structures transverse to Γ.

Proof (1) We denote by X the unit vector field tangent to Γ. Then

$$X \lrcorner \omega_0 = \omega_0(X) = 1 \quad \text{on} \quad \Gamma.$$

On the other hand, it is clear that: $X \lrcorner d\omega_0 = 0$ since

$$d\omega_0 = 2 \sum_{i=1}^{p} dx_i \wedge dy_i.$$

We will first show that the structure σ admits on a neighbourhood of Γ a lift ω such that

(a) $X \lrcorner \omega = 1$

(b) $X \lrcorner d\omega = 0$.

As σ is transverse to Γ, $\bar{\sigma}$ is transversely orientable on a neighbourhood of Γ; σ thus admits lifts there; whatever lift ω' is considered, we have $X \lrcorner \omega' \neq 0$ on Γ, by the hypothesis of transverality; we can thus suppose that conditon (a) is realised for a fixed lift ω'.

We will next find a function f, defined on a neighbourhood of Γ, equal to 1 on Γ, such that conditon (b) is satisfied for $\omega = f\omega'$. This is equivalent to $X \lrcorner d\omega = X \lrcorner (f d\omega' + df \wedge \omega') = f\, X \lrcorner d\omega' + (X \lrcorner df)\, \omega' - \omega'(X)\, df = 0$ at every point of Γ.

But $X \lrcorner df = 0$ since $f = 1$ on Γ and $\omega'(X) = 1$ by construction.

Thus we obtain

$$df = X \lrcorner d\omega' = \alpha$$

at every point of Γ. As $X \lrcorner \alpha = 0$, we trivially verify the existence of a function f satisfying the above condition at every point of Γ.

(2) We now consider the bundle $\bar{\sigma}$, *restricted to* Γ. As ω is a contact form, the alternating bilinear form $d\omega$ defines, for each $m \in \Gamma$, a *symplectic structure* on the fibre $\bar{\sigma}(m)$.

Similarly, the bundle $\bar{\sigma}_0 = S^1 \times \mathbb{R}^{2p}$ is furnished with a symplectic structure via $d\omega_0$.

The bundles $\bar{\sigma}$ and $\bar{\sigma}_0$ are trivial on Γ. On the other hand, we know that the symplectic forms on $\mathbb{R}^{2p}$ form an orbit of the canonical action of the group $GL(2p, \mathbb{R})$ on $\bigwedge^2 (\mathbb{R}^{2p})^*$ (cf. [2], Prop. I.4.2).

We deduce V: the symplectic group being connected, that there exists an isomorphism

$$h : \bar{\sigma}_0 \longrightarrow \bar{\sigma}$$

which, at each point of Γ, exchanges the symplectic structures.

(3) We now furnish the manifold $S^1 \times \mathbb{R}^{2p}$ with any Riemannian metric, and let exp be the corresponding exponential function.

The restriction

$$\phi_0 : \bar{\sigma}_0 \longrightarrow S^1 \times \mathbb{R}^{2p} \quad (\text{resp. } \phi : \bar{\sigma} \longrightarrow S^1 \times \mathbb{R}^{2p})$$

of exp to $\bar{\sigma}_0$ (resp. $\bar{\sigma}$), where $\bar{\sigma}_0$ and $\bar{\sigma}$ always denote bundles with base Γ, is a diffeomorphism from a neighbourhood of the zero section of $\bar{\sigma}_0$ (resp. $\bar{\sigma}$) to a neighbourhood of Γ.

Then

$$\psi = \phi \circ h \circ \phi_0^{-1}$$

defines a local diffeomorphism leaving fixed each point of Γ.

