# Chord diagrams, topological quantum field theory, and the sutured Floer homology of solid tori

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**Abstract**

We investigate contact elements in the sutured Floer homology of solid tori, as part of the (1+1)-dimensional TQFT defined by Honda–Kazez–Matić in [30]. We find that these sutured Floer homology vector spaces form a "categorification of Pascal's triangle", a triangle of vector spaces, with contact elements corresponding to chord diagrams and forming distinguished subsets of order given by the Narayana numbers. We find natural "creation and annihilation operators" which allow us to define a QFT-type basis consisting of contact elements. We show that sutured Floer homology in this case reduces to the combinatorics of chord diagrams. We prove that contact elements are in bijective correspondence with comparable pairs of basis elements with respect to a certain partial order, and in a natural and explicit way. We use this to extend Honda's notion of contact category to a 2-category. We also prove numerous results about the structure of contact elements, investigate various algebraic structures arising, and give numerous contact-geometric applications and interpretations.

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# Part I
# Introduction

## 1 Overview

### 1.1 Fun with chord diagrams

The main results of this paper can be described as elementary combinatorial results about chord diagrams, which have applications to contact geometry and sutured Floer homology.

**Definition 1.1 (Chord diagram)** *A chord diagram $(D, \Gamma)$ is a set of disjoint properly embedded arcs $\Gamma$ in a disc $D^2$, considered up to homotopy rel endpoints.*

We consider chord diagrams with $n$ chords, and hence $2n$ marked points on the boundary of the disc connected in pairs by disjoint chords. We consider one of those marked points on the boundary to be a base point, so that rotating a chord diagram will generally give a distinct chord diagram.

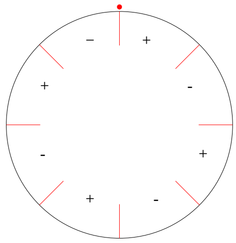



Figure 1: Base point and sign of regions.

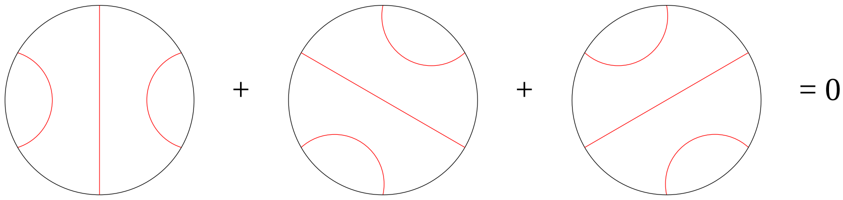



Figure 2: The bypass relation.

A chord diagram divides the disc $D$ into regions, which we alternately denote as positive or negative. The labelling is induced from a labelling on the arcs of $\partial D^2$ between marked points; we declare that the arc immediately clockwise of the base point is positive, and the arc immediately anticlockwise is negative. See figure 1.

**Remark 1.2 (Denoting base point)** *The base point will always be denoted by a red dot in our diagrams.*

**Definition 1.3 (Relative euler class of chord diagram)** *The* relative euler class *of a chord diagram* $(D, \Gamma)$ *is the sum of the signs of the regions of* $D - \Gamma$.

That is, a $+$ region counts as $+1$ and a $-$ region counts as $-1$.

We consider a certain vector space generated by chord diagrams.

**Definition 1.4 (Combinatorial *SFH*)** *The* $\mathbb{Z}_2$*-vector space generated by all chord diagrams of* $n$ *chords and euler class* $e$*, subject to the* bypass relation *in figure 2 is called* $SFH_{comb}(T, n, e)$*. The* $\mathbb{Z}_2$*-vector space generated by all chord diagrams of* $n$ *chords, subject to the same relation, is called* $SFH_{comb}(T, n)$*.*

We will show that these combinatorial objects are isomorphic to $SFH(T, n, e)$ and $SFH(T, n)$, which are objects defined by counting certain holomorphic curves in certain manifolds, in due course.

The bypass relation means that if we have three chord diagrams $\Gamma_1, \Gamma_2, \Gamma_3$ which are all identical, except in a sub-disc $D' \subset D$, on which $\Gamma_1, \Gamma_2, \Gamma_3$ each contains three arcs in the arrangements shown in figure 2, then we consider them to sum to zero.

The terminology "bypass" comes from contact geometry. We shall make the contact geometry clear as we go on, but the idea of "bypasses" here can be considered purely as a type of surgery on a chord diagram, which we call a "bypass move".

**Definition 1.5 (Arc of attachment)** *An arc of attachment in a chord diagram* $(D, \Gamma)$ *is an embedded arc which intersects the chords* $\Gamma$ *at precisely three points, namely, its two endpoints, and one interior point.*

A bypass move is something done along an arc of attachment, and it may be done upwards or downwards.

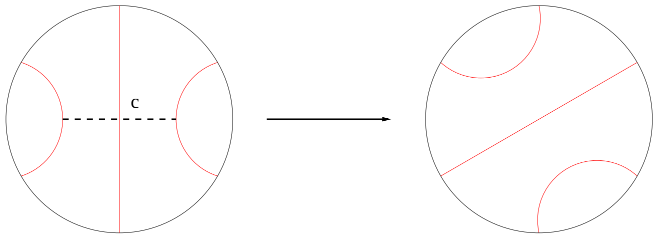



Figure 3: Upwards bypass move.

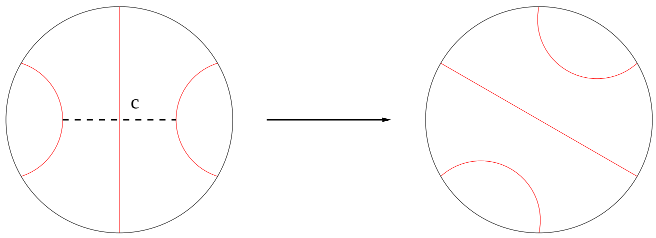



Figure 4: Downwards bypass move.

**Definition 1.6 (Bypass moves)** *Let $c$ be an arc of attachment in a chord diagram $(D, \Gamma)$.*

*(i) The* upwards bypass move *$Up(c)$ along $c$ on $(D, \Gamma)$ consists of removing a small disc neighbourhood of $c$ and replacing it with another disc with chords as shown in figure 3.*

*(ii) The* downwards bypass move *$Down(c)$ along $c$ on $(D, \Gamma)$ consists also of removing a small neighbourhood of $c$ and replacing it as shown in figure 4.*

We see that chord diagrams related by bypass moves naturally come in triples, and such triples are defined to sum to 0 in $SFH$.

The main combinatorial result of this paper is to give a nice basis for the vector spaces $SFH(T, n, e)$, and show that when chord diagrams are decomposed into a sum of basis elements, this decomposition has certain nice properties. There will be a partial order on this basis, and chord diagrams will correspond bijectively with pairs of basis elements which are comparable with respect to this partial order.

For instance, consider $SFH(T, 4, -1)$. This is the vector space generated by the 6 chord diagrams which have 4 chords and relative euler class $-1$: see figure 5.

We will show, and it was essentially known previously in [30], that $SFH(T, 4, -1) = \mathbb{Z}_2^3$. Our basis will consist of the three chord diagrams in the top row of figure 5, labelled with the words, respectively:

$$--+, \quad -+-, \quad +--.$$

Note that they are labelled by words $w$ on $\{-, +\}$ containing 2 minus signs and 1 plus sign. The number of such words is $\binom{3}{1}$; in general, the basis for $SFH(T, n+1, e)$ will be labelled by words on $\{-, +\}$ whose symbols sum to $e$, and the number of such words is $\binom{n}{k}$, where $k = (n+e)/2$.

On this set of words, there is a partial order defined by "all minus signs move right" (or stay where they are). (In this simple case, it is actually a total order; but this will not be true for words of longer length. For instance, in $SFH(T, 5, 0)$, we consider the words $-++-$ and $+--+$, which are not comparable.) Thus,

$$--+ \preceq -+- \preceq +--.$$

In terms of our basis, the 6 chord diagrams (arranged as in figure 5) are

$$\begin{array}{ccc} v_{--+} & v_{-+-} & v_{+--} \\ v_{--+} + v_{-+-} & v_{--+} + v_{+--} & v_{-+-} + v_{+--}. \end{array}$$

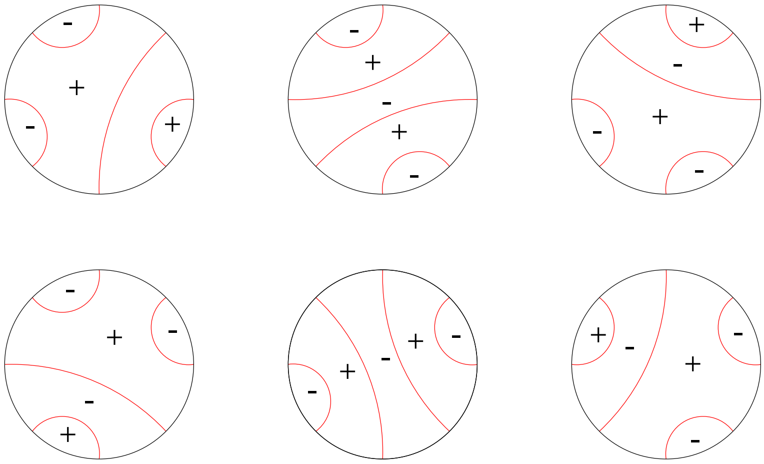


Figure 5: Chord diagrams with in $SFH(T,4,-1)$.

Moreover, there are six pairs of words $w_1, w_2$ which are *comparable* with respect to $\preceq$, namely three "doubles"

$$(--+,--+), \quad (-+-,-+-), \quad (+--,+--)$$

and three less trivial pairs

$$(--+,-+-), \quad (--+,+--), \quad (-+-,+--).$$

And in fact, for each pair, there is precisely one chord diagram having that pair as its first and last element.

That is, there is a bijection

$$\{\text{Chord diagrams}\} \leftrightarrow \{\text{Comparable pairs of words}\}$$

given by taking a chord diagram to the first and last basis elements in its basis decomposition.

This is a general fact, and our main theorem. Moreover, this bijection, and its inverse, can be described explicitly. That is, given a chord diagram, we can algorithmically extract its first and last basis elements, and they are comparable. Conversely, given two comparable words, we can algorithmically produce the chord diagram for which those words give its first and last basis elements. We will also say more about the set of basis chord diagrams that occur in a given chord diagram; as well as relationships between the various vector spaces $SFH(T,n,e)$.

An information-theoretic note from this result is that a chord diagram of 4 chords can be encoded in 6 bits, with the redundancy that the first 3 bits form a word lesser than the second 3 bits, with respect to $\preceq$. In general, a chord diagram of $n+1$ chords can be encoded in $2n$ bits, with a similar redundancy.

This particular example, with 4 chords and $e=-1$, is actually the essence of Honda's octahedral axiom [19].

While this result is largely combinatorics, the motivation, notation, methods, and applications comes from the theory of sutured Floer homology, and contact geometry.

## 1.2 Sutured Floer homology and contact structures

**Remark 1.7 (Notation: the letter $\Gamma$)** *This letter is often used to denote chord diagrams, and also to denote sutures on sutured manifolds. This is not unusual, because in the present context, both arise as dividing sets on convex surfaces in contact manifolds; and dividing sets are often denoted* $\Gamma$*. However, to avoid confusion, for now we shall use the letter* $\Gamma$ *to denote chord diagrams; to denote sutures, we shall use the boldface* $\mathbf{\Gamma}$*.*

We will review the theory of sutured Floer homology more fully in section 3. For the purposes of introduction, it is sufficient to note four facts about sutured Floer homology.

First, sutured Floer homology theory *associates to a sutured manifold* $(M, \mathbf{\Gamma})$ *a* $\mathbb{Z}_2$*-vector space* $SFH(M, \mathbf{\Gamma})$. For present introductory purposes, a sutured manifold can be thought of as a 3-manifold $M$ with boundary, with some disjoint simple closed curves $\mathbf{\Gamma}$ drawn on the boundary, dividing it into alternating "positive" and "negative" regions. We will review the definition of $SFH$ in section 3.

In the present paper, we consider our sutured manifold to be a solid torus, with $2n$ parallel longitudinal sutures. Its sutured Floer homology is known [30] to be $\mathbb{Z}_2^{2^{n-1}}$.

Second, sutured Floer homology *associates to a contact structure* $\xi$ *on* $(M, \mathbf{\Gamma})$ *an element* $c(\xi) \in SFH(-M, -\mathbf{\Gamma})$. Here $-M$ and $-\mathbf{\Gamma}$ refer to reversed orientation. A contact structure on a 3-manifold is a totally non-integrable 2-plane field. When we refer to a contact structure on a sutured 3-manifold $(M, \mathbf{\Gamma})$, we require it to be compatible with the sutures $\mathbf{\Gamma}$, in the sense that the boundary $\partial M$ is convex with dividing set $\Gamma$. Moreover, $c(\xi) = 0$ if $\xi$ is overtwisted; $c(\xi)$ can only be nonzero if $\xi$ is tight. We will review the notions of tight and overtwisted, and other relevant contact geometry, more fully in section 2.

In our case of a solid torus with longitudinal sutures, a tight contact structure can be described by examining the dividing set on a convex meridional disc of the solid torus, which is a chord diagram of $n$ chords. By [23] (see also [20], [21] but note [22]), the tight contact structures on $(T, n)$, up to isotopy rel boundary, are in bijective correspondence with chord diagrams of $n$ chords. That is, there is exactly one tight contact structure, up to isotopy rel boundary, for each such chord diagram. And, in a notationally-executed blatant cover-up of the unpleasant reversals of orientation and extra minus signs, we will write $(T, n)$ to denote the solid torus with $2n$ longitudinal sutures, after the reversals of orientation mentioned in the previous paragraph. We still have $SFH(T, n) = \mathbb{Z}_2^{2^{n-1}}$.

In any case, the upshot is that

*we may identify contact elements of* $SFH(T, n)$ *with chord diagrams of* $n$ *chords.*

Third, sutured Floer homology $SFH(M, \Gamma)$ *splits as a direct sum.* This direct sum is over spin-c structures on $(M, \Gamma)$.

In our simple case of $(T, n)$, for which $SFH(T, n+1) = \mathbb{Z}_2^{2^n}$, this sum over spin-c structures corresponds to a row of Pascal's triangle.

**Theorem 1.8 (Honda–Kazez–Matić [30], Juhász [33])** *$SFH(T, n+1) = \mathbb{Z}_2^{2^n}$ and splits as a sum over spin-c structures*

$$SFH(T, n+1) = \mathbb{Z}_2^{\binom{n}{0}} \oplus \cdots \oplus \mathbb{Z}_2^{\binom{n}{n}}.$$

*If $\xi$ is a contact structure on the sutured manifold $(T, n+1)$ with relative euler class $e$, then its contact element $c(\xi)$ lies in the summand $\mathbb{Z}_2^{\binom{n}{k}}$, where $k = (e+n)/2$.*

The *relative euler class* of a contact structure (evaluated in a meridional disc, which generates $H^2(T, \partial T)$) is precisely the relative euler class of the corresponding chord diagram. This is an integer $e$ of the same parity as $n$, and $-n \leq e \leq n$. The $n+1$ possible values of $e$ correspond precisely to the $n+1$ direct summands above. If $\xi$ has relative euler class $e$, then the corresponding contact element lies in the summand $\mathbb{Z}_2^{\binom{n}{k}}$, where $k = (e+n)/2$, which we denote $SFH(T, n+1, e)$.

Fourth, *an inclusion of sutured manifolds induces a map on* $SFH$ [30]. If the inclusion is $(M', \mathbf{\Gamma}') \hookrightarrow (M, \mathbf{\Gamma})$, and a contact structure $\xi''$ is specified on $(M' - M, \mathbf{\Gamma} \cup \mathbf{\Gamma}')$, then this map takes a contact structure $\xi'$ on $(M', \mathbf{\Gamma}')$ to the contact structure $\xi' \cup \xi''$ on $(M, \mathbf{\Gamma})$. This is a TQFT-type property, and we can call it *TQFT-inclusion.* We will use this principle to describe our basis for $SFH$, and many other things besides.

This indicates (but does not explain) the origin of the vector space defined combinatorially in section 1.1. That the original (holomorphically-defined) vector space $SFH$, and the combinatorially-defined vector space $SFH_{comb}$, are isomorphic, is not immediately clear. These facts will be explained later in section 4.

Our main goal, and motivation, in this paper is to answer the question:

*How do contact elements lie in sutured Floer homology?*

We now give some first propositions in this direction. The first, and in our case, in some sense the "only" relation between contact elements is the bypass relation. This was probably known to the authors of [30], although the whole of this result was not made explicit. The set of contact elements in $SFH(T,n,e)$ is not a subgroup under addition, but the extent to which it is closed under addition describes contact structures which are related by bypass moves, in the following sense.

**Proposition 1.9 (Addition of contact elements means bypasses)** *Suppose $a, b \in SFH(T,n,e)$ are contact elements. Then $a+b$ is a contact element if and only if $a, b$ are related by a bypass move. If so, then $a+b$ is the third element of their bypass triple.*

Recalling the bijection between chord diagrams and tight contact structures on $(T,n)$, we will find that all chord diagrams / contact elements are distinct in $SFH(T,n,e)$.

**Proposition 1.10 (Contact elements are distinct)** *Two distinct contact structures, or equivalently two distinct chord diagrams, give distinct contact elements of $SFH(T,n,e)$.*

The combinatorial version of sutured Floer homology described in section 1.1 seems to have been known in [30], although it was not made explicit; it also appears to be the origin of Honda's "contact category" [19]. In any case, the bypass relation alone does not show that $SFH$ is the combinatorial object described in section 1.1. But it is.

**Proposition 1.11 ($SFH$ is combinatorial)** *The $\mathbb{Z}_2$-vector space $SFH_{comb}(T,n)$ is isomorphic to $SFH(T,n)$. The isomorphism $SFH_{comb} \longrightarrow SFH$ takes a chord diagram to the contact element of the tight contact structure on $(T,n)$ with that chord diagram as its dividing set on a meridional disc. The isomorphism restricts also to the summands $SFH_{comb}(T,n,e) \cong SFH(T,n,e)$.*

## 1.3 Categorification of Pascal's triangle

If we consider all the sutured Floer homology groups $SFH(T,n+1,e)$, over all possible $n$ and $e$, we can arrange these in a triangle.

$$\begin{array}{ccccccc} & & & SFH(T,1,0) & & & \\ & & SFH(T,2,-1) & \oplus & SFH(T,2,1) & & \\ & SFH(T,3,-2) & \oplus & SFH(T,3,0) & \oplus & SFH(T,3,2) & \\ & & & \cdots & & & \end{array}$$

These $\mathbb{Z}_2$-vector spaces are isomorphic respectively to the following.

$$\begin{array}{ccccccc} & & & \mathbb{Z}_2^{\binom{0}{0}} & & & \\ & & \mathbb{Z}_2^{\binom{1}{0}} & \oplus & \mathbb{Z}_2^{\binom{1}{1}} & & \\ & \mathbb{Z}_2^{\binom{2}{0}} & \oplus & \mathbb{Z}_2^{\binom{2}{1}} & \oplus & \mathbb{Z}_2^{\binom{2}{2}} & \\ \mathbb{Z}_2^{\binom{3}{0}} & \oplus & \mathbb{Z}_2^{\binom{3}{1}} & \oplus & \mathbb{Z}_2^{\binom{3}{2}} & \oplus & \mathbb{Z}_2^{\binom{3}{3}} \end{array}$$

We can think of this as a "categorification of Pascal's triangle". Moreover, there are also maps between these vector spaces with properties analogous to the recursion in Pascal's triangle. These maps are denoted

$$\begin{array}{cccc} B_-, B_+ : & SFH(T,n) & \longrightarrow & SFH(T,n+1) \\ & \cong & & \cong \\ & \mathbb{Z}_2^{2^{n-1}} & \longrightarrow & \mathbb{Z}_2^{2^n} \end{array}$$

which we may call *creation* maps, defined by the picture in figure 6 of "creating a chord and adding a $\pm$ outermost region near the base point". They take chord diagrams to chord diagrams, i.e. contact elements to contact elements.

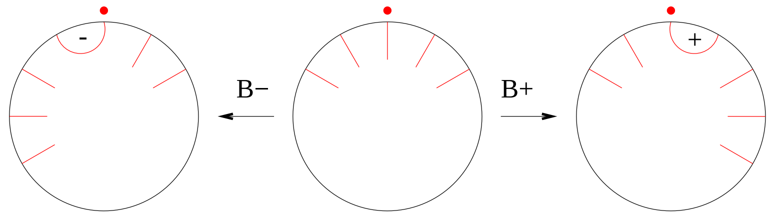



Figure 6: Creation maps $B_\pm$.

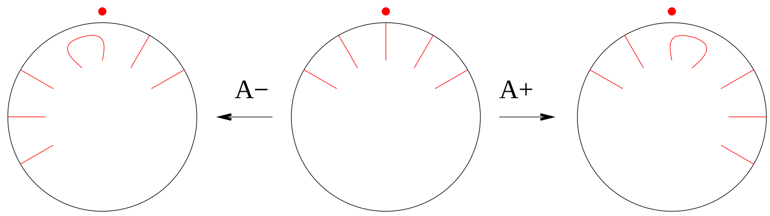



Figure 7: Annihilation maps $A_\pm$.

In the combinatorial definition of $SFH$, it is clear that this is linear. The fact that it is linear in bona fide sutured Floer homology comes from its TQFT-inclusion property.

The maps $B_-, B_+$ respectively subtract or add 1 to the relative euler class of the chord diagram / contact structure; so that, restricting to particular summands, $B_-, B_+$ respectively define maps

$$\begin{array}{ccc} SFH(T,n,e) & \stackrel{B_-}{\longrightarrow} & SFH(T,n+1,e-1), \\ \mathbb{Z}_2^{\binom{n-1}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}}, \end{array} \qquad \begin{array}{ccc} SFH(T,n,e) & \stackrel{B_+}{\longrightarrow} & SFH(T,n+1,e+1) \\ \mathbb{Z}_2^{\binom{n-1}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k+1}} \end{array}$$

where $k = (n+e-1)/2$.

These two maps "categorify the Pascal recursion".

**Proposition 1.12 (Creation operators, categorification of Pascal)** *There are creation operators $B_\pm : SFH(T,n,e) \longrightarrow SFH(T,n+1,e\pm 1)$ which correspond to "creating" a chord as in figure 6 above. These are injective linear maps and*

$$SFH(T,n+1,e) = B_+\left(SFH(T,n,e-1)\right) \oplus B_-\left(SFH(T,n,e+1)\right).$$

The correspondence to "creating a chord" described in the theorem refers to the TQFT-type inclusion property of $SFH$ [30]; we will define the operators more precisely in section 4.

Similarly, we can define two $\mathbb{Z}_2$-vector space maps

$$\begin{array}{cccc} A_-, A_+ : & SFH(T,n+1) & \longrightarrow & SFH(T,n) \\ & \cong & & \cong \\ & \mathbb{Z}_2^{2^n} & \longrightarrow & \mathbb{Z}_2^{2^{n-1}} \end{array}$$

which we may call *annihilation* maps, defined by the "closing off an outermost $\pm$ region near the base point". See figure 7.

Again, this is clearly linear in the combinatorial version of $SFH$; it is also linear as an application of the TQFT-property of $SFH$. The map $B_\pm$ corresponds to an inclusion of a solid torus with $2n$ boundary sutures, into a solid torus with $2n+2$ boundary sutures, with a specific contact structure in the thickened torus between them.

The maps $A_-$, $A_+$ respectively subtract or add 1 to the relative euler class of the chord diagram / contact structure; so that, restricting to these summands, again, we have respectively

$$\begin{array}{ccc} SFH(T,n+1,e) & \stackrel{A_-}{\longrightarrow} & SFH(T,n,e-1), \\ \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n-1}{k-1}}, \end{array} \qquad \begin{array}{ccc} SFH(T,n+1,e) & \stackrel{A_+}{\longrightarrow} & SFH(T,n,e+1) \\ \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}} \end{array}$$

where $k = (n+e-1)/2$.

The creation and annihilation operators satisfy some relations.

**Proposition 1.13 (Annihilation operators)** *There are operators*

$$A_\pm : SFH(T,n+1,e) \longrightarrow SFH(T,n,e\pm 1)$$

*which correspond to "annihilating" a chord in a chord diagram as in figure 7 above. These are surjective and satisfy*

$$A_+ \circ B_- = A_- \circ B_+ = 1 \quad \text{and} \quad A_+ \circ B_+ = A_- \circ B_- = 0.$$

Thus, the $A_\pm, B_\pm$ operators give a categorification of Pascal's triangle, in the sense of the following diagram:

$$\begin{array}{ccccccc}
 & & & \mathbb{Z}_2^{\binom{0}{0}} & & & \\
 & & & {\scriptstyle B_- A_+ A_- B_+} & & & \\
 & & & \swarrow \nearrow \nwarrow \searrow & & & \\
 & & \mathbb{Z}_2^{\binom{1}{0}} & & \mathbb{Z}_2^{\binom{1}{1}} & & \\
 & & {\scriptstyle B_- A_+ A_- B_+} & & {\scriptstyle B_- A_+ A_- B_+} & & \\
 & & \swarrow \nearrow \nwarrow \searrow & & \swarrow \nearrow \nwarrow \searrow & & \\
 & \mathbb{Z}_2^{\binom{2}{0}} & & \mathbb{Z}_2^{\binom{2}{1}} & & \mathbb{Z}_2^{\binom{2}{2}} & \\
 & {\scriptstyle B_- A_+ A_- B_+} & & {\scriptstyle B_- A_+ A_- B_+} & & {\scriptstyle B_- A_+ A_- B_+} & \\
 & \swarrow \nearrow \nwarrow \searrow & & \swarrow \nearrow \nwarrow \searrow & & \swarrow \nearrow \nwarrow \searrow & \\
\mathbb{Z}_2^{\binom{3}{0}} & & \mathbb{Z}_2^{\binom{3}{1}} & & \mathbb{Z}_2^{\binom{3}{2}} & & \mathbb{Z}_2^{\binom{3}{3}} \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

## 1.4 Basis, words, orderings, and quantum field theory

Denote by $v_\emptyset$ the nonzero element of $SFH(T,1,0) = SFH(T,1) = \mathbb{Z}_2$ (the "vacuum"), which corresponds to the unique chord diagram with 1 chord. Then in $SFH(T,n+1,e) \cong \mathbb{Z}_2^{\binom{n}{k}}$ there are $\binom{n}{k}$ contact elements of the form

$$B_\pm\ B_\pm\ \cdots\ B_\pm\ v_\emptyset$$

where there are $n_+$ of the $B_+$'s, and $n_-$ of the $B_-$'s, satisfying $k = n_+$, $n = n_+ + n_-$ and $e = n_+ - n_-$.

Denote by $W(n_-, n_+)$ the set of all words of plus and minus signs of length $n = n_+ + n_-$, with $n_-$ minus signs and $n_+$ plus signs; equivalently, which sum to $e = n_+ - n_-$. For every word $w \in W(n_-, n_+)$ there is a corresponding element $v_w = B_w v_\emptyset \in SFH(T, n+1, e)$. Here by $B_w$ we denote the string of $B_+$'s and $B_-$'s corresponding to $w$. Each $v_w \in SFH(T,n+1,e)$ is a contact element, corresponding to a chord diagram $\Gamma_w$ of $n+1$ chords and relative euler class $e$.

**Remark 1.14 (Convention with $n_-, n_+, n, e, k$)** *Unless mentioned otherwise, we will assume the relationships between variables $n_-, n_+, n, e, k$ so that $SFH(T,n+1,e) = \mathbb{Z}_2^{\binom{n}{k}}$ with basis indexed by $W(n_-, n_+)$. That is, they are related by*

$$k = (e+n)/2, \quad e = 2k - n, \quad n_+ = k, \quad n = n_+ + n_-, \quad e = n_+ - n_-.$$

The set $W(n_-, n_+)$ has some orderings:

**Definition 1.15 (Lexicographic ordering)** *There is a total order on $W(n_-, n_+)$ obtained from regarding $-$ as coming before $+$ in the dictionary. This also induces a total order on the elements $v_w \in SFH(T, n+1, e)$ and the chord diagrams $\Gamma_w$.*

We will usually read words from left to right, but we note that reading words from right to left also gives a total lexicographic order.

**Definition 1.16 (Partial ordering $\preceq$)** *There is a partial order $\preceq$ on $W(n_-, n_+)$ defined by: $w_1 \preceq w_2$ if and only if, for all $i = 1, \ldots, n_-$, the $i$'th $-$ sign in $w_1$ occurs to the left of (or in the same position as) the $i$'th $-$ sign in $w_2$. This also induces a partial order, also denoted $\preceq$, on the elements $v_w \in SFH(T, n+1, e)$ and the chord diagrams $\Gamma_w$.*

Thus the partial order $\preceq$ essentially says "all minus signs move right". This partial order is the intersection of lexicographic orderings in both directions.

In any case, $\preceq$ gives a partial order on words $w$, and also on the corresponding contact elements $v_w \in SFH(T, n+1, e)$ and chord diagrams $\Gamma_w$. It is clear that this is a sub-order of the lexicographic order, and a partial order.

Note that $|W(n_-, n_+)| = \binom{n}{n_+} = \binom{n}{k}$, so that there are as many $v_w \in SFH(T, n+1, e)$ as the dimension of $SFH(T, n+1, e)$. It is in fact a basis.

**Proposition 1.17 (QFT basis)** *There is a basis $\{v_w\}$ of $SFH(T, n+1, e)$, consisting of the $\binom{n}{k}$ possible contact elements in $SFH(T, n+1, e)$ obtained by applying creation operators to the nonzero element $v_\emptyset$ of $SFH(T, 1, 0) = \mathbb{Z}_2$. The basis is indexed by words $w \in W(n_-, n_+)$.*

The analogy, of course, is with operators for the creation and annihilation of particles in quantum field theory. We can think of $SFH(T, n+1)$ as the space generated by "$n$-particle states", and each chord diagram with $n+1$ chords and relative euler class $e$ as a particular "$n$-particle state of charge $e$". The chord diagram with 1 chord, we think of as "the vacuum". We think of $B_+$ as "creating a charge $+1$ particle" and $B_-$ as "creating a charge $-1$ particle"; and similarly, we think of $A_+$ as "annihilating a charge $-1$ particle" and $A_-$ "annihilating a charge $+1$ particle". The bypass relation can be thought of as "the superposition of two bypass-related states is the third state in their triple".

The fact that the $v_w$ form a basis says that "the space of $n$-particle states has a basis obtained by applying creation operators to the vacuum". This is usual in quantum field theory. However, for bosons, such creation operators commute; for fermions, they anti-commute; in our case, there is no commutation relation whatsoever, and any applications of creation operators in different orderings are independent. Perhaps our particles have "irrational spin", then.

## 1.5 Catalan and Narayana numbers

Since our goal is to understand how contact elements / chord diagrams lie in $SFH$, a simple first question is: *How many contact elements are there in $SFH(T, n)$?*

The number of distinct chord diagrams of $n$ chords is given by the *Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

of which the first few are:

$$1, 1, 2, 5, 14, 42, 132, \ldots$$

The Catalan numbers are classical combinatorial objects and have been studied extensively for centuries; they are prolific in combinatorics. For instance, the number of ways of arranging $n$ "open brackets" ( and $n$ "close brackets" ) meaningfully is $C_n$; it is not difficult to see that these are in bijective correspondence with the dividing sets we consider. They can also be defined recursively by $C_0 = 1$, $C_1 = 1$ and then

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0.$$

The number of distinct $n$-arc tight dividing sets is $C_n$; thus the contact elements form a distinguished subset of size $C_n$, among the $2^{2^{n-1}}$ elements of $SFH(T,n) \cong \mathbb{Z}_2^{2^{n-1}}$.

We can then refine our question, splitting according to relative euler class: *How many contact elements are there in $SFH(T, n+1, e)$?*

Let this number be $C_{n+1}^e$. So there are $C_{n+1}^e$ chord diagrams with $n+1$ chords and relative euler class $e$; and $e$ is an integer of the same parity as $n$ satisfying $-n \leq e \leq n$. It will also be useful to define $C_{n+1,k} = C_{n+1}^{2k-n} = C_{n+1}^e$, following our convention in remark 1.14; so that $k$ is an integer $0 \leq k \leq n$.

From counting chord diagrams of various relative euler classes, we have

$$C_{n+1} = C_{n+1,0} + C_{n+1,1} + \cdots + C_{n+1,n} = C_{n+1}^{-n} + C_{n+1}^{-n+2} + \cdots + C_{n+1}^{n}.$$

Our goal, in more refined form, is to find how these $C_{n+1}^e$ elements lie inside this $\mathbb{Z}_2$-vector space.

The numbers $C_{n,k}$ form a triangle, then, which form the *Catalan triangle* or the *Narayana numbers*.

$$\begin{array}{ccccccccc} &&&& C_1^0 &&&& \\ &&& C_2^{-1} && C_2^1 &&& \\ && C_3^{-2} && C_3^0 && C_3^2 && \\ & C_4^{-3} && C_4^{-1} && C_4^1 && C_4^3 & \\ C_5^{-4} && C_5^{-2} && C_5^0 && C_5^2 && C_5^4 \end{array} = \begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 3 && 1 && \\ & 1 && 6 && 6 && 1 & \\ 1 && 10 && 20 && 10 && 1 \end{array}$$

The Narayana numbers are usually given as $N_{n,k} = C_{n,k-1}$; we have shifted them for our purposes, since our $k$ is a useful quantity. They have an explicit formula, although we shall not use it:

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

or

$$C_{n+1}^e = C_{n+1,k} = N_{n+1,k+1} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}.$$

There is a substantial literature on the Narayana numbers (e.g. [1, 2, 3, 5, 10, 18, 31, 36, 41, 42, 43]); we will restate some of their basic properties.

**Proposition 1.18 (Narayana numbers, Catalan triangle)** *There is a "Catalan" or "Narayana triangle" consisting of $C_{n+1,k} = C_{n+1}^e$, satisfying the following relations:*

*(i)* $C_{n+1} = C_{n+1,0} + C_{n+1,1} + \cdots + C_{n+1,n}$.

*(ii)*

$$C_{n+1,k} = C_{n,k} + C_{n,k-1} + \sum_{\substack{n_1+n_2=n \\ k_1+k_2=k-1}} C_{n_1,k_1}\, C_{n_2,k_2},$$

*or equivalently,*

$$C_{n+1}^e = C_n^{e-1} + C_n^{e+1} + \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C_{n_1}^{e_1}\, C_{n_2}^{e_2}.$$

One can think of this recursion as a slightly more complicated version of the recursion in Pascal's triangle, incorporating the Catalan recursion. In fact, we will also show that there is a "categorification" of this recursion also, in a certain tenuous sense.

**Proposition 1.19 (Categorification of Catalan recursion)** *There is a "merge operator"*

$$M : SFH(T, n_1, e_1) \otimes SFH(T, n_2, e_2) \longrightarrow SFH(T, n_1 + n_2 + 1, e_1 + e_2)$$

*which, applied to contact elements $c(\xi_1) \otimes c(\xi_2)$, gives a contact element obtained by "merging" the corresponding dividing sets. The operator $M$ reduces to a creation operators $B_\pm$ in the case $n_1 = 0$ or*

$n_2 = 0$. *Evey contact element in* $SFH(T, n, e)$ *can be written uniquely as* $M(c(\xi_1), c(\xi_2))$, *where the* $c(\xi_i)$ *are contact elements in* $SFH(T, n_i, e_i)$ *for some* $n_i, e_i$ *with* $n_1 + n_2 = n$ *and* $e_1 + e_2 = e$. *That is,*

$$\left\{ \begin{array}{c} \textit{Contact elements} \\ \textit{in } SFH(T,n,e) \end{array} \right\} = B_+ \left( \left\{ \begin{array}{c} \textit{Contact elements in} \\ SFH(T,n-1,e-1) \end{array} \right\} \right) \sqcup B_- \left( \left\{ \begin{array}{c} \textit{Contact elements in} \\ SFH(T,n-1,e+1) \end{array} \right\} \right)$$
$$\sqcup \bigsqcup_{\substack{n_1+n_2=n \\ e_1+e_2=e}} M \left( \left\{ \begin{array}{c} \textit{Contact elements} \\ \textit{in } SFH(T,n_1,e_1) \end{array} \right\}, \left\{ \begin{array}{c} \textit{Contact elements} \\ \textit{in } SFH(T,n_2,e_2) \end{array} \right\} \right)$$

*In particular,*

$$SFH(T,n,e) = B_+(SFH(T,n-1,e-1)) + B_-(SFH(T,n-1,e+1)) + \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} M(SFH(T,n_1,e_1), SFH(T,n_2,e_2)).$$

Note this is by no means a direct sum; the equation simply says that $M$ is surjective onto $SFH(T, n, e)$.

We also have a crucial enumerative result for our main theorem: a bijection between comparable pairs and chord diagrams.

**Proposition 1.20 (Number of comparable pairs)** *Under the partial ordering* $\preceq$ *of* $W(n_-, n_+)$, *which indexes the basis elements of* $SFH(T, n + 1, e)$, *there are precisely* $C_{n+1}^e$ *comparable pairs, i.e. pairs* $(w_0, w_1)$ *with* $w_0 \preceq w_1$.

## 1.6 Contact elements and comparable pairs

Our main theorems flesh out the previous enumerative bijection. They describe in some detail how contact elements lie in $SFH(T, n+1, e)$, giving an explicit bijection between contact elements and comparable pairs of words. More precisely, a general superposition of states is determined by decomposing it in terms of basis vectors and looking at the first and last basis states.

Alternatively, we can think of every state as a morphism from a first state to a last state.

**Theorem 1.21 (Minimum/maximum basis vectors of contact element)** *Writing a contact element in* $SFH(T, n + 1, e)$ *as a sum of basis vectors* $v_w$, *where* $w \in W(n_-, n_+)$, *there is a lexicographically first* $v_{w_-}$ *and last* $v_{w_+}$ *basis vector amongst them. Then for every basis vector* $v_w$ *occurring in the sum,* $w_- \preceq w \preceq w_+$. *In particular,* $w_- \preceq w_+$.

**Theorem 1.22 (Contact elements and comparable pairs)** *The map*

$$\Phi : \left\{ \begin{array}{c} \textit{Chord} \\ \textit{diagrams} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Contact} \\ \textit{structures} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Comparable pairs of} \\ \textit{words } w_1 \preceq w_2 \end{array} \right\}$$

*given by taking a contact element* $v$, *and sending it to* $(w_-, w_+)$, *where* $v_{w_-}, v_{w_+}$ *are respectively the first and last basis vectors in the basis decomposition of* $v$, *is a bijection.*

*That is, given any comparable pair* $w_1 \preceq w_2$, *there is precisely one contact element which, when written as a sum of basis elements, has* $v_{w_1}$ *as its first and* $v_{w_2}$ *as its last.*

## 1.7 Moves on chord diagrams and words

The proofs of the main theorems are by an explicit construction. Given $w^- \preceq w^+$, we show how to construct a chord diagram whose decomposition has $v_{w^-}$ as its first and $v_{w^+}$ as its last element. Then by the enumerative bijection of proposition 1.20, this is shown to be a bijection.

