Chord diagrams, topological quantum field theory, and the sutured Floer homology of solid tori

arXiv:0903.1453v1 [math.GT]

Daniel Mathews

Stanford University
mathews@math.stanford.edu

Columbia University
17 April 2009

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Sutured manifolds & *SFH*

Juhász 2006, *Holomorphic discs and sutured manifolds*

Sutured manifold (M, Γ)

M 3-manifold with boundary.
Γ collection of disjoint simple closed curves on boundary, dividing ∂M into positive/negative regions.

(Balanced.)

$(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)$

- Take sutured Heegaard decomposition, symmetric product of Heegaard surface.
- Chain complex generated by intersection points of α, β tori.
- Differential counts certain holomorphic curves in symmetric product with certain boundary conditions.
- Invariant of (balanced) sutured manifold.

Sutured manifolds & *SFH*

Juhász 2006, *Holomorphic discs and sutured manifolds*

Sutured manifold (M, Γ)

M 3-manifold with boundary.
Γ collection of disjoint simple closed curves on boundary, dividing ∂M into positive/negative regions.

(Balanced.)

$(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)$

- Take sutured Heegaard decomposition, symmetric product of Heegaard surface.
- Chain complex generated by intersection points of α, β tori.
- Differential counts certain holomorphic curves in symmetric product with certain boundary conditions.
- Invariant of (balanced) sutured manifold.

Contact elements

Closed case Ozsváth–Szabó 2005, Honda–Kazez–Matić 2007; sutured case Honda–Kazez–Matić 2007, *The contact invariant in sutured Floer homology*

Contact structutre on sutured manifold

ξ contact structure on (M, Γ):

- ∂M convex
- Γ dividing set
- Positive/negative regions.

Theorem (Honda–Kazez–Matić)

A contact structure ξ on (M, Γ) gives a well-defined contact element $c(\xi) \in SFH(-M, -\Gamma)$.

We take $\mathbb{Z}_2$ coefficients throughout.
With $\mathbb{Z}$ coefficients, $c(\xi)$ subset of form $\{\pm x\}$.

Contact elements

Closed case Ozsváth–Szabó 2005, Honda–Kazez–Matić 2007; sutured case Honda–Kazez–Matić 2007, *The contact invariant in sutured Floer homology*

Contact structutre on sutured manifold

ξ contact structure on (M, Γ):

- ∂M convex
- Γ dividing set
- Positive/negative regions.

Theorem (Honda–Kazez–Matić)

A contact structure ξ on (M, Γ) gives a well-defined contact element $c(\xi) \in SFH(-M, -\Gamma)$.

We take $\mathbb{Z}_2$ coefficients throughout.
With $\mathbb{Z}$ coefficients, $c(\xi)$ subset of form $\{\pm x\}$.

Contact elements

Closed case Ozsváth–Szabó 2005, Honda–Kazez–Matić 2007; sutured case Honda–Kazez–Matić 2007, *The contact invariant in sutured Floer homology*

Contact structutre on sutured manifold

ξ contact structure on (M, Γ):

- ∂M convex
- Γ dividing set
- Positive/negative regions.

Theorem (Honda–Kazez–Matić)

A contact structure ξ on (M, Γ) gives a well-defined contact element $c(\xi) \in SFH(-M, -\Gamma)$.

We take $\mathbb{Z}_2$ coefficients throughout.
With $\mathbb{Z}$ coefficients, $c(\xi)$ subset of form $\{\pm x\}$.

Properties of *SFH*

Contact element properties:
(HKM 2007, *The contact invariant in sutured Floer homology*)

- ξ overtwisted $\Rightarrow c(\xi) = 0$
- (M, Γ, ξ) embeds in closed (N, ξ') with $c(\xi') \neq 0 \;\Rightarrow\; c(\xi) \neq 0$.

Every generator of chain complex has a spin-c structure $\mathfrak{s}$.
SFH splits over spin-c structures:

$$SFH(M, \Gamma) = \bigoplus_{\mathfrak{s}} SFH(M, \Gamma, \mathfrak{s}).$$

Properties of *SFH*

Contact element properties:
(HKM 2007, *The contact invariant in sutured Floer homology*)

- ξ overtwisted $\Rightarrow c(\xi) = 0$
- (M, Γ, ξ) embeds in closed (N, ξ') with $c(\xi') \neq 0 \;\Rightarrow\; c(\xi) \neq 0$.