Let $m \in \Gamma$; the tangent space T_m of $S^1 \times \mathbb{R}^{2p}$ at m can be written as

$$T_m = D_m \oplus \bar{\sigma}_0(m) \quad \text{on the one hand}$$
$$T_m = D_m \oplus \bar{\sigma}(m) \quad \text{on the other hand}$$

where D_m denotes the tangent at M to Γ.
We verify immediately that the derivative $T\psi(m)$ of ψ at m has the expression

$$T\psi(m) = 1_{D_m} \oplus h.$$

We then consider the contact structure

$$\sigma_1 = \psi^*\sigma.$$

It admits as a lift the form $\omega_1 = \psi^*\omega$, and it results from the properties of ω, $d\omega$, h and $T\psi$ at points of Γ that:

$$\omega_1 = \omega_0$$
$$\text{and} \quad d\omega_1 = d\omega_0$$

at every point of Γ.

(4) We finally conside the 1-parameter family

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) \quad t \in [0,1].$$

At each point $m \in \Gamma$, we have, by (3):

$$\omega_t(m) = \omega_0(m)$$
$$d\omega_t(m) = d\omega_0(m).$$

It follows that the family of Pfaffian equations σ_t defined by $\omega_t = 0$ is a family of contact structures on a neighbourhood of Γ, and that moreover

$$\dot{\sigma} = \frac{\partial \sigma_t}{\partial t} = 0 \text{ on } \Gamma, \text{ since } \frac{\partial \omega_t}{\partial t} = 0 \text{ is zero there.}$$

In applying Proposition 2.1, we thus obtain a 1-parameter family of vector fields X_t defined on a neighbourohod of Γ in $S^1 \times \mathbb{R}^{2p}$ such that:

$$X_t(m) = 0 \text{ for all } m \in \Gamma \text{ and all } t \in [0,1]$$
$$\theta(X_t)\sigma_t = \dot{\sigma}_t.$$

By integration of the differential equation $\frac{dx}{dt} = X_t(x)$, we obtain a 1-parameter family of germs of diffeomorphisms ϕ_t leaving fixed the points of Γ, such that, by lemma 1.3:

$$\phi_t^*\sigma_0 = \sigma_t.$$

Then, setting $\phi = \psi \circ \phi_1$ we have $\phi^*\sigma = \sigma_0$ and the proposition is proved.∎

Remark. The classification of germs of contact structures transverse to Γ up to *positive isomorphism* (i.e. via diffeomorphisms with positive Jacobian, that is orientation-preserving) is the following:

(i) If p is even, all such germs are positively isomorphic to σ_0.

(ii) If p is odd, we have two isomorphism classes, represented by the structures:

$$d\theta \pm \sum_{i=1}^{p} (x_i \, dy_i - y_i \, dx_i) = 0.$$

4 Remarkable contact forms on $S^1 \times D^2$

4.1

Let D^2 be the unit disc in the plane $\mathbb{R}^2$. Let $T = S^1 \times D^2 \subset S^1 \times \mathbb{R}^2$, and let ∂T be the boundary of the solid torus T.

Let ω (resp. $\tilde{\omega}$) be a differential form defined on a neighbourhood of ∂T (resp. of T); we say that $\tilde{\omega}$ is an *extension* of ω if ω and $\tilde{\omega}$ coincide on a neighbourhood of ∂T.

We propose to establish a criterion for when a *contact* form ω on a neighbourhood of ∂T is extendable to a *contact* form $\tilde{\omega}$ on T.

Let X be the unit vector field on $S^1 \times \mathbb{R}^2$, tangent at each point (a, b) to the curve $S^1 \times \{b\}$.

Let ω be a differential 1-form on an open set in $S^1 \times \mathbb{R}^2$; we say the form ω is *invariant* if $\theta(X)\omega = 0$.

This means that ω is invariant under the group of *rotations* $(\theta, x, y) \mapsto (\theta + \alpha, x, y)$ of $S^1 \times \mathbb{R}^2$; in other words, if $d\theta$ denotes the fundamental form on S^1, an invariant form can be expressed in the following manner:

$$\omega = \eta + f \, d\theta$$

where η is a differential form on an open set Ω of $\mathbb{R}^2$, and f is a numerically-valued function on Ω.