We will build up a method for performing bypass moves on basis chord diagrams, in order to turn any $\Gamma_{w_1}$ into $\Gamma_{w_2}$, whenever $w_1 \preceq w_2$, by upwards bypass moves. And conversely, we will show how to turn $\Gamma_{w_2}$ into $\Gamma_{w_1}$ by downwards bypass moves. This method will be explicitly analogous to certain combinatorial moves on the corresponding words.

Indeed, we will consider combinatorial moves on words $w \in W(n_-, n_+)$, consisting of moving certain blocks of $-$ signs past certain blocks of $+$ signs. We will show that single nontrivial upwards bypass moves on basis chord diagrams correspond perfectly to certain "elementary moves" on words. And then we will generalise to show that more complicated moves on words can be achieved by multiple bypass moves. We will build up to enough machinery to construct bypass moves to take us from any $w_1$ to $w_2$, for $w_1 \preceq w_2$. This will be called the *bypass system of the comparable pair* $w_1 \preceq w_2$. This bypass system is given explicitly.

Then, we will show that performing all these bypass moves *in the opposite direction*, gives us a chord diagram whose decomposition has $w_1, w_2$ as first and last elements. This arises from the bypass relation.

**Proposition 1.23 (Bypass system of a comparable pair)** *Suppose* $\Gamma_1 \preceq \Gamma_2$ *are basis chord diagrams.*

*(i) On* $\Gamma_1$*, there exists a bypass system* $FBS(\Gamma_1, \Gamma_2)$ *such that performing* upwards *bypass moves on it gives* $\Gamma_2$*.*

*(ii) On* $\Gamma_2$*, there exists a bypass system* $BBS(\Gamma_1, \Gamma_2)$ *such that performing* downwards *bypass moves on it gives* $\Gamma_1$*.*

**Proposition 1.24 (Bypass system of a pair, other way)** *Performing* downwards *bypass moves on* $FBS(\Gamma_1, \Gamma_2)$ *or* upwards *bypass moves on* $BBS(\Gamma_1, \Gamma_2)$ *gives the same chord diagram* $\Gamma$*. This* $\Gamma$ *has the property that in its basis decomposition,* $\Gamma_1$ *and* $\Gamma_2$ *appear, and for every basis element* $\Gamma'$ *in this decomposition,* $\Gamma_1 \preceq \Gamma' \preceq \Gamma_2$*. In particular,* $\Gamma_1$ *is a total minimum and* $\Gamma_2$ *a total maximum, with respect to* $\preceq$*, among all the basis elements occurring in the decomposition.*

The proofs of these proposition are based on a correspondence between the following notions, which we will define in due course.

$$
\begin{array}{rcl}
\left\{ \begin{array}{c} \text{elementary move} \\ \text{on a word} \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{bypass move} \\ \text{on an attaching arc} \end{array} \right\} \\
\left\{ \begin{array}{c} \text{generalised elementary} \\ \text{move on a word} \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{bypass moves on the} \\ \text{bypass system of a} \\ \text{generalised attaching arc} \end{array} \right\} \\
\left\{ \begin{array}{c} \text{nicely ordered sequence of} \\ \text{generalised elementary moves} \\ \text{on a word} \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{c} \text{bypass moves on the} \\ \text{bypass system of a} \\ \text{nicely ordered sequence of} \\ \text{generalised attaching arcs} \end{array} \right\}.
\end{array}
$$

The final correspondence is strong enough to give the constructions in the above two propositions, explicitly; which in turn give the main theorems.

## 1.8 Contact geometry, stacking, and bypass triples

Although none of the above mentions any contact geometry, the motivation is contact-geometric. Chord diagrams represent convex discs; sutured manifolds correspond to boundary conditions for contact structures; and a bypass move corresponds to attaching a bypass — which is a thickened half-disc with a particular contact structure — to a convex surface. Our results have not only have contact-geometric motivation, but contact-geometric application as well.

In particular, there is a construction we will call *stacking*. Given two chord diagrams $\Gamma_0, \Gamma_1$, we form a solid cylinder $\mathcal{M}(\Gamma_0, \Gamma_1)$ with sutures on the bottom and top being $\Gamma_0$ and $\Gamma_1$. We can ask whether these sutures correspond to a tight convex boundary of the cylinder: if so, we say $\Gamma_1$ is *stackable* on $\Gamma_0$. There is a corresponding map on $SFH$.

**Lemma 1.25 (Stackability map)** *There is a linear map*

$$m : SFH(T, n, e) \otimes SFH(T, n, e) \longrightarrow \mathbb{Z}_2$$

*which takes pairs of contact elements, corresponding to pairs of chord diagrams* $(\Gamma_0, \Gamma_1)$*, to* 0 *or* 1 *respectively as* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is overtwisted or tight.*

The answer to whether $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight, or equivalently when $m = 1$, is intimately related to the partial order $\preceq$; in fact, on basis chord diagrams $\Gamma_w$ for $w \in W(n_-, n_+)$, it is a complete answer. Then we can use this to obtain a result for general chord diagrams.

**Proposition 1.26 (Contact interpretation of $\preceq$)** *Let* $w_0, w_1 \in W(n_-, n_+)$*. Then* $\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ *is tight if and only if* $w_0 \preceq w_1$*.*

**Proposition 1.27 (General stackability)** *Let* $\Gamma_0, \Gamma_1$ *be chord diagrams with* $n$ *chords and relative euler class* $e$*. Then* $\Gamma_1$ *is stackable on* $\Gamma_0$ *(i.e.* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight) if and only if the cardinality of the set*

$$\left\{ (w_0, w_1) \; : \; \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in decomposition of } \Gamma_i \end{array} \right\}$$

*is odd.*

We will show various other properties of $m$ and $\mathcal{M}$:

(i) (Lemma 9.4) $m(\Gamma, \Gamma) = 1$, i.e. $\mathcal{M}(\Gamma, \Gamma)$ is tight.

(ii) (Lemma 9.5) If $\Gamma_0, \Gamma_1$ both have an outermost chord $\gamma_0, \gamma_1$ in the same position, then $m(\Gamma_0, \Gamma_1) = m(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1)$.

(iii) (Lemma 9.6) If $\Gamma_0, \Gamma_1$ are related by a bypass move then, when placed in the right order, $m(\Gamma_0, \Gamma_1)$ is tight. (The order is made explicit in lemma 9.6.)

The last lemma relates stackability to bypass moves. In fact, if $\Gamma_1$ can be obtained from $\Gamma_0$ by attaching bypasses on top of $\Gamma_0$, then we have a construction of $\mathcal{M}(\Gamma_0, \Gamma_1)$, along with a contact structure on the solid cylinder, not just the boundary. Under certain conditions, one can prove that this contact structure is tight: such questions were considered in [27], and the results of that paper are applied here. In fact, bypass triples naturally give rise to triples of tight contact solid cylinders.

More generally, we will also show that our explicit construction of bypass moves from $\Gamma_{w_1}$ to $\Gamma_{w_2}$, via the bypass system $FBS(w_1, w_2)$ for any $w_1 \preceq w_2$, when considered as actual contact-geometric bypass attachments, gives a tight contact structure on $\mathcal{M}(\Gamma_{w_1}, \Gamma_{w_2})$. For a given chord diagram $\Gamma$ with first and last basis elements $\Gamma^-, \Gamma^+$, this means that the triple $(\Gamma, \Gamma^-, \Gamma^+)$ can be considered as a "multiple bypass" bypass triple, and is a generalisation of a bypass triple. That is:

(i) (Lemma 10.4) $m(\Gamma, \Gamma^-) = m(\Gamma^+, \Gamma) = m(\Gamma^-, \Gamma^+) = 1$.

(ii) (Lemma 11.8) There are tight contact structures on $\mathcal{M}(\Gamma, \Gamma^-)$, $\mathcal{M}(\Gamma^+, \Gamma)$ and $\mathcal{M}(\Gamma^-, \Gamma^+)$ obtained by attaching bypasses along the bypass systems $FBS(\Gamma^-, \Gamma^+)$ and $BBS(\Gamma^-, \Gamma^+)$.

## 1.9 Contact elements are tangled

Finally, we will be able to use the above results to give some more information about contact elements. We know that their first and last basis elements correspond to comparable words, and that the first and last basis elements must determine the rest of the decomposition; but what about the basis elements between them?

First, we will prove that *contact elements have a total maximum and minimum.* In particular, if $\Gamma$ has first and last basis elements $v^-, v^+$ corresponding to words $w^-, w^+$, then for every $v_w$ occurring in the decomposition of $\Gamma$, $w^- \preceq w \preceq w^+$. Actually this is part of our main theorem 1.21.

Second, we will prove results about *the number of basis elements* in a contact element. For a basis element, this answer is clear: one — itself. In any other case, we will show that the answer is even.

**Proposition 1.28 (Size of basis decomposition)** *Every chord diagram which is not a basis element has an even number of basis elements in its decomposition.*

Third, we will exhibit some formulas expressing *relations between the basis elements* of a contact element (lemma 11.13):

$$\Gamma^+ = \sum_{\Gamma_w \in \Gamma} [\Gamma^-, \Gamma_w] \quad \text{and} \quad \Gamma^- = \sum_{\Gamma_w \in \Gamma} [\Gamma_w, \Gamma^+].$$

Here $\Gamma_w \in \Gamma$ means that the basis chord diagram $\Gamma_w$ appears in the decomposition of $\Gamma$.

Fourth, and perhaps our main result in this regard, is to show that basis elements of $\Gamma$ are "tangled up", in some sense. We have said that the first and last basis elements $v^-, v^+$ of $\Gamma$ are comparable to all others in the decomposition. We will show that no other basis elements in $\Gamma$ have this property.

**Theorem 1.29 (Not much comparability)** *Suppose $\Gamma_w$ occurs in the basis decomposition of $\Gamma$ and is comparable (with respect to $\preceq$) with every other basis element occurring in $\Gamma$. Then $\Gamma_w = \Gamma^+$ or $\Gamma^-$.*

We can think of this as saying that the basis elements of $\Gamma$ are "completely tangled". We cannot "untangle" the partial ordering among them, at any interior point, in any nice way.

More generally, we will show that for any basis element $v_w$ in the decomposition of $\Gamma$, other than $v^\pm$, the number of basis elements $v_{w'}$ of $\Gamma$ such that $w' \preceq w$ is even; and the number of $v_{w'}$ with $w \preceq w'$ is even also (proposition 11.15). This implies the above theorem.

## 1.10 Computation by rotation

None of the above gives a good way to *compute* all contact elements (not just basis elements). One good way to enumerate them is to use the fact that we may always rotate a chord diagram until there is an outermost region adjacent to the base point, and then it lies in the image of $B_\pm$. Such a rotation gives a linear operator in sutured Floer homology, and we will give a recursive formula (proposition 12.1) for this operator, as well as describing it explicitly (proposition 12.2).

## 1.11 Reading the symbolic encoding

Our main theorem, from an information theoretic perspective, gives an encoding of a chord diagram of $n+1$ chords in $2n$ bits, with redundancy. And the algorithm described above shows how to reconstruct a chord diagram from this $2n$-bit encoding. But some other properties can be seen directly:

(i) (Lemma 11.2) A chord diagram has an outermost region at the base point, if and only if $\Gamma^-, \Gamma^+$ begin with the same symbol, if and only if all basis elements of $\Gamma$ begin with the same symbol.

(ii) (Lemmas 11.3-11.7) Similarly, a chord diagram has an outermost regions at various other locations, if and only if $\Gamma^-, \Gamma^+$ have various other properties (ending with the same symbol; having the $j$'th $-$ sign not the first in its block; etc.), if and only if all basis elements of $\Gamma$ have the same properties.

## 1.12 A contact 2-category and a simplicial structure

We will also consider interesting algebraic structures which arise in our $SFH(T, n, e)$. Honda has defined a *contact category* closely related to $SFH$. The objects of his category are dividing sets on a given surface, and the objects are contact structures between them.

One interpretation of our main theorem is that in our simple case (i.e. a disc), the objects of this category can themselves be viewed themselves as morphisms. For chord diagrams are described by pairs $w_0 \preceq w_1$, and a partial order is nothing more than a category, with $\preceq$ giving the morphisms. Thus, we can obtain a *contact 2-category* $\mathcal{C}(n+1, e)$: the objects of Honda's contact category can be regarded themselves as its 1-morphisms, and the morphisms of that category become 2-morphisms, or "morphisms between morphisms".

**Proposition 1.30 (Contact 2-category)** *There is a 2-category $\mathcal{C}(n+1,e)$ such that:*

*(i) its objects are words $w \in W(n_-, n_+)$, or equivalently chord diagrams $\Gamma_w$;*

*(ii) its 1-morphisms are chord diagrams $\Gamma$ of $n+1$ chords with relative euler class $e$;*

*(iii) its 2-morphisms $\Gamma_0 \to \Gamma_1$ are contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$, with (vertical) composition of given by stacking contact structures.*

Honda's category obeys some of the properties of a *triangulated category*: there are certain triangles of morphisms called *distinguished*, and they obey certain properties. We thus obtain, for each pair $(n+1, e)$ a 2-category $\mathcal{C}(n+1, e)$ which has some of the properties of a triangulated category — and in a certain fairly trivial sense, even more such properties. It may even be that considering the situation over all values of $n$ and $e$, the contact category becomes a 3-category.

We will also show that there is a simplicial structure on the $SFH$ vector spaces forming the various diagonals of Pascal's triangle. We note that our creation and annihilation operators $A_\pm, B_\pm$ were defined at a particular point, namely the base point, but there are many other points. Choosing other points gives more creation and annihilation operators, with obey the same relations as face and degeneracy maps in simplicial structures. The associated boundary maps make the categorified Pascal's triangle into a double chain complex. We can compute the homology of all the complexes arising.

**Proposition 1.31 (Simplicial structure, Pascal's double complex)** *On each diagonal of Pascal's triangle, there are face and degeneracy maps giving it a simplicial structure, with boundary maps making each diagonal into a chain complex with trivial homology, and the whole triangle into a double complex. Its homology is zero, i.e. the complexes are all exact.*

## 1.13 Structure of this paper

We begin in sections 2–3 by giving preliminaries on contact geometry and sutured Floer homology. In section 2, we recall facts about convex surfaces, contact structures on balls and solid tori, edge-rounding, and bypasses. In section 3, we recall the definition of sutured Floer homology and briefly indicate the origin of the computation of theorem 1.8.

In part II, we prove the first elementary results. In section 4, we begin to build up our picture of $SFH$ groups categorifying Pascal's triangle. We prove the properties of creation and annihilation operators, the categorification of Pascal's triangle recursion, the bypass relation, the result that $SFH$ is combinatorial, the bijection between chord diagrams and contact elements, and the construction of a basis. In section 5, we return to enumerative combinatorics, proving the Catalan and Narayana recursions, and in so doing giving it a pictorial interpretation which allows us to construct an $SFH$ version of it, hence categorifying the Narayana recursion. We will also analyse the combinatorics of the partial order $\preceq$.

Part III contains the bulk of the technical results, and consists of a thorough study of our basis chord diagrams. In section 6, we show how to construct the chord diagram corresponding to a word, in multiple ways, giving two useful algorithms. The mechanics of these algorithms describe the chords in basis diagrams exactly, and are crucial for subsequent results. Section 7 gives the construction of the bypass systems taking $\Gamma_{w_0}$ to $\Gamma_{w_1}$, for any $w_0 \preceq w_1$. As noted above, we build up some machinery giving correspondences between combinatorial moves on words and bypass moves on chord diagrams, of more and more general type. This takes some time, and possibly some tedium, so the reader is encouraged to skip it on a first reading. However, with this machinery in place, the proof of our main theorem in section 8 is immediate.

Part IV then turns to contact geometry applications. In section 9 we introduce $\mathcal{M}(\Gamma_0, \Gamma_1)$ and $m(\Gamma_0, \Gamma_1)$ and prove various basic elementary properties. In section 10 we give the result relating stackability to the partial order $\preceq$, giving this partial order a contact interpretation. And then in section 11, we will use this to prove more advanced properties of $\mathcal{M}$ and $m$, such the number of basis elements in a decomposition, and relations within contact elements. In this section we will also give results on symbolic encoding of various properties of a chord diagram, and generalised bypass triples.

Finally, part V makes some further considerations. In section 12 we introduce the rotation operator, and compute it, recursively then explicitly. In section 13 we give the simplicial structure on the various diagonals of the categorified Pascal's triangle, arising from various creation and annihilation operators; we also give a symbolic interpretation; and we show how we then make the categorified Pascal's triangle into a double complex. And we close with section 14, in which we introduce our contact 2-category and make some remarks about improving it, and extending our results beyond discs.

# 2 Contact geometry preliminaries

We recall some basic facts about 3-dimensional contact geometry. In general, the reader is referred to [8, 24, 25, 34] for general introductions to the subject.

## 2.1 Fundamental facts

A contact structure $\xi$ on a 3-manifold $M$ is a totally non-integrable 2-plane field. A contact structure can always be described locally as the kernel of a 1-form $\alpha$; the non-integrability condition then becomes $\alpha \wedge d\alpha \neq 0$ everywhere.

A curve everywhere tangent to $\xi$ is called *legendrian*. A simple closed legendrian curve $C$ bounding a surface $\Sigma$ has a *Thurston-Bennequin number* $tb(C)$, which is given by how many times $\xi$ rotates along $C$, relative to the trivialization of the tangent bundle along $C$ given by $\Sigma$.

Contact structures naturally diverge into two types. *Overtwisted* contact structures are those which contain an *overtwisted disc*. An overtwisted disc is an embedded disc bounded by a legendrian curve with Thurston–Bennequin number 0. The classification of such contact structures is homotopy-theoretic [6]. A non-overtwisted contact structure is called *tight*, and their classification is much more subtle: see, e.g., [14, 15, 16, 20, 21].

An embedded surface $\Sigma$ in a contact manifold $(M, \xi)$ has a *characteristic foliation*, which is the singular foliation given by $T\Sigma \cap \xi$. The characteristic foliation on a surface in fact determines the germ of the contact structure along the surface [13].

## 2.2 Convex surfaces

A fundamental tool for understanding contact structures on 3-manifolds is that of *convex surfaces* [13]. A convex surface is an embedded surface $\Sigma$ in a contact manifold $(M, \xi)$ for which there exists a contact vector field $X$ transverse to $\Sigma$. (A contact vector field is a vector field whose flow preserves $\xi$.) If a convex surface has boundary, we usually require it to be legendrian.

Convex surfaces are *generic*. In particular, every closed embedded surface $\Sigma$ in $M$ is $C^\infty$ close to a convex surface; if $\Sigma$ has boundary, we may need to make a $C^0$ perturbation near the boundary.

The *dividing set* $\Gamma$ of a convex surface is the subset of $\Sigma$ where $X \in \xi$; we can think of this as where "$\xi$ is vertical". The dividing set of a convex surface is a properly embedded 1-manifold [13]. If we have a 1-form $\alpha$ for our contact structure, then $\Gamma$ divides $\Sigma$ into regions $R_+, R_-$ where $\alpha(X) > 0$ or $\alpha(X) < 0$ respectively. The dividing set "divides" the characteristic foliation in the sense that this foliation can be directed by a vector field which dilates an area form on $R_+$ and contracts it on $R_-$. In fact, given a dividing set $\Gamma$, *any* characteristic foliation on $\Sigma$ which is divided by $\Gamma$ can be taken to any other by a $C^0$-small isotopy of $\Sigma$ in $M$. Thus, in some sense, the dividing set is what fundamentally describes the contact structure near the surface.

It's easy to determine tightness near a convex surface: a convex surface $\Sigma \neq S^2$ has a tight neighbourhood if and only if it dividing set has no contractible components; and a convex $S^2$ has a tight neighbourhood if and only if its dividing set has a single component [13, 20].

For a 3-manifold with boundary, we can assume the boundary is convex, and then the dividing set will form sutures on the boundary. Hence, given a sutured manifold $(M, \Gamma)$, we will say that $\xi$ is a

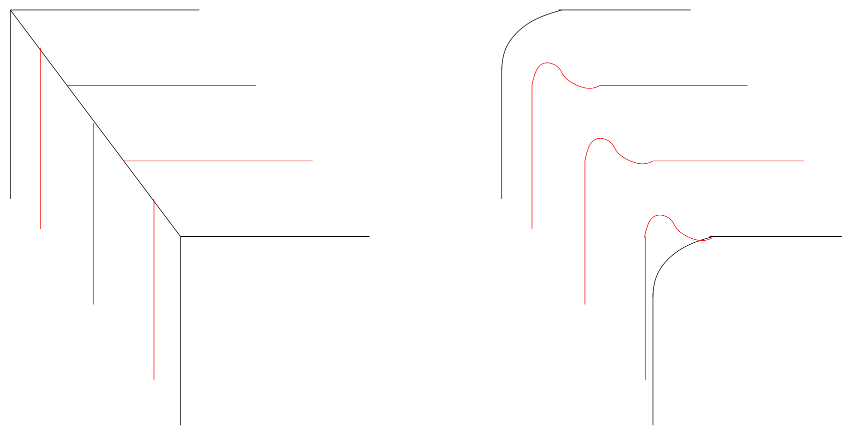


Figure 8: Edge rounding of convex surfaces.

*contact structure on the sutured manifold* $(M, \Gamma)$ if $\xi$ is a contact structure such that $\partial M$ is convex with dividing set $\Gamma$.

In the following, we will often consider two convex surfaces $\Sigma_1, \Sigma_2$ with dividing sets $\Gamma_1, \Gamma_2$ which meet along a common boundary $C$, forming a "corner". We require this common boundary to be legendrian. If so, the dividing sets $\Gamma_1, \Gamma_2$ must "interleave" along $C$, as shown on the left of figure 8. The number of intersections $|\Gamma_1 \cap C| = |\Gamma_2 \cap C|$ is precisely half $|tb(C)|$. We may round the corner to obtain a smooth surface. The dividing curves then behave as shown on the right of figure 8; we may think of the rule as "down and to the right; up and to the left".

### 2.3 Contact structures on balls and solid tori

The classification of tight contact structures on a ball is simple. We have already seen that on a convex boundary $S^2$, the contact structure in a neighbourhood is tight if and only if the dividing set $\Gamma$ is connected. It is a theorem of Eliashberg that in this case, the space of tight contact structures, fixed on the boundary sphere, is connected [7].

We next turn to tight contact structures $\xi$ on $(T, n)$, i.e. the solid torus with $2n$ longitudinal sutures. We can cut along a convex meridional disc $D$ with legendrian boundary, and obtain a 3-ball. Then on $D$, the dividing set $\Gamma_D$ gives a chord diagram. As it turns out, no matter what choice we take for our convex $D$, we obtain the same chord diagram: this follows from [21] or [23]. (This is not the case when the sutures are not longitudes; for other slopes, when $\Gamma$ has two components, the number of tight contact structures is related to the continued fraction expansion of the slope: see [20].)

Thus, the contact structure $\xi$ determines the chord diagram $\Gamma_D$. Conversely, the dividing set $\Gamma_D$ determines $\xi$ up to contact isotopy. Conversely, any chord diagram $\Gamma_D$ on $D$, taken as a dividing set, determines an $S^1$-invariant contact structure $\xi$ on the torus $D \times S^1 = T$. Hence there is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Chord diagrams} \\ \text{with } n \text{ chords} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Contact structures} \\ \text{on } (T, n) \end{array} \right\}$$

Moreover, the dividing set $\Gamma_D$ cuts $D$ into regions with signs, in the same way as noted above in section 1.1. The relative euler class $e(\xi)$ of $\xi$ takes the class of the meridional disc to the number obtained by summing the euler characteristics of these regions with sign. Since the regions are all discs, we simply add the number of positive regions and subtract the number of negative regions, giving the "relative euler class" described in 1.1.

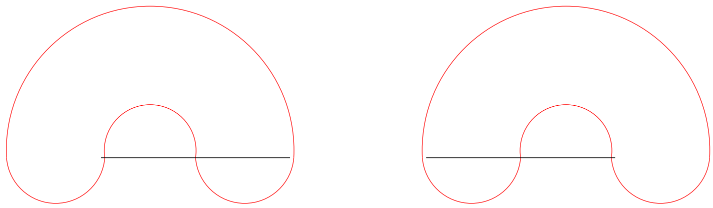


Figure 9: Possible attaching arcs on a tight $\partial B^3$. We view this as looking at $\partial B^3$ from the outside.

## 2.4 Bypasses

Our bypass moves come from actual contact geometric objects called *bypasses*. A bypass is "half an overtwisted disc"; attaching it to a surface along an attaching arc, above or below, and rounding gives the surgery on dividing sets we have called an upwards and downwards bypass move. See [20].

We will be interested in bypasses along an attaching arc on a tight 3-ball with convex boundary. We can ask two questions:

(i) After adding a bypass to the outside of the ball, does it remain tight? Since a bypass is "half an overtwisted disc", adding the bypass will often lead to an overtwisted contact structure. However in some cases we can guarantee it remains tight.

(ii) Does a bypass exist along our attaching arc, inside the manifold? If so, the manifold must remain tight; a submanifold of a tight contact manifold is again tight.

Topologically, an arc of attachment may be arranged in only two ways on a tight $S^2$ boundary: see figure 9.

In the first case, the answer to the first question is "yes" and the second question "no". To see this, we note that adding a bypass to the outside of the ball has no change on the effect of the topology of the dividing set; it is still connected. Thus, there is a tight contact structure on the enlarged ball with the bypass attached. Moreover, the specific contact structure on the enlarged ball, obtained by adding the bypass (with its standard contact structure) to the pre-existing tight contact structure on the ball, is again tight. This last sentence is an application of Honda's mischievously named "right to life principle", which essentially says that trivial bypass attachments (i.e. having trivial topological effect on the dividing set) have trivial effect on the contact structure [23, 26]. The second answer is "no", since if such a bypass existed, removing it would lead to a disconnected dividing set, contradicting the tightness of the contact structure.

In the second case we obtain precisely the opposite answers, and for the same reasons: "no" and "yes" respectively.

In the first case, we can call the arc of attachment *outer*, and in the second case *inner*.

# 3 Sutured Floer homology preliminaries

## 3.1 Brief overview of $SFH$

The theory of sutured Floer homology was introduced by Juhasz in [32]. It is an extension of Heegaard Floer homology theory [37, 38, 39, 40] for manifolds with boundary. We mention a few basic results of this theory, and refer to those papers for details and proofs.

Sutured Floer homology is an invariant of a *balanced sutured manifold*. Following the definitions of [32], a *sutured manifold* $(M, \mathbf{\Gamma})$ is a compact oriented 3-manifold with boundary $M$, with a set $\mathbf{\Gamma} \subset \partial M$ of disjoint annuli and tori on the boundary. The annuli in $\mathbf{\Gamma}$ have oriented core curves called *sutures*. Removing the set $\mathbf{\Gamma}$ from $\partial M$ breaks $\partial M - \mathbf{\Gamma}$ into connected components, which are oriented so that their boundaries agree with the sutures; in particular, orientations alternate on $\partial M$ as we cross a suture. The components of $\partial M - \mathbf{\Gamma}$ are given a sign, *positive* or *negative*, respectively as the normal vector defined by their orientation enters or exits $M$. The positive and negative components are denoted $R_+(\mathbf{\Gamma})$ and $R_-(\mathbf{\Gamma})$ respectively.

A sutured manifold $(M, \mathbf{\Gamma})$ is *balanced* if it satisfies the following conditions: $M$ has no closed components; $\chi(R_+(\mathbf{\Gamma})) = \chi(R_-(\mathbf{\Gamma}))$; and every boundary component of $M$ has an annular suture. (In particular, there are no toric sutures.)

As in from Heegaard Floer homology, we consider a Heegaard decomposition of our manifold. A *sutured Heegaard diagram* is a surface $\Sigma$ with boundary, with some simple closed curves $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_d\}$ drawn on it. The set $\alpha$ of $\alpha_i$ are pairwise disjoint; and the set $\beta$ of $\beta_i$ are pairwise disjoint.

From a sutured Heegaard diagram, one constructs a sutured manifold by taking $\Sigma \times I$ and gluing 2-handles to $\alpha_i \times \{0\}$ and $\beta_i \times \{1\}$. This is our 3-manifold with boundary $M$. The sutures are then defined by the annuli $\mathbf{\Gamma} = \partial\Sigma \times I$, with oriented core curves $\partial\Sigma \times \{1/2\}$. The balanced condition means that the following conditions hold: $|\alpha| = |\beta|$ (which we have already implicitly assumed); every component of $\Sigma \backslash \cup \alpha_i$ contains a boundary component of $\Sigma$; and every component of $\Sigma \backslash \cup \beta_i$ contains a boundary component of $\Sigma$. Every balanced sutured manifold has a sutured Heegaard diagram satisfying these conditions.

Again as in Heegaard Floer homology, one considers the symmetric product $\text{Sym}^d(\Sigma) = \Sigma^d / S_d$, where $d = |\alpha| = |\beta|$ and $S_d$ is the symmetric group acting by permuting coordinates. If $\Sigma$ has a complex structure, then so also does this $2d$-manifold. There are two totally real tori $\mathbb{T}_\alpha = (\alpha_1 \times \cdots \times \alpha_d)$ and $\mathbb{T}_\beta = (\beta_1 \times \cdots \times \beta_d)$ in $\text{Sym}^d(\Sigma)$.

Given a sutured Heegaard diagram $(\Sigma, \alpha, \beta)$, we consider a chain complex $CF(\Sigma, \alpha, \beta)$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, which correspond to $d$-tuples of intersection points of $\alpha_i$ and $\beta_i$, one on each $\alpha_i$ and one on each $\beta_i$. to define a differential $\partial$ on this complex, we consider holomorphic curves in $\text{Sym}^d(\Sigma)$ with certain boundary conditions.

First, we consider a more elementary notion. A *Whitney disc* from $\mathbf{x} \in \mathbb{T}_\alpha$ to $\mathbf{y} \in \mathbb{T}_\beta$ is a map $u$ from the unit disc $D \subset \mathbb{C}$ into $\text{Sym}^d(\Sigma)$ satisfying the following boundary conditions: $u(-i) = \mathbf{x}$; $u(i) = \mathbf{y}$; $u$ takes the "left side" of $\partial D$ (i.e. with real part $\leq 0$) into $\mathbb{T}_\alpha$; and $u$ takes the "right side" of $\partial D$ into $\mathbb{T}_\beta$. The set of homotopy classes of such discs is denoted $\pi_2(\mathbf{x}, \mathbf{y})$. We consider holomorphic discs which are Whitney discs.

We need to consider how our Whitney discs intersect the various regions of $\Sigma \backslash (\bigcup \alpha_i \cup \bigcup \beta_i)$. There is a well-defined intersection number with each such region, depending only on the homotopy class of a Whitney disc. We label the regions of $\Sigma \backslash (\bigcup \alpha_i \cup \bigcup \beta_i)$ as $D_1, \ldots, D_m$, and call linear combinations of the $D_i$ *domains*. To measure the intersection number with a region, we take a random point $z_i$ in each $D_i$; then a Whitney disc has a well-defined intersection number $n_{z_i}(u)$ with each $z_i$. This is the algebraic intersection number of $u$ with $\{z_i\} \times \text{Sym}^{d-1}(\Sigma)$, and it depends only on the homotopy class of $u$ and the component $D_i$ in which $z_i$ lies. The *domain of* $u$ is then $D(u) = \sum n_{z_i}(u) D_i$; since it depends only on the homotopy class of $u$, we may speak of $D(\phi)$, where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. When we have a Whitney disc connecting $\mathbf{x}$ to $\mathbf{y}$, its domain $D$ has boundary which (algebraically) runs from $\mathbf{x}$ to $\mathbf{y}$ along every $\alpha_i$ and $\beta_i$: that is, $\partial(\partial D \cap \alpha_i) = (\mathbf{x} \cap \alpha_i) - (\mathbf{y} \cap \alpha_i)$ and $\partial(\partial D \cap \beta_i) = (\mathbf{x} \cap \beta_i) - (\mathbf{y} \cap \beta - i)$. A domain $D$ satisfying these two equalities is called a *domain connecting* $\mathbf{x}$ *to* $\mathbf{y}$; the set of such domains is called $D(\mathbf{x}, \mathbf{y})$.

For a domain $D$ connecting $\mathbf{x}$ to $\mathbf{y}$, we may define $\mathcal{M}(D)$ to be the moduli space of holomorphic Whitney discs with domain $D$. Our differential will count index-1 families of such discs. Since there is an $\mathbb{R}$-action on $D$ preserving $-i$ and $i$, we may quotient an index-1 family of curves by this action and obtain $\widehat{\mathcal{M}}(D)$. Our differential $\partial$ on $CF(\Sigma, \alpha, \beta)$ is then defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{D \in D(\mathbf{x}, \mathbf{y}) \\ \text{Index}=1}} \# \hat{\mathcal{M}}(D) \, \mathbf{y}.$$

Our sutured Heegaard diagram is also required to be *admissible*. An admissible diagram is one for which every periodic domain has positive and negative coefficients; a domain is *periodic* if its boundary is a sum of closed curves $\alpha_i$ and $\beta_i$ (i.e. each arc in each $\alpha_i$ and $\beta_i$ covered an equal number of times). One can show that any balanced sutured Heegaard diagram is isotopic to an admissible one. Admissibility guarantees that the set of positive domains (i.e. having all coefficients positive) connecting $\mathbf{x}$ to $\mathbf{y}$ is finite: since holomorphic discs have positive domains by positivity of intersection, this means that the above sum is finite.

One can show, using Gromov compactness, that $\partial^2 = 0$. The homology of the complex is called *sutured Floer homology* $SFH(M, \mathbf{\Gamma})$; one can show that it is invariant of the choice of admissible balanced sutured Heegaard diagram.

## 3.2 Spin-c structures

We will now briefly explain how $SFH$ splits as a direct sum over spin-c structures.

For our purposes, again following [32], we can regard a spin-c structure on $(M, \mathbf{\Gamma})$ as a vector field, satisfying certain boundary conditions, up to homotopy relative to the boundary in the complement of a ball.

More precisely, we require a vector field on $(M, \mathbf{\Gamma})$ to point out of $M$ along $R_+(\mathbf{\Gamma})$, in along $R_-(\mathbf{\Gamma})$, and along the annuli $\mathbf{\Gamma}$ to be tangent to $\partial M$, as the gradient of the height function $S^1 \times I \longrightarrow I$. We will say such a vector field is *compatible with* $\mathbf{\Gamma}$. Two vector fields $v, w$ on $M$, compatible with $\mathbf{\Gamma}$, are said to be *homologous* if there exists an open ball $B$ in the interior of $M$ such that $v, w$ are homotopic in $M - B$.

A *spin-c structure* on $(M, \mathbf{\Gamma})$ is a homology class of vector fields on $M$ compatible with $\mathbf{\Gamma}$. We call the set of such homology classes $\text{Spin}^c(M, \mathbf{\Gamma})$. And given $\mathfrak{s} \in \text{Spin}^c(M, \mathbf{\Gamma})$, its *first chern class* $c_1(\mathbf{s})$ is defined as follows. Take a vector field $v$ representing $\mathfrak{s}$, and form the perpendicular 2-plane field $v^\perp$. Then $c_1(\mathbf{s}) = c_1(v^\perp) \in H^2(M; \mathbb{Z})$.

Note that $c_1(\mathbf{s})$ cannot be any element of $H_2(M; \mathbb{Z})$; since $v$ is compatible with $\mathbf{\Gamma}$, $c_1(v^\perp)$ restricts to a particular homology class in $H^2(\partial M; \mathbb{Z})$. Since $c_1$ only depends only on the homotopy class of $v^\perp$ over a 2-skeleton, altering $v^\perp$ inside a ball has no effect, and homotopy outside a ball has no effect either; so that $c_1(\mathbf{s})$ is well defined.

Each generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of $CF(\Sigma, \alpha, \beta)$ gives a spin-c structure $\mathfrak{s}(\mathbf{x})$ as follows. The sutured Heegaard diagram gives a Morse function $f : M \longrightarrow \mathbb{R}$ for which $\Sigma$ is a level set and the $\alpha, \beta$ curves are intersections of stable and unstable manifolds with this level set. Moreover, $f$ is easily chosen so that $\text{grad} f$ is compatible with $\mathbf{\Gamma}$. The point $\mathbf{x}$ is a $d$-tuple of intersection points of $\alpha_i$ and $\beta_i$ curves, one on each curve; and each such intersection point corresponds to a trajectory $\gamma_{\mathbf{x}}$ between an index one and index two critical point of $f$. Thus $\mathbf{x}$ gives a set of pairwise disjoint trajectories of $\text{grad} f$ connecting all the index one and index two critical points of $f$ in pairs. We may modify $\text{grad} f$ on a neighbourhood of each of these trajectories to give a nowhere zero vector field $v$. Then $\mathfrak{s}(\mathbf{x})$ is the spin-c structure represented by $v$.

Given two points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we may join the corresponding trajectories as $\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}$ to obtain a collection of oriented simple closed curves in $M$, which we denote $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(M; \mathbb{Z})$. We can then prove that $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(M; \mathbb{Z})$ is Poincaré dual to $\mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y})$. Note that $\epsilon(\mathbf{x}, \mathbf{y})$ can be homotoped to lie entirely on the $\alpha$ and $\beta$ curves, and hence corresponds to some curves in $\mathbb{T}_\alpha \cup \mathbb{T}_\beta \subset \text{Sym}^d(\Sigma)$. We note that $H_1(\Sigma; \mathbb{Z}) = H_1(M; \mathbb{Z})$ under inclusion, and we may regard $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(\Sigma; \mathbb{Z})$ also.

Now, if $\mathfrak{s}(\mathbf{x}) \neq \mathfrak{s}(\mathbf{y})$ then $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0 \in H_1(\Sigma; \mathbb{Z})$; this curve is not a boundary. But any Whitney disc connecting $\mathbf{x}$ to $\mathbf{y}$ must have such a boundary; so there are no Whitney discs in this case, and in particular, no holomorphic Whitney discs.

Thus, the differential $\partial$ on $CF(\Sigma, \alpha, \beta)$ takes $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ having the same spin-c structure. Hence $SFH$ splits as a sum over spin-c structures:

$$SFH(M, \mathbf{\Gamma}) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(M, \mathbf{\Gamma})} SFH(M, \mathbf{\Gamma}, \mathfrak{s}).$$

### 3.3 Contact elements and TQFT

We now briefly explain how a contact structure $\xi$ on $(M, \mathbf{\Gamma})$ gives rise to an element of $SFH(-M, -\mathbf{\Gamma})$. If we use $\mathbb{Z}$-coefficients, there is a $(\pm 1)$ ambiguity; but with $\mathbb{Z}_2$ coefficients the contact element is a well-defined single element. Here we follow the definition of Honda–Kazez–Matić in [28], and we only consider $\mathbb{Z}_2$ coefficients. It is an extension of the definition of a contact class in the Heegaard Floer homology of a closed manifold, as defined in [39] and reformulated in [29].

The central concept in this definition is that of a *partial open book decomposition* $(S, R_+(\mathbf{\Gamma}), h)$ of a sutured manifold $(M, \mathbf{\Gamma})$. Here $S$ is a surface with boundary, $R_+(\mathbf{\Gamma}) \subset S$ is a subsurface and $h$ is the *partial monodromy map*, $h : S - \overline{R_+(\mathbf{\Gamma})} \longrightarrow S$. We first consider $S \times [-1, 1]/\sim$, thickening $S$ and taking the quotient by the relation $\sim$, which identifies all boundary points $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 1]$: this is the "binding" of the open book.