Every generator of chain complex has a spin-c structure $\mathfrak{s}$.
SFH splits over spin-c structures:

$$SFH(M, \Gamma) = \bigoplus_{\mathfrak{s}} SFH(M, \Gamma, \mathfrak{s}).$$

TQFT property

Honda–Kazez–Matić 2008, *Contact structures, sutured Floer homology and TQFT*

Theorem

Given

- $(M', \Gamma') \hookrightarrow (M, \Gamma)$ *inclusion of sutured manifolds.*
- ξ'' *contact structure on* $(M - M', \Gamma \cup \Gamma')$

there is a natural map

$$SFH(M', \Gamma') \longrightarrow SFH(M, \Gamma).$$

Further

$$c(\xi') \mapsto c(\xi' \cup \xi'').$$

"TQFT-inclusion".
(Actually $\longrightarrow SFH(M, \Gamma) \otimes V^m$ where m is number of "isolated" components).

TQFT property

Honda–Kazez–Matić 2008, *Contact structures, sutured Floer homology and TQFT*

Theorem

Given

- $(M', \Gamma') \hookrightarrow (M, \Gamma)$ *inclusion of sutured manifolds.*
- ξ'' *contact structure on* $(M - M', \Gamma \cup \Gamma')$

there is a natural map

$$SFH(M', \Gamma') \longrightarrow SFH(M, \Gamma).$$

Further

$$c(\xi') \mapsto c(\xi' \cup \xi'').$$

"TQFT-inclusion".

(Actually $\longrightarrow SFH(M, \Gamma) \otimes V^m$ where m is number of "isolated" components).

The question

Motivating question:

How do contact elements lie in SFH?

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Solid tori

Only sutured 3-manifolds we consider are *solid tori*.

Sutured manifold (T, n)

- Solid torus $D^2 \times S^1$
- Convex boundary $\partial D^2 \times S^1$
- Longitudinal dividng set $F \times S^1$, F finite, $|F| = 2n$.

(Notational cover-up: $(T, n) = (-(D^2 \times S^1), -(F \times S^1))$.)

- Part of the $(1+1)$-dimensional TQFT discussed in HKM 2008.

To classify contact structures:

- consider dividing sets on convex meridian disc and boundary torus

Solid tori

Only sutured 3-manifolds we consider are *solid tori*.

Sutured manifold (T, n)

- Solid torus $D^2 \times S^1$
- Convex boundary $\partial D^2 \times S^1$
- Longitudinal dividng set $F \times S^1$, F finite, $|F| = 2n$.

(Notational cover-up: $(T, n) = (-(D^2 \times S^1), -(F \times S^1))$.)

- Part of the $(1+1)$-dimensional TQFT discussed in HKM 2008.

To classify contact structures:

- consider dividing sets on convex meridian disc and boundary torus

Convex surfaces

Giroux 1991, *Convexité en topologie de contact*

Generic property for embedded surface in contact 3-manifold.

Convex surface S

There exists a contact vector field X transverse to S.

"Invariant vertical direction".

Dividng set

$$\Gamma = \{x \in S \ : \ X(x) \in \xi\}.$$

"Where ξ is perpendicular".
Dividing set *divides* S into positive/negative regions $S_\pm$.
Euler class evaluation:

$$e(\xi)[S] = \chi(S_+) - \chi(S_-).$$

Convex surfaces

Giroux 1991, *Convexité en topologie de contact*

Generic property for embedded surface in contact 3-manifold.

Convex surface S

There exists a contact vector field X transverse to S.

"Invariant vertical direction".

Dividng set

$$\Gamma = \{x \in S \ : \ X(x) \in \xi\}.$$

"Where ξ is perpendicular".

Dividing set *divides* S into positive/negative regions $S_\pm$.

Euler class evaluation:

$$e(\xi)[S] = \chi(S_+) - \chi(S_-).$$

Convex surfaces

Giroux 1991, *Convexité en topologie de contact*

Generic property for embedded surface in contact 3-manifold.