In particular, if ω is an invariant form on a neighbourhood of ∂T, η (resp. f) is a differential form (resp. a function) defined on a neighbourhood of the circle $C = \partial D^2$ in $\mathbb{R}^2$.

Proposition 4.1 *Let ω be an invariant differential form on a neighbourhood of ∂T, such that f has no zeroes on $C = \partial D^2$. Then, if ω is contact, it admits an extension $\tilde{\omega}$ over T which is invariant and contact.*

4.2

To prove this proposition, (essentially due to R. Lutz) we will use the following auxiliary result:

Let $\Delta \subset \mathbb{R}^2$ be a compact, connected set, with boundary a disjoint union of finitely many simple closed curves γ_i, $i = 1, \dots, p$, without double points.

The domain Δ is oriented by the 2-form $\Omega = dx \wedge dy$ (x, y coordinates on $\mathbb{R}^2$); $\partial\Delta = \sum_{i=1}^{p} \gamma_i$ will denote the *oriented* boundary of Δ.

Now let μ be a differential 1-form, defined on a neighbourhood of $\partial\Delta$ in $\mathbb{R}^2$, such that, if $d\mu = h\, dx \wedge dy$, we have $h > 0$ on $\partial\Delta$.

Lemma 4.2 *The following conditions are equivalent:*

(a)

$$\int_{\partial\Delta} \mu > 0.$$

(b) There exists a form $\tilde{\mu}$ defined on a neighbourhood of Δ, extending μ (i.e. $\tilde{\mu} = \mu$ on a neighbourhood of $\partial\Delta$), such that, if $d\tilde{\mu} = \tilde{h}\, dx \wedge dy$, we have $\tilde{h} > 0$ on Δ.

PROOF $b \Rightarrow a$. By Stokes' theorem, we have

$$\int_{\partial\Delta} \mu = \int_{\partial\Delta} \tilde{\mu} = \int_{\Delta} d\tilde{\mu} > 0.$$

$a \Rightarrow b$. We easily show the existence of a function $\bar{h}$, C^∞ and *strictly positive* on Δ, such that:

(1) $\bar{h} = h$ on a neighbourhood of $\partial\Delta$.

(2) $\int_{\Delta} \bar{h}\, dx \wedge dy = \int_{\partial\Delta} \mu$.

Then, the differential form $\beta = \bar{h}\, dx \wedge dy$ is exact, since $H^2(\Delta, \mathbb{R}) = 0$. Thus there exists, by De Rham's theorem and the equality (2), a differential 1-form α on Δ such that:

(3) $d\alpha = \beta$

(4) $\int_{\gamma_i} \alpha = \int_{\gamma_i} \mu$ for all $i = 1, \dots, p$.

By (1) and (3), we have

$$d(\mu - \alpha) = d\mu - \beta = 0 \text{ in a neighbourhood of } \partial\Delta.$$

By (4), $\mu - \alpha$ is exact on a neighbourhood of γ_i, for all i; we thus have

$$\mu = \alpha + df_i$$

where f_i is a function defined on a neighbourhood of γ_i.

There then exists a function f on Δ such that $f = f_i$ in a neighbourhood of each γ_i. It then suffices to set

$$\tilde{\mu} = \alpha + df$$

and the lemma is proved. ∎

4.3 Proof of proposition 4.1

4.3.1

Consider first any invariant form

$$\omega = \eta + f\ d\theta.$$

Thus

$$d\omega = d\eta + df \wedge d\theta$$

and

$$\omega \wedge d\omega = (\eta \wedge df + f\ d\eta) \wedge d\theta$$

since $\eta \wedge d\eta = 0$, η being a differential form in the plane.