The manifold $M$ is then given by gluing $(x, 1)$ to $(h(x), -1)$, for $x \in S - \overline{R_+(\mathbf{\Gamma})}$. This manifold has boundary consisting of $R_+(\mathbf{\Gamma}) \times \{1\}$, which becomes $R_+(\mathbf{\Gamma})$ in the sutured manifold; also $(S - \mathrm{Im}\,(h)) \times \{-1\}$, which becomes $R_-(\mathbf{\Gamma})$ in the sutured manifold; and their boundaries $\partial(R_+(\mathbf{\Gamma})) \times \{1\}$, $\partial((S - \mathrm{Im}\,(h)) \times \{-1\}$, which have been glued together, form the sutures.

Now, we recall Giroux's theorem [17] (see also [9]) that isotopy classes of contact structures on a (closed) 3-manifold correspond precisely to open book decompositions modulo positive stabilization. In their paper [28], Honda–Kazez–Matić extend this result to sutured manifolds and partial open books.

Thus, given a contact structure on $(M, \mathbf{\Gamma})$, we take a corresponding partial open book decomposition. A partial open book decomposition then gives rise to a sutured Heegaard splitting along a surface $\Sigma$, obtained by gluing together two "opposite pages" of the partial open book. (In this partial open book, however, two opposite pages will not usually be homeomorphic surfaces.) We take a *basis* for $(S, R_+(\mathbf{\Gamma}))$, which is a set of pairwise disjoint properly embedded arcs $a_i$ in $S - \overline{R_+(\mathbf{\Gamma})}$, so that after cutting $S$ along the $a_i$, the surface deformation retracts to $\overline{R_+(\mathbf{\Gamma})}$. From such a basis, it is possible to obtain some $\alpha$ and $\beta$ curves on our sutured Heegaard surface, giving rise to a balanced sutured Heegaard diagram. For $\alpha$ and $\beta$ curves obtained by their method, there is a canonical pairing between $\alpha_i$ and $\beta_i$ curves, and there is a canonical intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of each pair near the boundary of $S$.

In this construction, it turns out that $\partial\Sigma$ and $\partial(R_+(\mathbf{\Gamma}))$, which are equal as sets, have opposite orientation. Thus we end up with $\mathbf{x} \in CF(\Sigma, \beta, \alpha)$, rather than $CF(\Sigma, \alpha, \beta)$.

It is then shown that this point $\mathbf{x}$ satisfies $\partial \mathbf{x} = 0$, so we may consider $\mathbf{x}$ as an element of $SFH$. However, because of the orientation issue, it turns out that $\mathbf{x} \in SFH(-M, -\mathbf{\Gamma})$. Moreover, for a different choice of partial open book decomposition or basis curves, this element behaves naturally under corresponding isomorphisms of $SFH$. Thus we may speak of *the contact element* $c(\xi) \in SFH(-M, -\Gamma)$.

The contact class is known to satisfy various properties, also noted in [28]: for instance, $c(\xi) = 0$ when $\xi$ is overtwisted, or when the partial monodromy $h$ is not "right-veering".

In [30], Honda–Kazez–Matić proved that $SFH$ has some of the properties of a topological quantum field theory. We give a $\mathbb{Z}_2$ version.

**Theorem 3.1 (Honda–Kazez–Matić [30])** *Let $(M', \mathbf{\Gamma}')$ be a sutured submanifold of $(M, \mathbf{\Gamma})$, and let $\xi$ be a contact structure on $(M - Int(M'), \mathbf{\Gamma} \cup \mathbf{\Gamma}')$. Let $M - Int(M')$ have $m$ components which are isolated, i.e. components which do not intersect $\partial M$. Then $\xi$ induces a natural map*

$$\Phi_\xi : SFH(-M', -\mathbf{\Gamma}') \longrightarrow SFH(-M, -\Gamma) \otimes V^m,$$

*where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \widehat{HF}(S^1 \times S^2)$. This map has the property that for any contact structure $\xi'$ on $(M', \mathbf{\Gamma}')$,*

$$\Phi_\xi\left(c(\xi')\right) = c(\xi' \cup \xi) \otimes x^{\otimes m},$$

*where $x$ is the contact class of the standard tight contact structure on $S^1 \times S^2$.*

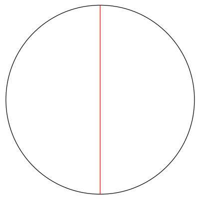


Figure 10: Chord diagram with 1 chord.

### 3.4 Solid tori

Juhasz [33] and Honda–Kazez–Matić [28, 30] have proved theorems to calculate $SFH$ when two sutured manifolds are glued together in certain ways. One immediate corollary, given in [30], is a computation of $SFH$ for solid tori with longitudinal sutures. We have seen in the introduction that

$$SFH(T,n) = \mathbb{Z}_2^{2^{n-1}}$$

and the direct sum over spin-c structures is

$$SFH(T,n) = \mathbb{Z}_2^{\binom{n-1}{0}} \oplus \mathbb{Z}_2^{\binom{n-1}{1}} \oplus \cdots \oplus \mathbb{Z}_2^{\binom{n-1}{n-1}} = \bigoplus_e SFH(T,n,e)$$

The relative euler class of the chord diagram, or contact structure, corresponds precisely to these summands [28, 30, 33]. Recall that contact structures on $(T,n)$ are in bijective correspondence with chord diagrams of $n$ chords; and the relative euler class of the contact structure is the relative euler class of the chord diagram. Thus, a chord diagram $\Gamma$ with $n$ chords and euler class $e$ gives rise to an element of $SFH(T,n,e)$. This is theorem 1.8 from section 1.2 in our overview.

## Part II
# First steps

## 4 First observations in $SFH$

### 4.1 The vacuum

Let us begin by considering the case of $(T,1)$. We have $SFH(T,1) = SFH(T,1,0) = \mathbb{Z}_2$. Contact elements in $SFH(T,1)$ correspond to chord diagrams with 1 chord. There are not many of these! See figure 10.

As we have seen, this corresponds to an $S^1$-invariant contact structure on $D^2 \times S^1$ with boundary dividing set / sutures consisting of two longitudinal curves. It is the unique tight contact structure on this sutured manifold; and it is a standard neighbourhood of a closed legendrian curve. Such a contact manifold can be contact embedded in the standard contact $S^3$; or even in a slightly smaller sutured manifold $S^3 - B^3 = B^3$ with one suture; it is generally true that $\widehat{HF}(M) = SFH(M - B^3, \mathbf{\Gamma})$ where $\mathbf{\Gamma}$ is a single curve on the sphere. In either of these cases, the contact element for the standard contact structure is nonzero; by Stein fillability of $S^3$, for instance. By TQFT-inclusion, we have the following lemma.

**Lemma 4.1** *The contact element of the unique tight contact structure on* $(T,1)$ *is the nonzero element* $v_\emptyset \in \mathbb{Z}_2 = SFH(T,1)$. ∎

We think of this contact element $v_\emptyset$ as "the vacuum", in our tenuous quantum field theory interpretation. The vacuum state in quantum field theory is not zero.

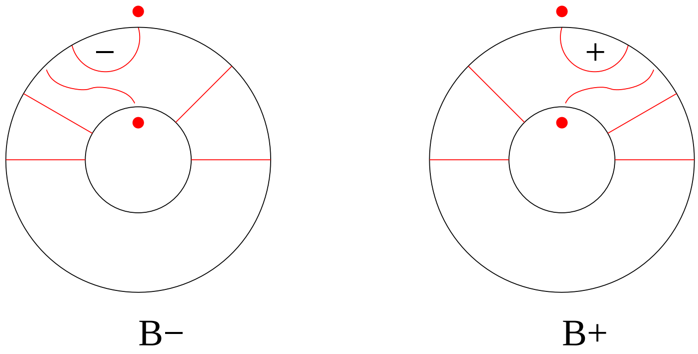



Figure 11: Inclusion of sutured manifolds for $B_\pm$.

## 4.2 Creation and annihilation

We have defined orientation and sign conventions on chord diagrams in section 1.1. We now define creation and annihilation operators properly.

We consider an embedding of $(T, n)$ in $(T, n+1)$, and a contact structure on $(T, n+1) - (T, n)$, in order to use TQFT-inclusion. Such an embedding is given by embedding a disc inside a larger disc, all times $S^1$. On the intermediate manifold $(T, n+1) - (T, n)$, which is an annulus times $S^1$ with $2n+2$ longitudinal sutures "on the outside" and $2n$ longitudinal sutures "on the inside", we can specify an $S^1$-invariant contact structure by drawing a dividing set on the annulus, which is assumed to be convex. We use the following two dividing sets, respectively for positive and negative creation operators. Note we must mark basepoints; recall these are denoted by a solid red dot.

**Definition 4.2 (Creation operators)** *The* ±-creation operators *are the* $\mathbb{Z}_2$*-vector space maps*

$$B_-, B_+ : SFH(T, n) \longrightarrow SFH(T, n+1)$$

*given by TQFT-inclusion, from the inclusion* $(T, n) \hookrightarrow (T, n+1)$ *together with the contact structures on* $(T, n+1) - (T, n)$ *described by the dividing sets given in figure 11.*

Given a contact structure on $(T, n)$, described by a chord diagram $\Gamma$ of $n$ chords, applying $B_\pm$ to its contact element gives a chord diagram with $(n+1)$ chords, as described in the introduction.

Moreover, if $\Gamma$ has relative euler class $e$, then after applying $B_\pm$, we have a chord diagram with euler class $e \pm 1$. So $B_\pm$ takes contact elements in $SFH(T, n, e)$ to contact elements in $SFH(T, n+1, e \pm 1)$.

As an aside, note that $B_+$ and $B_-$ "create" an extra chord by the "creation" of an extra piece on our solid torus. But they can also be viewed as "creating" an extra chord by "annihilating" part of the manifold as shown in figure 12. This is a direct proof that they take tight contact structures to tight contact structures.

Similarly, we can define annihilation maps corresponding to a similar inclusion of sutured manifolds $(T, n+1) \hookrightarrow (T, n)$, with certain dividing sets specified on an annulus in $(T, n) - (T, n+1)$.

**Definition 4.3 (Annihilation operators)** *The* ±-annihilation operators *are the* $\mathbb{Z}_2$*-vector space maps* $A_+, A_- : SFH(T, n+1) \longrightarrow SFH(T, n)$ *given by TQFT-inclusion, from the inclusion* $(T, n+1) \hookrightarrow (T, n)$ *together with the contact structures on* $(T, n) - (T, n+1)$ *described by the dividing sets given in figure 13.*

It's clear that $A_\pm$ has the effect on chord diagrams described in the introduction. And $A_\pm$ takes contact elements in $SFH(T, n+1, e)$ to contact elements in $SFH(T, n, e \pm 1)$.

Note that if $A_+$ is applied to a chord diagram with an outermost positive region at the base point, then we do not obtain a chord diagram, but a diagram with a closed loop. The corresponding contact structure is overtwisted, and the contact element is zero.

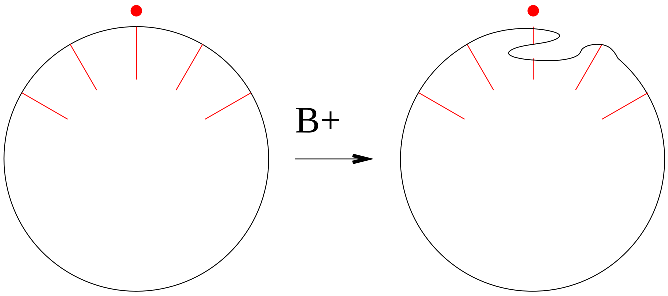



Figure 12: "Creating by annihilating".

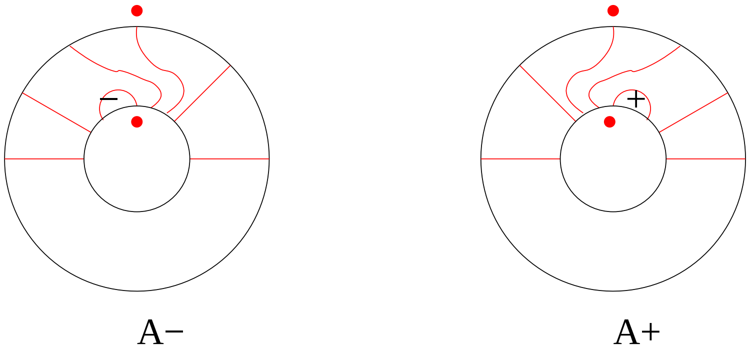



Figure 13: Inclusion of sutured manifolds for $A_\pm$.

Again as an aside, $A_+$ and $A_-$ actually "annihilate" a chord by the "creation" of an extra piece on our solid torus. (While we can "create by creating" and "create by annihilating", we can only "annihilate by creating" — we cannot "annihilate by annihilating".)

Note that there are actually "creation" and "annihilation" operators which we can consider, not just near the basepoint, but in any specific location. We will refer specifically to some of these later; for now we note that they exist.

It's also now clear that the creation and annihilation effects have the relations $A_+ \circ B_- = A_- \circ B_+ = 1$ and $A_+ \circ B_+ = A_- \circ B_- = 0$, when applied to contact elements. Just place these figures of annuli together.

## 4.3 Nontriviality and uniqueness

We can now see that contact classes are distinct and nonzero; this argument also appeared in Honda–Kazez–Matić [30].

**Lemma 4.4** *Any tight contact structure $\xi$ on $(T, n)$, corresponding to a chord diagram $\Gamma$, has nonzero contact element $c(\xi)$.*

Proof For any such contact element $c(\xi)$, corresponding to a chord diagram $\Gamma$, at least one of the annihilation operators $A_+$ or $A_-$ reduces it to a chord diagram with fewer chords (i.e. at most one of these can create a closed loop). By repeatedly applying annihilation operators in this way we may reduce the chord diagram to one chord, i.e. the vacuum $v_\emptyset \neq 0$. The composition of these annihilation operators is a linear map which takes the $c(\xi)$ to $v_\emptyset \neq 0$. Hence $c(\xi) \neq 0$. ■

The following argument also appeared in [30]: this is proposition 1.10.

**Proposition (Contact elements are distinct)** *For any two distinct dividing sets $\Gamma_1, \Gamma_2$ consisting of $n$ disjoint properly embedded chords, the corresponding contact elements are distinct.*

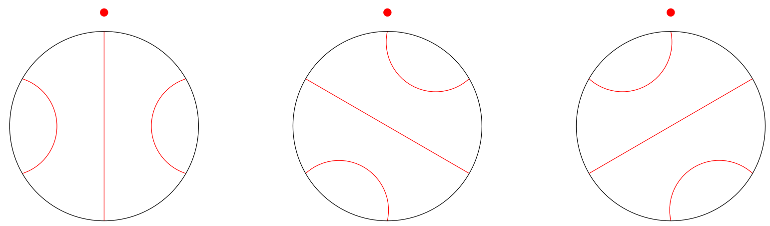


Figure 14: Chord diagrams in $SFH(T,3,0)$.

PROOF There is a sequence of annihilation operators which reduces $\Gamma_1$ to the vacuum state but which, when applied to $\Gamma_2$, at some point creates a closed curve in the dividing set. These annihilation operators might not be applied in the positions of $A_+$, $A_-$, but may be at other positions; we have noted that there is nothing special about annihilating at the base point. The composition of these operators takes the contact element $c(\Gamma_1)$ to $v_\emptyset$ but takes $c(\Gamma_2)$ to 0; hence they cannot be equal. ■

**Remark 4.5 (Chord diagrams / contact elements: lax notation)** *We will often denote by $\Gamma$ a chord diagram, or its corresponding contact element, and drop the notation $c(\xi)$ or the already lax $c(\Gamma)$. The meaning should be clear.*

## 4.4 Bypasses and addition

The simplest case for which there is more than one chord diagram is 3 chords and $e = 0$. Since $SFH(T,3,0) = \mathbb{Z}_2^2$ and $C_3^0 = 3$, there are 3 chord diagrams giving 3 distinct elements of $\mathbb{Z}_2^2$: see figure 14.

These three dividing sets are related by bypass moves, as described in section 1.1. They form the simplest possible bypass triple.

The 3 nonzero elements of $\mathbb{Z}_2^2$ have the property that the sum of any two of them is equal to the third; or equivalently in mod 2 arithmetic, the sum of all three is zero. Hence the 3 contact elements have the same property. Thus in this case, bypasses do mean addition. The result in general follows from TQFT-inclusion.

**Proposition 4.6** *Suppose that $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple. Then $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$.*

*Conversely, suppose three chord diagrams $\Gamma_1, \Gamma_2, \Gamma_3$ satisfy $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. Then $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple.*

PROOF Suppose $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple in $SFH(T,n,e)$. Then they are obtained by taking the three chord diagrams of 3 chords in figure 14, and adding an annulus to the outside of the disc containing the same diagram. This gives an inclusion of one solid torus inside another, with a contact structure specified in the intermediate region, and hence by TQFT-inclusion we obtain a linear map $\mathbb{Z}_2^2 = SFH(T,3,0) \longrightarrow SFH(T,n,e)$. Since the sum of the three contact classes in $SFH(T,3,0)$ is zero, after applying this linear map, the sum of $\Gamma_1, \Gamma_2, \Gamma_3$ is zero also.

For the converse: proof by induction on the number of chords $n$. For $n = 3$ it is clear. Note that if three chord diagrams sum to zero then they all have the same relative euler class. Furthermore, if the $\Gamma_i$ then they are all distinct.

We use the following fact: given any two distinct chord diagrams with $n \geq 3$ chords and the same relative euler class, there exists an annihilation operator, annihilating at some location (possibly not at the base point), that creates no closed curves on either. If annihilating at every position creates a closed curve on at least one of the diagrams, then the two chord diagrams consist entirely of outermost chords, enclosing all positive regions on one chord diagram, and enclosing all negative regions on the other. Thus the chord diagrams have distinct relative euler class, a contradiction.

Applying this to $\Gamma_1$ and $\Gamma_2$, we find an annihilation operator $A$ which reduces the number of chords by 1, and such that $A\Gamma_1, A\Gamma_2$ are nonzero. If $A\Gamma_1$ and $A\Gamma_2$ are distinct then from linearity of $A$ and

$\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ we find that $A\Gamma_3$ is also nonzero; and thus we have reduced to a smaller case and are done by induction. If $A\Gamma_1$ and $A\Gamma_2$ are equal, then we have the situation that an annihilation operator takes two distinct dividing sets and outputs the same dividing set. It follows that the situation must be as in the diagram below; and hence $\Gamma_1, \Gamma_2$ are related by a bypass. Annihilating all the other chords, we reduce to the 3-arc case and we are done. ■

This proposition is simply a reformulation of proposition 1.9, which is now also proved. So, the set of contact elements does not form an additive subgroup; but the extent to which it is closed under addition precisely describes the existence of bypasses.

## 4.5 The basis

We now show that the elements $v_w$, for $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$. Recall $v_w$ is obtained from applying $B_\pm$ to $v_\emptyset \in SFH(T, 1, 0)$ repeatedly, according to the word $w$.

We can now prove proposition 1.17:

**Proposition (QFT basis)** *The $v_w$ for $w \in W(n_-, n_+)$ form a basis for $SFH(T, n+1, e)$.*

Proof First we show the $v_w$ are linearly independent. For this suppose that some $v_{w_1} + \cdots + v_{w_j} = 0$. Then, to this sum apply a sequence of annihilation operators which undoes the creation operators in the definition of $w_1$. The composition $A$ of these operators takes $e_{w_1}$ to the vacuum $v \neq 0$ and every other $v_{w_i}$ to 0; hence $A(v_{w_1} + \cdots + v_{w_j}) = v_\emptyset = 0$, a contradiction.

The number of $v_w$ is the number of $w \in W(n_-, n_+)$, which is $\binom{n}{k}$, which is the dimension of $SFH(T, n+1, e)$. Hence they form a basis. ■

Moreover, since we now know that contact elements span every $SFH(T, n+1, e)$, we know that the creation operators can be refined to maps $B_\pm : SFH(T, n, e) \longrightarrow SFH(T, n+1, e \pm 1)$ and the annihilation operators to maps $A_\pm : SFH(T, n+1, e) \longrightarrow SFH(T, n, e \pm 1)$; previously we only knew such refinements on relative euler classes could be done on that part of $SFH$ spanned by contact elements. Also, the relationships $A_+ \circ B_- = A_- \circ B_+ = 1$ and $A_+ \circ B_+ = A_- \circ B_- = 0$ are true for all $SFH(T, n, e)$ as well.

So we have proved proposition 1.13 on annihilation operators. We can also prove proposition 1.12.

**Proposition (Categorification of Pascal recursion)** *$B_\pm$ is injective and*

$$SFH(T, n+1, e) = B_+ SFH(T, n, e-1) \oplus B_- SFH(T, n, e+1).$$

Proof The basis of $SFH(T, n+1, e)$ consists of elements $v_w$. If $w$ begins with a $+$, $w = +w'$, then $v_w = B_+ v_{w'} \in B_+ SFH(T, n, e-1)$. If $w$ begins with a $-$, $w = -w'$, then $v_w = B_- v_{w'} \in B_- SFH(T, n, e+1)$. ■

Given any chord diagram / contact element, there is a simple algorithm (actually several simple algorithms!) to determine its decomposition as a sum of basis elements. This will be described in detail in section 6. Essentially, there is either an outermost region at the base point, or there is not. If there is an outermost region, then we can factor out a $B_\pm$ and reduce to a smaller chord diagram. If there is no such outermost region, then we can perform upwards and downwards bypass moves near the basepoint to write our chord diagram a sum of two dividing sets, each of which containing an outermost region at the base point. We can proceed in this way until we reach the vacuum; and at this point we have our decomposition.

We illustrate with an example: see figure 15.

We can now prove proposition 1.11.

**Proposition (SFH is combinatorial)** *There is an isomorphism*

$$SFH_{comb}(T, n, e) \cong SFH(T, n, e).$$

*This isomorphism takes a chord diagram to the contact element of the tight contact structure on $(T, n)$ with that chord diagram as its dividing set on a meridional disc.*

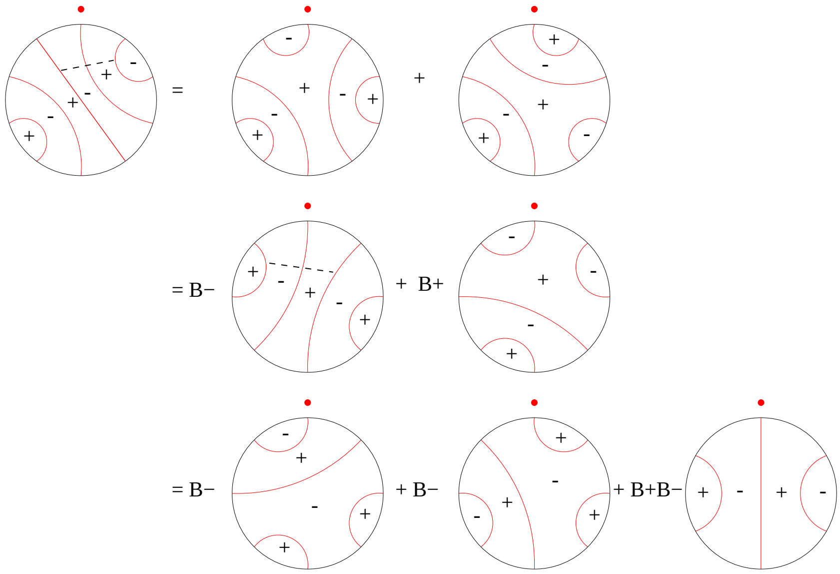



Figure 15: Decomposition of a basis element. From here we have $B_- B_- B_+ B_+ v_\emptyset$, $B_- B_+ B_+ B_- v_\emptyset$, $B_+ B_- B_- B_+ v_\emptyset$ and $B_+ B_- B_+ B_- v_\emptyset$, hence $v_{--++} + v_{-++-} + v_{+--+} + v_{+-+-}$.

Proof Recall that $SFH_{comb}(T, n, e)$ is defined as the $\mathbb{Z}_2$-vector space generated by appropriate chord diagrams, say $\mathbb{Z}_2[V]$, modulo bypass relations; let the subspace of $\mathbb{Z}_2[V]$ generated by bypass relations be $\mathcal{B}$. There is certainly a map $\phi : \mathbb{Z}_2[V] \longrightarrow SFH(T, n, e)$, taking chord diagrams to the corresponding contact elements in $SFH$. Moreover, this map takes $\mathcal{B} \mapsto \{0\}$, and so descends to a map $\phi : SFH_{comb} \longrightarrow SFH$. Since $SFH$ is generated by chord diagrams, $\phi$ is surjective. Now in $SFH_{comb}(T, n, e)$, the basis of $SFH$ is not necessarily a basis (yet!), but it is a spanning set: every chord diagram can be decomposed, using the bypass relation, as a sum of chord diagrams $\Gamma_w$ for words $w \in W(n_-, n_+)$. Thus the dimension of $SFH_{comb}$ is $\leq \binom{n}{k}$, while the dimension of $SFH$ is $\binom{n}{k}$. However $\phi$ is surjective, so the dimension of $SFH$ must actually be $\binom{n}{k}$ and $\phi$ must be an isomorphism. ∎

## 4.6 The octahedral axiom

We now pause briefly to return to the example of $SFH(T, 4, -1) = \mathbb{Z}_2^3$ discussed in section 1.1. Recall we have 6 chord diagrams, 3 basis elements $v_{--+}$, $v_{-+-}$ and $v_{+--}$, and the 6 contact elements are

$$v_{--+}, \quad v_{-+-}, \quad v_{+--}, v_{--+} + v_{-+-}, \quad v_{--+} + v_{+--}, \quad v_{-+-} + v_{+--}$$

Thus the 6 contact elements in $\mathbb{Z}_2^3$ consist of the 3 basis elements and all sums of them in pairs. We may consider the elements of $\mathbb{Z}_2^3$ as the vertices of a cube with coordinates $(x, y, z)$, with planes through the origin corresponding to 2-dimensional subspaces and lines through the origin corresponding to 1-dimensional subspaces. We see that each 2-dimensional subspace generated by two basis elements (with equations $x = 0$, $y = 0$ and $z = 0$) contains 3 contact elements which form a bypass triple; and also the subspace $x + y + z = 0$. Thus, the 6 vertices of the cube which are contact elements contain between them 4 triangles which are bypass triples. We can thus take these 6 vertices and arrange them as an octahedron; then 4 of the 8 faces are exact. This is the arrangement which appears in the octahedral axiom of Honda [19].

One can think of every $SFH(T, n, e)$ as containing contact elements which describe some higher-order version of the octahedral axiom.

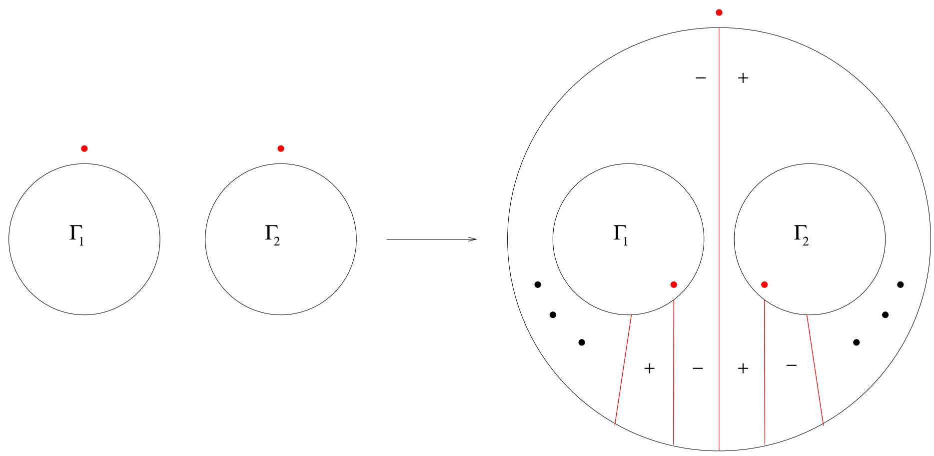



Figure 16: Merging operation.

# 5 Enumerative combinatorics

In this section we collect some enumerative results that we will need later.

## 5.1 Catalan, Narayana, and merging

We will not discuss much of the combinatorics in these numbers, and limit ourselves to what we need. We will define the Catalan numbers recursively by $C_0 = 1$, $C_1 = 1$ and then

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0. \tag{1}$$

We will define our (shifted) Narayana numbers $C_{n+1,k} = C^e_{n+1}$ recursively by $C^0_1 = 1$ and

$$C^e_{n+1} = C^{e-1}_n + C^{e+1}_n + \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C^e_{n_1} C^e_{n_2} \tag{2}$$

We will show that the number of chord diagrams with $n$ chords satisfies the Catalan recursion, and the number of chord diagrams with $n$ chords and relative euler class $e$ satisfies the Narayana recursion. The initial values are clearly right.

One easy way to see this recursion is a *merging operation*. Given two chord diagrams $\Gamma_1, \Gamma_2$ with $n_1, n_2$ chords and relative euler classes $e_1, e_2$, we can combine them into a larger chord diagram with $n_1 + n_2 + 1$ chords, and relative euler class $e_1 + e_2$, as shown in figure 16.

Note the specification of base points. The merging operation is also defined when $n_1 = 0$ or $n_2 = 0$; in this case the chord diagram added on one side is null; and the operation reduces to the effect of $B_+$ or $B_-$. It's easy to see that any given chord diagram with at least 2 chords can be expressed as the merge of two smaller (possibly null) chord diagrams, and in precisely one way.

Counting the number of chord diagrams of $n$ chords without regard to relative euler class gives the Catalan recursion, where each term $C_k C_{n-k}$ counts the number of merged chord diagrams with $k$ chords in the left diagram and $n-k$ in the right. Doing the same thing but keeping track of relative euler class, we obtain the Narayana recursion.

And it is now clear from this interpretation that

$$C_n = \sum_e C^e_n.$$

We have now proved proposition 1.18.

We further note that the "merging" operation precisely describes an inclusion of sutured manifolds — in this case, $(T, n_1) \sqcup (T, n_2) \to (T, n_1 + n_2 + 1)$ — together with a contact structure on the intermediate manifold $(T, n_1 + n_2 + 1) - ((T, n_1) \sqcup (T, n_2))$. Thus TQFT-inclusion applies. Note $SFH((T, n_1) \sqcup (T, n_2)) = SFH(T, n_1) \otimes SFH(T, n_2)$.

**Definition 5.1 (Merge operator)** *The inclusion of sutured manifolds, and contact structure on the intermediate manifold, described by the merging of two chord diagrams as above, gives a linear map*

$$M : SFH(T, n_1) \otimes SFH(T, n_2) \longrightarrow SFH(T, n_1 + n_2 + 1)$$

*which restricts to a map*

$$M : SFH(T, n_1, e_1) \otimes SFH(T, n_2, e_2) \longrightarrow SFH(T, n_1 + n_2 + 1, e_1 + e_2)$$

*on each summand.*

Having defined $M$ in this way, we see that $M$ restricts to $B_\pm$ when $n_1 = 0$ or $n_2 = 0$. And our counting argument makes clear that every contact element in $SFH(T, n, e)$ can be written in precisely one way as $M$ of two contact elements. Since every contact element in $SFH(T, n, e)$ lies in the image of $M$, and contact elements span $SFH(T, n, e)$, then the sum of $M$ applied to all appropriate $SFH(T, n_i, e_i)$, we must obtain all of $SFH(T, n, e)$. This proves proposition 1.19.

## 5.2 The baseball interpretation of $\preceq$

We will give a sporting interpretation of our partial order, which will be useful later.

**Definition 5.2 (Word as baseball score)** *Given a word $w \in W(n_-, n_+)$, call the $m$'th symbol from the left the $m$'th* innings. *Call the sum of the first $m$ symbols the* score after $m$ innings.

**Lemma 5.3 (Baseball interpretation of $\preceq$)** *Take $w_0, w_1 \in W(n_-, n_+)$. The relation $w_0 \preceq w_1$ is true if and only if after every innings, $w_1$ has a score higher than (or equal to) $w_0$.*

This is a low-scoring version of baseball: in every innings, each team scores $\pm 1$ run. If is also a fixed version of baseball: since $w_0, w_1 \in W(n_-, n_+)$, the scores at the end of the game, after all $n$ innings, are equal. In any case, an innings where the lead changes from one team to the other is precisely the case when the corresponding words are not comparable; these are uninteresting as spectator sport, unfortunately. Two words are comparable only if they describe a low-scoring, fixed, and uninteresting baseball game!

PROOF If $w_0 \preceq w_1$, then $w_1$ can be obtained from $w_0$ by performing the operation of replacing a substring $-+$ with $+-$, in various locations, finitely many times. Every time we perform that operation, we increase $w_1$'s lead over $w_0$ for one innings where previously the lead did not increase. After performed all the operations, we have $w_1$ ahead (or equal) at every innings.

Suppose conversely that $w_0$ does not precede $w_1$ in $\preceq$. Then there is some $i$ for which the $i$'th $-$ sign in $w_1$ lies to the left of the $i$'th $-$ sign in $w_0$. Let the position of the $i$'th $-$ sign in $w_1$ be $m$. Then after $m$ innings, $w_1$ has a lower score than $w_0$. ∎

## 5.3 Counting comparable pairs

We consider words $w_0, w_1 \in W(n_-, n_+)$, i.e. of length $n$ with sum $e$. This is proposition 1.20.

**Proposition** *The number of pairs $w_0, w_1 \in W(n_-, n_+)$ with $w_0 \preceq w_1$ is $C_{n+1}^e = C_{n+1,k}$*

Proof First, there is a bijection between pairs of comparable words of length $n$ with $k$ plus signs, and functions $f : \{1, \ldots, n+1\} \longrightarrow \{1, \ldots, n+1\}$ satisfying $f(i) \leq i$ for all $i$ and taking $k+1$ distinct values. The bijection is described as follows. Given a pair of comparable words $w_0 \preceq w_1$, we know that for all $j$, the $j$'th + sign in $w_0$ is to the right of the $j$'th + sign in $w_1$. We use this to define a decreasing function.

Insert a + at the start of $w_0$ and $w_1$ to obtain $w_0', w_1'$, so these are now words of length $n+1$ with $k+1$ plus signs. For $i \in \{1, \ldots, n+1\}$, let the number of + signs up to and including the $i$'th symbol of $w_0$ be $j(i)$; then define $f(i)$ to be the position of the $j(i)$'th + sign in $w_1$.

Conversely, given such a function, we can easily reconstruct the words $w_0, w_1$. The positions of the + signs in $w_1'$ are precisely the values of $f$. And the positions of the + signs in $w_0'$ are precisely those $i$ for which $f(i)$ jumps, $f(i) > f(i-1)$.

The number of such functions $f : \{1, \ldots, n+1\} \longrightarrow \{1, \ldots, n+1\}$ with $k+1$ distinct values is well known to be $N_{n+1,k+1} = C_{n+1,k}$. ∎

# Part III
# Combinatorics of chord diagrams

## 6 Construction of basis elements

We now examine *basis chord diagrams* in detail.

On a chord diagram, we have our basepoint. When that chord diagram is a basis chord diagram, we will also define another sort of base point, which we will call the *root point*, as well.

### 6.1 An example

Consider the basis element $v_{-+-++}$. Suppose we want to draw the corresponding chord diagram.

One way to proceed is to note that by definition $v_{-+-++} = B_-(v_{+-++})$. Hence there is an outermost chord which encloses a negative region and which lies immediately to the "left" of the basepoint; one of its endpoints is the basepoint. If we were to consider removing this outermost chord, including its endpoints (including the basepoint), and placing a new basepoint to its "left" (i.e. "jumping over" the location of the previous outermost chord), we should then have $v_{+-++}$. But $v_{+-++} = B_+(v_{-++})$, hence $v_{+-++}$ has an outermost chord at the base point enclosing a positive region; this also tell us the location of a chord on $v_{+-++}$.

We can then repeat. Since $v_{-++} = B_-(v_{++})$, there must be an outermost chord at the base point in this new chord diagram, enclosing a negative region; and hence we may locate chord on $v_{-++}$, $v_{+-++}$ and $v_{-+-++}$. Similarly, we can locate the remaining chords, until we reduce to the vacuum diagram $v_\emptyset$. See figure 17.

### 6.2 The base point construction algorithm

This can be formalised into an algorithm to construct the chord diagram of a basis element $v_w$. It will construct the diagram, starting from the base point; hence the name.

**Algorithm 6.1 (Base point construction algorithm)** *Let $w$ be a word of length $n$ in the symbols + and −. Consider a disc with $2n+2$ points marked on the boundary, and one of those marked points called the* base point. *Proceed through the word from left to right, and at each stage draw a chord and move to a new, temporary base point as follows. Once there is a chord ending at a marked point, that marked point is called* used.

*(i) If the symbol is −, draw a chord from the current temporary base point to the next unused marked point anticlockwise ("left" in the diagram) from it. After drawing this chord, move the temporary*

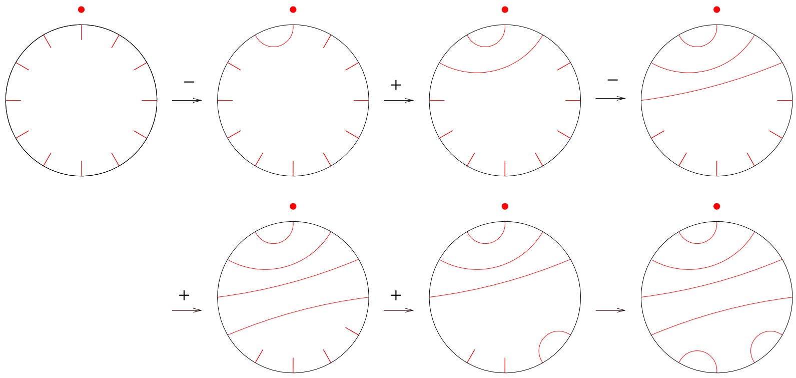


Figure 17: Construction of $v_{-+-++}$.

*base point to the next unused marked point in the anticlockwise direction ("left" in the diagram). (I.e., immediately anticlockwise of the new chord.)*

*(ii) If the symbol is $+$, draw a chord from the current temporary base point to the next unused marked point clockwise ("right") from it. After drawing this chord, move the temporary base point to the next unused marked point in the clockwise direction. (I.e., immediately clockwise of the new chord.)*

*This constructs $n$ chords connecting $2n$ marked points. Finally, connect the remaining two marked points with a chord. The base point returns to its initial position, which is the "permanent" base point.*

The stages in the construction of $v_{-+-++}$ are depicted in figure 17.

The words "left" and "right" to describe directions around a circle may seem horrendously bad as terminology, only being appropriate near the basepoint, and eventually conflicting around the other side of the circle! But we will see there is a reason for it, and it does make some sense.

That this produces $v_w$ may appear rather obvious, but since the details will be needed later we provide them. In fact, we will need to describe the mechanics of this algorithm in gruesome detail.

**Proposition 6.2 (Base point construction algorithm works)** *The base point construction algorithm is well-defined; in particular, at every stage of this algorithm, the chord described can be made disjoint from all previously drawn chords; and uniquely up to homotopy rel endpoints within the disc minus the previously drawn chords.*

*Moreover, it actually produces the chord diagram $\Gamma_w$.*

First, a little notation.