Convex surface S

There exists a contact vector field X transverse to S.

"Invariant vertical direction".

Dividng set

$$\Gamma = \{x \in S \ : \ X(x) \in \xi\}.$$

"Where ξ is perpendicular".
Dividing set *divides* S into positive/negative regions $S_\pm$.
Euler class evaluation:

$$e(\xi)[S] = \chi(S_+) - \chi(S_-).$$

Contact structure near a convex surface

Giroux 1991, *Convexité en topologie de contact*;
Honda 2000, *On the classification of tight contact structures. I*

Theorem (Giroux)

The dividing set essentially determines the contact structure near a convex surface.

Given S, Γ, is the nearby contact structure tight?

For $S \neq S^2$:

Contact structure is tight iff Γ has no contractible components.

For $S = S^2$:

Contact structure is tight iff Γ has one component.

If $S^2 = \partial B^3$, tight contact structure near boundary extends uniquely over ball (Eliashberg).

Contact structure near a convex surface

Giroux 1991, *Convexité en topologie de contact*;
Honda 2000, *On the classification of tight contact structures. I*

Theorem (Giroux)

The dividing set essentially determines the contact structure near a convex surface.

Given S, Γ, is the nearby contact structure tight?

For $S \neq S^2$:

Contact structure is tight iff Γ has no contractible components.

For $S = S^2$:

Contact structure is tight iff Γ has one component.

If $S^2 = \partial B^3$, tight contact structure near boundary extends uniquely over ball (Eliashberg).

Corners & rounding

Honda 2000, *On the classification of tight contact structures. I*

- When convex surfaces meet transversely along a legendrian curve, dividing sets interleave.
- Corners can be rounded in a standard way.

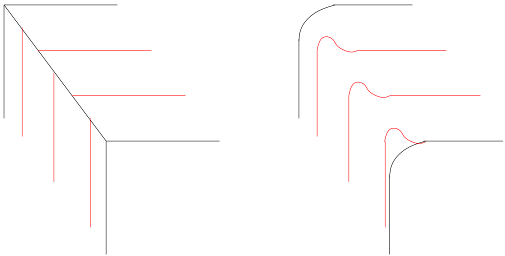


Figure: Rounding corners.

Contact structures on (T, n) and chord diagrams

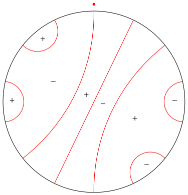
Dividing set Γ on meridional disc (convex, leg. b'dy)

- Interleaves with sutures $F \times S^1$ on boundary; $2n$ endpoints.
- For tight contact structure, Γ has no closed components.

Chord diagram

Collection of disjoint properly embedded arcs on disc.
Up to homotopy rel endpoints.

E.g.



Contact structures on (T, n) and chord diagrams

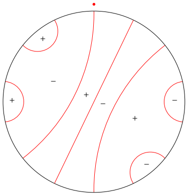
Dividing set Γ on meridional disc (convex, leg. b'dy)

- Interleaves with sutures $F \times S^1$ on boundary; $2n$ endpoints.
- For tight contact structure, Γ has no closed components.

Chord diagram

Collection of disjoint properly embedded arcs on disc.
Up to homotopy rel endpoints.

E.g.



Euler class of chord diagram

Chord diagram has relative euler class e.

$$|e| \leq n - 1, \quad e + n \equiv 1 \bmod 2.$$

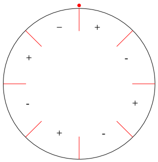


Figure: Basepoint, convention for signs of regions.

Contact structures on (T, n) *are* chord diagrams

Honda 2000, *On the classification of tight contact structures. II*;
Honda 2002, *Gluing tight contact structures*;
Giroux 2001, *Structures de contact sur les variétés fibrées en cercles...*

Chord diagram determines at most one tight contact structure on $D^2 \times S^1$:

- Cut into solid cylinder, round corners of D^3

For solid tori in general:

- Chord diagrams on D may give overtwisted contact structure on $D^2 \times S^1$
- Distinct chord diagrams may give isotopic contact structures.

However with *longitudinal sutures* of (T, n), neither occurs.