The form ω is thus contact if and only if:

$$\eta \wedge df + f\ d\eta \neq 0 \quad \text{at every point.} \tag{1}$$

At a point where $f = 0$, (1) is equivalent to

$$\eta \wedge df \neq 0.$$

This implies $df \neq 0$, thus the set of zeroes of f is a *curve*; moreover, the restriction of η to this curve has no zeroes; in particular, if f is zero along a closed curve γ, we have $\int_\gamma \eta \neq 0$.

Note on the other hand that, if $f \neq 0$, we have the identity

$$\eta \wedge df + f\ d\eta = f^2\ d\left(\frac{\eta}{f}\right). \tag{2}$$

4.3.2

Let us now return to the given form ω on a neighbourhood of ∂T in $S^1 \times \mathbb{R}^2$, invariant and contact.

We may assume $f > 0$ on C; it suffices to substitute ω for $-\omega$. By hypothesis, and considering (1) and (2), we have

$$\eta \wedge df + f\ d\eta = f^2\ d\left(\frac{\eta}{f}\right) \neq 0 \text{ at every point of } C.$$

Let (x, y) be coordinates on $\mathbb{R}^2$ such that

$$d\left(\frac{\eta}{f}\right) = \frac{1}{f^2}\left(\eta \wedge df + f\ d\eta\right) = h\ dx \wedge dy \text{ with } h > 0 \text{ on } C.$$

The plane $\mathbb{R}^2$ is then oriented by the form $dx \wedge dy$.

There are two cases to consider:

(a) $\int_C \frac{\eta}{f} > 0$ where $C = \partial D^2$ is the *oriented* boundary of D^2.

In this case, we extend f to a function $\tilde{f}$ *strictly positive* on D^2. Then, setting $\mu = \frac{\eta}{f}$, we note that $d\mu$ is positive on a neighbourhood of C by hypothesis, and that $\int_{\partial D^2} \mu > 0$.

By lemma 4.2, there exists a form $\tilde{\mu}$ on D^2 extending μ, such that $d\tilde{\mu}$ is positive on D^2. We set $\tilde{\eta} = \tilde{f}\,\tilde{\mu}$; it is clear, considering (2), that the form

$$\tilde{\omega} = \tilde{\eta} + \tilde{f}\,d\theta$$

answers the question.

(b) $\int_C \frac{\eta}{f} \leq 0$.

We consider the form

$$\omega_1 = \eta_1 + f_1\,d\theta = -x\,dy + y\,dx + \left(r^2 - \frac{1}{4}\right)\,d\theta \quad \text{where} \quad r^2 = x^2 + y^2.$$

We have $d\omega_1 = -2\,dx \wedge dy + 2r\,dr \wedge d\theta$; thus

$$\omega_1 \wedge d\omega_1 = \left(-r^2 + \frac{1}{2}\right) dx \wedge dy \wedge d\theta.$$

Let γ_ϵ be the circle with centre 0, of radius $\frac{1}{2} + \epsilon$ $(0 < \epsilon < \frac{1}{\sqrt{2}})$ in the (x, y) plane. We immediately verify that

$$\int_{\gamma_\epsilon} \frac{\eta_1}{f_1} \longrightarrow -\infty \quad \text{when} \quad \epsilon \longrightarrow 0.$$

We fix ϵ sufficiently small that

$$\int_C \frac{\eta}{f} - \int_{\gamma_\epsilon} \frac{\eta_1}{f_1} > 0.$$

We then consider the annulus Δ_ϵ situated between C and γ_ϵ. As f (resp. f_1 is strictly positive on C (resp. on γ_ϵ), there exists a *strictly positive* function $\bar{f}$ on Δ_ϵ extending f and f_1. We set

$$\begin{aligned} \mu &= \frac{\eta}{\bar{f}} \quad \text{in a neighbourhood of } C \\ &= \frac{\eta_1}{\bar{f}} \quad \text{in a neighbourhood of } \gamma_\epsilon. \end{aligned}$$