**Definition 6.3 (Labels of marked points relative to basepoint.)** *We label the $2n+2$ marked points with integers modulo $2n+2$. The base point is labelled $0$ and the numbering proceeds clockwise. (So the marked point immediately "right" / clockwise of the base point $0$ is the point $1$.)*

**Remark 6.4 (Labelling convention)** *Marked points will always be labelled with respect to the (permanent) base point, unless otherwise noted. In the various steps of the base point algorithm, as the "temporary base point" moves, the numbering of marked points does not change. We will have another labelling system subsequently, but this one will remain the usual one.*

Note that with this choice of labelling, a chord connecting two consecutive points $(2j-1, 2j)$ encloses a *negative* region; while an chord connecting two consecutive points $(2j, 2j+1)$ encloses a *positive* region.

**Definition 6.5 (Discrete interval)** *We define a* discrete interval *of marked points $[a,b]$ on the circle to be a set of marked points of the form $\{a, a+1, \ldots, b\}$.*

**Definition 6.6 (Substring)** *A* substring *of a word/string $w$ is a set of adjacent/contiguous symbols of $w$.*

So, for instance, $-+$ is a substring of $--++--++$ but $---+$ is not.

**Definition 6.7 (Block)** *Maximal substrings of identical symbols are called* blocks.

**Definition 6.8 (Leading and following symbols)** *A symbol in $w$ which is the first in its block (read left to right) we shall call a* leading *symbol. Non-leading symbols are called* following.

Now we can describe the mechanics of the algorithm precisely. We can locate how the temporary base point moves at each step of the algorithm; and where the chord is drawn at every step. As it turns out, the *odd*-numbered marked points serve as useful indicators of where we are up to in the base point algorithm. We consider the base point construction algorithm for a word $w \in W(n_-, n_+)$, describing a basis element of $SFH(T, n+1, e)$, with our usual notation conventions.

**Lemma 6.9 (Mechanics of base point construction algorithm)**

*(i) Consider the stage of the base point algorithm which processes the $i$'th $-$ sign ($1 \leq i \leq n_-$) in $w$. Let $i_+$ be the number of $+$ symbols processed up to this point. At this stage:*

*(a) A chord is drawn with endpoints:*

$$\begin{array}{ll} (1-2i, 2i_+) & \text{if the present (i.e. } i\text{'th) } - \text{ sign is leading} \\ (1-2i, 2-2i) & \text{if the present } - \text{ sign is following} \end{array}$$

*In particular, the chord has an endpoint at the odd-numbered marked point $1-2i$.*

*(b) The temporary base point then moves to the marked point $-2i$.*

*(c) The set of used marked points is the discrete interval $[1-2i, 2i_+]$.*

*(ii) Consider the stage of the base point algorithm which processes the $j$'th $+$ sign ($1 \leq j \leq n_+$) in $w$. Let $j_-$ be the number of $-$ signs processed up to this point. At this stage:*

*(a) A chord is drawn with endpoints:*

$$\begin{array}{ll} (-2j_-, 2j-1) & \text{if the present (i.e. } j\text{'th) } + \text{ sign is leading} \\ (2j-2, 2j-1) & \text{if the present } + \text{ sign is following} \end{array}$$

*In particular, the chord has an endpoint at the odd-numbered marked point $2j-1$.*

*(b) The temporary base point then moves to the marked point $2j$.*

*(c) The set of used marked points is the discrete interval $[-2j_-, 2j-1]$.*

Proof (Of proposition 6.2 and lemma 6.9) Proof by induction on the number of symbols processed in $w$. We consider processing $+$ signs; $-$ signs are obviously similar. The result is clear as we process the first symbol in $w$. A leading $+$ sign will "switch sides" and connect a "negatively labelled", or "left", or anticlockwise-of-the-basepoint marked point to a "positively labelled", "right" or clockwise-of-the-basepoint marked point. A following $+$ sign will give a chord enclosing a positive outermost interval clockwise from the temporary base point.

At each stage before termination, the set of used marked points is then as given by 6.9, as is the temporary base point. Hence the next chord can always be drawn, and uniquely up to homotopy in the disc (minus the previous chords) rel endpoints. At the final stage before termination, the discrete interval of used marked points consists of all but two of the marked points; hence the remaining two must be adjacent, and connecting them is possible (and uniquely up to homotopy in the same way).

That the algorithm produces $v_w$ is easily seen by induction on the length of the word. For words of length 1 (or even 0) it is clear. Let $w = sw'$ where $s$ is a symbol (i.e. $+$ or $-$) and $w'$ is a word one symbol shorter than $w$. Then by induction the algorithm applied to $w'$ produces $v_{w'}$; and it's also clear that the algorithm for $w$ produces $B_s(v_{w'})$ as claimed. ∎

**Lemma 6.10 (Existence of root point)** *The position of the final chord constructed in the base point construction algorithm only depends on $n, e$ (or equivalently $n_-, n_+$) and the final symbol of $w$. The final chord encloses an outermost region of sign given by that final symbol, and has an endpoint at the point numbered $2n_+ + 1 = -2n_- - 1 = e + n + 1 = e - (n+1)$ with respect to the basepoint.*

PROOF After having processed all symbols, from the previous lemma, the used marked points form the discrete interval

$$\begin{array}{rl} [-2n_-, 2n_+ - 1] & \text{if the final symbol is a + sign} \\ [1 - 2n_-, 2n_+] & \text{if the final symbol is a − sign} \end{array}$$

Hence if $w$ ends in a $+$, then the final chord connects the two remaining unused points

$$(2n_+, 2n_+ + 1) = (-2n_- - 2, -2n_- - 1)$$

and hence encloses an outermost positive region; while if $w$ ends in a $-$, then the final chord connects

$$(2n_+ + 1, 2n_+ + 2) = (-2n_- - 1, -2n_-)$$

and encloses an outermost negative region. ∎

**Definition 6.11 (Root point)** *The marked point numbered $2n_+ + 1 = -2n_- - 1 = e \pm (n+1)$ with respect to the basepoint is called the* root point.

**Remark 6.12 (Denoting root point)** *In our pictures, the root point will be denoted by a hollow red dot.*

We now see that our terminology of "left" and "right" is not so horrendous after all. For as we perform the base point algorithm, all chord additions are done so that the discrete interval of used marked points never crosses the root point; and hence talking about "left" and "right" of the base point (i.e. anticlockwise and clockwise) never ceases to make sense. We can then define the terminology properly.

**Definition 6.13 (Left/west and right/east side)** *The marked points forming the discrete interval $[e - n - 1, 0]$ are called the* left side *or* westside *of the circle. The marked points forming the discrete interval $[0, e + n + 1]$ are called the* right side *or* eastside.

Thus, to move left from the base point is to move anticlockwise; but to move left from the root point is to move clockwise. We can now use this algorithm to number chords and regions in the chord diagram $\Gamma_w$ corresponding to $v_w$.

**Definition 6.14 (Base-$\pm$ numbering of chords and regions)**

*(i) The chord created in the base point construction algorithm by processing the $i$'th $-$ sign of $w$ is called the* base-$i$'th $-$ chord. *It encloses a $-$ region, which is also a region of the completed chord diagram $\Gamma_w$, which we call the* base-$i$'th $-$ region.

*(ii) The chord created in the base point construction algorithm by processing the $j$'th $+$ sign of $w$ is called the* base-$j$'th $+$ chord. *It encloses a $+$ region, which is also a region of the completed chord diagram $\Gamma_w$, which we call the* base-$j$'th $+$ region.

Note that every chord has a base-$\pm$ numbering, except the final one constructed in our algorithm, i.e. the chord at the root. And every region has a base-$\pm$ numbering, except the two regions adjacent to the root point, of which there is one positive and one negative.

We remark that this consideration makes it explicit how the relative euler class of the chord diagram $\Gamma_w$ is $e = n_+ - n_-$; every $-$ sign creates a $-$ region, and every $+$ sign creates a $+$ region, in the algorithm; with two regions left at the end, of opposite sign.

## 6.3 The root point construction algorithm

The above algorithm takes a word $w$ and constructs the corresponding chord diagram, starting from the basepoint, reading $w$ from left to right. But equally, we can construct the chord diagram from the root point, reading $w$ from right to left. In some sense this is more natural, since the word denotes a composition of the creation operators $B_\pm$, and compositions proceed from right to left.

The algorithm is basically identical; except that, because we proceed from the "bottom" of our chord diagram to the "top", clockwise and anticlockwise are reversed. However, if we draw our diagrams as we do, then "left" and "right" are not reversed — and if we choose our labellings of marked points appropriately, then the various expressions above do not change at all.

**Algorithm 6.15 (Root point construction algorithm)** *We process a word $w \in W(n_-, n_+)$, from right to left. Begin with a disc with $2n+2$ marked points on its boundary, and one of those points called the* root point*. At each stage draw a chord and move to a new, temporary root point as follows.*

*(i) For a $-$ symbol, draw a chord from the current root point to the next unused marked point* clockwise*/left from it. After drawing this chord, move the temporary root point to the next unused marked point in the* clockwise*/left direction. (I.e., immediately clockwise of the new chord.)*

*(ii) For a $+$ symbol, draw a chord from the current basepoint to the next unused marked point* anticlockwise*/right from it. After drawing this chord, move the basepoint to the next unused marked point in the anticlockwise/right direction. (I.e., immediately anticlockwise of the new chord.)*

*This constructs $n$ chords connecting $2n$ marked points. Finally, connect the remaining two marked points with a chord. The root point now moves back to its initial, permanent position.*

We remark that there are similar results to proposition 6.2 and lemma 6.9 for this algorithm: it works, and actually constructs $\Gamma_w$.

We also have root-numberings of chords and regions, which we will need later, in analogy to our base-numberings.

**Definition 6.16 (Root-$\pm$ numbering of chords and regions)**

*(i) The chord created in the root point construction algorithm by processing the $i$'th $-$ sign of $w$ (from the* left*) is called the* root-$i$'th $-$ chord*. It encloses a $-$ region, which is also a region of the completed chord diagram $\Gamma_w$, which we call the* root-$i$'th $-$ region*.*

*(ii) The chord created in the base point construction algorithm by processing the $j$'th $+$ sign of $w$ (from the* left*) is called the* root-$j$'th $+$ chord*. It encloses a $+$ region, which is also a region of the completed chord diagram $\Gamma_w$, which we call the* root-$j$'th $+$ region*.*

Note especially that the root point construction algorithm processes $w$ from *right to left*: but when we speak of the root-$i$'th $\pm$ region we are reading $w$ from *left to right.* This is confusing, but makes subsequent considerations easier.

Also note that every chord has a root-$\pm$ numbering, except the chord at the base point. And every region has a root-$\pm$ numbering, except the two regions adjacent to the base point, of which there is one positive and one negative.

Thus, *every chord* has some numbering, whether from the base or the root.

## 6.4 Basis decomposition of a chord diagram

We saw in section 4.5 that there is a natural way to expand out a given chord diagram as a sum of basis elements. We now formalise this procedure. As with constructing basis elements, this algorithm may be carried out "from the base point" or "from the root point". The procedure may seem obvious, from the example, but will actually give us interesting information about the elements that occur in the decomposition of a chord diagram.

Given a diagram $\Gamma$ with basepoint and root point identified, we label the marked points with respect to the root point. We will successively obtain sets of chord diagrams

$$\{\Gamma\} = \Upsilon_0 \rightsquigarrow \Upsilon_1 \rightsquigarrow \cdots \rightsquigarrow \Upsilon_n$$

where $\Upsilon_k$ is the set of all diagrams obtained at the $k$'th stage, and obtained from decomposing the diagrams in $\Upsilon_{k-1}$. The final set $\Upsilon_n$ will contain precisely the elements of the basis decomposition of $\Gamma$. In particular, $|\Upsilon_k| > |\Upsilon_{k-1}|$ and

$$\Gamma = \sum_{\Gamma' \in \Upsilon_0} \Gamma' = \sum_{\Gamma' \in \Upsilon_1} \Gamma' = \cdots = \sum_{\Gamma' \in \Upsilon_n} \Gamma'.$$

In fact, each $\Upsilon_k$ consists of chord diagrams $\Gamma_{w\cdot}$; each $\Gamma_{w\cdot}$ is the sum of those basis elements of $\Gamma$ whose words begin with $w$.

To make this precise, note that the first $k$ steps of the base point construction algorithm depend only on the first $k$ symbols of the word $w$; and the first $k$ steps of the root point construction algorithm depend only on the last $k$ symbols of $w$.

**Definition 6.17 (Partial chord diagrams)** *Let $w$ be a word of length $k$.*

*(i) The* partial chord diagram for $w\cdot$ *is a disc with $2n+2$ marked points, including a base and root point, and the first $k$ chords drawn in processing any word of length $n$ beginning with $w$ in the base point construction algorithm.*

*(ii) The* partial chord diagram for $\cdot w$ *is a disc with $2n+2$ marked points, including a base and root point, and the first $k$ chords drawn in processing any word of length $n$ ending in $w$ in the root point construction algorithm.*

The dots in $w\cdot$ and $\cdot w$ describe "where the rest of the word goes".

We will label the elements of $\Upsilon_k$ as $\Gamma_{w\cdot}$, where $w$ is a word of length $k$. The chord diagram $\Gamma_{w\cdot}$ will be the sum of all basis elements of $\Gamma$ whose words begin with $w$, and it will contain the partial chord diagram for $w\cdot$.

We may think of $\Gamma$ as corresponding to the empty word, $\Gamma = \Gamma_{\emptyset\cdot}$.

**Algorithm 6.18 (Base point decomposition algorithm)** *Let $\Upsilon_0 = \{\Gamma\} = \{\Gamma_{\emptyset\cdot}\}$. At the $k$'th step, we take $\Upsilon_{k-1}$, and for each element $\Gamma_{w\cdot}$ of $\Upsilon_{k-1}$, corresponding to the word $w$ of length $k-1$, we do the following.*

*(i) If there exists a word $w' = w+$ or $w-$ such that $\Gamma_{w\cdot}$ contains the partial chord diagram for $w'\cdot$, then we place $\Gamma_{w\cdot}$ in $\Upsilon_k$ and name it $\Gamma_{w'\cdot}$.*

*(ii) Otherwise, there is no such word. Hence neither of the two chords added in the $k$'th stage of the base point construction algorithm for the words $w\pm$ lie in $\Gamma_{w\cdot}$. Equivalently, we consider the location of the base point after $k-1$ stages of the base point construction algorithm for $w$; then, on the unused disc of $\Gamma_w$, there is no outermost chord at the base point.*

*We then consider an arc of attachment which runs close to the boundary of the unused disc, which is centred on the chord emanating from the base point (as shifted after $k-1$ stages of the base point construction algorithm for $w$), and which has its two ends on the two chords emanating from the marked points adjacent to the base point on the unused disc. We perform the two possible bypass moves, obtaining two distinct chord diagrams. One of these contains the partial chord diagram for $w - \cdot$, and the other contains the partial chord diagram for $w + \cdot$. We label them $\Gamma_{w-\cdot}$ and $\Gamma_{w+\cdot}$ and place them in $\Upsilon_k$.*

*This constructs $\Upsilon_k$ from $\Upsilon_{k-1}$.*

It's clear from the algorithm that the $\Upsilon_k$ have the desired properties. Precisely, we have the following.

**Lemma 6.19** *For each $k$, the elements of $\Upsilon_k$ obtained in the base point decomposition algorithm can be grouped in some fashion so as to be summable, and they sum to $\Gamma$. The basis decomposition of $\Gamma_{w\cdot}$ contains all the basis elements of $\Gamma$ whose words begin with $w$, and contains the partial chord diagram for $w\cdot$. The elements of $\Upsilon_n$ are basis elements and are precisely those occurring in the decomposition of $\Gamma$.* ∎

In fact, to see how to sum the elements of $\Upsilon$, we bracket them exactly according to how they came from $\Upsilon_{k-1}$; the decomposition process actually gives us a directed binary tree of chord diagrams, equivalent to a bracketing. To see that $\Upsilon_n$ consists of basis elements, note that they contain the partial chord diagrams for words of length $n$; this leaves only one possible place for the remaining chord.

We may apply the same idea from the root point rather than the base point. We obtain $\Upsilon_k$ containing $\Gamma_{\cdot w}$, where $w$ is a word of length $k$, which is the sum of all basis elements of $\Gamma$ ending in $w$.

**Algorithm 6.20 (Root point decomposition algorithm)** *Let $\Upsilon_0 = \{\Gamma\} = \{\Gamma_{\cdot\emptyset}$. At the $k$'th step, we take $\Upsilon_{k-1}$, and for each element $\Gamma_{\cdot w}$ of $\Upsilon_{k-1}$, corresponding to the word $w$ of length $k-1$, we do the following.*

*(i) If there exists a word $w' = -w$ or $+w$ such that $\Gamma_{\cdot w}$ contains the partial chord diagram for $\cdot w'$, then we place $\Gamma_{\cdot w}$ in $\Upsilon_k$ and name it $\Gamma_{w'}$.*

*(ii) Otherwise, there is no such word. Hence neither of the two chords added in the $k$'th stage of the root point construction algorithm for the words $\pm w$ lie in $\Gamma_{\cdot w}$. Equivalently, we consider the location of the base point after $k-1$ stages of the root point construction algorithm for $w$; then, on the unused disc of $\Gamma_{\cdot w}$, there is no outermost chord at the base point.*

*We then consider an arc of attachment which runs close to the boundary of the unused disc, which is centred on the chord emanating from the base point (as shifted after $k-1$ stages of the root point construction algorithm for $w$), and which has its two ends on the two chords emanating from the marked points adjacent to the base point on the unused disc. We perform the two possible bypass moves, obtaining two distinct chord diagrams. One of these contains the partial chord diagram for $\cdot - w$, and the other contains the partial chord diagram for $\cdot + w$. We label them $\Gamma_{\cdot -w}$ and $\Gamma_{\cdot +w}$ and place them in $\Upsilon_k$.*

*This constructs $\Upsilon_k$ from $\Upsilon_{k-1}$.*

**Lemma 6.21** *For each $k$, the elements of $\Upsilon_k$ obtained in the root point decomposition algorithm can be grouped in some fashion so as to be summable, and they sum to $\Gamma$. The basis decomposition of $\Gamma_{\cdot w}$ contains all the basis elements of $\Gamma$ whose words end with $w$, and contains the partial chord diagram for $\cdot w$. The elements of $\Upsilon_n$ are basis elements and are precisely those occurring in the decomposition of $\Gamma$.* ∎

# 7 Bypass systems on basis chord diagrams

## 7.1 The idea

In this section we will investigate performing multiple bypass moves on basis chord diagrams. We will show that by using bypasses in a controlled way, we can go from a given basis chord diagram to many others — in particular, to any other chord diagram to which it is comparable under the partial order $\preceq$.

**Definition 7.1 (Bypass system)** *A* bypass system *on a chord diagram is a finite set of disjoint nontrivial arcs of attachment.*

Here, as usual, a nontrivial arc of attachment is one whose three points of intersection with chords all lie on distinct chords.

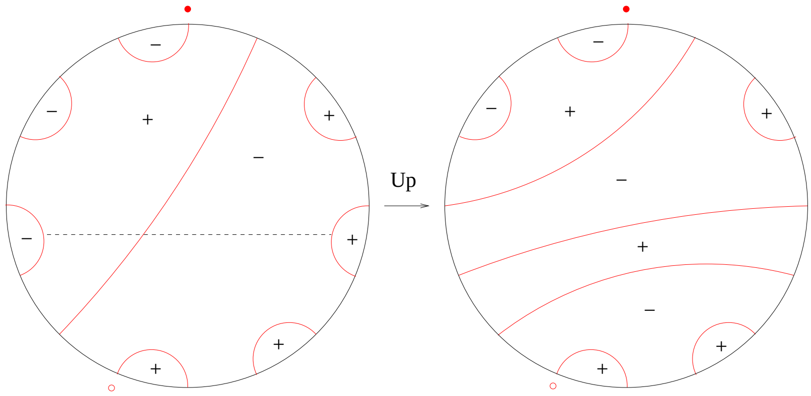



Figure 18: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$.

In this section we will prove proposition 1.23, constructing bypass systems $FBS(\Gamma_1, \Gamma_2)$ and $BBS(\Gamma_1, \Gamma_2)$ that take $\Gamma_1$ to $\Gamma_2$ and vice versa. And we will prove proposition 1.24, describing how performing bypass moves in the opposite direction along such bypass systems gives a chord diagram with a prescribed minimum and maximum. The construction will be explicit.

This is the bijection between comparable pairs of basis elements, and chord diagrams, claimed in theorem 1.22. We will prove theorem 1.22 in section 8.

Our approach will be to develop a series of increasingly involved analogies between combinatorial manipulations on words $w \in W(n_-, n_+)$, and bypass systems on basis chord diagrams $\Gamma_w$. We will start with single bypass moves, and proceed to general bypass systems.

## 7.2 A menagerie of examples

We begin with some illustrative examples of increasing difficulty. These illustrate various phenomena one observes when combining several bypass moves on a chord diagram. We will give a set of attaching arcs to show how we can take a basis chord diagram to another.

First, we show how to go from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$. Here we "move the third $-$ sign past the first two $+$ signs". Moves like this are called *forwards elementary moves* and they are obtained by single upwards bypasses. See figure 18.

Next, we show how to go from the same starting diagram $\Gamma_{---++++}$ to $\Gamma_{-++--++}$, moving both the second and third $-$ sign past the first two $+$ signs. This is also a forwards elementary move, and is obtained from a single upwards bypass. In general, an elementary move consists of taking a string of contiguous $-$ symbols and moving them to the right, past an adjacent string of contiguous $+$ symbols. See figure 19.

Note that here we included one bypass move which can be thought of as encoding the instruction "move the second $-$ sign past the first $+$ sign". If we think of the $-$ signs as remaining in order, then in this process, the third $-$ sign must be "brought along for the ride", past the first $+$ sign as well. Alternatively, if we "treat the two $-$ signs individually", and perform one bypass move for each, respectively encoding the instruction to move them past the first $+$ sign, we obtain figure 20.

We see it gives the same result. This is an instance of a more general phenomenon of redundancy of bypasses, when they occur in the following arrangements. See figure 21

Next, we show how to go from $\Gamma_{--++--++}$ to $\Gamma_{++--++--}$. Here we "move the first and second $-$ signs past the first and second $+$ signs, and move the third and fourth $-$ signs past the third and fourth $+$ signs". There are two forwards elementary moves involved, but in some sense they do not interfere with each other; this is obtained by two upwards bypasses. See figure 22.

While the position of each of these bypass arcs, taken individually, might seem clear now from the foregoing, note that there are actually *two distinct* ways to place them relative to the other. If

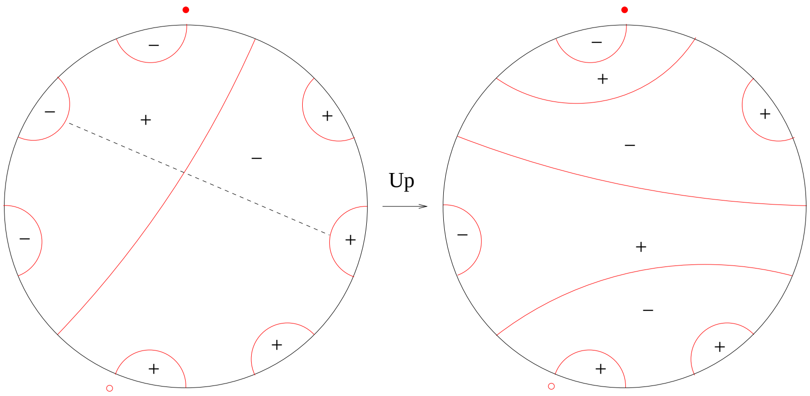



Figure 19: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{-++--++}$.

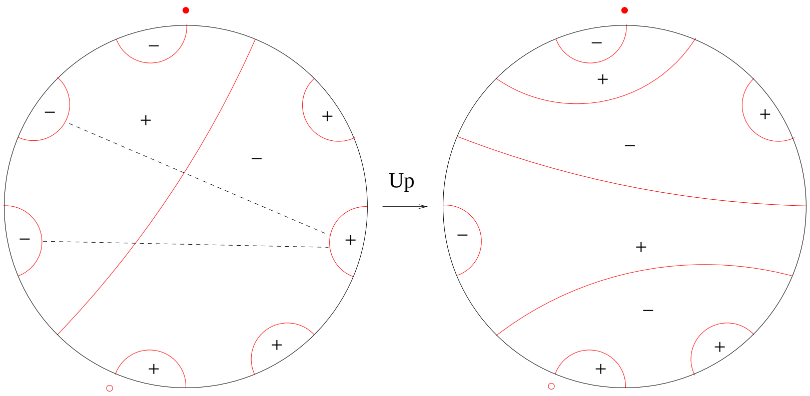



Figure 20: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{-++--++}$.

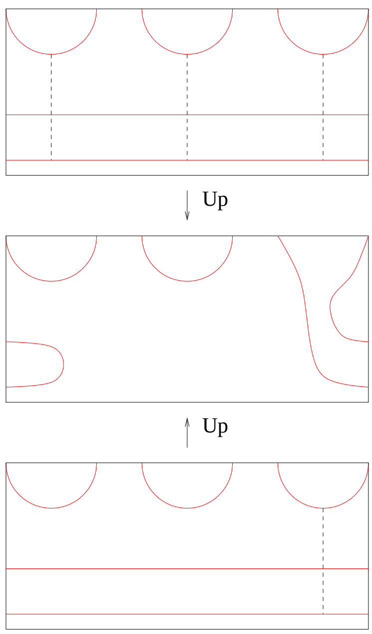



Figure 21: Redundancy of bypasses.

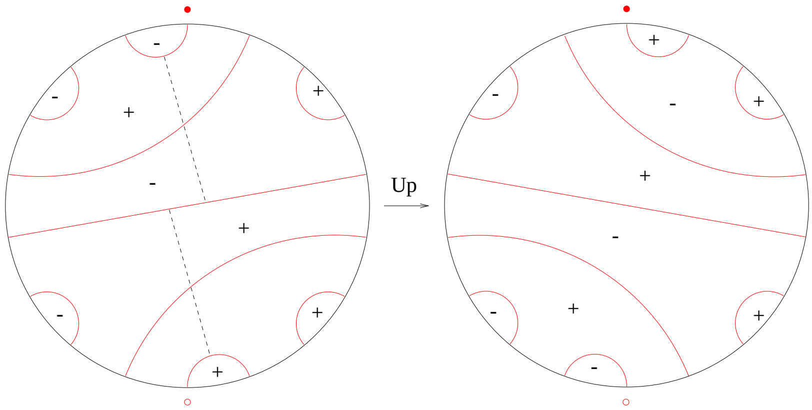



Figure 22: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++--++--}$.

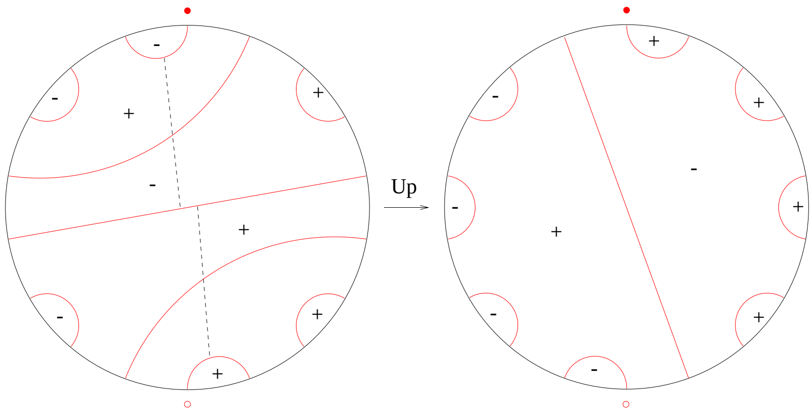



Figure 23: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$.

we consider these attaching arcs, placed in the other possible arrangement, we obtain a drastically different result: we go from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$. See figure 23.

Thus, the relative positioning of bypass arcs in this way corresponds to some sort of "carrying" or "compounding" phenomenon. Each arc itself moves some string of $-$ signs past some string of $+$ signs. But if two arcs are in this arrangement, those $-$ signs moved right by the first elementary move are then carried in the second also. Alternatively, the "treating each $-$ sign individually" approach here, from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$ requires six bypass arcs: 2 for the first $-$ sign, 2 for the second, 1 for the third, and 1 for the fourth. See figure 24.

In general, in the following, we will apply the "take individual care" approach, because it is easier to formalise, even though the sets of bypass moves so obtained often contain massive redundancy. This will lead to the notion of the *bypass system of a pair* of comparable basis chord diagrams.

We will spend the rest of this section formalizing all the above constructions.

## 7.3 Elementary moves on words

Given a word $w$, recall we group it into *blocks* of $+$ and $-$ symbols and hence may write

$$w = (-)^{a_1}(+)^{b_1} \cdots (-)^{a_k}(+)^{b_k}.$$

Possibly $k = 1$; possibly $a_1 = 0$; possibly $b_k = 0$. But every other $a_i$ and $b_i$ is nonzero. So $w$ as written above has $2k$ blocks (or $2k-1$ or $2k-2$ blocks if $a_1 = 0$ or/and $b_k = 0$.)

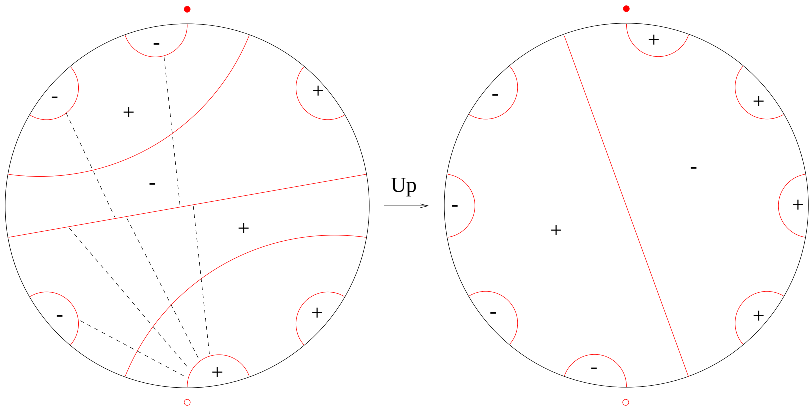



Figure 24: "Individual care" approach to $\Gamma_{--++--++} \to \Gamma_{++++----}$.

We now make a combinatorial definition of moves on words of $+$ and $-$ symbols.

**Definition 7.2 (Elementary moves on words)** *Let $w$ be a word in the symbols $\{+, -\}$.*

*(i) A* forwards elementary move *consists of taking a contiguous substring of $w$ of the form $(-)^a(+)^b$ and replacing it with $(+)^b(-)^a$.*

*(ii) A* backwards elementary move *consists of taking a contiguous substring of $w$ of the form $(+)^b(-)^a$ and replacing it with $(-)^a(+)^b$.*

*Collectively we call these* elementary moves.

The effect of an elementary move is therefore to "slide some $-$'s past some $+$'s". The forwards or backwards nature of the move corresponds to the $-$ signs moving forwards or backwards, as the word is read from left to right. Note that if $w'$ is obtained from $w$ by a forwards elementary move, then $w \preceq w'$; while if $w'$ is obtained from $w$ by a backwards elementary move, then $w' \preceq w$.

**Lemma 7.3 (Number of elementary moves)** *The word $w = (-)^{a_1}(+)^{b_1} \cdots (-)^{a_k}(+)^{b_k}$ has precisely*

$$a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$$

*forward elementary moves and*

$$b_1 a_2 + b_2 a_3 + \cdots + b_{k-1} a_k$$

*backward elementary moves, for a total of*

$$a_1 b_1 + b_1 a_2 + a_2 b_2 + \cdots + b_{k-1} a_k + a_k b_k$$

*elementary moves.*

Proof For a forward move, we must choose a substring of $(-)^{a_i}(+)^{b_i}$ for some $i$ of the form $(-)^A(+)^B$, and move them past each other. There are $a_i b_i$ such substrings. For a backward move, we must choose a substring of $(+)^{b_i}(-)^{a_{i+1}}$ for some $i$ of the form $(+)^B(-)^A$, and move them past each other. There are $b_i a_{i+1}$ such substrings. ■

**Definition 7.4 (Denoting elementary moves)**

*(i) The $(i, j)$* forwards elementary move *$FE(i, j)$ moves the $i$'th $-$ sign (from the left), and all the minus signs to its right in the same block, to the position immediately to the right of the $j$'th $+$ sign (from the left).*

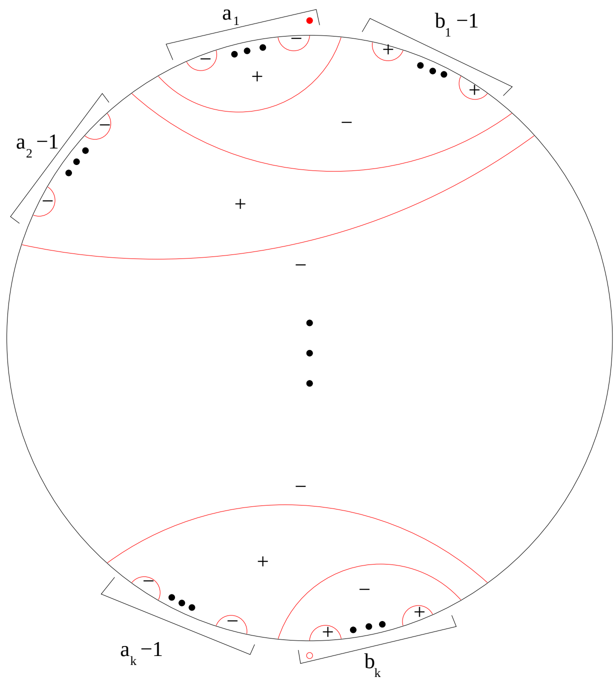



Figure 25: General basis chord diagram $(-)^{a_1} \cdots (+)^{b_k}$.

*(ii) The* $(i, j)$ backwards elementary move $BE(i, j)$ *moves the* $j$*'th* $+$ *sign (from the left), and all the plus signs to its right in the same block, to the position immediately to the right of the* $i$*'th* $-$ *sign (from the left).*

Note that we must have $1 \leq i \leq n_-$ and $1 \leq j \leq n_+$ in this definition. But for $i, j$ satisfying these inequalities, $FE(i, j)$ is not always defined; $FE(i, j)$ is only defined if the $i$'th $-$ sign is to the left of the $j$'th $+$ sign, and the $j$'th $+$ sign lies in block of $+$ symbols to the immediate right of the block with the $i$'th $-$ sign. For any pair $(i, j)$, at most one of $FE(i, j)$ or $BE(i, j)$ can be defined.

## 7.4 Anatomy of attaching arcs on basis chord diagrams

We now give a complete description of attaching arcs on basis chord diagrams. As it turns out, these correspond in a precise way to elementary moves on words. Again writing a word $w$ as

$$w = (-)^{a_1} (+)^{b_1} \cdots (-)^{a_k} (+)^{b_k},$$

the corresponding chord diagram $\Gamma_w$ is as shown in figure 25.

**Lemma 7.5 (Number of arcs of attachment)** *There are precisely*

$$a_1 b_1 + b_1 a_2 + a_2 b_2 + \cdots + b_{k-1} a_k + a_k b_k$$

*topologically distinct possible nontrivial arcs of attachment for* $\Gamma_w$*. (We consider arcs of attachment to be topologically equivalent if they are homotopic in the disc, subject to the constraint that the three intersection points must lie on chords.)*

Proof The proof is based on the observation that the non-outermost chords neatly compartmentalise the disc into pieces.

We will give the proof when $a_1 \neq 0$ and $b_k \neq 0$; the cases where one or both of these are zero is similar. An arc of attachment intersects the chord diagram $\Gamma_w$ in three points; for a nontrivial arc of attachment, the middle of these must lie on a non-outermost chord. There are precisely $2k-1$ non-outermost chords, corresponding to the $2k-1$ leading symbols in $w$ (other than the very first symbol). Let these non-outermost chords be $c_1, d_1, c_2, d_2, \ldots, c_{k-1}, d_{k-1}, c_k$, respectively from base to root. We count the number of nontrivial arcs of attachment with its middle intersection point lying on each of these $2k-1$ chords.

So $c_i$ separates two regions; one of these (towards the base) has boundary with $a_i$ other components of $\Gamma_w$; and the other (towards the root) has boundary with $b_i$ other components of $\Gamma_w$. Thus there are $a_i b_i$ possible arcs of attachment centred on $c_i$.

Similarly, $d_i$ separates two regions; one of these (towards the base) has boundary with $b_i$ other components of $\Gamma_w$; the other (towards the root) has boundary with $a_{i+1}$. This gives $b_i a_{i+1}$ possible arcs of attachment centred on $d_i$. ∎

Suspiciously, the number of elementary moves on $w$ equals the number of nontrivial arcs of attachment on $\Gamma_w$. There is indeed a nice bijection between them.

**Definition 7.6 (Prior and latter chords of arc of attachment)** *Given an arc of attachment, consider the chords on which its endpoints lie.*

*(i) The chord which was created first in the base point construction algorithm, we call the* prior chord *of the arc.*

*(ii) The chord created later, we call the* latter chord *of the arc.*

A nontrivial arc of attachment on a disc intersects three distinct chords which are adjacent to four distinct regions; two of these do not intersect the arc at all.

**Definition 7.7 (Outer regions of an arc of attachment)** *Let $(D, \Gamma)$ be a chord diagram and $c$ an arc of attachment. The two regions of $D - \Gamma$ which are adjacent to chords intersecting $c$, but which do not themselves intersect $c$, are called the* outer regions *of $c$.*

*(i) The outer region of $c$ adjacent to its prior chord is called the* prior outer region *of $c$.*

*(ii) The outer region of $c$ adjacent to its latter chord is called the* latter outer region *of $c$.*

Note that the prior outer region and latter outer region necessarily have different signs. Considering the signs of these regions will be useful.

**Definition 7.8 (Forwards and backwards arcs of attachment)**

*(i) An arc of attachment whose prior outer region is negative (and latter outer region positive) is called a* forwards *arc of attachment.*

*(ii) An arc of attachment whose prior outer region is positive (and latter outer region negative) is called a* backwards *arc of attachment.*

Now, for any nontrivial arc of attachment, the prior outer region is certainly not adjacent to the root point, and the latter outer region is not adjacent to the base point. Hence we may make the following definition.

**Definition 7.9 (Denoting forwards and backwards arcs of attachment)**

*(i) The forwards attaching arc whose prior outer region is the base-$i$'th $-$ region and whose latter outer region is the root-$j$'th $+$ region is called* the forwards $(i,j)$-attaching arc $FA(i,j)$.

*(ii) The backwards attaching arc whose prior outer region is the base-$j$'th $+$ region and whose latter outer region is the root-$i$'th $-$ region is called* the backwards $(i,j)$ attaching arc $BA(i,j)$.

Note that these attaching arcs do not exist for all $(i,j)$. The next lemma answers precisely when they do.