Theorem (Honda, Giroux)

$$\left\{ \begin{array}{c} \textit{Tight contact structures} \\ \textit{on } (T, n) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Chord diagrams} \\ \textit{with } n \textit{ chords} \end{array} \right\}$$

Contact structures on (T, n) *are* chord diagrams

Honda 2000, *On the classification of tight contact structures. II*;
Honda 2002, *Gluing tight contact structures*;
Giroux 2001, *Structures de contact sur les variétés fibrées en cercles...*

Chord diagram determines at most one tight contact structure on $D^2 \times S^1$:

- Cut into solid cylinder, round corners of D^3

For solid tori in general:

- Chord diagrams on D may give overtwisted contact structure on $D^2 \times S^1$
- Distinct chord diagrams may give isotopic contact structures.

However with *longitudinal sutures* of (T, n), neither occurs.

Theorem (Honda, Giroux)

$$\left\{ \begin{array}{c} \textit{Tight contact structures} \\ \textit{on } (T, n) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Chord diagrams} \\ \textit{with } n \textit{ chords} \end{array} \right\}$$

Contact structures on (T, n) *are* chord diagrams

Honda 2000, *On the classification of tight contact structures. II*;
Honda 2002, *Gluing tight contact structures*;
Giroux 2001, *Structures de contact sur les variétés fibrées en cercles...*

Chord diagram determines at most one tight contact structure on $D^2 \times S^1$:

- Cut into solid cylinder, round corners of D^3

For solid tori in general:

- Chord diagrams on D may give overtwisted contact structure on $D^2 \times S^1$
- Distinct chord diagrams may give isotopic contact structures.

However with *longitudinal sutures* of (T, n), neither occurs.

Theorem (Honda, Giroux)

$$\left\{ \begin{array}{c} \textit{Tight contact structures} \\ \textit{on } (T, n) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Chord diagrams} \\ \textit{with } n \textit{ chords} \end{array} \right\}$$

Catalan and Naryana numbers

$$\#\left\{\begin{array}{c}\text{Tight contact}\\ \text{structures on } (T,n)\end{array}\right\} = \#\left\{\begin{array}{c}\text{Chord diagrams}\\ n \text{ chords}\end{array}\right\}$$

Catalan numbers $C_n = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$$\#\left\{\begin{array}{c}\text{Tight contact}\\ \text{structures on } (T,n)\\ \text{euler class } e\end{array}\right\} = \#\left\{\begin{array}{c}\text{Chord diagrams}\\ n \text{ chords}\\ \text{euler class } e\end{array}\right\}$$

Narayana numbers C_n^e:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 6 & & 6 & & 1 & \\ 1 & & 10 & & 20 & & 10 & & 1 \end{array}$$

Catalan and Naryana numbers

$$\#\left\{\begin{array}{c}\text{Tight contact}\\ \text{structures on } (T,n)\end{array}\right\} = \#\left\{\begin{array}{c}\text{Chord diagrams}\\ n \text{ chords}\end{array}\right\}$$

Catalan numbers $C_n = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$$\#\left\{\begin{array}{c}\text{Tight contact}\\ \text{structures on } (T,n)\\ \text{euler class } e\end{array}\right\} = \#\left\{\begin{array}{c}\text{Chord diagrams}\\ n \text{ chords}\\ \text{euler class } e\end{array}\right\}$$

Narayana numbers C_n^e:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 1 & & \\
 & 1 & & 6 & & 6 & & 1 & \\
1 & & 10 & & 20 & & 10 & & 1
\end{array}$$

The Catalan disease
and Narayana symptoms

Catalan:

- tight ct. str's on (T, n)
- chord diagrams, n chords
- pairings of n brackets
- Dyck paths length $2n$
- rooted planar bin. trees
- Recursion

$$C_{n+1} = \sum_{n_1+n_2=n} C_{n_1} C_{n_2}.$$

Narayana:

- # with euler class e
- # with euler class e
- # with k occurrences of "()"
- # with k peaks
- # with k "left" leaves
- Recursion:

$$C^e_{n+1} = \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C^{e_1}_{n_1} C^{e_2}_{n_2}.$$

The Catalan disease

and Narayana symptoms

Catalan:

- tight ct. str's on (T, n)
- chord diagrams, n chords
- pairings of n brackets
- Dyck paths length $2n$
- rooted planar bin. trees
- Recursion

$$C_{n+1} = \sum_{n_1+n_2=n} C_{n_1} C_{n_2}.$$

Narayana:

- # with euler class e
- # with euler class e
- # with k occurrences of "()"
- # with k peaks
- # with k "left" leaves
- Recursion:

$$C^e_{n+1} = \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C^{e_1}_{n_1} C^{e_2}_{n_2}.$$

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Computation of $SFH(T, n)$

Juhász 2008, "Floer homology and surface decompositions"
Honda–Kazez–Matić 2008, *Contact structures, sutured Floer homology and TQFT*

Theorem

$$SFH(T, n+1) = \mathbb{Z}_2^{2^n}.$$

Split over spin-c structures:

$$SFH(T, n+1) = \bigoplus_k \mathbb{Z}_2^{\binom{n}{k}}.$$

For ξ with euler class e,

$$c(\xi) \in \mathbb{Z}_2^{\binom{n}{k}} \quad \text{where} \quad k = \frac{e+n}{2}$$

so let

$$SFH(T, n+1, e) = \mathbb{Z}_2^{\binom{n}{k}}.$$

Computation of $SFH(T, n)$

Juhász 2008, “Floer homology and surface decompositions”
Honda–Kazez–Matić 2008, *Contact structures, sutured Floer homology and TQFT*

Theorem

$$SFH(T, n+1) = \mathbb{Z}_2^{2^n}.$$

Split over spin-c structures:

$$SFH(T, n+1) = \bigoplus_k \mathbb{Z}_2^{\binom{n}{k}}.$$

For ξ with euler class e,

$$c(\xi) \in \mathbb{Z}_2^{\binom{n}{k}} \quad \text{where} \quad k = \frac{e+n}{2}$$

so let

$$SFH(T, n+1, e) = \mathbb{Z}_2^{\binom{n}{k}}.$$

"Categorified Pascal triangle"

$$
\begin{array}{lcccccccc}
SFH(T,1) & = & & & & SFH(T,1,0) & & & \\
SFH(T,2) & = & & & SFH(T,2,-1) & \oplus & SFH(T,2,1) & & \\
SFH(T,3) & = & & SFH(T,3,-2) & \oplus & SFH(T,3,0) & \oplus & SFH(T,3,2) & \\
SFH(T,4) & = & SFH(T,4,-3) & \oplus & SFH(T,4,-1) & \oplus & SFH(T,4,1) & \oplus & SFH(T,4,3) \\
\cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

$$
\begin{array}{lcccccccc}
SFH(T,1) & = & & & & \mathbb{Z}_2^{\binom{0}{0}} & & & \\
SFH(T,2) & = & & & \mathbb{Z}_2^{\binom{1}{0}} & \oplus & \mathbb{Z}_2^{\binom{1}{1}} & & \\
SFH(T,3) & = & & \mathbb{Z}_2^{\binom{2}{0}} & \oplus & \mathbb{Z}_2^{\binom{2}{1}} & \oplus & \mathbb{Z}_2^{\binom{2}{2}} & \\
SFH(T,4) & = & \mathbb{Z}_2^{\binom{3}{0}} & \oplus & \mathbb{Z}_2^{\binom{3}{1}} & \oplus & \mathbb{Z}_2^{\binom{3}{2}} & \oplus & \mathbb{Z}_2^{\binom{3}{3}}
\end{array}
$$

Catalan and Pascal triangle

Contact elements in each $SFH(T, n, e)$ form a distinguished subset of order C_n^e.

$$\left\{ \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 6 & & 6 & & 1 \end{array} \right\} \subset \left\{ \begin{array}{ccccccc} & & & \mathbb{Z}_2^1 & & & \\ & & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^1 & & \\ & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^2 & \oplus & \mathbb{Z}_2^1 & \\ \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^1 \end{array} \right\}$$

Question:

How do the C_n^e contact elements lie in $\mathbb{Z}_2^{\binom{n}{k}}$?