We have by hypothesis $d\mu > 0$ in a neighbourhood of $\partial\Delta_\epsilon$, and $\int_{\partial\Delta_\epsilon} \mu > 0$. By lemma 4.2, there exists a form $\tilde{\mu}$, such that $d\tilde{\mu} > 0$ on Δ_ϵ, and extending μ. The differential form

$$\begin{aligned} \tilde{\omega} &= \bar{f}\,\tilde{\mu} + \bar{f}\,d\theta && \text{on } S^1 \times \Delta_\epsilon \\ &= \omega_1 && \text{on } S^1 \times \big(D^2 - \Delta_\epsilon\big) \end{aligned}$$

then answers the question, and proposition 4.1 is proved.

Remark. Proposition 4.1 represents a particular case of a theorem used by Lutz to establish the results of [5].

4.4

We consider the manifold with boundary

$$N = S^1 \times S^1 \times [1, \infty) = \left\{ \left(e^{i\theta}, e^{i\theta'}, r \right) \ : \ \theta, \theta' \in \mathbb{R};\ 1 \le r < \infty \right\}.$$

We give N the contact structure defined by the equation

$$\omega_0 = d\theta + r^2\ d\theta' = 0.$$

Note that this is an expression of the "canonical" form $\omega_0 = d\theta + x\ dy - y\ dx$ introduced in 3.1, when using the polar coordinates defined by $x + iy = r\ e^{i\theta'}$.

Now let

$$\bar{\phi}_A : S^1 \times S^1 \longrightarrow S^1 \times S^1$$

a *unimodular automorphism* of the torus in 2 dimensions, defined by

$$\theta = a\bar{\theta} + b\bar{\theta}'$$
$$\theta' = c\bar{\theta} + d\bar{\theta}'$$

where the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a unimodular matrix, that is, $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

We finally consider the manifold $T \cup_{\bar{\phi}_A} N$ obtained by gluing the solid torus $T = S^1 \times D^2$ and the manifold N along their boundaries, via the diffeomorphism

$$\bar{\phi_A} : \partial T = S^1 \times S^1 \longrightarrow \partial N = S^1 \times S^1.$$

Proposition 4.3 *There exists on $T \cup_{\bar{\phi}_A} N$ a contact form $\tilde{\omega}_0$ equal to ω_0 on N.*

Proof This is a corollary of proposition 4.1. It is effectively clear that the form defined by

$$\omega_A = d\left(a\bar{\theta} + b\bar{\theta}'\right) + r^2\ d\left(c\bar{\theta} + d\bar{\theta}'\right)$$

and contact on

$$\dot{T} = S^1 \times \left(D^2 - \{0\}\right) = \left\{ \left(e^{i\bar{\theta}}, \bar{x} + i\bar{y} = re^{i\bar{\theta}'} \right)\ : \bar{\theta}, \bar{\theta}' \in \mathbb{R},\ 0 < r < 1 \right\}$$

extends the given form ω_0 on N.

But $\omega_A = \left(b + dr^2\right) d\bar{\theta}' + \left(a + cr^2\right) d\bar{\theta} = \eta + f\ d\bar{\theta}$ is *invariant* under rotations in $\bar{\theta}$. More, the polynomial $f = a + cr^2$ is not identically zero; therefore let r_0 be such that $0 < r_0 < 1$ and $a + cr_0^2 \neq 0$; the result is established by applying proposition 4.1 to ω_A and ot the solid torus $S^1 \times D^2_{r_0}$, where $D^2_{r_0}$ denotes the disc with centre 0 and radius r_0. ■

5 Theorem

Theorem 5.1 *Let M be a compact, connected, orintable 3-dimensional manifold. There exists on M a contact form.*

5.1

We first reformulate a result of Lickorish [6].