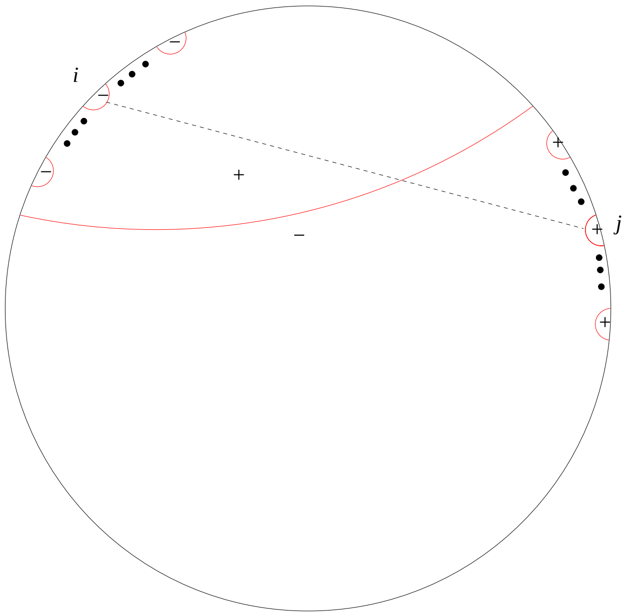



Figure 26: Forwards attaching arc $FA(i,j)$.

**Lemma 7.10 (Existence of attaching arcs)** *For a given word $w$, there is a forwards (resp. backwards) $(i,j)$ attaching arc $FA(i,j)$ (resp. $BA(i,j)$) on $\Gamma_w$ if and only if there is a forwards (resp. backwards) $(i,j)$ elementary move $FE(i,j)$ (resp. $BE(i,j)$) in $w$.*

Proof We prove the forwards case; the backwards case is similar. Suppose there exists such an elementary move; so that the $i$'th $-$ sign and $j$'th $+$ sign appear as desired; within blocks of the form $(-)^a(+)^b$. Then, considering the base point construction algorithm, we see that the chord diagram for $w$ contains an arrangement as shown in figure 26.

Thus there is a forwards or backwards $(i,j)$ attaching arc, as desired. Conversely, any forwards or backwards $(i,j)$ attaching arc comes in this arrangement, and hence there is a forwards or backwards $(i,j)$ elementary move, as desired. ∎

**Remark 7.11 (Forwards and backwards; notation)** *Throughout this section, we will have constructions and lemmas which come in two varieties, one "forwards version" and one "backwards version". For the most part the backwards versions are entirely analogous to the forwards versions. To save space, we will often give arguments, and sometimes statements, for the forwards version only; but state the final results for both forwards and backwards versions.*

## 7.5 Single bypass moves and elementary moves

We now give a complete description of bypass moves on basis chord diagrams. They are in analogy with elementary moves.

**Lemma 7.12 (Bypass moves and elementary moves)** *The chord diagram obtained from $\Gamma_w$ by an upwards bypass move along $FA(i,j)$ (resp. downwards along $BA(i,j)$) is the basis chord diagram $\Gamma_{w'}$, where $w' = FE(i,j)(w)$ (resp. $BE(i,j)(w)$).*

$$\begin{array}{ccc} w & \stackrel{FE(i,j)}{\mapsto} & w' \\ \downarrow & & \downarrow \\ \Gamma_w & \stackrel{Up\,(FA(i,j))}{\mapsto} & \Gamma_{w'} \end{array} \quad \text{and} \quad \begin{array}{ccc} w & \stackrel{BE(i,j)}{\mapsto} & w' \\ \downarrow & & \downarrow \\ \Gamma_w & \stackrel{Down\,(BA(i,j))}{\mapsto} & \Gamma_{w'} \end{array}$$

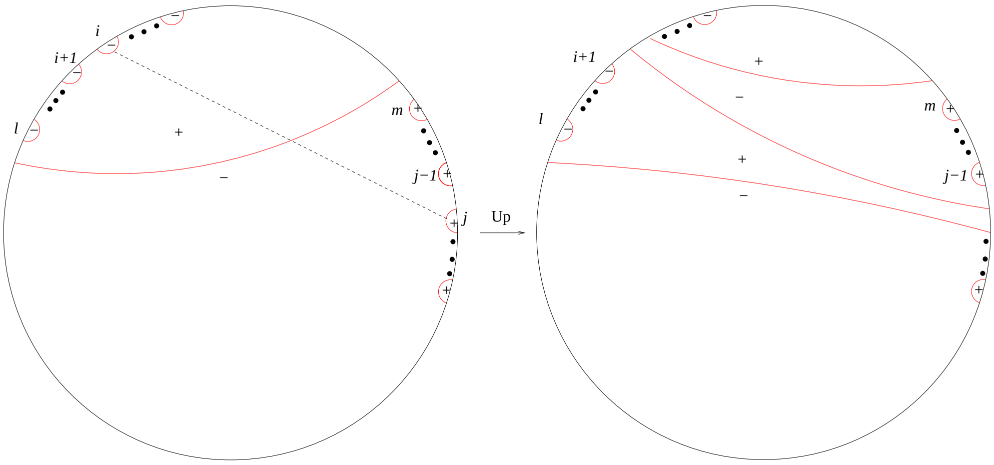



Figure 27: Effect of bypass move along $FA(i,j)$.

*In the other direction, a downwards bypass move along $FA(i,j)$ (resp. upwards along $BA(i,j)$) gives $\Gamma_w + \Gamma_{w'}$.*

Proof If we have an $(i,j)$ forwards elementary move $FE(i,j)$, corresponding to the $(i,j)$ forwards attaching arc $FA(i,j)$, then the $i$'th $-$ sign and $j$'th $+$ sign occur in consecutive blocks, with the $j$'th $+$ sign in the block to the right of the block containing the $i$'th $-$ sign. In our chord diagram, then, we have the situation depicted in figure 27, where we number the base-$i$'th $-$ region (and adjacent base $-$ regions), and we number the root-$j$'th $+$ region (and adjacent root $+$ regions). Let the last $-$ sign in the block with the $i$'th $-$ sign be the $l$'th $-$ sign in $w$ (so $l \geq i$, possibly $l = i$), and let the first $+$ sign in the block with the $j$'th $+$ sign be numbered $m$ (so $m \leq j$, possibly $m = j$).

An upwards bypass move along $FA(i,j)$ then has the effect shown. This has the effect of producing a basis chord diagram for the word $w'$, where $w'$ is obtained from $w$ by swapping the string of $i$'th thru $l$'th $-$ signs with the string of $m$'th thru $j$'th $+$ signs, $(-)^{l-i+1}(+)^{j-m+1} \mapsto (+)^{j-m+1}(-)^{l-i+1}$. Thus $w'$ is precisely the word obtained from $w$ by moving the $i$'th $-$ sign (and all $-$ signs to the right of the $i$'th one, in the same block) past the $j$'th $+$ sign, i.e. by $FE(i,j)$.

A similar argument works for backwards arcs of attachment and backwards elementary moves. The bypass relation then gives the final statement. ■

In particular, performing a bypass move on a basis chord diagram gives either a basis diagram or a sum of two basis diagrams.

## 7.6 Bypass systems in general

We can now give some basic results about performing bypass moves along bypass systems.

**Lemma 7.13** *Suppose we have a bypass system consisting entirely of forwards (resp. backwards) attaching arcs $c_1, \ldots, c_m$. After performing an upwards (resp. downwards) bypass move along $c_1$, then for each other $c_i$, either:*

*(i) $c_i$ remains a forwards (resp. backwards) attaching arc, or*

*(ii) $c_i$ becomes trivial, in the sense that performing an upwards (resp. downwards) bypass move along $c_i$ now has trivial effect on the chord diagram.*

Proof The arc $c_i$ intersects either $0, 1, 2$ or $3$ of the same three chords as $c_1$. If it intersects none of the same chords, then their order of construction in the root point algorithm remains unchanged,

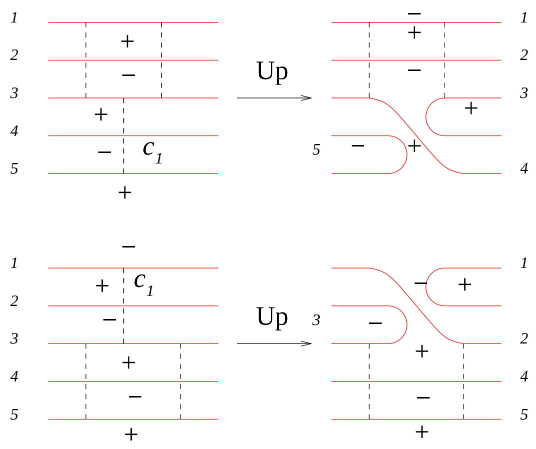



Figure 28: Effect of single bypass move and order of chords.

so that $c_i$ remains a forwards attaching arc. If it intersects one of the same chords, and the situation must be one of those shown in figure 28, with order of the chords as shown; the result then holds.

If $c_i$ intersects two of the same chords as $c_1$, then it becomes trivial, as discussed above, see figure 21. If it intersects all three of the same chords, then it clearly becomes trivial. ∎

**Lemma 7.14** *The effect of performing upwards (resp. downwards) bypass moves on a basis chord diagram $\Gamma_{w_1}$ along a bypass system consisting entirely of forwards (resp. backwards) attaching arcs is again a chord diagram $\Gamma_{w_2}$, with $w_1 \preceq w_2$ (resp. $w_1 \preceq w_1$).*

Proof Let the attaching arcs of the bypass system be $c_1, \ldots, c_m$. After performing an upwards bypass move along a forwards attaching arc on a basis chord diagram, by lemma 7.12 we obtain another basis chord diagram; and by lemma 7.13, all the other arcs of attachment either become trivial or remain forwards attaching arcs. Remove the trivial ones and repeat. By induction we eventually obtain a basis chord diagram $\Gamma_{w_2}$. At the performance of each bypass move we move ahead with respect to our partial order, by lemma 7.12. ∎

## 7.7 Generalised elementary moves on words

So far, we have defined *elementary moves* $FE(i,j)$ and $BE(i,j)$ on a word $w$. The forwards move $FE(i,j)$ moves the $i$'th $-$ sign (and all $-$ signs to its right, in the same block) to the right, past the $j$'th $+$ sign: *provided that the $j$'th $+$ sign is in the block immediately to the right of the $i$'th $-$ sign.* This is a useful notion because it corresponds precisely to bypass moves on the chord diagram $\Gamma_w$. We now generalise this notion to what we call *generalised elementary moves.* These are basically identical to elementary moves, without the proviso in italics; an extension of the same definition.

**Definition 7.15 (Generalised elementary move)** *Let $w$ be a word on $\{-,+\}$ with $n_-$ $-$ signs and $n_+$ $+$ signs. Let $1 \leq i \leq n_-$ and $1 \leq j \leq n_+$.*

*(i) If the $i$'th $-$ sign in $w$ occurs to the left of the $j$'th $+$ sign, we define the* forwards generalised elementary move *$FE(i,j)$ to take the $i$'th $-$ sign, and all $-$ signs between it and the $j$'th $+$ sign, and move them to a position immediately after the $j$'th $+$ sign.*

*(ii) If the $j$'th $+$ sign in $w$ occurs to the left of the $i$'th $-$ sign, we define the* backwards generalised elementary move *$BE(i,j)$ to take the $j$'th $+$ sign, and all $+$ signs between it and the $i$'th $-$ sign, and move them to a position immediately after the $i$'th $-$ sign.*

It's clear that when the $i$'th $-$ sign and the $j$'th $+$ sign are in adjacent blocks, then the definitions of forwards and backwards generalised elementary moves reduce to forwards and backwards elementary moves. Thus this definition is indeed a generalization of elementary moves, and we may use the same notation without contradiction.

So, for instance, if $w = - - - + + - ++$, then $FE(2,3)$ produces the word $- + + + - - - + +$. The generalised elementary move $BE(4,1)$ produces the word $- - - - + + ++$.

Since forwards generalised elementary moves move $-$ signs to the right, and backwards generalised elementary moves move $+$ signs to the right, the following lemma is clear.

**Lemma 7.16 (Forwards moves move forward)** *If $w'$ is obtained from $w$ by a generalised forwards (resp. backwards) elementary move, then $w \preceq w'$ (resp. $w' \preceq w$).* ■

We have seen that a (forwards or backwards) elementary move on a word can be effected on a basis chord diagram by a single upwards bypass move along a (forwards or backwards) arc of attachment. It is also true that a (forwards or backwards) generalised elementary move can be effected by upwards bypass moves along (forwards or backwards) arcs of attachment — but more than one is required. We will now see how.

## 7.8 Generalised arcs of attachment

Recall that we have denoted forwards and backwards arcs of attachment $FA(i,j)$, $BA(i,j)$. The forward arc $FA(i,j)$ connects the base-$i$'th $-$ region to the root-$j$'th $+$ region. The backward arc $BA(i,j)$ connects the base-$j$'th $+$ region to the root-$i$'th $-$ region. We will now generalise this notion.

**Definition 7.17 (Generalised arc of attachment)** A generalised arc of attachment *in a chord diagram $(D, \Gamma)$ is an arc which intersects $\Gamma$ in an odd number of points. A generalised arc of attachment is* nontrivial *if all its intersection points with $\Gamma$ lie on different components of $\Gamma$. Two generalised arcs of attachment are considered equivalent if they are homotopic, with their intersection points with $\Gamma$ constrained to lie on $\Gamma$.*

For chord diagrams on a disc $(D, \Gamma)$, it is clear that for any two chords, there is at most one nontrivial generalised arc of attachment between them, up to equivalence. We will usually implicitly consider generalised arcs of attachment up to equivalence. We will therefore speak of *the* generalised arc of attachment between two chords.

We have notions of prior and latter chords, prior and latter outer regions, and forwards and backwards, as for bona fide arcs of attachment.

**Definition 7.18 (Prior and latter chords)** *Given a generalised arc of attachment on a basis chord diagram, consider the chords on which its endpoints lie.*

*(i) The chord which was created first in the base point construction algorithm, we call the* prior chord *of the generalised attaching arc.*

*(ii) The chord created later, we call the* latter chord *of the generalised attaching arc.*

**Definition 7.19 (Outer regions of a generalised arc of attachment)** *Let $(D, \Gamma)$ be a chord diagram and $c$ a generalised arc of attachment. The two regions of $D - \Gamma$ which are adjacent to chords of $\Gamma$ intersecting $c$, but which do not themselves intersect $c$, are called the* outer regions *of $c$.*

*(i) The outer region of $c$ adjacent to its prior chord is called the* prior outer region *of $c$.*

*(ii) The outer region of $c$ adjacent to its latter chord is called the* latter outer region *of $c$.*

Given a prior outer region and a latter outer region for $c$, it's clear that there is at most one chord between them. Hence we may speak of *the* generalised arc of attachment between the two regions.

**Definition 7.20 (Forwards and backwards generalised attaching arcs)**

*(i) A generalised arc of attachment whose prior outer region is negative, and latter outer region positive, is called a* forwards *generalised arc of attachment.*

*(ii) A generalised arc of attachment whose prior outer region is positive, and latter outer region negative, is called a* backwards *generalised arc of attachment.*

We can use similar notation — in fact, generalise the same notation — to describe generalised arcs of attachment. For a generalised arc of attachment, as for any nontrivial arc of attachment, the prior outer region is not the outermost region at the root point, and the latter outer region is not the outermost region at the base point.

**Definition 7.21 (Denoting generalised attaching arcs)**

*(i) The forwards generalised attaching arc whose prior outer region is the base-$i$'th $-$ region and whose latter outer region is the root-$j$'th $+$ region is called* the forwards $(i, j)$ generalised attaching arc $FA(i, j)$.

*(ii) The backwards generalised attaching arc whose prior outer region is the base-$j$'th $+$ region and whose latter outer region is the root-$i$'th $-$ region is called* the backwards $(i, j)$ generalised attaching arc $BA(i, j)$.

It's clear that this notation generalises the notation for attaching arcs. In fact, for any meaningful pair $(i, j)$, precisely one of $FA(i, j)$ or $BA(i, j)$ exists, as the next lemma makes clear.

**Lemma 7.22 (Existence of generalised attaching arcs)**

*(i) There is an $FA(i, j)$ in $\Gamma_w$ if and only if the $i$'th $-$ sign in $w$ occurs before the $j$'th $+$ sign.*

*(ii) There is a $BA(i, j)$ in $\Gamma_w$ if and only if the $j$'th $+$ sign in $w$ occurs before the $i$'th $-$ sign.*

*Thus, for every $i, j$ with $1 \leq i \leq n_-$ and $1 \leq j \leq n_+$, precisely one of $FA(i, j)$ or $BA(i, j)$ exists. Moreover, every nontrivial generalised attaching arc in $(D, \Gamma)$ is of the form $FA(i, j)$ or $BA(i, j)$ for some $(i, j)$.*

Proof Certainly, if $FA(i, j)$ exists, then its prior outer region is the base-$i$'th $-$ region, and its latter outer region is the root-$j$'th $+$ region; and by definition of prior and latter, and a little consideration of the relationship between the base and root algorithms, we see that the $i$'th $-$ sign must occur before the $j$'th $+$ sign.

Conversely, suppose the $i$'th $-$ sign occurs before the $j$'th $+$ sign. Then in the base point construction algorithm, the $i$'th $-$ sign produces a chord $\gamma_i$, enclosing a negative region $r_i$. The $j$'th $+$ sign produces a chord $\gamma'$, and the next symbol in $w$ (or the final chord drawn in the algorithm) produces a chord $\gamma_j$; this chord is produced in the root point algorithm by the $j$'th $+$ sign, and encloses a positive region $r_j$ in that algorithm. Since the base point algorithm produces $\gamma_i$ before $\gamma'$ before $\gamma_j$, it cannot be that $r_i$ and $r_j$ are adjacent. Therefore, there is a generalised attaching arc connecting $\gamma_i$ to $\gamma_j$.

The proof is similar for backwards attaching arcs. For the final statement, suppose we have a generalised attaching arc. It has a prior outer region and a latter outer region, one positive and one negative, which have numberings. ■

Clearly, a generalised attaching arc is not something that we can perform a bypass move on. But from it, we can obtain attaching arcs, forming a bypass system, and then we can perform bypass moves on them.

## 7.9 Bypass system of a generalised attaching arc

We will need now to consider multiple bypass moves on chord diagrams. Recall that a generalised attaching arc intersects the chords of our diagram at an odd number of points. Hence it may be broken into several bona fide attaching arcs, in a unique way. However they are not disjoint: they overlap at the endpoints. We can, however, break a generalised attaching arc into attaching arcs, and then perturb their endpoints in a specified way so that they are disjoint.

To make this precise, let $c$ be a generalised attaching arc in a basis chord diagram $(D, \Gamma_w)$. Let $p$ be an intersection point of $c$ with a chord $\gamma$ of $\Gamma_w$, at an interior point of $c$. Then there is a "prior" and a "latter" direction along $c$ from $p$, towards the endpoints of $c$ on its prior and latter chords, respectively. Also, since $c$ intersects distinct chords of $\Gamma_w$, other than $\gamma$, in both directions from $p$, $\gamma$ cannot be an outermost chord. Thus by our classification of chords in basis chord diagrams (lemma 6.9), $\gamma$ runs from the westside to the eastside of $(D, \Gamma)$. Hence, from $p$, there is a well-defined "west" and "east" direction along $\gamma$.

**Definition 7.23 (Bypass system of a generalised attaching arc)** *Let $c$ be a nontrivial generalised attaching arc in a basis chord diagram $(D, \Gamma_w)$ which intersects $\Gamma_w$ in $2m+1$ points. Then there is a unique way to split $c$ into a series of attaching arcs $c_1, \ldots, c_m$, labelled from prior chord to latter chord, which intersect each other only at the endpoints. The* bypass system *of $c$ is given as follows.*

*(i) If $c$ a forwards generalised attaching arc, then $c_1, \ldots, c_m$ are forwards attaching arcs, and we perturb them as follows. At the intersection point $p$ of $c_i$ and $c_{i+1}$ on a non-outermost chord $\gamma$ of $\Gamma_w$, we move the endpoint of $c_i$ slightly west of $p$, and the endpoint of $c_{i+1}$ slightly east of $p$.*

*(ii) If $c$ a backwards generalised attaching arc, then $c_1, \ldots, c_m$ are backwards attaching arcs, and we perturb them as follows. At the intersection point $p$ of $c_i$ and $c_{i+1}$ on a non-outermost chord $\gamma$ of $\Gamma_w$, we move the endpoint of $c_i$ slightly east of $p$, and the endpoint of $c_{i+1}$ slightly west of $p$.*

It's clear that this is indeed a bypass system. See figure 29 for an example. We now show that this edifice of definitions (and we have more to come!) is meaningful.

**Lemma 7.24 (Generalised attaching arcs and elementary moves)**

*(i) If we perform upwards bypass moves on $\Gamma_w$ along the bypass system of $FA(i,j)$, then we obtain $\Gamma_{w'}$, where $w' = FE(i,j)(w)$ .*

*(ii) If we perform downwards bypass moves on $\Gamma_w$ along the bypass system of $BA(i,j)$ then we obtain $\Gamma_{w'}$, where $w' = BE(i,j)(w)$*

$$\begin{array}{ccc} w & \overset{FE(i,j)}{\mapsto} & w' \\ \downarrow & & \downarrow \\ \Gamma_w & \overset{Up\,(FA(i,j))}{\mapsto} & \Gamma_{w'} \end{array} \quad \text{and} \quad \begin{array}{ccc} w & \overset{BE(i,j)}{\mapsto} & w' \\ \downarrow & & \downarrow \\ \Gamma_w & \overset{Down\,(BA(i,j))}{\mapsto} & \Gamma_{w'} \end{array}$$

PROOF From lemma 7.22, $FA(i,j)$ exists precisely when the $i$'th $-$ sign occurs before the $j$'th $+$ sign in $w$. First suppose that $j < n_+$, so there is a $(j+1)$'th $+$ sign in $w$. Let the substring of $w$ between the $i$'th $-$ sign and the $(j+1)$'th $+$ sign be $(-)^{a_1}(+)^{b_1} \cdots (+)^{b_{k-1}}(-)^{a_k}(+)^{b_k}$. Then the situation appears as shown in figure 29. (Note that the chords constructed prior to the base-$i$'th $-$ chord lie in regions $A$ or $B$ accordingly as the $i$'th $-$ sign is following or leading; similarly, the chords constructed after the root-$j$'th $+$ chord lie in $C$ or $D$ accordingly as the $j$'th $+$ sign is the last in its block, or not.)

Performing upwards bypass moves along the arcs of attachment produces the result shown, which corresponds to replacing the substring

$$(-)^{a_1}(+)^{b_1} \cdots (+)^{b_{k-1}}(-)^{a_k}(+)^{b_k}$$

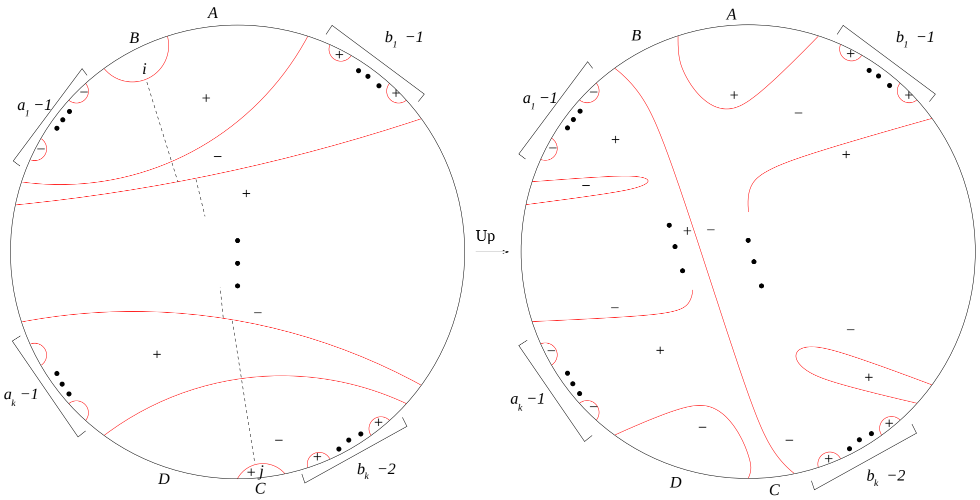



Figure 29: Effect of upwards bypass moves along a forwards generalised attaching arc.

of $w$ with

$$(+)^{b_1+\cdots+b_k-1}(-)^{a_1+\cdots+a_k}(+).$$

That is, all the minus signs from the $i$'th $-$ sign, up to the $j$'th $+$ sign, have been moved to the immediate right of the $j$'th $+$ sign.

If $j = n_+$, the the picture is similar; the generalised arc of attachment has an endpoint on the chord created in the root point algorithm as the rightmost $+$ sign in $w$ is processed. The effect is to move all minus signs, from the $i$'th onwards, to the end of the word.

The backwards case is similar. ∎

Loosely, the effect of performing upwards (resp. downwards) bypass moves along the bypass system of a forwards (resp. backwards) generalised attaching arc is to create one "long chord" running all along the generalised attaching arc, and "closing off" all the chords on either side of it.

## 7.10 Anatomy of multiple generalised arcs of attachment

We now consider taking several disjoint generalised arcs of attachment, and performing bypass moves along their bypass systems.

Note that for any two given forwards arcs of attachment $FA(i_1, j_1)$ and $FA(i_2, j_2)$, it is not always the case that there is only one way to place their bypass systems. A unique bypass system is not necessarily well defined. For one thing, the two generalised arcs might intersect. Even if they do not intersect, it might be that having placed $FA(i_1, j_1)$, we can place $FA(i_2, j_2)$ on either side of $FA(i_1, j_1)$; and the results of performing bypass moves along the resulting bypass systems might be different.

Thus, in placing several generalised arcs of attachment, we need to specify precisely how they are placed. To this end, let us make some definitions. We note that a forward generalised arc of attachment $FA(i, j)$, taken together with its prior and latter chords, split the disc $D$ into four regions; we group these into "southwest" and "northeast" halves as shown.

**Definition 7.25 (Compass points for forward generalised arc)** *A forward generalised arc of attachment $FA(i, j)$, taken together with its prior and latter chords, can be taken to split the disc $D$ into four pieces, proceeding clockwise around the disc:*

*(i) The piece containing the prior outer region of $FA(i, j)$.*

*(ii) The piece which contains the marked points on the eastside immediately anticlockwise/right of the latter chord of $FA(i, j)$*

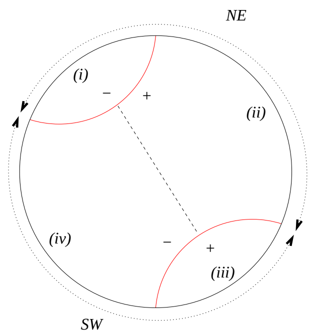



Figure 30: Compass points.

(iii) *The piece containing the latter outer region of $FA(i, j)$.*

(iv) *The piece which contains the marked points on the westside immediately anticlockwise/left of the prior chord of $FA(i, j)$.*

*Pieces (i) and (ii) are called the* northeast *of $FA(i, j)$. Pieces (iii) and (iv) are called the* southwest *of $FA(i, j)$.*

Note that all chords and regions constructed in the base point construction algorithm prior to the base-$i$'th $-$ region lie in the northeast of $FA(i, j)$, and all chords and regions created after the root-$j$'th $+$ region lie to the southwest.

Similar definitions exist in the backwards case. The compass points are chosen with our westside/eastside in mind, thinking of the base point as the "north pole" and the root point as the "south pole". See figure 30.

## 7.11 Placing two generalised arcs of attachment

These compass points can be used to define how to place multiple generalised arcs of attachment disjointly. We will consider the case of $FA(i_1, j_1)$ and $FA(i_2, j_2)$, where $i_1 < i_2$ and $j_1 \leq j_2$.

Recall that $FA(i_2, j_2)$ joins the base-$i_2$'th $-$ region to the root-$j_2$'th $+$ region. Now from the base point construction algorithm, since $i_1 < j_1$, the base-$i_2$'th $-$ region either lies

(i) entirely in the southwest of $FA(i_1, j_1)$; in this case there is no choice for the prior endpoint of $FA(i_2, j_2)$, up to equivalence; or

(ii) in both the southwest and northeast regions of $FA(i_1, j_1)$, and there is a choice: the prior endpoint of $FA(i_2, j_2)$ may be southwest or northeast of $FA(i_1, j_1)$.

If there is a choice, we choose southwest. See figure 31 showing case (ii).

Similarly, since $j_1 \leq j_2$, the root-$j_2$'th $+$ region either lies

(i) entirely in the southwest of $FA(i_1, j_1)$, and identical to the root-$j_1$'th $+$ region (i.e. $j_1 = j_2$); so there is a choice how to place the latter endpoint of $FA(i_2, j_2)$, which may be southwest or northeast of $FA(i_1, j_1)$ along the same latter chord;

(ii) entirely in the southwest of $FA(i_1, j_1)$ in region (iv) of figure 30; so there is no choice how to place the latter endpoint of $FA(i_2, j_2)$; or

(iii) entirely in the southwest of $FA(i_1, j_1)$ in region (iii) of figure 30, so $FA(i_2, j_2)$ might lie southwest or northeast of $FA(i_1, j_1)$.

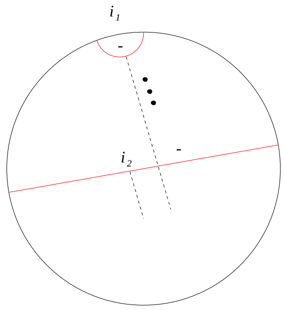



Figure 31: Placing two arcs: $i_1 < i_2$.

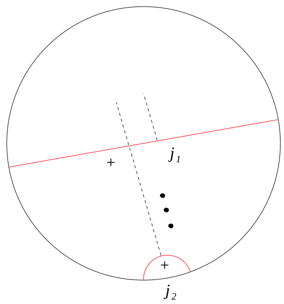



Figure 32: Placing two arcs: $j_1 < j_2$.

If there is a choice, we again take the southwest choice. See figure 32 showing case (iii).

Note, further, that the conditions $i_1 < i_2$ and $j_1 \leq j_2$, with our "southwest choices", ensure that the arcs can be drawn disjoint from each other: the inequalities ensure that the prior and latter chords at least partially lie in the southwest of $FA(i_1, j_1)$.

So, with the above choices if necessary, we have a well-defined way to draw $FA(i_2, j_2)$. Moreover, having drawn $FA(i_2, j_2)$, it is "southwestmost" in a certain sense.

**Lemma 7.26 (Placing two generalised arcs)** *Let $w$ be a word such that forward generalised elementary moves $FE(i_1, j_1)$ and $FE(i_2, j_2)$ exist, where $i_1 < i_2$ and $j_1 \leq j_2$. There is one, and only one way (up to equivalence) to draw $FA(i_1, j_1)$ and $FA(i_2, j_2)$ on $\Gamma_w$ so that*

*(i) $FA(i_1, j_1)$ and $FA(i_2, j_2)$ are disjoint;*

*(ii) for every chord of $\Gamma_w$ which intersects both $FA(i_1, j_1)$ and $FA(i_2, j_2)$, the intersection point with $FA(i_2, j_2)$ lies southwest of $FA(i_1, j_1)$.* ■

There is a similar result for backwards generalised attaching arcs also.

## 7.12 Placing nicely ordered multiple generalised arcs of attachment

We actually wish to consider multiple generalised arcs of attachment, which are "nicely ordered" in a similar way.

**Definition 7.27 (Nicely ordered generalised arcs of attachment)**

*(i) A sequence of forward generalised arcs of attachment*

$$FA(i_1, j_1),\ FA(i_2, j_2), \ldots, FA(i_m, j_m)$$

*on* $\Gamma_w$ *is* nicely ordered *if*

$$i_1 < i_2 < \cdots < i_m \quad \textit{and} \quad j_1 \leq j_2 \leq \cdots \leq j_m,$$

*(ii) A sequence of backward generalised arcs of attachment*

$$BA(i_1, j_1),\ BA(i_2, j_2), \ldots, BA(i_m, j_m)$$

*on* $\Gamma_w$ *is* nicely ordered *if*

$$j_m < j_{m_1} < \cdots < j_1 \quad \textit{and} \quad i_m \leq i_{m-1} \leq \cdots \leq i_1.$$

Given a set of generalised attaching arcs, which are nicely ordered in this way, we now describe how we should like to place them, and obtain a bypass system. The idea is that, for forwards arcs, if we keep making the "southwest" choice any time we are faced with a choice, then there is only one possibility for placing all the generalised arcs of attachment. For backwards arcs the result is similar, placing arcs from $BA(i_m, j_m)$ to $BA(i_1, j_1)$, always taking the "northwest" choice.

**Lemma 7.28** *Let* $FA(i_1, j_1), FA(i_2, j_2), FA(i_3, j_3)$ *be a nicely ordered sequence of forwards generalised attaching arcs. Suppose both* $FA(i_1, j_1)$ *and* $FA(i_3, j_3)$ *intersect a chord of* $\Gamma_w$*. Then so does* $FA(i_2, j_2)$*.*

PROOF In general, $FA(i, j)$ intersects precisely the following chords:

(i) The chord created by processing the $i$'th $-$ in $w$ in the base point construction algorithm (or the symbol immediately preceding it, in the root point construction algorithm). This is the prior chord.

(ii) The chord created by processing the $j$'th $+$ in $w$ in the root point construction algorithm (or the symbol immediately following it, in the base point construction algorithm). This is the latter chord.

(iii) The non-outermost chords created by processing "changes of sign" in $w$ in between: more precisely, processing the leading $-$ and $+$ signs strictly after the $j$'th $+$ sign and up to and including the $i$'th $-$ sign in the base point construction algorithm;

If both $FA(i_1, j_1)$ and $FA(i_3, j_3)$ intersect a chord $\gamma$, then in the base point construction algorithm, it is created in the base point construction algorithm by processing some symbol, which is one of:

(i) (from $FE(i_1, j_1)$) either the $i_1$'th $-$ sign, or a leading $-$ or $+$ sign after the $i_1$'th minus sign, up to and including the $j_1$'th $+$ sign, or the symbol immediately after the $j_1$'th $+$ sign;

(ii) (from $FE(i_3, j_3)$) either the $i_3$'th $-$ sign, or a leading $-$ or $+$ sign after the $i_3$'th minus sign, up to and including the $j_3$'th $+$ sign, or the symbol immediately after the $j_3$'th $+$ sign.

Combining these, since $i_1 < i_3$, and $j_1 \leq j_3$, $\gamma$ must be created by a symbol which is:

(i) the $i_3$'th $-$ sign, which is leading;

(ii) a leading $-$ or $+$ sign after the $i_3$'th $-$ sign, up to and including the $j_1$'th $+$ sign; or

(iii) the symbol immediately after the the $j_3$'th $+$ sign, and $j_1 = j_3$.

In every case, since $i_1 < i_2 < i_3$ and $j_1 \leq j_2 \leq j_3$, this chord $\gamma$ also intersects $FA(i_2, j_2)$. ■

So now, suppose we have a nicely ordered sequence of forwards generalised arcs of attachment

$$FA(i_1, j_1), \ldots, FA(i_m, j_m).$$

We know that we can place $FA(i_1, j_1)$, and then place $FA(i_2, j_2)$ "southwest" of it, as described in lemma 7.26. Then, we wish to place $FA(i_3, j_3)$. Let $\gamma$ be a chord of $\Gamma_w$ that intersects $FA(i_3, j_3)$. We note that the previous lemma governs how the forwards generalised arcs of attachment can intersect it. In particular, if either of $FA(i_1, j_1)$ or $FA(i_2, j_2)$ intersects $\gamma$, then $FA(i_2, j_2)$ certainly does. So if we require the intersection point of $FA(i_3, j_3)$ with $\gamma$ to be southwest of $FA(i_2, j_2)$, then it is also southwest of $FA(i_1, j_1)$.

Proceeding inductively, we obtain the following lemma.

**Lemma 7.29 (Placing multiple generalised attaching arcs)** *Let $FA(i_1, j_1)$, …, $FA(i_m, j_m)$ be a nicely ordered sequence of forwards generalised arcs of attachment. They can be placed on $\Gamma_w$, so that:*

*(i) they are disjoint;*

*(ii) for any chord $\gamma$ of $\Gamma_w$ which nontrivially intersects at least one of these arcs, the set of $FA(i_k, j_k)$ intersecting $\gamma$ is a discrete interval of $k$, of the form*

$$FA(i_s, j_s), FA(i_{s+1}, j_{s+1}), \ldots, FA(i_t, j_t)$$

*and moreover, for $u < v$, the intersection $\gamma \cap FA(i_v, j_v)$ lies southwest of $\gamma \cap FA(i_u, j_u)$;*

*(iii) none of the $FA(i_k, j_k)$ intersect the southwest region of $FA(i_m, j_m)$.*

*Moreover, there is only one way to place the arcs satisfying these conditions, up to equivalence.*

There is also a backwards version of this result.

PROOF For small $m$, we have proved the lemma. We show that we can inductively add a further $FA(i_m, j_m)$ to a previously constructed $FA(i_1, j_1), \ldots, FA(i_{m-1}, j_{m-1})$. We place $FA(i_m, j_m)$ to lie southwest of $FA(i_{m-1}, j_{m-1})$ as described in lemma 7.26. We note that, by inductive assumption (iii), there are no other arcs southwest of $FA(i_{m-1}, j_{m-1})$, and hence this specifies a unique way to place $FA(i_m, j_m)$; and disjointly, so (i) is true. Moreover, there are now no arcs southwest of $FA(i_m, j_m)$; so (iii) is true.

For every chord $\gamma$ nontrivially intersecting one of these arcs, if it does not intersect $FA(i_m, j_m)$, then (ii) is true by inductive assumption. Otherwise, it intersects $FA(i_m, j_m)$, and the intersection point $FA(i_m, j_m)$ is southwest of all others; so (ii) is true again. ∎

We give a name to this construction.

**Definition 7.30 (Bypass system of nicely ordered generalised attaching arcs)** *Let*

$$FA(i_1, j_1), \ldots, FA(i_m, j_m)$$

*be a nicely ordered sequence of forwards generalised arcs of attachment. The* bypass system of *this sequence is the bypass system obtained from placing these arcs as described in lemma 7.29.*

## 7.13 Generalised elementary moves of a comparable pair

We first extend the notion of "nicely ordered" to elementary moves.

**Definition 7.31 (Nicely ordered generalised elementary moves)**

*(i) A sequence of forward generalised elementary moves*

$$FE(i_1, j_1),\ FE(i_2, j_2), \ldots, FE(i_m, j_m)$$

*on $w$ is* nicely ordered *if*

$$i_1 < i_2 < \cdots < i_m \quad \textit{and} \quad j_1 \leq j_2 \leq \cdots \leq j_m,$$

*(ii) A sequence of backward generalised elementary moves*

$$BE(i_1, j_1),\ BE(i_2, j_2), \ldots, BE(i_m, j_m)$$

*on $w$ is* nicely ordered *if*

$$j_m < j_{m_1} < \cdots < j_1 \quad \textit{and} \quad i_m \leq i_{m-1} \leq \cdots \leq i_1.$$

Suppose we have two words comparable words $w_1, w_2 \in W(n_-, n_+)$. Then, for every $1 \leq i \leq n_-$, the $i$'th $-$ sign in $w_1$ lies to the left of the $i$'th $-$ sign in $w_2$. That is, the $i$'th $-$ sign in $w_1$ has fewer $+$ signs to its left, than does the $i$'th $-$ sign in $w_2$. Suppose that the $i$'th $-$ sign has $\alpha_i$ $+$ signs to its left in $w_1$, and $\beta_i$ $+$ signs to its left in $w_2$; so $\alpha_i \leq \beta_i$. So we have two sequences

$$\alpha_1 \leq \alpha_2 \leq \cdots \alpha_{n_-} \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \cdots \beta_{n_-},$$

which respectively describe $w_1$ and $w_2$.