Catalan and Pascal triangle

Contact elements in each $SFH(T, n, e)$ form a distinguished subset of order C_n^e.

$$\left\{ \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 6 & & 6 & & 1 \end{array} \right\} \subset \left\{ \begin{array}{ccccccc} & & & \mathbb{Z}_2^1 & & & \\ & & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^1 & & \\ & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^2 & \oplus & \mathbb{Z}_2^1 & \\ \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^1 \end{array} \right\}$$

Question:

How do the C_n^e contact elements lie in $\mathbb{Z}_2^{\binom{n}{k}}$?

Addition and bypasses

Are the contact elements a subgroup?

- No.
- Closure under addition described by bypasses.

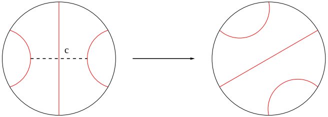


Figure: Upwards bypass surgery along arc c.

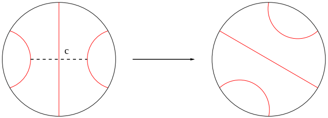


Figure: Downwards bypass surgery along arc c.

Addition and bypasses

Are the contact elements a subgroup?

- No.
- Closure under addition described by bypasses.

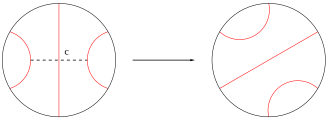


Figure: Upwards bypass surgery along arc c.

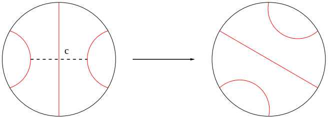


Figure: Downwards bypass surgery along arc c.

Addition and bypasses

Are the contact elements a subgroup?

- No.
- Closure under addition described by bypasses.

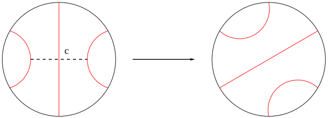


Figure: Upwards bypass surgery along arc c.

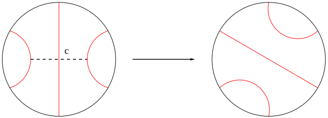


Figure: Downwards bypass surgery along arc c.

The bypass relation

Bypass-related chord diagrams naturally come in triples.

Proposition

Suppose $a, b \in SFH(T, n, e)$ are contact elements. Then $a + b$ is a contact element if and only if a, b are related by a bypass surgery. In this case, $a + b$ is the third chord diagram in the triple.

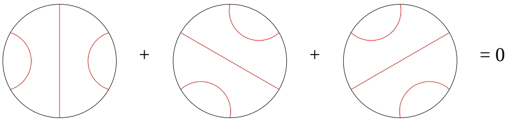


Figure: Bypass relation.

The bypass relation

Bypass-related chord diagrams naturally come in triples.

Proposition

Suppose $a, b \in SFH(T, n, e)$ are contact elements.
Then $a + b$ is a contact element if and only if a, b are related by a bypass surgery.
In this case, $a + b$ is the third chord diagram in the triple.

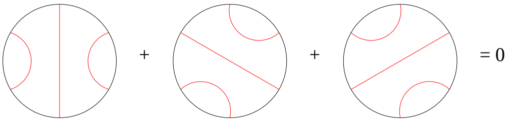


Figure: Bypass relation.

Reduction to kindergarten

In fact one can show:

Proposition (SFH is combinatorial)

$$SFH(T, n, e) = \frac{\mathbb{Z}_2 \langle \text{Chord diag's, } n \text{ chords, euler class } e \rangle}{\text{Bypass relation}}$$

Also:

- There is a basis of contact elements.
- Distinct contact structures / chord diagrams all give distinct contact elements.

Reduction to kindergarten

In fact one can show:

Proposition (SFH is combinatorial)

$$SFH(T, n, e) = \frac{\mathbb{Z}_2 \langle \textit{Chord diag's, n chords, euler class e} \rangle}{\textit{Bypass relation}}$$

Also:

- There is a basis of contact elements.
- Distinct contact structures / chord diagrams all give distinct contact elements.

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Creation operators

A well-defined way to create create chords, enclosing positive/negative regions at the basepoint.

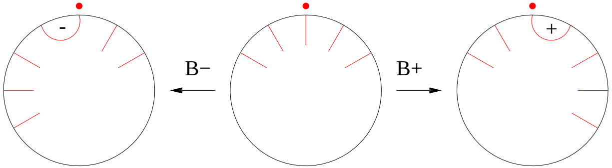


Figure: Creation operators.