Always let $T = S^1 \times D^2$ the solid torus in 3 dimensions. A *great circle* $\gamma \subset T$ is, by defintion, the graph in T of a function from S^1 to $\mathring{D}^2$ = interior of D^2; and, any tubular neighbourhood of Γ is differphic to a solid torus.

Let $\Gamma = (\gamma_i)$, $i = 1, \ldots, p$, a finite family of disjoint great circles in T, and let $\mathcal{T} = (T_i)$, $i = 1, \ldots, p$, be a family of compact tubular neighbourhoods of the γ_i, such that $t_i \cap t_j = \emptyset$ if $i \neq j$. Then $T - \bigcup_{i=1}^{p} \mathring{T}_i$ (where $\mathring{T}_i$ = interior of T_i) is a manifold with boundary, and

$$\partial \left(T - \bigcup_{i=1}^{p} \mathring{T}_i \right) = \left(\bigcup_{i=1}^{p} \partial T_i \right) \cup \partial T$$

is the disjoint union of $p+1$ two-dimensional tori.

Finally let $\tilde{T}_0, \tilde{T}_1, \ldots, \tilde{T}_p$ be $p+1$ copies of T. Let

$$\phi_0 \; : \; \partial \tilde{T}_0 \longrightarrow \partial T, \quad \phi_i \; : \; \partial \tilde{T}_i \longrightarrow \partial T_i \quad (i = 1, \ldots, p)$$

be diffeomorphisms; we denote by Φ the family $\phi_0, \ldots, \phi_p$.

We set

$$M_{\Gamma, \mathcal{T}, \Phi} = \left(T - \bigcup_{i=}^{p} \mathring{T}_i \right) \cup_{\phi_0} \tilde{T}_0 \cup_{\phi_1} \tilde{T}_1 \cup \cdots \cup_{\phi_p} \tilde{T}_p.$$

Then $M_{\Gamma, \mathcal{T}, \Phi}$ is a compact, connected, orientable 3-dimensional manifold.

Theorem 5.2 *(Lickorish [6]). For every compact, connected, orientable 3-dimensional manifold M, there exists a triple $\Gamma, \mathcal{T}, \Phi$ such that M is diffeomorphic to $M_{\Gamma, \mathcal{T}, \Phi}$.*

Remarks.

(1) The family Γ being fixed, we can choose the tubular neighbourhoods T_i sufficiently small as we like, in the following sense: given any pair $(\mathcal{T}, \Phi)$ associated to Γ, and neighbourhoods U_i of the circles γ_i, there exists a pair $(\mathcal{T}', \Phi')$ with $T'_i \subset U_i$ for each i, such that $M_{\Gamma, \mathcal{T}', \Phi'}$ is diffeomorphic to $M_{\Gamma \mathcal{T}, \Phi}$.

(2) Let V be a C^1-neighbourhood of the core $S^1 \times \{0\}$ of the solid torus T; I mean by this a set of great circles which are graphs of functions from S^1 to D^2 belonging to a given C^1-neighbourhood of the zero function. One easily shows that the previous theorem still holds true if we require that the families Γ

5.2

Now suppose that the families Γ and $\mathcal{T}$ are fixed. Suppose further we have diffeomorphisms

$$\psi_i \;:\; T_i \longrightarrow T \quad \text{for all} \quad i = 1, \ldots, p.$$

For any family Φ, set

$$\begin{array}{llll} \bar{\phi}_i & = \phi_i \circ \phi_i & : \partial \tilde{T}_i \longrightarrow \partial T & i = 1, \ldots, p \\ \bar{\phi}_0 & = \phi_i & : \partial \tilde{T}_0 \longrightarrow \partial T. & \end{array}$$

The family Φ is determined by the family $\bar{\phi} = \bar{\phi}_0, \ldots, \bar{\phi}_p$, which is a family of automorphisms of the 2-dimensional torus.