Now, we can consider the generalised forwards elementary moves

$$FE(1, \beta_1), FE(2, \beta_2), \ldots, FE(n_-, \beta_{n_-})$$

on $w_1$. Note this is a nicely ordered sequence. The first move $FE(1, \beta_1)$ moves the first $-$ sign, and all $-$ signs between it and the $\beta_1$'th $+$ sign, to the immediate right of the $\beta_1$'th $+$ sign. If $\beta_1 = \beta_2$, then in a vague sense, "the move $FE(2, \beta_2)$ has already been done": all $-$ signs between the second $-$ sign and the $\beta_1 = \beta_2$'th $+$ sign have already been moved to the immediate right of the $\beta_1 = \beta_2$'th $+$ sign. If, however, $\beta_1 < \beta_2$, then the generalised forwards elementary move $FE(2, \beta_2)$ is still well-defined, and we may perform it. In fact, in this case, we note that, from $w_1$, the two moves $FE(1, \beta_1)$ and $FE(2, \beta_2)$ *commute* — they may be applied, one after the other, to $w_1$, being well-defined in every case, and yielding the same result, which is $(+)^{\beta_1}(-)(+)^{\beta_2 - \beta_1}(-)\cdots$.

These considerations immediately yield the following results; and similar results for backwards moves.

**Lemma 7.32 (Redundancy of generalised moves)** *Let $w$ be a word. Let $FE(i_1, j)$ and $FE(i_2, j)$ be well-defined forwards generalised elementary moves on $w$, with $i_1 < i_2$. Then:*

*(i) After applying $FE(i_1, j)$, then $FE(i_2, j)$ is no longer well-defined.*

*(ii) After applying $FE(i_2, j)$, then $FE(i_1, j)$ is still well-defined, and after applying it, the result is the same as simply applying $FE(i_1, j)$ alone:*

$$FE(i_1, j)\ (w) = FE(i_1, j) \circ FE(i_2, j)\ (w)$$ ■

In the case where $FE(i_2, j)$ is no longer well-defined, we may regard it as "the null move" and having trivial effect. Thus, we can extend the definition of $FE(i_2, j)$, to be the identity, where otherwise it is not defined. With this definition, we see that $FE(i_1, j)$ and $FE(i_2, j)$ commute, and their composition in either order is equal to $FE(i_1, j)$.

**Lemma 7.33 (Non-redundancy of generalised moves)** *Let $w$ be a word on which $FE(i_1, j_1)$ and $FE(i_2, j_2)$ are well-defined nontrivial forwards generalised elementary moves, with $i_1 < i_2$ and $j_1 < j_2$. Then:*

*(i) After applying either of $FE(i_1, j_1)$ or $FE(i_2, j_2)$ to $w$, the other is still well-defined and nontrivial.*

*(ii) The effect of applying both $FE(i_1, j_1)$ and $FE(i_2, j_2)$ to $w$, in either order, is identical:*

$$FE(i_1, j_1) \circ FE(i_2, j_2)\ (w) = FE(i_2, j_2) \circ FE(i_1, j_1)\ (w). \qquad \blacksquare$$

With the extended definition of $FE(i, j)$ and $BE(i, j)$ to be trivial when not otherwise defined, we obtain a general commutativity result:

**Lemma 7.34 (General commutativity of generalised moves)** *For any $i_1 \leq i_2$ and $j_1 \leq j_2$, then $FE(i_1, j_1)$ and $FE(i_2, j_2)$ commute.* ■

Thus, for any nicely ordered set of generalised forwards elementary moves, they all commute; and hence we may speak of applying them to a word, without regard to their order. There are forwards and backwards versions.

**Definition 7.35 (Generalised forwards elementary moves of a comparable pair)** *Let $w_1 \preceq w_2$.*

*(i) Let $\beta_i$ denote the number of $+$ signs to the left of the $i$'th $-$ sign in $w_2$. Then the* generalised forwards elementary moves of the pair *$(w_1, w_2)$ are*

$$FE(1, \beta_1), FE(2, \beta_2), \ldots, FE(n_-, \beta_{n_-}).$$

*(ii) Let $\alpha_j$ denote the number of $-$ signs to the left of the $j$'th $+$ sign in $w_1$. Then the* generalised backwards elementary moves of the pair *$(w_1, w_2)$ are*

$$BE(\alpha_1, 1), BE(\alpha_2, 2), \ldots, BE(\alpha_{n_+}, n_+).$$

Note that while $w_1 \preceq w_2$ means that "all $-$ signs move to the right", some may not move at all. In particular, if the $i$'th $-$ sign does not move, then the forwards generalised elementary move $FE(i, \beta_i)$ is trivial. If we like, we could delete such moves from the sequence; but since we define such moves to be trivial, it is easier to keep such moves in the sequence. (This is our way of "taking individual care" of each symbol, even though it might be redundant.)

Since we have a nicely ordered sequence of moves, by lemma 7.34, we may apply them to $w_1$ in any order and obtain the same result.

**Lemma 7.36 (Elementary moves between comparable words)**

*(i) The result of applying the generalised forwards elementary moves of the pair $(w_1, w_2)$, to $w_1$, is $w_2$.*

*(ii) The result of applying the generalised backwards elementary moves of the pair $(w_1, w_2)$, to $w_2$, is $w_1$.*

Proof After applying all the forwards moves, in any order, to $w_1$, then for all $1 \leq i \leq n_-$, the number of $+$ signs to the left of the $i$'th $-$ sign is $\beta_i$; thus we must have $w_2$. Similarly for backwards moves. ■

Thus we can go from $w_1$ to $w_2$ (and vice versa) by a well-defined nicely ordered sequence of generalised elementary moves. It now remains to show that this can be paralleled by bypass moves along the bypass system of a well-defined nicely ordered sequence of generalised arcs of attachment.

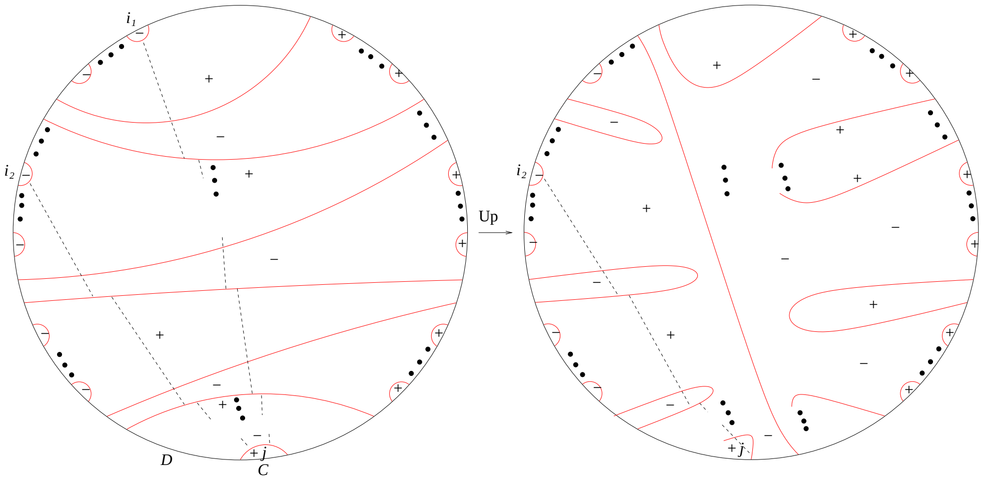



Figure 33: Redundancy with two generalised arcs.

## 7.14 Mechanics of bypass systems of nicely ordered sequences

We now consider in more detail the effect of performing bypass moves along the bypass system of a nicely ordered sequence of generalised attaching arcs. As we know from lemma 7.24, if we restrict to a single generalised attaching arc, then we obtain the basis chord diagram for the word obtained by performing a corresponding generalised elementary move.

We now see, in this subsection, that if we perform bypass moves along the bypass system of *multiple* generalised attaching arcs, when they are nicely ordered (as in definition 7.27) and then placed appropriately (as in definition 7.30), then we also obtain a basis chord diagram, and in particular the basis chord diagram for the word obtained by performing a composition of generalised elementary moves on the original word. More generally, all the lemmata of the previous section are paralleled by bypass systems of nicely ordered sequences of generalised attaching arcs.

In particular, in the previous section we proved (lemma 7.34) that generalised elementary moves in nicely ordered sequences commute — once we expand the definition a little to say that "when a generalised elementary move does not exist, it has trivial effect". A corresponding result is obvious for bypass moves on bypass systems: in a bypass system, the arcs of attachment are all disjoint, so the bypass moves on them obviously commute.

First, we consider redundancy.

**Lemma 7.37 (Redundancy of generalised attaching arcs)** *Let $\Gamma_w$ be a basis chord diagram on which forwards generalised arcs of attachment $FA(i_1, j)$ and $FA(i_2, j)$ exist, with $i_1 < i_2$. Then:*

*(i) After performing upwards bypass moves along the bypass system of $FA(i_1, j)$, then the bypass system of $FA(i_2, j)$ consists entirely of trivial bypasses. That is, performing upwards bypass moves along the bypass system $FA(i_2, j)$ has trivial effect.*

*(ii) After performing upwards bypass moves along the bypass system of $FA(i_2, j)$, then $FA(i_1, j)$ is still well-defined, and after applying it, the result is the same as simply applying $FE(i_1, j)$ alone:*

$$Up(FA(i_1, j))\ (\Gamma_w) = Up(FA(i_1, j)) \circ Up(FA(i_2, j))\ (\Gamma_w)$$

Proof This is a proof by picture; we draw the picture for the forwards case, and the backwards case is similar. See figure 33. ∎

We now consider this redundancy in more detail, so that it can be extended to the case of multiple attaching arcs.

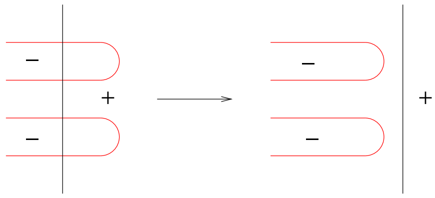



Figure 34: Pushing off trivial parts of a generalised attaching arc.

**Lemma 7.38 (Detailed redundancy in two generalised arcs)** *Let $FA(i_1, j_1)$, $FA(i_2, j_2)$ be a nicely ordered sequence of two forwards generalised attaching arcs on the basis chord diagram $\Gamma_w$. We write $FA_w(i_1, j_1), FA_w(i_2, j_2)$ to denote that they refer to $\Gamma_w$. After performing upwards bypass moves along $FA_w(i_1, j_1)$, we obtain a basis chord diagram $\Gamma_{w'}$, where $w' = FE(i_1, j_1)(w)$, by lemma 7.24.*

*On $\Gamma_{w'}$, $FA_w(i_2, j_2)$ may no longer be a nontrivial generalised attaching arc; but it is* equivalent *to the generalised attaching arc $FA_{w'}(i_2, j_2)$ on $\Gamma_{w'}$, in the following sense. If on $\Gamma_{w'}$, there is no forwards generalised attaching arc $FA_{w'}(i_2, j_2)$, then $j_1 = j_2$ and the bypass system of $FA_w(i_2, j_2)$ consists entirely of trivial arcs of attachment. Otherwise:*

*(i) The generalised attaching arc $FA_w(i_2, j_2)$ can be homotoped, rel endpoints, to $FA_{w'}(i_2, j_2)$.*

*(ii) This homotopy consists of finitely many (possibly none) local "pushing off" moves, of the sort depicted in figure 34.*

*(iii) Performing upwards bypass moves along the bypass system of $FA_w(i_2, j_2)$ or $FA_{w'}(i_2, j_2)$ gives the same chord diagram.*

Proof This is largely a proof by picture. Note that the chord created by the processing the $i_2$'th $-$ sign in $w$ or $w'$ (or any word for that matter), in the base point algorithm, is the chord emanating from the marked point $1 - 2i_2$, by lemma 6.9. Thus, even after performing bypass moves along the bypass system of $FA_w(i_1, j_1)$, $FA_w(i_2, j_2)$ still has an endpoint on the chord created by processing the $i_2$'th $-$ sign in $w$ in the base point construction algorithm; and similarly, it still has an endpoint on the chord created by processing the $j_2$'th $+$ sign in the root point construction algorithm. In particular, $FA_w(i_2, j_2)$ remains adjacent to the same marked points: it has the same "west end" for its prior chord and the same "east end" for its latter chord.

If $FE(i_2, j_2)$ does not exist in $w'$, but did exist in $w$, then we must have $j_1 = j_2$; otherwise the move $FE(i_1, j_1)$ would not move the $i_2$'th $-$ sign far enough, and $FE(i_2, j_2)$ would still have nontrivial effect. This is precisely the case when $FA(i_2, j_2)$ becomes redundant, as described above in lemma 7.37.

As we have seen, the effect of performing upwards bypass moves along the bypass system of a forwards generalised attaching arc is to create a "long chord", to close off outermost negative regions to the southwest, and to close off outermost positive regions to the northeast. Some of these outermost negative regions now have parts of $FA_w(i_2, j_2)$ inside them, and they are pushed off. After performing this homotopy, we certainly have $FA_{w'}(i_2, j_2)$.

As for the final claim, the effect of upwards bypass moves along the bypass systems before and after the homotopy are also best conveyed by picture; as in figure 33, there are many trivial bypasses, and by the principle expressed in figure 35, the effect is the same, after the "pushing off" homotopy of figure 34. ∎

We now consider a general nicely ordered sequence of forwards generalised attaching arcs.

**Lemma 7.39** *Consider the bypass system of the nicely ordered sequence of forwards generalised attaching arcs on the basis chord diagram $\Gamma_w$*

$$FA_w(i_1, j_1),\ FA_w(i_2, j_2),\ FA_w(i_m, j_m).$$

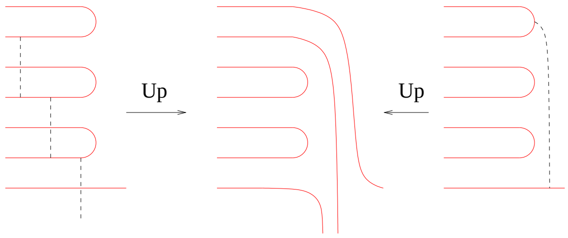



Figure 35: Pushing off makes no difference to effect of bypass moves.

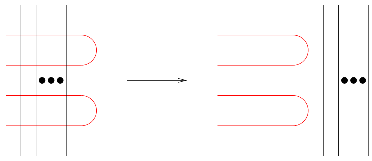


Figure 36: Pushing off trivial parts of multiple generalised attaching arcs.

*We use the subscript $w$ to denote that they refer to $\Gamma_w$. After performing upwards bypass moves along the bypass system of $FA_w(i_1, j_1)$, we obtain a basis chord diagram $\Gamma_{w'}$, where $w' = FE(i_1, j_1)(w)$, by lemma 7.24.*

*On $w'$, each of $FA_w(i_2, j_2), \ldots, FA_w(i_m, j_m)$ may no longer be a generalised attaching arc. If the forwards generalised elementary move $FE(i_k, j_k)$ does not exist on $w'$ then the bypass system of $FA_w(i_k, j_k)$ consists entirely of trivial arcs of attachment. But the other, nontrivial arcs are* equivalent *to the nontrivial generalised attaching arcs among $FA_{w'}(i_2, j_2), \ldots, FA_{w'}(i_m, j_m)$ on $\Gamma_{w'}$, in the following sense.*

*(i) The nontrivial arcs among $FA_w(i_2, j_2) \cup \cdots \cup FA_w(i_m, j_m)$ can be simultaneously homotoped, rel endpoints, to $FA_{w'}(i_2, j_2), \ldots, FA_{w'}(i_m, j_m)$, placed "northeast to southwest" as described in definition 7.30.*

*(ii) This homotopy consists of finitely many (possibly none) local "pushing off" moves, possibly "pushing several arcs off several chords at once", of the sort depicted in figure 36.*

*(iii) Performing upwards bypass moves on $\Gamma_{w'}$ along the bypass system of $FA_w(i_2, j_2)$, ..., $FA_w(i_m, j_m)$, or of $FA_{w'}(i_2, j_2)$, ..., $FA_{w'}(i_m, j_m)$, has the same effect.*

Proof This is again a proof by picture, except the pictures are a little more complicated than in the previous lemma. Again, the chords created by the processing the $i_2$'th $-$ sign in $w$ or $w'$ (or any word for that matter), in the base point algorithm, emanate from the same marked point $1 - 2i_2$, by lemma 6.9, so even after performing bypass moves along the bypass system of $FA_w(i_1, j_1)$, all the other $FA_w(i_k, j_k)$ have endpoints on the appropriate chords.

The picture of the local homotopy is similar, as now the outermost regions closed off in performing bypass moves along the bypass system of $FA_w(i_1, j_1)$ may now have parts of several $FA_w(i_k, j_k)$ inside them, but they can all be pushed off simultaneously; and after performing this homotopy, we have all the $FA_{w'}(i_k, j_k)$.

As for the final claim, it is again best conveyed by picture. The general arrangement is shown in figure 37.

We now easily obtain a complete analogy between multiple generalised elementary moves and multiple generalised attaching arcs.

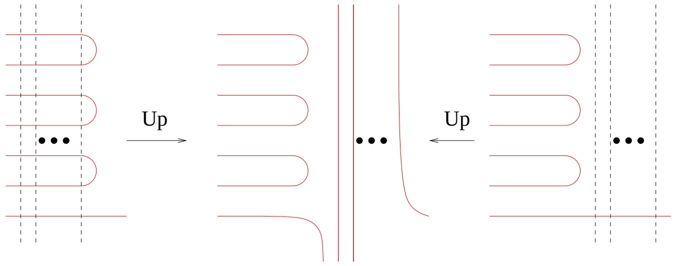



Figure 37: Pushing off several generalised attaching arcs makes no difference to effect of bypass moves.

**Lemma 7.40 (Multiple elementary moves and bypass systems)** *Suppose there is a nicely ordered sequence of forwards generalised elementary moves*

$$FE(i_1, j_1), \ldots, FE(i_m, j_m)$$

*on $w$, and equivalently, a nicely ordered sequence of forwards generalised attaching arcs*

$$FA(i_1, j_1), \ldots, FA(i_m, j_m)$$

*on $\Gamma_w$. If we perform upwards bypass moves along the bypass system of this nicely ordered sequence of forwards generalised attaching arcs, then we obtain $\Gamma_{w'}$, where $w'$ is obtained from $w$ by performing the above forwards generalised elementary moves.*

$$\begin{array}{ccc} w & \overset{FE(i_1,j_1)\circ\cdots\circ FE(i_m,j_m)}{\mapsto} & w' \\ \downarrow & & \downarrow \\ \Gamma_w & \overset{Up(FA(i_1,j_1),\ldots,FA(i_m,j_m))}{\mapsto} & \Gamma_{w'} \end{array}$$

Proof Proof by induction on $m$. For $m = 1$ it is true by lemma 7.24; and now consider the case for general $m$. We know again from lemma 7.24 that performing the bypass moves of $FA(i_1, j_1)$, we obtain $\Gamma_{w''}$, where $w'' = FE(i_1, j_1)(w)$. Then, by lemma 7.38, on $\Gamma_{w''}$, some of the arcs $FA_w(i_2, j_2)$, $\ldots$, $FA_w(i_m, j_m)$ are trivial (namely those with $j_k = j_1$), resulting in only trivial bypass moves; and the rest can be simultaneously homotoped to $FA_{w''}(i_2, j_2), \ldots, FA_{w''}(i_m, j_m)$, again placed properly "northeast to southwest", and in such a way that the effect of bypass moves along their bypass systems is unchanged. Then we are done by induction. ■

The same all applies in a backwards version.

## 7.15 Bypass system of a comparable pair

We have now built so much superstructure that we can almost use it. Consider two comparable words $w_1 \preceq w_2$. We have a nicely ordered sequence of generalised elementary moves of the pair; now we define some the generalised arcs and then a bypass system.

**Definition 7.41 (Generalised attaching arcs of a comparable pair)** *Let $w_1, w_2 \in W(n_-, n_+)$ with $w_1 \preceq w_2$.*

(i) *Let $\beta_i$ denote the number of $+$ signs to the left of the $i$'th $-$ sign in $w_2$. Then the* nicely ordered sequence of forwards generalised attaching arcs of the pair $(w_1, w_2)$ *is*

$$FA(1, \beta_1),\ FA(2, \beta_2),\ \ldots,\ FA(n_-, \beta_{n_-}).$$

(ii) *Let $\alpha_j$ denote the number of $-$ signs to the left of the $j$'th $+$ sign in $w_1$. Then the* nicely ordered sequence of backwards generalised attaching arcs of the pair $(w_1, w_2)$ *is*

$$BA(\alpha_1, 1),\ BA(\alpha_2, 2),\ \ldots,\ BA(\alpha_{n_+}, n_+).$$

It's clear that these are indeed nicely ordered sequences.

As noted for elementary moves, while "all $-$ signs move right", some may not move at all. If the $i$'th $-$ sign does not move, then we consider $FA(i, \beta_i)$ to be a null arc, with a null bypass system. For such $-$ signs at the start of the words, we might have $\beta = 0$. From all these generalised attaching arcs, we obtain a bypass system.

**Definition 7.42 (Bypass system of a comaparable pair)** *Let $w_1 \preceq w_2$ be comparable words.*

(i) *The* forwards bypass system $FBS(w_1, w_2)$ *of the pair $(w_1, w_2)$ is the bypass system of the nicely ordered sequence of forwards generalised attaching arcs of $(w_1, w_2)$.*

(ii) *The* backwards bypass system $BBS(w_1, w_2)$ *of the pair $(w_1, w_2)$ is the bypass system of the nicely ordered sequence of backwards generalised attaching arcs of $(w_1, w_2)$.*

Now, at last, we can prove proposition 1.23, that performing bypass moves along these bypass systems turns $\Gamma_{w_1}$ into $\Gamma_{w_2}$ and vice versa.

Proof (of proposition 1.23) By lemma 7.36, the corresponding sequences of generalised elementary moves take $w_1$ to $w_2$ and vice versa. The corresponding effect on basis chord diagrams is now immediate from lemma 7.40. ∎

And, we can now prove proposition 1.24; that performing bypass moves on these systems in the other direction gives a chord diagram whose basis decomposition has minimum $\Gamma_{w_1}$ and maximum $\Gamma_{w_2}$, with respect to $\preceq$.

Proof (of proposition 1.24) We first consider the forwards bypass system $FBS(w_1, w_2)$. From proposition 1.23, we know that performing upwards bypass moves along this system on $\Gamma_{w_1}$ gives $\Gamma_{w_2}$. As we perform these bypass moves, we may find that some are trivial. It is clear, however, that there is a *minimal* bypass system, consisting of $c_1, \ldots, c_m$, such that

(i) this bypass system contains no trivial attaching arcs,

(ii) performing upwards bypass moves along these attaching arcs gives $\Gamma_{w_2}$, and

(iii) performing upwards bypass moves along any proper subset of them does not give $\Gamma_{w_2}$.

(We say nothing about the uniqueness of this minimal bypass system.)

Now let $\Gamma$ be the chord diagram obtained by performing downwards bypass moves on $FBS(w_1, w_2)$. By the addition relation, the chord diagram obtained by a downwards bypass move is the sum of the chord diagram with an upwards bypass move, and the chord diagram with no bypass move. Thus $\Gamma$ is a sum of $2^m$ chord diagrams, corresponding to the $2^m$ subsets of $\{c_1, \ldots, c_m\}$, each a bypass system of forwards attaching arcs, upon which we perform upwards bypass moves. Each of these $2^m$ chord diagrams is a basis chord diagram, by lemma 7.14. They are not necessarily distinct. However, since each of the arcs of attachment is nontrivial, the chord diagram $\Gamma_{w_1}$ appears exactly once; and by minimality, $\Gamma_{w_2}$ appears exactly once also. For every other basis chord diagram $\Gamma_w$ which appears, it is obtained by $\Gamma_{w_1}$ by some sequence of upwards bypass moves along forwards attaching arcs; and then by attaching some more upwards bypass moves along forwards attaching arcs, we can obtain $\Gamma_{w_2}$; thus $w_1 \preceq w \preceq w_2$.

The effect of performing upwards bypass moves on $BBS(w_1, w_2)$ is similar. ∎

**Corollary 7.43** *For every pair $w_1 \preceq w_2$, there is a chord diagram such that, if we write it as a sum of basis chord diagrams, then $\Gamma_{w_1}$ is lexicographically the first and $\Gamma_{w_2}$ is lexicographically the last.* ∎

## 8 Proof of main theorems

We now prove theorem 1.22, that there is a bijection between chord diagrams and pairs of comparable words $w_1 \preceq w_2$. This map takes a chord diagram to the lexicographically first and last elements occurring in its basis decomposition.

Corollary 7.43 above shows that there is a map

$$\left\{ \begin{array}{c} \text{Comparable pairs of} \\ \text{words } w_1 \preceq w_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Chord} \\ \text{Diagrams} \end{array} \right\}$$

taking $(w_1, w_2)$ to a chord diagram in which $\Gamma_{w_1}$ is lexicographically the first in its basis decomposition, and $\Gamma_{w_2}$ the last. Since basis decompositions are unique, this map is clearly injective. Moreover, by proposition 1.20, proved in section 5, these two sets have the same cardinality. Thus we have a bijection.

This proves the main theorem 1.22. Moreover, it shows that every chord diagram can be constructed by the methods of the previous section. In particular, if we take any contact element and write it as a sum of basis vectors $v_w$, and take the lexicographically first and last basis elements $v_{w_-}, v_{w_+}$ among them, we have $w_- \preceq w \preceq w_+$. This proves theorem 1.21.

Having proved this, we may make a definition.

**Definition 8.1 ( Chord diagram notation )** *For any two basis chord diagrams $\Gamma^- \preceq \Gamma^+$, or words $w^-, w^+$, we write $[\Gamma^-, \Gamma^+]$ or $[w^-, w^+]$ to denote the unique chord diagram which has $\Gamma^-$ and $\Gamma^+$ respectively as first and last basis element.*

The theorem says that every chord diagram can be written uniquely as $[\Gamma^-, \Gamma^+]$ or $[w^-, w^+]$.

Note that in this definition, "first and last" could be according to the lexicographic order, the partial order $\preceq$, or even the right-to-left lexicographic order (reversing $\Gamma_-, \Gamma_+$). For $\preceq$ is just the intersection of a left-to-right and a right-to-left lexicographic ordering.

The theorem, along with the idea of proposition 1.24, give the following corollary.

**Corollary 8.2 (Upwards vs. downwards bypass moves)**

*(i) Suppose there is a bypass system $B$ on $\Gamma_{w_1}$, and $Up(B)(\Gamma_{w_1}) = \Gamma_{w_2}$. Then $Down(B)(\Gamma_{w_1}) = [\Gamma_{w_1}, \Gamma_{w_2}]$.*

*(ii) Suppose there is a bypass system $B$ on $\Gamma_{w_2}$, and $Down(B)(\Gamma_{w_2}) = \Gamma_{w_1}$. Then $Up(B)(\Gamma_{w_2}) = [\Gamma_{w_1}, \Gamma_{w_2}]$.*

PROOF As in the proof of proposition 1.24, take a minimal version of $B$. We have immediately $w_1 \preceq w_2$. Summing the downwards bypass system as a sum over all subsets of upwards bypasses, we see that we have $w_1$ as the minimal element occurring in this sum and $w_2$ the maximum. ∎

# Part IV
# Contact geometry applications

## 9 Contact-geometric criteria for chord diagrams

In this part, we give a construction from contact geometry, and we use what we now know about the combinatorics of chord diagrams, in order to obtain contact-geometric information.

### 9.1 Definition of $\mathcal{M}(\Gamma_0, \Gamma_1)$ and $m(\Gamma_0, \Gamma_1)$

Now, suppose we have two chord diagrams $\Gamma_0, \Gamma_1$, both with the same number of chords $n$, and with marked base points (and hence root points). Then we consider the cylinder $D \times I$, where $D$ is a disc

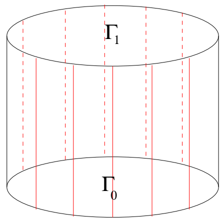



Figure 38: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

and $I = [0,1]$. Its boundary is $(D \times \{0\}) \cup (\partial D \times I) \cup (D \times \{1\})$. We now draw some curves on this boundary. We draw the chord diagram $\Gamma_0$ on $D \times \{0\}$, and we draw $\Gamma_1$ on $D \times \{1\}$. We do this so that the marked points are aligned at points $\{p_i\} \times \{0,1\}$, and the base points are aligned at points $\{p_0\} \times \{0,1\}$. We then choose $2n$ points $\{q_i\}$ on $\partial D$, evenly spaced between successive $p_i$; and we draw the $2n$ curves $\{q_i\} \times [0,1]$ on $\partial D \times I$.

We will usually think of the $[0,1]$ direction as the "up / down" direction, where the positive direction is up and the negative direction is down.

We now note that $D \times I$ can be considered as a contact 3-ball, with boundary dividing set given by $\Gamma$. The "corners" along $\partial D \times \{0,1\}$ can be made Legendrian, and the two surfaces intersecting along these corners have interleaving dividing sets as required by the contact geometry. Thus we may round the corners and obtain a 3-ball $B$ with a dividing set (still denoted $\Gamma$ in abusive notation). If $\Gamma$ on this rounded ball has is connected, then we can take the unique tight contact structure (up to isotopy rel boundary) on $B$ with dividing set $\Gamma$. If $\Gamma$ is disconnected, then any contact structure on the ball with this boundary dividing set is overtwisted. Such contact structures can be considered equally as structures on a rounded ball, or the cylinder ("ball with corners") $D \times I$.

The manifold so obtained is really a sutured manifold, but we can call it "tight" or "overtwisted".

**Definition 9.1 ($\boldsymbol{\mathcal{M}(\Gamma_0, \Gamma_1)}$)** *Given two chord diagrams $\Gamma_0, \Gamma_1$ with $n$ chords, the sutured manifold $\mathcal{M}(\Gamma_0, \Gamma_1)$ is $D \times I$ with sutures*

$$(\Gamma_0 \times \{0\}) \ \cup \ ((\cup\{q_i\}) \times I) \ \cup \ (\Gamma_1 \times \{1\}).$$

*See figure 38.*

**Definition 9.2 (Tight/overtwisted $\boldsymbol{\mathcal{M}(\Gamma_0, \Gamma_1)}$)** *If $\mathcal{M}(\Gamma_0, \Gamma_1)$ is* tight *if it admits a tight contact structure, i.e. if after rounding corners, the sutures of $\mathcal{M}(\Gamma_0, \Gamma_1)$ are connected. Otherwise we say $\mathcal{M}(\Gamma_0, \Gamma_1)$ is* overtwisted.

**Definition 9.3 (Stackable)** *We say that a chord diagram $\Gamma_1$ is* stackable *on another chord diagram $\Gamma_0$ if $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight.*

Note that these definitions are purely combinatorial, but they have contact geometric meaning. (In some sense, all 3-dimensional contact geometry is combinatorics — at least, to the extent that 3-dimensional contact geometry is about convex surfaces and dividing sets, and to the extent convex surfaces and dividing sets are combinatorics.)

We can also define a map $m$ from the TQFT-property of $SFH$. We note that the sutures of $\mathcal{M}(\Gamma_0, \Gamma_1)$ lie on a $\partial B^3 = S^2$ and are simply a gluing of the two chord diagrams $\Gamma_0, \Gamma_1$, with some curves running between them, as shown in figure 38.

Now suppose we remove a small neighbourhood of a point on one of the curves running between $\Gamma_0, \Gamma_1$. Then in our $(1+1)$-dimensional "almost TQFT", taking the product of all these with $S^1$, we can regard this construction as arising from an inclusion of two solid tori (each with $2n$ boundary sutures) into a single solid torus (with 2 boundary sutures), i.e.

$$(T, n) \cup (T, n) \hookrightarrow (T, 1),$$

and with a specified contact structure on the intermediate (pants) $\times S^1$. Hence there is a map

$$m : SFH(T, n, e) \otimes SFH(T, n, e) \longrightarrow SFH(T, 1, 0) = \mathbb{Z}_2.$$

Two chord diagrams $\Gamma_0, \Gamma_1$ correspond to contact elements in $SFH(T, n, e)$ and hence give a contact element in $SFH(T, 1, 0)$. This is an overtwisted contact structure, in the case that $\mathcal{M}(\Gamma_0, \Gamma_1)$ has disconnected sutures, i.e. is overtwisted, and hence gives contact element 0. Otherwise, it is the unique tight contact structure on $(T, 1)$, in the case that $\mathcal{M}(\Gamma_0, \Gamma_1)$ has connected sutures, i.e. is tight, and then gives the contact element 1.

That is, $m(\Gamma_0, \Gamma_1) \in \mathbb{Z}_2$ is 0 or 1, respectively as $\mathcal{M}(\Gamma_0, \Gamma_1)$ is overtwisted or tight. In fact, the map $m$ could also be defined purely combinatorially with this as the definition, using the combinatorial definition of $SFH$. We have therefore proved lemma 1.25.

**Lemma (Stackability map)** *There is a linear map*

$$m : SFH(T, n, e) \otimes SFH(T, n, e) \longrightarrow \mathbb{Z}_2$$

*which takes pairs of contact elements, corresponding to pairs of chord diagrams* $(\Gamma_0, \Gamma_1)$*, to* 0 *or* 1 *respectively as* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is overtwisted or tight.* ∎

Note that although $m$ is bilinear, it is not symmetric. There are many pairs of chord diagrams $\Gamma_0, \Gamma_1$ such that $m(\Gamma_0, \Gamma_1) = m(\Gamma_1, \Gamma_0) = 0$; we will shortly see that $m(\Gamma, \Gamma) = 1$; and we will also see that among the basis elements, there are many pairs for which $m(\Gamma_0, \Gamma_1) = 0$ and $m(\Gamma_1, \Gamma_0) = 1$.

## 9.2 Properties of $\mathcal{M}(\Gamma, \Gamma)$ and $m(\Gamma, \Gamma)$

**Lemma 9.4 ($\mathcal{M}(\Gamma, \Gamma)$ tight)** *For any chord diagram* $\Gamma$*,* $\mathcal{M}(\Gamma, \Gamma)$ *is tight. That is,* $m(\Gamma, \Gamma) = 1$*.*

We give two proofs. We include the second, less elegant one, because it is similar in spirit to some subsequent proofs.

Proof (# 1) Since the chord diagram has no closed loops, let alone closed loops bounding discs, there is a tight $I$-invariant contact structure on $D^2 \times I$ such that each slice $D^2 \times \{\cdot\}$ is convex with dividing set $\Gamma$, and such that $\partial D^2 \times \{\cdot\}$ is legendrian. This contact structure on $D^2 \times I$ can be given convex boundary, so that the dividing set is identical to that of $\mathcal{M}(\Gamma, \Gamma)$; and hence $\mathcal{M}(\Gamma, \Gamma)$ is just a standard tight neighbourhood of the convex disc $(D^2, \Gamma)$ ∎

We can alternatively prove the result by looking at the dividing set on $\mathcal{M}(\Gamma, \Gamma)$ the in more detail, and showing explicitly that it is connected.

Proof (# 2) Proof by induction on $|\#\Gamma|$, the number of components in $\Gamma$, i.e. the number of chords in the chord diagram. For $|\#\Gamma| = 1$, there is only one chord diagram possible on the disc; and the dividing set on $\partial \mathcal{M}(\Gamma, \Gamma)$, after rounding corners, is obviously connected.

Now consider a general $\Gamma$. Let $\gamma$ be an outermost chord of $\Gamma$. Thus we may consider $\Gamma - \gamma$ to be a chord diagram with $|\Gamma| - 1$ components. We reduce the case of $\mathcal{M}(\Gamma, \Gamma)$ to the case of $M(\Gamma - \gamma, \Gamma - \gamma)$, hence proving the result by induction.

Note that $\gamma \times \{1\}$, on the boundary of $\mathcal{M}(\Gamma, \Gamma)$ has two endpoints; and after rounding corners, one of these is connected to an endpoint of $\gamma \times \{0\}$. Thus on the rounded ball we have a connected arc $c$, part of the dividing set, of the form $c = c_1 \cup (\gamma \times \{1\}) \cup c_2 \cup (\gamma \times \{0\}) \cup c_3$, where the $c_i$ are (rounded versions of) arcs $q_i \times [0, 1]$ of $M(\Gamma, \Gamma)$. But now folding corners in a slightly different way, we see that this is equivalent to the dividing set of $M(\Gamma - \gamma, \Gamma - \gamma)$; where all of $c$ becomes one of the $q_i \times [0, 1]$ arcs. See figure 39.

That is, the contact manifolds $\mathcal{M}(\Gamma, \Gamma)$ and $\mathcal{M}(\Gamma - \gamma, \Gamma - \gamma)$ are contactomorphic. ∎

The argument of this proof is useful in its own right, so we record it.

**Lemma 9.5 (Cancelling corresponding outermost chords)** *Suppose* $\Gamma_0, \Gamma_1$ *each has an outermost chord* $\gamma_0, \gamma_1$ *in the same position. Then* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight iff* $\mathcal{M}(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1)$ *is tight. That is,* $m(\Gamma_0, \Gamma_1) = m(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1)$*.* ∎

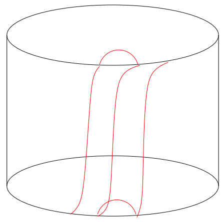


Figure 39: $\mathcal{M}(\Gamma, \Gamma)$ with some edge-rounding.

## 9.3 Bypass-related $\Gamma_0, \Gamma_1$

If our $\Gamma_0, \Gamma_1$ are related by a bypass move, then we can easily determine whether $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight or overtwisted. Bypass-related chord diagrams come naturally in triples, and there are 6 possibilities among them for our construction. We can determine all of them.

**Lemma 9.6** *Let $c$ be a nontrivial arc of attachment on a chord diagram $\Gamma$. Then*

*(i) all of $\mathcal{M}(\Gamma, Up_c(\Gamma))$, $\mathcal{M}(Up_c(\Gamma), Down_c(\Gamma))$ and $\mathcal{M}(Down_c(\Gamma), \Gamma)$ are tight,*

$$m(\Gamma, Up_c(\Gamma)) = m(Up_c(\Gamma), Down_c(\Gamma)) = m(Down_c(\Gamma), \Gamma) = 1;$$

*(ii) all of $\mathcal{M}(\Gamma, Down_c(\Gamma))$, $\mathcal{M}(Down_c(\Gamma), Up_c(\Gamma))$, $\mathcal{M}(Up_c(\Gamma), \Gamma)$ are overtwisted,*

$$m(\Gamma, Down_c(\Gamma)) = m(Down_c(\Gamma), Up_c(\Gamma)) = m(Up_c(\Gamma), \Gamma) = 0.$$

Proof We note that bypasses come in triples: with the three chord diagrams under consideration we have the following.

$$\begin{array}{ccccc} & & \Gamma & & \\ & \nearrow & & \searrow & \\ \mathrm{Down}_c(\Gamma) & & \leftarrow & & \mathrm{Up}_c(\Gamma) \end{array}$$

Here the arrows denote upwards bypass moves. So in fact it is sufficient to prove $\mathcal{M}(\Gamma, \mathrm{Up}_c(\Gamma))$ is tight, since all three claimed tight manifolds are of this form; and that $\mathcal{M}(\Gamma, \mathrm{Down}_c(\Gamma))$ is overtwisted, since all three claimed overtwisted manifolds are of this form.

If $\Gamma$ has only 3 chords, then there is only one possible arrangement (up to rotation of the cylinder) for both $\mathcal{M}(\Gamma, \mathrm{Up}_c(\Gamma))$ and $\mathcal{M}(\Gamma, \mathrm{Down}_c(\Gamma))$; the result is then true by inspection. If there are more than three chords, then we see that there must be some outermost chord $\gamma$ which is disjoint from $c$; hence it appears in all of $\Gamma$, $\mathrm{Up}_c(\Gamma)$ and $\mathrm{Down}_c(\Gamma)$. Applying lemma 9.5, the claimed manifold is tight if and only if the same is true for $\Gamma - \gamma$, $\mathrm{Up}_c(\Gamma) - \gamma$ and $\mathrm{Down}_c(\Gamma) - \gamma$. Repeatedly applying lemma 9.5 we reduce to the case of 3 chords. ∎

This lemma has a more purely contact interpretation. Recall that $\mathcal{M}(\Gamma, \Gamma)$ is a tight contact 3-ball. The lemma says that every nontrivial arc of attachment $c$ on $\Gamma \times \{0\}$ or $\Gamma \times \{1\}$ is outer, according to the definition of section 2.4.

A weaker statement can be proved purely algebraically: that if $\Gamma_0, \Gamma_1$ are bypass-related then precisely one of $\mathcal{M}(\Gamma_0, \Gamma_1)$, $\mathcal{M}(\Gamma_1, \Gamma_0)$ is tight. This statement is equivalent to

$$m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0) = 1.$$

For the third bypass in their triple is $\Gamma_0 + \Gamma_1$, and from lemma 9.4 we have $m(\Gamma_0, \Gamma_0) = m(\Gamma_1, \Gamma_1) = m(\Gamma_0 + \Gamma_1, \Gamma_0 + \Gamma_1) = 1$. So

$$\begin{aligned} 1 &= m(\Gamma_0 + \Gamma_1, \Gamma_0 + \Gamma_1) \\ &= m(\Gamma_0, \Gamma_0) + m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0) + m(\Gamma_1, \Gamma_1) \\ &= m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0). \end{aligned}$$

# 10 Contact-geometric interpretation of the partial order $\preceq$

We now consider $\mathcal{M}(\Gamma_0, \Gamma_1)$ where each $\Gamma_i$ is a *basis* chord diagram. We shall also require that $\Gamma_0, \Gamma_1$ have the same number $n+1$ of chords and the same relative euler class $e$. So, each chord diagram corresponds to a word $w \in W(n_-, n_+)$.

We ask when $\mathcal{M}(w_0, w_1)$ is tight, i.e. when $\Gamma_{w_1}$ is stackable on $\Gamma_{w_0}$. In this section, we will prove proposition 1.26:

**Proposition** *$\mathcal{M}(w_0, w_1)$ is tight if and only if $w_0 \preceq w_1$.*

## 10.1 Easy direction

**Lemma 10.1** *If $w_0$ does not precede $w_1$ with respect to $\preceq$, then $\mathcal{M}(w_0, w_1)$ is overtwisted.*

Proof By the "baseball interpretation", there is some point in the game, playing innings from left to right, where team 0 takes the lead. Hence there is a point in the game, after the $m$'th innings, where team 0 moves precisely one step ahead. That is, there is some $m$ such that in $w_0$, there are $i$ minus signs and $j$ plus signs up to the $m$'th position, but in $w_1$ there are $i+1$ minus signs and $j-1$ plus signs up to the $m$'th position. Moreover, since team 0 just took the lead, the $m$'th symbol in $w_0$ is a $+$, while in $w_1$ the $m$'th symbol is a $-$.

By lemma 6.9 then, after the $m$'th stage of the base point algorithm, in $\Gamma_0$ the discrete interval of used marked points is $[-2i, 2j-1]$, while in $\Gamma_1$ the discrete interval of used marked points is $[1-2(i+1), 2(j-1)] = [-2i-1, 2j-2]$. After rounding corners, the chords with endpoints in these intervals precisely match. Since the $m$'th stage is not the final stage, the curves on the rounded ball have several components. Thus $\mathcal{M}(w_0, w_1)$ is overtwisted. ∎

## 10.2 Preliminary cases

**Lemma 10.2** *If $\mathcal{M}(w_0, w_1)$ is overtwisted, then a separate component of the dividing set can be observed in constructing the basis chord diagrams $\Gamma_0, \Gamma_1$ with the base point construction algorithm, before the final step.*

Proof We know that, after rounding, we have a system of curves on $S^2$ and the total euler class is 0; hence there is an odd number of components; hence at least 3 components. Thus there is some component $\gamma$ that intersects neither the root point on $\Gamma_0$ nor the root point on $\Gamma_1$. On $\mathcal{M}(w_0, w_1)$, $\gamma$ contains some of the chords on $\Gamma_0$, and some on $\Gamma_1$, but no chords with endpoints at either root point. Thus, a separate component $\gamma$ can be observed at some stage of the base point algorithm before the final step. ∎

**Lemma 10.3** *The proposition for $w_0, w_1$ beginning with the same symbol, $w_0 = sw_0'$, $w_1 = sw_1'$, where $s \in \{+, -\}$, reduces to the proposition for $w_0', w_1'$, i.e. shorter words.*

Proof We note that, by lemma 9.5, $\mathcal{M}(w_0, w_1)$ is contactomorphic to $\mathcal{M}(w_0', w_1')$, through rounding and re-folding. And clearly $w_0 \preceq w_1$ iff $w_0' \preceq w_1'$. ∎

Thus, we may assume that $w_0$ begins with a $-$ and $w_1$ begins with a $+$.

## 10.3 Proof of proposition

We now suppose that $w_0 \preceq w_1$, and show that $\mathcal{M}(w_0, w_1)$ is tight.

We will prove this by induction on the length $n$ of the words $w_0$ and $w_1$. It is clearly true by inspection for words of length 1 and 2; we now assume it is true for all lengths less than $n$, and consider words $w_0, w_1$ of length $n$.

By lemma 10.2, we know that if $\mathcal{M}(w_0, w_1)$ is overtwisted, we will see a closed loop before the final stage of the base point construction algorithm.

We will show by induction on $m$, that no closed loop appears at the $m$'th stage of the base point construction algorithm, before the final step. By lemma 10.3 and lemma 10.1, we can assume $w_0$ begins with a $-$ and $w_1$ begins with a $+$; so no closed loop appears at the first stage; the result is true for $m = 1$. At the $m$'th stage of the algorithm, let $[a_m, b_m]$ denote the discrete interval of used marked points on $\Gamma_0$ and $[c_m, d_m]$ on $\Gamma_1$. The hypothesis $w_0 \preceq w_1$ means that for all $m$, $a_m - 1 < c_m$ and (equivalently) $b_m - 1 < d_m$.

So now suppose that there is no closed loop at any stage before $m$, but a closed loop appears at stage $m$. At the previous $(m-1)$'th stage, we had discrete intervals of used marked points $[a_{m-1}, b_{m-1}]$ and $[c_{m-1}, d_{m-1}]$. Let us examine what can happen at the $m$'th stage.

On $\Gamma_0$, there are three possible positions for the chord added at the $m$'th step of the algorithm. These three possibilities connect the pairs of marked points $(a_{m-1}-2, a_{m-1}-1)$, $(a_{m-1}-1, b_{m-1}+1)$, or $(b_{m-1}+1, b_{m-1}+2)$. Similarly, on $\Gamma_1$ there are three possible positions for the new chord. At least one of these new chords must form part of the new closed loop; else it would have appeared earlier. We will assume that the new chord on $\Gamma_0$ is part of the new closed loop; the case where the new chord lies on $\Gamma_1$ is similar.

Let this new chord on $\Gamma_0$, added at the $m$'th stage, be $\gamma_m$. Note $\gamma_m$ cannot include the marked point $a_m$, since $a_m$ on $D \times \{0\}$ connects to $a_m - 1$ on $D \times \{1\}$, and $a_m - 1 < c_m$, so this is left of all used points of $\Gamma_1$ at this stage, and cannot form a closed loop. Thus, the $\gamma_m$ is $(b_{m-1}+1, b_{m-1}+2) = (b_m - 1, b_m)$, and it forms part of a closed loop.

We see that $\gamma_m$ encloses an outermost region on the eastside of $\Gamma_0$, hence a positive outermost region. Hence it is constructed by processing a following $+$ sign in $w_0$. (If $w_0$ begins with a $+$, this also creates a positive outermost region, but we have dealt with the case $m = 1$.) Let this be the $j$'th $+$ sign in $w$, so using lemma 6.9, $(b_m - 1, b_m) = (2j-2, 2j-1)$ Thus $w_0 = u + +v$, where $u$ (possibly empty) contains $j - 2$ plus signs.

Now, the endpoints of $\gamma_m$ on $\Gamma_0$ connects to the two marked points $\{b_m - 2, b_m - 1\} = \{2j-3, 2j-2\}$ on $\Gamma_1$. We have $d_m > b_m - 1$, so these are not the rightmost points among the used points on $\Gamma_1$, at this $m$'th stage. Moreover, the closed loop we have just created cannot involve any of the points right of $2j - 2 = b_m - 1$ on $\Gamma_1$, since these points cannot to marked points right of $b_m$ on $\Gamma_0$, which have not been used yet.

Thus, the chord emanating from $2j - 2$ on $\Gamma_1$ must go to the westside, enclosing a $-$ region, and must be created by processing a leading $-$ symbol in $w_1$. And the chord emanating from $2j - 3$ on $\Gamma_1$, by lemma 6.9, is created by processing the $(j-1)$'th $+$ sign in $w_1$. Thus $w_1 = y + -z$, where $y$ (possibly empty) contains $j - 2$ plus signs.

Now, rounding the ball and refolding, we may perform a "finger move", pushing the whole new chord $\gamma_m$ off $D \times \{0\}$ and down to $D \times \{1\}$, which has the effect of removing $\gamma_m$ from $D \times \{0\}$, and connecting up the marked points labelled $2j-3, 2j-2$ on $D \times \{1\}$. See figure 40.

The chord diagram on $D \times \{0\}$ then reduces to the chord diagram for $w_0' = u + v$, deleting the $(j-1)$'th $+$ sign from $w_0$. The chord diagram on $D \times \{1\}$ reduces to the chord diagram for $w_1' = y - z$, deleting the $(j-1)$'th $+$ sign. Thus the situation reduces to $\mathcal{M}(w_0', w_1')$ for two smaller words obtained from deleting the $(j-1)$'th $+$ sign both $w_0$ and $w_1$; since we deleted the same numbered $+$ signs, $w_0' \preceq w_1'$. But we know that the proposition is true for all smaller length words, so $\mathcal{M}(w_0', w_1')$ is tight; so there cannot be any closed loop, and we have a contradiction.

Thus, adding the $m$'th chord in the base point construction algorithm for $\Gamma_0$ and $\Gamma_1$, we never see a closed loop. By induction, at every stage before the end of the algorithm, we never see a closed loop. Hence there are no closed loops, and $\mathcal{M}(w_0, w_1)$ is tight.

This concludes the proof of proposition 1.26

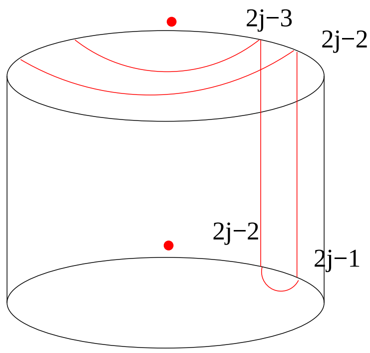



Figure 40: Finger move on $\mathcal{M}(\Gamma_0, \Gamma_1)$.

## 10.4 Which chord diagrams are stackable?

Notice that the proposition of the previous section gives the value of $m$ on all basis elements. This defines $m$ precisely, and as discussed in section 9.1, $m$ describes stackability. Thus, we can put this together to give an answer to the general question: given two chord diagrams $\Gamma_0, \Gamma_1$, is $\Gamma_1$ stackable on $\Gamma_0$?

We give an answer by proving proposition 1.27:

**Proposition (General stackability)** *Let $\Gamma_0, \Gamma_1$ be chord diagrams with $n$ chords and relative euler class $e$. Then $\Gamma_1$ is stackable on $\Gamma_0$ (i.e. $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight) if and only if the cardinality of the set*

$$\left\{ (w_0, w_1) \; : \; \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in decomposition of } \Gamma_i \end{array} \right\}$$

*is odd.*

Proof The tightness of $\mathcal{M}(\Gamma_0, \Gamma_1)$ is equivalent to $m(\Gamma_0, \Gamma_1) = 1$, and we simply now expand this out as a sum over basis elements. ∎

## 10.5 Contact interpretation of $\Gamma^-, \Gamma^+$

Let us consider the decomposition algorithms of section 6.4. Note that these algorithms decompose $\Gamma$ into its basis elements by performing bypass moves. In the algorithm, these bypass moves were considered purely combinatorially; but of course we can consider the contact manifolds obtained by attaching such bypasses. Every basis element is obtained by attaching some bypasses to $\Gamma$ along disjoint arcs of attachment, i.e. along a basis system. Most basis elements are obtained by attaching some bypasses above and some below. But $\Gamma^-$ is obtained by *attaching only upwards bypasses*, and is the only such basis element in $\Gamma$. Similarly, $\Gamma^+$ is the one and only basis element in $\Gamma$ obtained by *attaching only downwards bypasses*. Thus the basis decomposition algorithm (either one) naturally constructs $\mathcal{M}(\Gamma, \Gamma^-)$ and $\mathcal{M}(\Gamma^+, \Gamma)$. However it does this by attaching bypasses along arcs that may in general intersect; it is not a nice bypass system.

Nonetheless, we immediately see that there are stackability relations:

**Lemma 10.4** $\mathcal{M}(\Gamma, \Gamma^-)$ *and* $\mathcal{M}(\Gamma^+, \Gamma)$ *are tight,* $m(\Gamma, \Gamma^-) = m(\Gamma^+, \Gamma) = 1$.

Proof Expand out over basis elements: the only $\Gamma_w$ in the decomposition of $\Gamma$ satisfying $\Gamma_w \preceq \Gamma^-$ is $\Gamma_w = \Gamma^-$ itself, and similarly for $\Gamma^+$. ∎

This result can also be proved directly by considering the dividing set on $\mathcal{M}(\Gamma, \Gamma^-)$ and successively performing rounding and un-rounding of corners and isotopies. Each chord created in the base point construction algorithm for $\Gamma^-$ can be isotoped off the top of the cylinder, and pushed down the cylinder into the bottom disc, where it simplifies $\Gamma$ to the chord diagram on the unused disc of an appropriate $\Gamma_{w^\cdot}$.

# 11 Properties of contact elements

Based on the above results, we can now give some properties of contact elements. In particular, we can say some things about the basis elements which occur in the decomposition of a contact element.

## 11.1 How many basis elements occur in a decomposition?

A natural first question is just how many basis elements there can be. If $\Gamma$ is a basis chord diagram, then obviously there is one element in its decomposition, namely itself. Otherwise, the answer is: an even number. That is, we prove proposition 1.28:

**Proposition** *Every chord diagram which is not a basis element has an even number of basis elements in its decomposition.*

PROOF Consider a non-basis chord diagram $\Gamma$. Then its basis decomposition algorithm is nontrivial, i.e. some decomposition actually occurs. As we perform the decomposition algorithm, we obtain chord diagrams $\Gamma_w$ in $\Upsilon_k$ associated to words of length $k$. For each basis element in the decomposition of $\Gamma$, it appears at some stage of this algorithm (possibly not the last). But when it does appear, it comes from a non-basis chord diagram which is related to it by a bypass move. However, by lemma 7.12, the only non-basis chord diagrams related by a bypass move to a basis chord diagram, are sums of two basis chord diagrams. It must be this pair of basis chord diagrams which appears; and so the basis elements come in pairs. ∎

In fact, we have proved a little more: if we write out the basis elements of $\Gamma$ in lexicographic order, then the $(2j-1)$'th and $2j$'th are bypass-related; equivalently (lemma 7.12), their words are related by an elementary move.

**Lemma 11.1 (Test for basis element)** *For any chord diagram $\Gamma$ with $n$ chords and euler class $e = n_+ - n_-$,*

$$\begin{aligned} m\left(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma\right) &= m\left(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}}\right) \\ &= \begin{cases} 0 & \text{if } \Gamma \text{ is a basis chord diagram} \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

PROOF Every word $w \in W(n_-, n_+)$ satisfies $(-)^{n_-}(+)^{n_+} \preceq w \preceq (+)^{n_+}(-)^{n_-}$. We expand out in terms of basis elements. These expressions simply count the number of basis elements in the decomposition of $\Gamma$, mod 2. ∎

One can also prove directly, by examination of chord diagrams, that if $\Gamma$ is a basis chord diagram, then $\mathcal{M}(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma)$ and $\mathcal{M}(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}})$ are both tight; and otherwise they are both overtwisted.

## 11.2 Symbolic interpretation of outermost regions

As it turns out, certain symbols appearing in both $\Gamma^-, \Gamma^+$ imply that certain symbols are repeated in all the basis elements of $\Gamma$, and mean that $\Gamma$ has an outermost chord in a specific place.

**Lemma 11.2 (Outermost regions at base point)** *Let $\Gamma = [\Gamma^-, \Gamma^+] = [w^-, w^+]$. The following are equivalent.*

*(i)* $\Gamma$ *has an outermost chord enclosing a negative region (resp. positive region) at the base point.*

*(ii)* *For every* $\Gamma_w$ *in the basis decomposition of* $\Gamma$*,* $w$ *begins with a* $-$ *(resp.* $+$*).*

*(iii)* $\Gamma^-$ *and* $\Gamma^+$ *both have outermost chords enclosing a negative region (resp. positive region) at the base point, i.e.* $w^-, w^+$ *both begin with a* $-$ *(resp. begin with a* $+$*).*

PROOF That (i) implies (ii) follows immediately from considering the decomposition algorithm. That (ii) implies (iii) is obvious. That (iii) implies (i) follows immediately from

$$B_-[\Gamma_{w_1}, \Gamma_{w_2}] = [B_-\Gamma_{w_1}, B_-\Gamma_{w_2}] = [\Gamma_{-w_1}, \Gamma_{-w_2}].$$ ∎

There is a similar result at the root point.

**Lemma 11.3 (Outermost regions at root point)** *Let* $\Gamma = [\Gamma^-, \Gamma^+] = [w^-, w^+]$ *The following are equivalent.*

*(i)* $\Gamma$ *has an outermost chord enclosing a negative region (resp. positive region) at the root point.*

*(ii)* *For every* $\Gamma_w$ *in the basis decomposition of* $\Gamma$*,* $w$ *ends with a* $-$ *(resp.* $+$*).*

*(iii)* $\Gamma^-$ *and* $\Gamma^+$ *both have outermost chords enclosing a negative region (resp. positive region) at the root point, i.e.* $w^-, w^+$ *both end with a* $-$ *(resp. begin with a* $+$*).* ∎

We can detect other outermost chords in a similar way. In particular, we can detect negative outermost chords on the westside, and positive outermost chords on the eastside. First, we note from lemma 6.9 that a basis chord diagram $\Gamma_w$ has a negative outermost chord on the westside, at $(-2j-1, -2j)$, if and only if the $(j+1)$'th $-$ sign in $w$ is following. Similarly, $\Gamma_w$ has a positive outermost chord on the eastside, at $(2j, 2j+1)$, if and only if the $(j+1)$'th $+$ sign in $w$ is following.

As mentioned in sections 1.12 and 4.2, creation and annihilation operators can be defined, not just at the base point, but elsewhere. One may define linear operators on $SFH$ which have the effect of adding or removing an outermost chord at any given site.

**Definition 11.4 (Eastside/westside creation operators)**

*(i)* *For each* $i = 0, \ldots, n_-$*, the operator*

$$B_-^{west,i} : SFH(T, n+1, e) \rightarrow SFH(T, n+2, e-1)$$

*takes a chord diagram with* $n+1$ *chords and relative euler class* $e$*, and produces a chord diagram with* $n+2$ *chords and relative euler class* $e-1$*, adding an outermost chord enclosing a negative region on the westside, between points* $(-2i-3, -2i-2)$ *(as labelled on the chord diagram with* $n+2$ *chords).*

*(ii)* *For each* $j = 0, \ldots, n_+$*, the operator*

$$B_+^{east,j} : SFH(T, n+1, e) \rightarrow SFH(n+2, e+1)$$

*takes a chord diagram with* $n+1$ *chords and relative euler class* $e$*, and produces a chord diagram with* $n+2$ *chords and relative euler class* $e+1$*, adding an outermost chord enclosing a positive region on the eastside, between points* $(2j+2, 2j+3)$*.*

**Definition 11.5 (Eastside/westside annihilation operators)**

*(i)* *For each* $i = 0, \ldots, n_-$*, the operator,*

$$A_+^{west,i} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e+1)$$

*takes a chord diagram with* $n+1$ *chords and relative euler class* $e$*, and produces a chord diagram with* $n$ *chords and relative euler class* $e+1$*, by joining the chords at positions* $(-2i-2, -2i-1)$*.*

*(ii) For each $j = 0, \ldots, n_+$, the operator*

$$A_-^{east,j} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e-1)$$

*takes a chord diagram with $n+1$ chords and relative euler class $e$, and produces a chord diagram with $n$ chords and relative euler class $e-1$, by joinint the chords at positions $(2j+1, 2j+2)$.*

Note that the numbering of these operators may seem a little strange. There are however good reasons for it; see section 13.

We see that, like our original annihilation and creation operators and the base point, we have

$$A_+^{west,j} \circ B_-^{west,j} = 1, \quad A_-^{east,j} \circ B_+^{east,j} = 1,$$

and we will investigate further relations in section 13.

It's easy from lemma 6.9 to see that $B_-^{west,j}$ has the effect on $\Gamma_w$ of producing $\Gamma_{w'}$, where $w'$ is obtained from $w$ as follows. If $0 \leq j \leq n_- - 1$, then we insert a $-$ sign in $w$ immediately after the $(j+1)$'th $-$ sign. If $j = n_-$, then we add a $-$ sign at the end of $w$. Similarly, $B_+^{east,j}$ adds a $+$ sign immediately after the $(J+1)$'th $+$ sign, if $0 \leq j \leq n_+ - 1$; and adds a $+$ sign at the end, if $j = n_+$.

We can do the same for annihilation operators. The operator $A_+^{west,j}$ has the effect of deleting the $(j+1)$'th $-$ sign, for $0 \leq j \leq n_- - 1$; and for $j = n_-$, it deletes the $-$ sign at the end of the word, or returns 0 if the word ends in a $+$. The operator $A_-^{east,j}$ has the effect of deleting the $(j+1)$'th $+$ sign, for $0 \leq j \leq n_+ - 1$; and for $j = n_+$, it deletes the $+$ sign at the end of the word (if there is one), else returns 0.

**Lemma 11.6 (Outermost negative regions on westside)** *Let $\Gamma = [\Gamma^-, \Gamma^+] = [w^-, w^+]$. Let $j$ be an integer from 1 to $n_- - 1$. The following are equivalent.*

*(i) $\Gamma$ has an outermost chord enclosing a negative region between the points $(-2j-1, -2j)$.*

*(ii) For every $\Gamma_w$ in the basis decomposition of $\Gamma$, $\Gamma_w$ has an outermost chord enclosing a negative region between $(-2j-1, -2j)$. Equivalently, every such $w$ has the $(j+1)$'th $-$ sign following (i.e. not the first in its block).*

*(iii) Both $\Gamma^-$ and $\Gamma^+$ have an outermost chord enclosing a negative region between $(-2j-1, -2j)$, i.e. has $(j+1)$'th $-$ sign following..*

Proof The proof is similar to the previous two lemmas, after noting

$$B_-^{west,j} \Gamma = B_-^{west,j} [\Gamma^-, \Gamma^+] = [B_-^{west,j} \Gamma^-, B_-^{west,j} \Gamma^+],$$

adding a $-$ sign immediately after the $j$'th $-$ sign in every word occurring in the decomposition of $\Gamma$. ■

There is a similar lemma on the eastside.

**Lemma 11.7 (Outermost positive regions on eastside)** *Let $\Gamma = [\Gamma^-, \Gamma^+] = [w^-, w^+]$. Let $j$ be an integer from 1 to $n_+ - 1$. The following are equivalent.*

*(i) $\Gamma$ has an outermost chord enclosing a positive region between the points $(2j, 2j+1)$..*

*(ii) For every $\Gamma_w$ in the basis decomposition of $\Gamma$, $\Gamma_w$ has an outermost chord enclosing a positive region between $(2j, 2j+1)$. Equivalently, every such $w$ has the $(j+1)$'th $-$ sign following (i.e. not the first in its block).*

*(iii) Both $\Gamma^-$ and $\Gamma^+$ have an outermost chord enclosing a negative region between $(2j, 2j+1)$, i.e. have $(j+1)$'th $+$ sign following.* ■

All the lemmas in this section say that, if a chord diagram has an outermost region in a specific place, then so do all the basis chord diagrams in its decomposition. In particular, as we proceed through the decomposition algorithm, there is no decomposition at that chord. In fact, this can be seen explicitly from the decomposition algorithm.

## 11.3 Generalised bypass triples

We have seen that bypass-related chord diagrams naturally come in triples. We now consider a "multiple bypass" version of this phenomenon, namely the triple $(\Gamma^+, \Gamma, \Gamma^-)$

Recall that for $[\Gamma^-, \Gamma^+] = [w^-, w^+]$, we have a bypass system $c^- = BS(w^-, w^+)$ on $\Gamma^-$ such that $\text{Up}\,(c^-)\Gamma^- = \Gamma^+$ and $\text{Down}\,(c^-)\Gamma^- = [\Gamma^-, \Gamma^+]$. Taking the corresponding bypass systems $c^+$ on $\Gamma^+$ and $c$ on $[\Gamma^-, \Gamma^+]$, we see that these bypass systems take the three chord diagrams to each other:

$$\begin{array}{ccccc} & & \Gamma & & \\ & \text{Up}\,(c) \swarrow\!\nearrow \text{Down}\,(c^-) & & \text{Up}\,(c^+) \nwarrow\!\searrow \text{Down}\,(c) & \\ \Gamma^- & & \begin{array}{c}\text{Up}\,(c^-)\\ \rightleftarrows \\ \text{Down}\,(c^+)\end{array} & & \Gamma^+ \end{array}$$

Moreover, we have seen that the cylinders obtained by stacking them in the three ways indicated are all tight:

$$m(\Gamma, \Gamma^-) = m(\Gamma^-, \Gamma^+) = m(\Gamma^+, \Gamma) = 1.$$

All this is analogous to a bypass triple. In fact it is a generalisation of a bypass triple: when $\Gamma^-, \Gamma^+$ are related by a single bypass, or equivalently when $w^-, w^+$ are related by an elementary move, this situation reduces to a bypass triple.

Actually, more is true. Obtaining one chord diagram from another by a sequence of bypass moves, all in the same direction, and along a bypass system, i.e. a set of disjoint attaching arcs, gives more than just a dividing set on the boundary of a cylinder. It actually gives a *contact structure* on the solid cylinder. This contact structure is given by the standard neighbourhood of the convex disc, with actual contact-geometric bypasses attached to it (all above or all below). Since we have a bypass system, the attaching arcs are all disjoint, so there is no ambiguity about what order to add bypasses. So we have well-defined contact structures on $\mathcal{M}(\Gamma, \Gamma^-)$, $\mathcal{M}(\Gamma^-, \Gamma^+)$ and $\mathcal{M}(\Gamma^+, \Gamma)$; the fact that $m = 1$ for all of them means that they are tight in a neighbourhood of the boundary; but in fact the whole manifolds are tight.

**Proposition 11.8 (Contact generalised bypass triple)** *The contact structures on*

$$\mathcal{M}(\Gamma, \Gamma^-), \quad \mathcal{M}(\Gamma^-, \Gamma^+) \quad \textit{and} \quad \mathcal{M}(\Gamma^+, \Gamma)$$

*obtained by attaching bypasses are all tight.*

As part of this proof, we will show that it does not matter whether we add the proposed bypasses up from the bottom or down from the top.

In this proposition, we are asking when a contact manifold obtained by attaching bypasses to a convex disc along a bypass system produces a tight contact structure. This question has been completely answered by Honda, Kazez and Matić in [27]: the key indicator is a *pinwheel*.

**Theorem 11.9 (Honda, Kazez, Matić [27])** *Let $D$ be a convex disc with legendrian boundary and $c$ a bypass system. The contact structure on $D \times I$ obtained by attaching bypasses on top of the product contact neighbourhood of $D$ along all the attaching arcs of $c$ is tight if and only if there are no pinwheels in $D$.*

Thus, to prove our proposition, we only need to check there are no pinwheels.

**Definition 11.10 (Pinwheel)** *A* pinwheel *is an embedded polygonal region $P$ in $(D, \Gamma)$ satisfying the following conditions.*

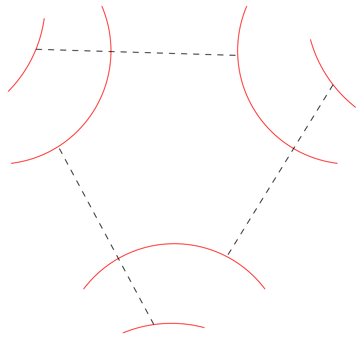


Figure 41: A pinwheel.

*(i) The boundary of $P$ consists of $2k$ ($k \geq 1$) consecutive sides*

$$\gamma_1, \alpha_1, \gamma_2, \alpha_2, \ldots, \gamma_k, \alpha_k,$$

*labelled anticlockwise, where $\gamma_i$ is an arc on a chord of the dividing set $\Gamma$, and $\alpha_i$ is half of an arc of attachment $c_i$.*

*(ii) For each $i$, $c_i$ extends past $\alpha_i$ in the direction shown in figure 41, and does not again intersect $P$.*

Proof (Of proposition 11.8) Since adding bypasses above $\Gamma^-$ along $FBS(w^-, w^+)$ gives $\Gamma^+$, we show that $FBS(w^-, w^+)$ has no pinwheels. Suppose there were a pinwheel $P$ and consider the requirement for attaching arcs to extend past $P$ in the direction shown. Recall that the chords of $\Gamma^-$ are ordered by the stage of the base point construction algorithm at which they are constructed; let $s_i$ denote the stage at which $\gamma_i$ is constructed.

Now, recall that $FBS(w^-, w^+)$ is a forwards bypass system and hence each attaching arc has a negative prior outer region. Moreover, proceeding along each attaching arc from prior endpoint to latter endpoint, the numberings of the chords of $\Gamma^-$ it intersects strictly increases. If $P$ is a negative region, this implies $s_k < s_{k-1} < \cdots < s_1 < s_k$, a contradiction. Similarly, if $P$ is a positive region, we have $s_1 < s_2 < \cdots < s_k < s_1$, also a contradiction.

Thus the contact structure on $\mathcal{M}(\Gamma^-, \Gamma^+)$ by attaching upwards bypasses along $c^-$ to a standard neighbourhood of $(D, \Gamma^-)$ is tight. Similarly, attaching upwards or downwards bypasses along $c^+$ to a standard neighbourhood of $(D, \Gamma^+)$ gives a tight contact manifold. Since these two sets of bypasses lie in the same places and "undo each other", they give the same contact structure on $\mathcal{M}(\Gamma^-, \Gamma^+)$, which is tight.

By the same argument, attaching downwards bypasses along $c^-$ to $(D, \Gamma^-)$ is gives a tight contact structure on $\mathcal{M}(\Gamma, \Gamma^-)$, which is the same contact structure as obtained by attaching bypasses to $c$ below $\Gamma$. And a similar argument shows that both contact structures on $\mathcal{M}(\Gamma^+, \Gamma)$ are also equivalent, and tight. ■

## 11.4 Relations within contact elements

We now examine the various basis elements within the decomposition of a chord diagram / contact element $\Gamma = [\Gamma^-, \Gamma^+] = [w^-, w^+]$, using the bypass systems discussed above.

**Definition 11.11 (Basis chord diagram in Γ)** *If $\Gamma_w$ appears in the decomposition of $\Gamma$, we write $\Gamma_w \in \Gamma$.*

After all, any sum with $\mathbb{Z}_2$ coefficients can be regarded as a subset.

Moreover, we have many more tight cylinders.

**Lemma 11.12 (More tight cylinders)** *For every $\Gamma_w$ obtained by performing upwards bypass moves along some subset $A^-$ of of $c^-$ on $\Gamma^-$ (or equivalently, by attaching downwards bypasses along some subset of $c^+$ on $\Gamma^+$),*

$$\begin{aligned} m(\Gamma^-, \Gamma_w) &= m(\Gamma_w, \Gamma^+) \\ &= m(\Gamma, [\Gamma^-, \Gamma_w]) = m([\Gamma^-, \Gamma_w], \Gamma^-) \\ &= m(\Gamma^+, [\Gamma_w, \Gamma^+] = m([\Gamma_w, \Gamma^+], \Gamma) = 1. \end{aligned}$$

*In fact, the corresponding solid cylinders all have tight contact structures given by attaching bypasses to one of $\Gamma^-, \Gamma^+, \Gamma$ along some subset of $c^-, c^+, c$ respectively.*

Proof We rely heavily on proposition 11.8, and show that these bypass attachments give contact manifolds which can be embedded inside the tight contact manifolds constructed in that proposition. For instance, we saw in proposition 11.8 that the contact structure on $\mathcal{M}(\Gamma^-, \Gamma^+)$ obtained by attaching bypasses above $\Gamma^-$ along $c^-$ is tight. Attaching only bypasses along the subset $A$ then gives the contact manifold $\mathcal{M}(\Gamma^-, \Gamma_w)$ as a contact submanifold of this tight $\mathcal{M}(\Gamma^-, \Gamma^+)$; hence it is also tight.

Similarly, a contact structure on $\mathcal{M}(\Gamma_w, \Gamma^+)$ is obtained by attaching bypasses below $\Gamma^+$ along the subset $A^+$ of $c^+$ corresponding to the complement of $A$. This is also a contact submanifold of our tight $\mathcal{M}(\Gamma^-, \Gamma^+)$.

By corollary 8.2, since performing downwards bypass moves on $\Gamma^+$ along $A^+$ gives $\Gamma_w$, then upwards bypass moves give $[\Gamma_w, \Gamma^+]$; so these attachments of bypasses give a contact structure on $\mathcal{M}(\Gamma^+, [\Gamma_w, \Gamma^+])$. This contact manifold embeds inside the tight $\mathcal{M}(\Gamma^+, \Gamma)$ obtained by attaching bypasses along all of $c^+$. This same chord diagram is given by downwards bypass moves on $\Gamma$ along $A$, the subset of $c$ corresponding to the complement of $A^+$; and so we obtain a tight contact structure on $\mathcal{M}([\Gamma_w, \Gamma^+], \Gamma)$.

Similarly, upwards bypass moves on $\Gamma^-$ along $A^-$ give $\Gamma_w$, so downwards moves give $[\Gamma^-, \Gamma_w]$; we similarly obtain a tight contact structure on $\mathcal{M}([\Gamma^-, \Gamma_w], \Gamma^-)$. This same chord diagram is given by upwards bypass moves on $\Gamma$ along $A$. Thus we obtain a tight contact structure on $\mathcal{M}(\Gamma, [\Gamma^-, \Gamma_w])$. ■

Now some algebraic shenanigans gives us a couple of interesting formulas.

**Lemma 11.13 (Relations between basis elements)** *The chord diagrams $\Gamma^+, \Gamma^-$ satisfy*

$$\Gamma^+ = \sum_{\Gamma_w \in \Gamma} [\Gamma^-, \Gamma_w] \quad \text{and} \quad \Gamma^- = \sum_{\Gamma_w \in \Gamma} [\Gamma_w, \Gamma^+].$$

Proof We have $\text{Down}\,(c)(\Gamma) = \Gamma^+$, but let us instead expand this as a sum over subsets of upwards bypass moves $\sum_{A_i \subseteq c} \text{Up}\,(A_i)(\Gamma)$ over subsets $A_i$ of the bypass system $c$. But then $\text{Up}\,(A_i)(\Gamma) = \text{Down}\,(A_i')(\Gamma^-)$, where $A_i'$ is the complement of the corresponding subset of the bypass system of $c^-$. But $\text{Up}\,(A_i')(\Gamma^-)$ is a basis chord diagram $\Gamma_{w_i}$ for some $w^- \preceq w_i \preceq w^+$. Hence by corollary 8.2, $\text{Down}\,(A_i') = [w^-, w_i]$. Putting this together,

$$\Gamma^+ = \text{Down}\,(c)(\Gamma) = \sum_{A_i \subseteq c} \text{Up}\,(A_i)(\Gamma) = \sum_{A_i' \subseteq c^-} \text{Down}\,(A_i')(\Gamma^-) = \sum_{A_i' \subseteq c^-} [\Gamma^-, \text{Up}\,(A_i')(\Gamma^-)]$$

Now, some of these terms may be identical: they cancel as identical whenever

$$\text{Up}\,(A_i')(\Gamma^-) = \text{Up}\,(A_j')(\Gamma^-)$$

for some subsets $A_i', A_j' \subset c^-$. But note that the sum

$$\sum_{A_i' \subseteq c^-} \text{Up}\,(A_i')(\Gamma^-) = \text{Down}\,(c^-)(\Gamma^-) = \Gamma,$$

so the terms are identical precisely when they cancel in the expansion of $\Gamma$. In other words, the sum over $A_i' \subseteq c^-$ may be regarded as the sum over basis elements which occur in the decomposition of $\Gamma$. This argument, and a similar argument for $c^+$, give us the result. ■

This result says something about the set of basis elements $\Gamma_w$ occurring in $\Gamma$: in the sum involved, there must be a great deal of cancellation. In particular, we have the following consequences.

**Proposition 11.14** *For* $\Gamma_w \in \Gamma = [\Gamma^-, \Gamma^+]$,

$$m(\Gamma, \Gamma_w) = \begin{cases} 0 & \Gamma_w = \Gamma^- \\ 1 & \textit{otherwise} \end{cases}$$

$$m(\Gamma_w, \Gamma) = \begin{cases} 0 & \Gamma_w = \Gamma^+ \\ 1 & \textit{otherwise} \end{cases}$$

Proof Applying the previous lemma to $[\Gamma^-, \Gamma_w]$ gives

$$\Gamma_w = \sum_{\Gamma_{w'} \in [\Gamma^-, \Gamma_w]} [\Gamma^-, \Gamma_{w'}].$$

Hence

$$m(\Gamma, \Gamma_w) = \sum_{\Gamma_{w'} \in [\Gamma^-, \Gamma_w]} m(\Gamma, [\Gamma^-, \Gamma_{w'}]).$$

Now $\Gamma_w$ occurs in the basis decomposition of $[\Gamma^-, \Gamma^+]$, hence is obtained by performing upwards bypass moves along some subset $A^-$ of the bypass system $c^- = BS(\Gamma^-, \Gamma^+)$ on $\Gamma^-$. By corollary 8.2, performing downwards bypass moves on $\Gamma^-$ along $A^-$ gives $[\Gamma^-, \Gamma_w]$. To see what basis elements $\Gamma_{w'}$ occur in $[\Gamma^-, \Gamma_w]$, we expand the downwards bypass moves on $\Gamma^-$ along $A^-$ as a sum over upwards bypass moves on $\Gamma^-$ along subsets of $A^-$. This certainly gives $[\Gamma^-, \Gamma_w]$ as a sum of basis chord diagrams; so every $\Gamma_{w'} \in [\Gamma^-, \Gamma_w]$ can be obtained by performing upwards bypass moves along some subset of $A^-$, which in turn is a subset of $c^-$. In particular, proposition 11.8 applies to $\Gamma_{w'}$ and every term $m(\Gamma, [\Gamma^-, \Gamma_{w'}]) = 1$.