We know that, if $\bar{\Phi}$ and $\bar{\Phi}'$ are isotopic, that is, for all $i = 0, 1, \ldots, p$, $\bar{\phi}_i$ and $\bar{\phi}'_i$ are isotopic automorphisms of $S^1 \times S^1$, then $M_{\Gamma,\mathcal{T},\Phi}$ and $M_{\Gamma,\mathcal{T},\Phi'}$ are diffeomorphic.

Recall finally that any automorphism $\bar{\phi}$ of $S^1 \times S^1$ is isotopic to a unique unimodular transformation (for the definition and notation, see 4.4); let A be the matrix representing the automorphism

$$\bar{\phi}_* \;:\; \pi_1\left(S^1 \times S^1\right) \longrightarrow \pi_1\left(S^1 \times S^1\right)$$

induced by $\bar{\phi}$ on the fundamental group of the torus; we have $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$, and A is a unimodular matrix; then the automorphism $\bar{\phi}_A^{-1} \circ \phi$ is homotopic to the identity, thus is an isotopy (we know that, if S is a compact orienable surface, the connected component of the identity in Diff(S) consists of the diffeomorphisms homotopic to the identity: see for example [7]).

5.3 Proof of theorem 5

(1) Consider on the solid torus $T = S^1 \times D^2$ the "canonical" contact strucutre σ_0 (cf 3.1) defined by

$$\omega_0 = d\theta + x\ dy - y\ dx = 0.$$

As σ_0 is transverse to the core $S^1 \times \{0\}$ of T, there exists a C^1-neighbourhood V of $S^1 \times \{0\}$ such that σ_0 is transverse to every great circle of V.

(2) There then exist families $\Gamma, \mathcal{T}, \Phi$ and ψ such that

   (a) M is diffeomorphic to $M_{\Gamma,\mathcal{T},\Phi}$ (by theorem 5.1)

   (b) The γ_i are great circles of V (5.1 remark 2)

   (c) $\psi_i : T_i \longrightarrow T$, $i = 1, \ldots, p$, is a diffeomorphism such that $\psi_i^* \sigma_0 = \sigma_{0,i}$, where $\sigma_{0,i}$ denotes the restriction of σ_0 to T_i. This is deduced from remark 1 of 5.1 and from proposition 3.2, considering (b).

   (d) The diffeomorphisms $\bar{\phi}_i$, $i = 0, \ldots, p$, are unimodular automorphisms of the torus $S^1 \times S^1$ (by 5.2).

(3) Consider now

$$M \simeq M_{\Gamma,\mathcal{T},\Phi} = \left(T - \bigcup \mathring{T}_i\right) \cup_{\bar{\phi}_0} \tilde{T}_0 \cup_{\bar{\phi}_1} \tilde{T}_1 \cdots \cup_{\bar{\phi}_p} \tilde{T}_p.$$

By (c), (d) and proposition 4.4, the contact structure σ_0 on $T - \bigcup \mathring{T}_i$ extends to a contact structure on each solid torus T_i, and we have constructed a contact structure σ on M.

(4) We easily verify that the contact structure σ thus constructed is *transversally orientable*; it remains therefore to define a global contact form ω on M (see 1.4) and the theorem is proved.

5.4 Remarks

(1) We know, by a classical theorem of Haefliger, that in general there does not exist a differential form which is *completely integrable* (i.e. $\omega \wedge d\omega = 0$ at every point) and *analytic* on a compact real analytic 3-dimensional manifold.

(2) By contrast, the existence of an analytic contact form is evident: we must construct a contact form ω which is at least C^∞; the analytic forms being dense in the set of C^∞ forms (furnished with the C^1-topology), there exists an analytic form sufficiently close to ω as also to be contact.

(3) If we effect the construction shown in 5.3 starting from the form

$$\omega_0' = d\theta - x\ dy + y\ dx$$

we obtain on M a contact form ω' such that $\omega' \wedge d\omega'$ defines the opposite orientation to $\omega \wedge d\omega$ (see 3.2 remark).

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