Thus $m(\Gamma, \Gamma_w)$ is simply equal to the number of terms in the above sum. If $\Gamma_w = \Gamma^-$ then $[\Gamma^-, \Gamma_w] = \Gamma^-$ has one term, and we obtain 1. Otherwise $[\Gamma^-, \Gamma_w]$ is not a basis element, hence by proposition 1.28 has an even number of terms, and the sum is zero. ∎

**Proposition 11.15** *For every* $\Gamma_w$ *occurring in the basis decomposition of* $\Gamma$*, other than* $\Gamma^\pm$*, the number of basis elements of* $\Gamma$ *which precede* $\Gamma_w$ *(with respect to* $\preceq$*) is even, and the number of basis elements which follow it (with respect to* $\preceq$*) is also even.*

Proof Expand out $m(\Gamma, \Gamma_w) = 0$ and $m(\Gamma_w, \Gamma) = 0$ over the basis elements of $\Gamma$. ∎

We can now prove theorem 1.29:

**Theorem** *Suppose* $\Gamma_w$ *occurs in the basis decomposition of* $\Gamma$ *and is comparable (with respect to* $\preceq$*) with every other basis element occurring in* $\Gamma$*. Then* $\Gamma_w = \Gamma^+$ *or* $\Gamma^-$*.*

Proof If $\Gamma$ is a basis element, it is clear. Otherwise, the number of elements comparable to $\Gamma_w$ is $m(\Gamma, \Gamma_w) + m(\Gamma_w, \Gamma) + 1$. (We overcount $\Gamma_w$ in the sum, so correct by adding 1.) If $\Gamma_w$ is comparable to every basis element in $\Gamma$ then this number must be even, since $\Gamma$ contains an even number of basis elements (proposition 1.28). But by proposition 11.14 it is odd. ∎

# Part V
# Further considerations

## 12 The rotation operator

We now consider the operation of rotating chord diagrams, or equivalently, moving the basepoint. If we are to keep a negative region anticlockwise from the base point, and a positive region clockwise, then we must move the base point by two marked points.

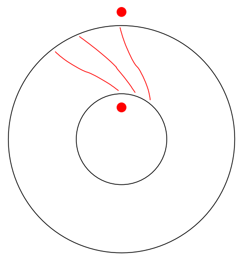


Figure 42: The rotation operator.

Such a rotation corresponds to the inclusion of sutured manifolds $(T, n) \hookrightarrow (T, n)$ given by thickening the solid torus along its boundary. On the intermediate manifold, which is an annulus $\times S^1$, we specify an $S^1$-invariant contact structure by giving a dividing set on the annulus, as shown in figure 42.

By TQFT-inclusion, we obtain a linear operator

$$\begin{array}{rccc} R: & SFH(T, n+1, e) & \longrightarrow & SFH(T, n+1, e) \\ & \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}} \end{array}$$

Obviously $R^n$ is the identity, takes contact elements to contact elements, and rotates chord diagrams anticlockwise (or, equivalently, moves the base point 2 marked points clockwise). When we wish to refer to a particular $n, e, k$ we write $R_{n,k}$ for the above map on $\mathbb{Z}_2^{\binom{n}{k}}$

## 12.1 Small cases

For $SFH(T, 1, 0) = \mathbb{Z}_2$, there is only one nonzero element, corresponding to the vacuum $v_\emptyset$, and it is fixed by rotation. Thus $R$ is a $1 \times 1$ identity matrix in this case.

Similarly, for an extremal euler class $SFH(T, n+1, e = \pm n) = \mathbb{Z}_2$ there is only one nonzero element, corresponding to $\Gamma_{(\pm)^n}$, consisting only of outermost chords. Again $R$ is the identity in this case.

The smallest non-identity case is $SFH(T, 3, 0) = \mathbb{Z}_2^{\binom{2}{1}} = \mathbb{Z}_2^2$. And $C_3^0 = 3$, with the three chord diagrams being a bypass triple. We easily obtain

$$v_{-+} \mapsto v_{+-} \mapsto v_{+-} + v_{+-}$$

and hence

$$R_{2,1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

We write matrices using the lexicographically ordered basis

In a similar way we obtain

$$R_{3,1} = R_{3,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad R_{4,1} = R_{4,3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note it appears from these examples that perhaps $R_{n,k} = R_{n,n-k}$; that is not the case as we find here.

$$R_{5,2} = \left[\begin{array}{c|cc|ccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right], \quad R_{5,3} = \left[\begin{array}{c|ccc|cccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right],$$

(We have suggestively boxed this matrix in a way that will become useful.)

## 12.2 Computation of $R$

We now compute $R$. The computation is recursive: we define $R_{n,k}$ in terms of smaller $R$ matrices. We will need to choose basis elements with certain properties: if $x$ is a word in $+$ and $-$, we will say *$x$-basis elements* to mean those $v_w$ where the string $w$ begins with the string $x$ (and possibly $w = x$). We will also say *$x$-rows* or *$x$-columns* to mean the columns correspond to all $x$-basis elements. And for two strings $x$ and $y$, we can take the *$x \times y$ minor* of $R$ to be the submatrix consisting of the intersection of the $x$-rows with the $y$-columns.

For example, in our computation of $R_{5,2}$ and $R_{5,3}$ above the boxes have been added to suggest the following decomposition:

$R_{5,2} =$

| | $---$ | $--+$ | $-+$ | $+$ |
|---|---|---|---|---|
| $---$ | 0 | $R_{1,1}$ \| 0 | 0 | 0 |
| $--+$ | 0 | 0 | $R_{2,1}$ \| 0 | 0 |
| $-+$ | 0 | 0 | 0 | $R_{3,1}$ \| 0 |
| $+$ | $(---)$-cols of $R_{4,1}$ | $(--)$-cols of $R_{4,1}$ | $(-)$-cols of $R_{4,1}$ | $R_{4,1}$ |

$R_{5,3} =$

| | $--$ | $-+$ | $+$ |
|---|---|---|---|
| $--$ | 0 | $R_{2,2}$ \| 0 | 0 |
| $-+$ | 0 | 0 | $R_{3,2}$ \| 0 |
| $+$ | $(--)$-cols of $R_{4,2}$ | $(-)$-cols of $R_{4,2}$ | $(+)$-cols of $R_{4,2}$ |

We will now prove that something similar occurs for all $R_{n,k}$.

**Proposition 12.1 (Recursive computation of $R$)** *$R_{n,k}$ is described as follows.*

*(i) The $(+) \times (+)$ minor of $R_{n,k}$ is $R_{n-1,k-1}$.*

*(ii) The $(+)\times(-+)$ minor of $R_{n,k}$ is the $(-)$-columns of $R_{n-1,k-1}$. More generally, the $(+)\times((-)^j+)$ minor of $R_{n,k}$ is the $((-)^j)$-columns of $R_{n-1,k-1}$, for $j = 1, \ldots, n-k$.*

*(iii) The $(-+)\times(+-)$ minor of $R_{n,k}$ is $R_{n-2,k-1}$. More generally, the $((-)^j-+)\times((-)^j+-)$ minor of $R_{n,k}$ is $R_{n-j-2,k-1}$.*

*(iv) All other entries are zero. To write these remaining entries out exhaustively (with some overlap):*

*(a) (Below and on the diagonal, in the $(-)$ rows.) The $(-+) \times (-)$ minor is zero. More generally, the $((-)^j+) \times ((-)^j)$ minor is zero, for $j = 1, \ldots, n-k$.*

(b) *(Above the diagonal and the submatrices $R_{n-j-2,k-1}$.) The $(--)\times(+)$ minor is zero. More generally, the $((-)^j--)\times((-)^j+)$ minor is zero, for $j=0,\dots,n-k-2$.*

(c) *(The pieces in the $(-)$ rows just to the right of the submatrices $R_{n-j-2,k-1}$.) The $(-)\times(++)$ minor is zero. More generally, the $(-)\times((-)^j++)$ minor is zero, for $j=0,\dots,n-k$.*

Proof We simply verify all these conditions. The conditions given are equivalent to the following equations on operators:

(i) $A_-RB_+=R$.

(ii) $A_-R(B_-)^jB_+=A_-RB_+(B_-)^j$, for $j=1,\dots,n-k$.

(iii) $A_-A_+(A_+)^jR(B_-)^jB_+B_-=R$, for $j=0,\dots,n-k-1$.

(iv) (a) $A_-(A_+)^jR(B_-)^j=0$, for $j=1,\dots,n-k$.
(b) $A_-A_+(A_+)^jR(B_-)^jB_+=0$, for $j=0,\dots,n-k-2$.
(c) $A_+R(B_-)^jB_+B_+=0$, for $j=0,\dots,n-k$.

These are now easily proved by examining the corresponding chord diagrams. ■

These matrices have interesting combinatorial properties: for instance, for every row, there is precisely one column which has its highest nonzero element in that row.

## 12.3 An explicit description

From the above recursive form of the matrix for $R$, we can write down a recursive formula.

$$\begin{aligned} R &= \sum_{n=0}^{\infty} B_+RB_-^nA_-A_+^n + B_-^{n+1}B_+RA_+A_-A_+^n \\ &= \sum_{n=0}^{\infty} \left(B_+RB_-^n + B_-^{n+1}B_+RA_+\right)A_-A_+^n \\ &= \sum_{n=0}^{\infty} \left[B_+RA_+, B_-^{n+1}\right]A_-A_+^n \end{aligned}$$

We can also describe combinatorially $R(v_w)$ for each basis vector $v_w$. We will write $w$ in the form

$$w=(-)^a_1(+)^{b_1}\cdots(-)^{a_k}(+)^{b_k}$$

where possibly $a_1=0$ or $b_k=0$, but all other $a_i,b_i$ are nonzero. This follows from interpreting the formula for $R$ above as a set of instructions for operating on $w$, removing or adding $+$ and $-$ signs.

**Proposition 12.2 (Explicit computation of $R$)** *If $k\geq 2$ then $R(v_w)$ is given by taking*

$$\begin{aligned} &(+)^{b_1-1}(-)^{a_1+1}(+)^{b_2}(-)^{a_2}\cdots(+)^{b_{k-1}}(-)^{a_{k-1}}(+)^{b_k+1}(-)^{a_k-1} \\ &=(+)^{b_1-1}(-)^{a_1+1}\left(\prod_{j=2}^{k-1}(+)^{b_j}(-)^{a_j}\right)(+)^{b_k+1}(-)^{a_k-1} \end{aligned}$$

*and then, for each possible way of grouping $(1,2,\dots,k)$ into the form*

$$((1,2,\dots,l_1),(l_1+1,l_1+2,\dots,l_2),\dots,(l_{T-1}+1,l_{T-1}+1,\dots,l_T=k)),$$

*taking the term*

$$(+)^{b_1+\cdots+b_{l_1}-1}(-)^{a_1+\cdots+a_{l_1}+1}(+)^{b_{l_1+1}+\cdots+b_{l_2}}(-)^{a_{l_1+1}+\cdots+a_{l_2}}\cdots$$
$$\cdots(+)^{b_{l_{T-2}+1}+\cdots+b_{l_{T-1}}}(-)^{b_{l_{T-2}+1}+\cdots+b_{l_{T-1}}}(+)^{b_{l_{T-1}+1}+\cdots+b_{l_T}+1}(-)^{a_{l_{T-1}+1}+\cdots+a_{l_T}-1}$$

$$= (+)^{b_1+\cdots+b_{l_1}-1}(-)^{a_1+\cdots+a_{l_1}+1}\left(\prod_{m=2}^{T-1}(+)^{b_{l_{m-1}+1}+\cdots+b_{l_m}}(-)^{a_{l_{m-1}+1}+\cdots+a_{l_m}}\right)$$
$$(+)^{b_{l_{T-1}+1}+\cdots+b_{l_T}+1}(-)^{a_{l_{T-1}+1}+\cdots+a_{l_T}-1}$$

*(including the trivial grouping $((1),(2),\ldots,(k))$ corresponding to the first term above) and summing all the corresponding basis elements.*

*If $k=1$, so that $w$ is of the form $(-)^a$ or $(+)^b$ or $(-)^a(+)^b$, then $R(v_w)$ is given by a single term $v_{w'}$ where $w'$ is given by:*

*(i) for $w=(-)^a$, $w'=(-)^a$ also;*

*(ii) for $w=(+)^a$, $w'=(+)^a$ also;*

*(iii) for $w=(-)^a(+)^b$, $w'=(+)^b(-)^a$;* ■

Note that every chord diagram has an outermost region: after some rotation, every chord diagram has an outermost region at the base point. And a chord diagram at the base point is of the form $B_\pm\Gamma$, for some smaller $\Gamma$. Thus, these rotation matrices give a quick way to compute all the contact elements in $SFH(T,n+1,e)$ recursively. If we know all the contact elements of $SFH(T,n,e\pm1)$, then we apply $B_-$ to all contact elements in $SFH(T,n,e+1)$ and $B_+$ to all contact elements in $SFH(T,n,e-1)$. Applying $B_-$ to a contact element (or any element) simply prepends a $-$ to all of the words in its basis decomposition. And then, applying $R$ will generate all contact elements in $SFH(T,n+1,e)$. In fact, when the euler class is not extremal, there are outermost regions of both signs; so it is sufficient to look at only one of $B_\pm$.

# 13 Simplicial structures

Recall that in section 11.2, we defined eastside and westside creation and annihilation operators

$$B_-^{west,i},\ A_+^{west,i},\ B_+^{east,j},\ A_-^{east,j}$$

for $0\le i\le n_-$ and $0\le j\le n_+$, where:

(i) $B_-^{west,i}$ inserts a chord $(-2i-3,-2i-2)$.

(ii) $A_+^{west,i}$ joins the chords at positions $(-2i-2,-2i-1)$

(iii) $B_+^{east,j}$ inserts a chord $(2i+2,2i+3)$.

(iv) $A_-^{east,j}$ joins the chords at positions $(2j+1,2j+2)$

Note that it is perfectly compatible with these conditions to take $i$ or $j=-1$ and obtain our original operators,

$$B_-=B_-^{west,-1},\ A_+=A_+^{west,-1},\ B_+=B_+^{east,-1},\ A_-=A_-^{east,-1}$$

We have seen that

$$B_-^{west,j}\circ A_+^{west,j}=1,\quad B_+^{east,j}\circ A_-^{east,j}=1,$$

but moreover, they satisfy stronger relations for various $j$. We see that:

**Lemma 13.1 (Westside simplicial structure)** *For all* $0 \le i, j \le n_-$*, we have*

$$\begin{aligned}
A_+^{west,i} \circ A_+^{west,j} &= A_+^{west,j-1} \circ A_+^{west,i} \quad i < j \\
A_+^{west,i} \circ B_-^{west,j} &= \begin{cases} B_-^{west,j-1} \circ A_+^{west,i} & i < j \\ 1 & i = j, j+1 \\ B_-^{west,j} \circ A_+^{west,i-1} & i > j+1 \end{cases} \\
B_-^{west,i} \circ B_-^{west,j} &= B_-^{west,j+1} \circ B_-^{west,i} \quad i \le j
\end{aligned}$$

Proof Clear. Perhaps only the cases involving the extremal $A_+^{west,n_-}$ requires some explanation, since the operator $A_+^{west,n_-}$ joins the points $(-2n_- - 2, -2n_- - 1)$, where $-2n_- - 1$ is the root point, and $-2n_- - 2$ is on the eastside. If $n_+ > 0$, then the relations clearly follow, but if $n_+ = 0$, then $A_+^{west,n_-}$ actually connects the root point to the base point. However in the case $n_+ = 0$ we have $e = -n$, and there is only one possible chord diagram, i.e. the one with $n+1$ outermost negative regions. With only one chord diagram to check, the relations are easily verified. ■

It follows that there is a *simplicial structure* on our vector spaces $SFH(T, n+1, e)$, which we think of as having dimension $n_- = (n-e)/2$, with face maps $d_i^+ = A_+^{west,i}$ and degeneracy maps $s_j^+ = B_-^{west,j}$ for $0 \le i, j \le n_-$. Hence the map $d^+ = \sum_i d_i^+ = \sum_i A_+^{west,i}$ satisfies $(d^+)^2 = 0$ (recall we have $\mathbb{Z}_2$ coefficients), and we obtain chain complexes

$$SFH(T, n+1, e) \xrightarrow{d^+} SFH(T, n, e+1) \xrightarrow{d^+} \cdots \xrightarrow{d^+} SFH\left(T, \frac{n+e}{2} + 1, \frac{n+e}{2}\right);$$

equivalently, the pairs $(n_-, n_+)$ proceed $(n_-, n_+) \mapsto (n_- - 1, n_+) \mapsto (n_- - 2, n_+) \mapsto \cdots \mapsto (0, n_+)$. This is a "northeast–southwest" diagonal of Pascal's triangle. We can call the chain complex $C_*^{+,n_+}$, so the $i$-dimensional part (dimension is $i = n_-$) is

$$\begin{aligned}
C_i^{+,n_+} = C_{n_-}^{+,n_+} &= SFH\left(T, \frac{n+e}{2} + 1 + i, \frac{n+e}{2} - i\right) \\
&= SFH\left(T, n_+ + n_- + 1, n_+ - n_-\right).
\end{aligned}$$

Similarly, we have a simplicial structure on the eastside:

**Lemma 13.2 (Eastside simplicial structure)** *For all* $0 \le i, j \le n_+$*, we have*

$$\begin{aligned}
A_-^{east,i} \circ A_-^{east,j} &= A_-^{east,j-1} \circ A_-^{east,i} \quad i < j \\
A_-^{east,i} \circ B_+^{east,j} &= \begin{cases} B_+^{east,j-1} \circ A_-^{east,i} & i < j \\ 1 & i = j, j+1 \\ B_+^{east,j} \circ A_-^{east,i-1} & i > j+1 \end{cases} \\
B_+^{east,i} \circ B_+^{east,j} &= B_+^{east,j+1} \circ B_+^{east,i} \quad i \le j
\end{aligned}$$

This similarly gives rise to another simplicial structure, where we consider $SFH(T, n+1, e)$ to have dimension $n_+ = (n+e)/2$. We take $d_i^- = A_-^{east,i}$ and and $s_j^- = B_+^{east,j}$, and boundary map $d^- = \sum d_i^-$, to obtain the chain complex

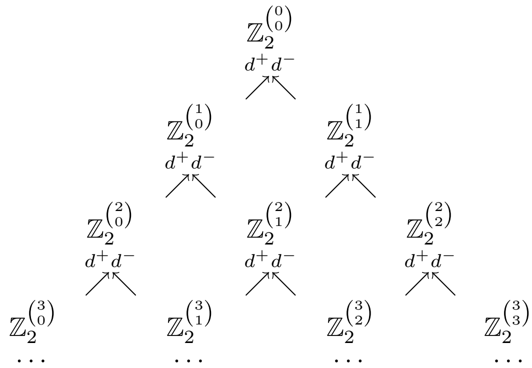
$$SFH(T, n+1, e) \xrightarrow{d^-} SFH(T, n, e-1) \xrightarrow{d^-} \cdots \xrightarrow{d^-} SFH\left(T, \frac{n-e}{2} + 1, \frac{-n+e}{2}\right)$$

where the pairs $(n_-, n_+)$ proceed $(n_-, n_+) \mapsto (n_-, n_+ - 1) \mapsto \cdots \mapsto (n_-, 0)$. This is a "northwest–southeast" diagonal of Pascal's triangle. The homology can be denoted $C_*^{-,n_-}$, so that

$$\begin{aligned}
C_i^{-,n_-} = C_{n_+}^{-,n_-} &= SFH\left(T \frac{n-e}{2} + 1 + i, \frac{-n+e}{2} + i\right) \\
&= SFH\left(T, n_- + n_+ + 1, -n_- + n_+\right).
\end{aligned}$$

Thus the chain complex groups $C_{n_-}^{+,n_+}$ and $C_{n_+}^{-,n_-}$ are both equal to the $SFH$ vector space corresponding to $(n_-, n_+)$.

It is not difficult to see that the two boundary operators $d^-, d^+$ commute. Thus we obtain a double complex structure on the categorified Pascal's triangle:



It is also not too difficult to see that the homology of the chain complexes is rather uninteresting.

**Proposition 13.3 (Westside/eastside homology)**

*(i) The homology of the complex* $\left(C_*^{+,n_+}, d^+\right)$ *is*

$$H_i\left(C_*^{+,n_+}, d^+\right) = 0$$

*for all* $i$

*(ii) The homology of the complex* $\left(C_*^{-,n_-}, d^-\right)$ *is*

$$H_i\left(C_*^{-,n_-}, d^-\right) = 0$$

*for all* $i$.

Here we use a similar argument to Frabetti in [11], in the context of planar binary trees. We will remark on the relationship with planar binary trees in section 14.4.

Proof We note that our original creation operator $B_- = B_-^{west,-1} : C_*^{+,n_+} \longrightarrow C_{*+1}^{+,n_+}$ satisfies

$$A_+^{west,0} B_- = 1, \quad A_+^{west,i} B_- = B_- \text{ for } i > 0.$$

Hence $B_- d^+ + d^+ B_- = 1$. Then $B_-$ is a chain homotopy from the chain maps 1 to 0 on $C_*^{-,n_-}$. ■

Recall that these simplicial face and degeneracy and boundary maps all have symbolic meanings on words:

(i) $B_-^{west,j}$ insert a $-$ sign in $w$ immediately after the $(j+1)$'th $-$ sign, if $0 \le j \le n_- - 1$; and adds a $-$ sign at the end, if $j = n_-$.

(ii) $B_+^{east,j}$ inserts a $+$ sign immediately after the $(j+1)$'th $+$ sign, if $0 \le j \le n_+ - 1$; and adds a $+$ sign at the end, if $j = n_+$.

(iii) $A_+^{west,j}$ deletes the $(j+1)$'th $-$ sign, if $0 \le j \le n_- - 1$; and for $j = n_-$, deletes the $-$ sign at the end of the word (if there is one), else returns 0.

(iv) $A_-^{east,j}$ deletes the $(j+1)$'th $+$ sign, if $0 \leq j \leq n_+ - 1$; and for $j = n_+$, deletes the $+$ sign at the end of the word (if there is one), else returns 0.

Thus $d^+ = \sum A_+^{west,j}$ gives a sum over deleting the various $-$ signs in a word, counting every $-$ sign once, and counting a $-$ sign at the end of the word separately. Similarly, $d^-$ gives a sum over deleting the various $+$ signs in a word, counting a $+$ sign at the end of the word separately.

The simplicial structure on our $SFH$ vector spaces can therefore be considered to arise from symbolic manipulations on words. The fact that $(d^+)^2 = 0$ then has an easy combinatorial proof.

# 14 QFT and categorical considerations

## 14.1 Dimensionally-reduced TQFT

We have seen that sutured Floer homology obeys some of the properties of a topological quantum field theory.

Moreover, in the case of sutured manifolds of the type $(\Sigma \times S^1, F \times S^1)$, where $\Sigma$ is a surface with boundary, and $F$ is a finite collection of points on $\partial\Sigma$, sutured Floer homology can be regarded as a $(1+1)$-dimensional TQFT via dimensional reduction. Clearly, in the case that $\Sigma = D^2$, these sutured manifolds are precisely our $(T, n)$.

In [30], it is noted that

$$SFH(\Sigma \times S^1, F \times S^1) = \mathbb{Z}_2^{2^{n-\chi(\Sigma)}},$$

where $|F| = 2n$ is the number of boundary sutures. As in the case $\Sigma = D^2$, contact structures on such sutured manifolds correspond bijectively to dividing sets $K$ drawn on $\Sigma$ without any contractible components: see [21, 17]. Note, however, that in higher genus surfaces, $K$ may have closed components.

The dimensionally-reduced TQFT has the following properties:

(i) To every pair $(\Sigma, F)$, where $F$ divides $\partial\Sigma$ into positive and negative arcs, we associate the vector space $V(\Sigma, F) = \mathbb{Z}_2^{n-\chi(\Sigma)}$.

(ii) To every properly embedded 1-manifold $K \subset \Sigma$ with boundary $F$, dividing $\Sigma$ into positive and negative regions, consistent with the signs on $\partial\Sigma - F$, we associate an element $c(K)$ of this vector space $V(\Sigma, F)$.

(iii) We consider a certain class of gluings of $\Sigma$: we take $\gamma, \gamma' \subset \partial\Sigma$, disjoint 1-submanifolds, with endpoints not in $F$; we suppose that we have a gluing $\tau$ which glues $\gamma$ to $\gamma'$ to preserve all relevant sign labellings, to obtain $(\Sigma', F')$. To such a gluing we a associate a map $\Phi_\tau : V(\Sigma, F) \longrightarrow V(\Sigma', F')$, and it takes any $c(K) \mapsto c(\bar{K})$, where $\bar{K}$ is obtained by performing the gluing $\tau$ on $K$.

See [30] for further details.

We have given a fairly explicit description of the mechanics of this topological quantum field theory, in the case where $\Sigma$ is a disc.

In their paper [30], Honda–Kazez–Matić prove some further properties of this TQFT:

(i) $V(\Sigma, F)$ is generated by contact elements;

(ii) $c(K) = 0$ if and only if $K$ is *separating* in the sense that $\Sigma - K$ has components which do not intersect $\partial\Sigma$.

## 14.2 A contact 2-category

Honda [19] has introduced the notion of a *contact category* for a pair $(\Sigma, F)$. The objects of this category are the various $K$ in the dimensionally-reduced TQFT described above, i.e. properly embedded 1-manifolds with boundary $F$, i.e. dividing sets corresponding to $S^1$-invariant contact structures on

$(\Sigma \times S^1, F \times S^1)$. The morphisms $K_0 \to K_1$ in this category are contact structures on $\Sigma \times I$ with dividing set $K_0$ on $\Sigma \times \{0\}$ and dividing set $K_1$ on $\Sigma \times \{1\}$. This category obeys many of the properties of a *triangulated category*. In particular, there are distinguished triangles, arising from bypass additions, and these obey an octahedral axiom.

We note that in our case $\Sigma = D^2$, the various $K$ corresponding to nontrivial contact structures are just chord diagrams $\Gamma$. And we have shown that such $\Gamma$ are naturally described by pairs $(\Gamma^-, \Gamma^+)$, corresponding to words $w^- \preceq w^+$. But a partial order can itself be considered as a category! In particular, a partial order is a category where every pair of objects has at most one morphism between them. This leads us to the idea of a 2-category.

We will define a "contact 2-category", which is a generalisation of Honda's contact category, in the sense that the objects of Honda's category become our 1-morphisms; and its 1-morphisms become our 2-morphisms. It is a specialisation of Honda's contact category, in the sense that it only applies to $\Sigma = D^2$. (There are isomorphisms between $V(\Sigma, F)$ and $V(D^2, F')$, see section 14.4 below, but there is no canonical isomorphism; and for any given gluing of $D^2$ to obtain $\Sigma$, there may be contact elements in $V(\Sigma, F)$ for which this gluing does not apply. Hence, for the time being at least, we restrict ourselves to discs.)

**Definition 14.1 (Contact 2-category)** *The* contact 2-category $\mathcal{C}(n+1, e)$ *is defined as follows.*

*(i) The objects (or 0-cells) $Ob(\mathcal{C})$ are words on $\{-,+\}$ with $n_-$ $-$ signs and $n_+$ $+$ signs.*

*(ii) The 1-morphisms $w_0 \to w_1$ are those arising from the partial order $\preceq$. There is one 1-morphism $w_0 \to w_1$ if $w_0 \preceq w_1$, and none otherwise.*

- *The composition of two morphisms $w_0 \to w_1 \to w_2$ is the unique morphism $w_0 \to w_2$.*
- *Thus the 1-morphisms correspond precisely to the chord diagrams $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$ on the disc with $n+1$ chords; the composition of the two chord diagrams $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$ and $\Gamma' = [\Gamma_{w_1}, \Gamma_{w_2}]$ is*
$$\Gamma \circ \Gamma' = [\Gamma_{w_0}, \Gamma_{w_2}].$$

*(iii) The 2-morphisms $\Gamma_0 \to \Gamma_1$ are the tight contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$, along with one extra 2-morphism $\{*\}$ for overtwisted contact structures. There are two types of composition of 2-morphisms.*

- *Given two 2-morphisms*
$$\Gamma_0 \stackrel{\xi_0}{\to} \Gamma_1 \stackrel{\xi_1}{\to} \Gamma_2,$$
*their* vertical composition $\xi_0 \cdot \xi_1$ *is the 2-morphism $\Gamma_0 \to \Gamma_2$ is the contact structure on $\mathcal{M}(\Gamma_0, \Gamma_2)$ obtained by stacking $\mathcal{M}(\Gamma_0, \Gamma_1)$ and $\mathcal{M}(\Gamma_1, \Gamma_2)$.*
- *Given three objects $w_0, w_1, w_2$, two pairs of 1-morphisms between them*
$$w_0 \stackrel{\Gamma_0, \Gamma_0'}{\longrightarrow} w_1, \quad w_1 \stackrel{\Gamma_1, \Gamma_1'}{\longrightarrow} w_2,$$
*and two* 2*-morphisms*
$$\Gamma_0 \stackrel{\xi_0}{\longrightarrow} \Gamma_0', \quad \Gamma_1 \stackrel{\xi_1}{\longrightarrow} \Gamma_1',$$
*the* horizontal composition $\xi_0 \xi_1$ *is a morphism $(\Gamma_0 \circ \Gamma_1) \to (\Gamma_0' \circ \Gamma_1)$ defined as follows. Since the 1-morphisms arise from a partial order, $\Gamma_0 = \Gamma_0'$ and $\Gamma_1 = \Gamma_1'$. Thus $\xi_0$ is a contact structure on $\mathcal{M}(\Gamma_0, \Gamma_0)$ and $\xi_1$ on $\mathcal{M}(\Gamma_1, \Gamma_1)$. If these are both the unique tight contact structures, then we define $\xi_0 \xi_1$ to be the unique tight contact structure on $\mathcal{M}(\Gamma_0 \circ \Gamma_1, \Gamma_0 \circ \Gamma_1)$. Otherwise $\xi_0 \xi_1 = \{*\}$.*

**Lemma 14.2 (Existence of contact 2-category)** $\mathcal{C}(n+1, e)$ *is a 2-category.*

PROOF That the objects and 1-morphisms form a category is clear. That vertical composition is associative is clear, since it just corresponds to a union of contact structures. Note that $\{*\}$ acts as a zero for this composition; any composition involving $\{*\}$ is again $\{*\}$.

That horizontal composition is associative is also clear: if any of the $\xi_i$ being composed is overtwisted $\{*\}$, then the horizontal composition is $\{*\}$; else associativity follows immediately since 1-morphisms arise from a partial order. Again $\{*\}$ acts as a zero.

There is an identity 2-morphism for each 1-morphism $\Gamma$: the identity 2-morphism $\Gamma \xrightarrow{1_\Gamma} \Gamma$ is the tight contact structure on $\mathcal{M}(\Gamma, \Gamma)$. This is just a thickened neighbourhood of a convex surface, so its vertical composition is indeed the identity; and since it is not $\{*\}$, its horizontal composition is also the identity.

The "interchange law"

$$(\xi_1 \cdot \xi_2)(\xi_3 \cdot \xi_4) = (\xi_1 \xi_3) \cdot (\xi_2 \cdot \xi_4)$$

is only defined when the 2-morphisms $\xi_1, \xi_3$ are contact structures on some $\mathcal{M}([\Gamma, \Gamma])$, where $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$; and similarly the 2-morphisms $\xi_2, \xi_4$ are contact structures on some $\mathcal{M}(\Gamma', \Gamma')$, where $\Gamma' = [\Gamma_{w_1}, \Gamma_{w_2}]$. If any of these is $\{*\}$, we have $\{*\}$ on both sides. If not, then $\xi_1 = \xi_3 = 1_\Gamma$ and $\xi_2 = \xi_4 = 1_{\Gamma'}$, being standard neighbourhoods of chord diagrams; thus both sides are equal to $\xi_1 \cdot \xi_2 = 1_{\Gamma \circ \Gamma'}$, the unique tight contact structure on $\mathcal{M}(\Gamma'', \Gamma'')$, where $\Gamma'' = [\Gamma_{w_0}, \Gamma_{w_2}]$. ■

We have now proved proposition 1.30.

## 14.3 Improving the 2-category structure

Honda has shown that the morphisms of his contact category satisfy some, but not all, of the axioms of a triangulated category. The 2-morphisms of our contact 2-category, therefore obey the same axioms. In fact, in our simpler case of a disc, more of the axioms are obeyed.

A triangulated category is an additive category with certain triples of morphisms $A \to B \to C \to A$ called *distinguished triangles* satisfying various properties.

The first defect of the contact category from being a triangulated category is that it is not additive. However, in our case of a disc, it is: there are at most two morphisms between two chord diagrams $\Gamma_0, \Gamma_1$, namely $\{*\}$, and the unique tight contact structure, if $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight. We can simply "add" morphisms by regarding the tight contact structure as 1 and $\{*\}$ as 0. (When replacing $D$ with a more complicated surface, there may be many more contact structures.)

Note also that for our contact 2-category, there are *no triangles* among 1-morphisms, since they come from a partial order. Hence, if our 2-category is to satisfy anything like the axioms of a triangulated category, it is only meaningful to look at the 2-morphisms.

In Honda's contact category, the distinguished triangles are contact structures arising from bypass attachments. We could do the same in our case, obtaining distinguished triangles of 2-morphisms; but in our case we also have more complicated bypass *systems*, such as bypass systems of generalised arcs of attachment in section 7, and bypass systems of well-placed sequences of nicely ordered generalised arcs of attachment. We also have our "generalised bypass triples" arising from a choice of 1-morphism $w_0 \preceq w_1$, which is the triple of chord diagrams (1-morphisms) $\Gamma_{w_0}, \Gamma_{w_1}, [\Gamma_{w_0}, \Gamma_{w_1}]$. We could potentially regard any of these as distinguished triangles.

The second way in which Honda's contact category fails to be a triangulated category is that not every morphism can be completed to a distinguished triangle. In our case, with any of the above choices of definition of distinguished triangles, this also fails. For instance, two basis chord diagrams for non-comparable words have no bypass systems of well-placed nicely ordered generalised arcs of attachment to obtain one from the other.

We can ask: Is there a more general definition of distinguished triangle, or improvement of the category structure, for which this axiom is true?

Honda shows that $SFH$ gives a functor from his contact category to the category of vector spaces. Here, of course, since our 1- and 2-morphisms are the objects and morphisms of that category. In this functor, our objects map to basis elements of those vector spaces.

As a final categorical comment, note that our contact 2-category $\mathcal{C}(n+1,e)$ is specific to an $n$ and $e$; in fact, all its objects and morphisms relate to $SFH(T,n,e)$.

If we consider these $SFH(T,n,e)$ over all $n$ and $e$, we obtain a family of 2-categories. Moreover, the 0-cells of this 2-category are words on $\{-,+\}$. But these can themselves be regarded as *paths* on Pascal's triangle.

This suggests the construction of a 3-category where:

(i) objects are points of Pascal's triangle, pairs $(n+1,e)$, or perhaps more generally the integer lattice, or perhaps just a point;

(ii) 1-morphisms are finite paths on Pascal's triangle, or the lattice, generated by unit southeast and southwest moves on the triangle;

(iii) 2-morphisms are generated by the partial order $\preceq$; equivalently, paths on Pascal's triangle from the origin to the same endpoint, one always lying left of the other; equivalently, chord diagrams $\Gamma$ or contact elements;

(iv) 3-morphisms are contact structures on $\mathcal{M}(\Gamma_0,\Gamma_1)$;

This is a question for further investigation.

## 14.4 QFT remarks

The dimensionally-reduced TQFT described above has certain gluing isomorphisms, proved in [30].

**Theorem 14.3 ([30], 7.9)** *Suppose that $\gamma,\gamma'$ are disjoint 1-submanifolds of $\partial\Sigma$, with endpoints not in $F$, and each intersecting $F$ precisely once. Then the gluing $\tau$ produces $(\Sigma',F')$, and*

$$\Phi_\tau : V(\Sigma,F) \longrightarrow V(\Sigma',F')$$

*is an isomorphism.*

Such a gluing decreases $|F|=2n$ by 2 and decreases $\chi$ by 1. Since the dimension of $V(\Sigma,F)$ is $2^{n-\chi(\sigma)}$, the two vector spaces $V(\Sigma,F)$ and $V(\Sigma',F')$ are the same, as we should expect.

By repeated application of such gluing, or the reverse procedure of cutting, we can obtain many isomorphisms between different $V(\Sigma,F)$. In particular, if we have any $(\Sigma,F)$ and $K$, and we can cut $\Sigma$ into a disc along properly embedded arcs or closed curves, each of which intersects $K$ precisely once, then we have an isomorphism $V(D^2,F')\cong V(\Sigma,F)$.

In [30] it is proved that contact elements $c(K)$ generate $V(\Sigma,F)$. In fact, it's easy to see from our isomorphism that $V(\Sigma,F)$ is generated by contact elements which are obtained by gluing contact elements in $V(D^2,F')$.

However, in general, although there may be an isomorphism between any $V(\Sigma,F)$ and some $V(D^2,F')$, this need not not give a bijection between contact elements. Every contact element in $V(D^2,F')$ gives a corresponding contact element in $V(\Sigma,F)$; but not all contact elements in $V(\Sigma,F)$ arise in this way. That is, the isomorphism $V(D^2,F')\to V(\Sigma,F)$ induces an injective but not surjective map on contact elements.

In particular, such an isomorphism exists for any $(\Sigma,F)$, where $\Sigma$ is an $m$-times punctured disc ($m$ any nonnegative integer), and $F$ has two points on the boundary of every puncture. We may simply glue up every puncture and obtain a disc. Such a punctured disc can be regarded as a cobordism from $m$ "trivial circles", with two points marked on each, to a "non-trivial circle" with $2n$ points marked.

In a standard topological quantum field theory picture, this is a good reason why a chord diagram with 1 chord can be regarded as "the vacuum". It may be glued up, or "filled in", or "capped off", without any effect. It is, effectively, not there. A cobordism from $m$ vacua, to a higher particle state, is equivalent to a cobordism from the empty set, in this TQFT.

We may also remark that our chord diagrams are bijective, in an explicit fashion, with *planar binary trees*, and the vector space generated by such objects has been considered previously; they have also been considered in physical contexts. See, e.g., [4, 11, 12, 18, 35, 36]. The bypass relation translates into a similar linear relation on trees, which appears not to have been considered previously, so far as the author could find.

The upshot is that the story is not finished yet, and discovering much of the structure of this TQFT still lies ahead.

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