We obtain maps

$$B_{\pm} : SFH(T, n, e) \longrightarrow SFH(T, n+1, e \pm 1).$$

Creation operators

A well-defined way to create create chords, enclosing positive/negative regions at the basepoint.

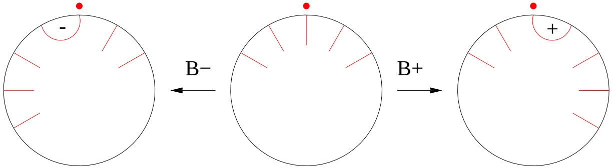


Figure: Creation operators.

We obtain maps

$$B_{\pm} : SFH(T, n, e) \longrightarrow SFH(T, n+1, e \pm 1).$$

Origin of creation operators

$B_{\pm}$ arise from TQFT-inclusion

$$(T, n) \hookrightarrow (T, n+1)$$

with intermediate contact structure

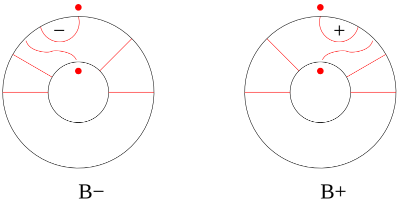


Figure: Creation operator inclusion.

Annihilation operators

Similarly

$$A_{\pm} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e \pm 1)$$

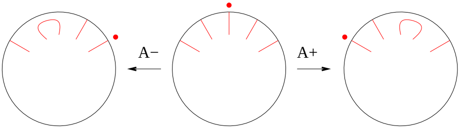


Figure: Annihilation operators.

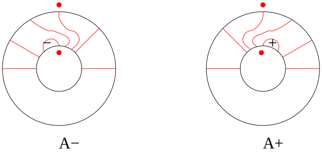


Figure: Annihilation operator inclusion.

Morphisms in Pascal's triangle

"Full categorification"

$$\mathbb{Z}_2^{\binom{0}{0}}$$
$$B_- A_+ A_- B_+$$
$$\swarrow \quad \nearrow \nwarrow \quad \searrow$$

$$\mathbb{Z}_2^{\binom{1}{0}} \qquad\qquad \mathbb{Z}_2^{\binom{1}{1}}$$
$$B_- A_+ A_- B_+ \qquad\qquad B_- A_+ A_- B_+$$
$$\swarrow \quad \nearrow \nwarrow \quad \searrow \qquad\qquad \swarrow \quad \nearrow \nwarrow \quad \searrow$$

$$\mathbb{Z}_2^{\binom{2}{0}} \qquad\qquad \mathbb{Z}_2^{\binom{2}{1}} \qquad\qquad \mathbb{Z}_2^{\binom{2}{2}}$$
$$B_- A_+ A_- B_+ \qquad\qquad B_- A_+ A_- B_+ \qquad\qquad B_- A_+ A_- B_+$$
$$\swarrow \quad \nearrow \nwarrow \quad \searrow \qquad\qquad \swarrow \quad \nearrow \nwarrow \quad \searrow \qquad\qquad \swarrow \quad \nearrow \nwarrow \quad \searrow$$

$$\mathbb{Z}_2^{\binom{3}{0}} \qquad\qquad \mathbb{Z}_2^{\binom{3}{1}} \qquad\qquad \mathbb{Z}_2^{\binom{3}{2}} \qquad\qquad \mathbb{Z}_2^{\binom{3}{3}}$$
$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

Operator relations, Pascal recursion

Proposition (Creation/annihilation relations)

$$A_+ \circ B_- = A_- \circ B_+ = 1$$
$$A_+ \circ B_+ = A_- \circ B_- = 0$$

Proposition (Categorification of Pascal recursion)

$$SFH(T, n+1, e) = B_+ SFH(T, n, e-1) \oplus B_- SFH(T, n, e+1)$$

I.e.:

$$\mathbb{Z}_2^{\binom{n+1}{k}} = B_+ \mathbb{Z}_2^{\binom{n}{k-1}} \oplus B_- \mathbb{Z}_2^{\binom{n}{k}}$$

Operator relations, Pascal recursion

Proposition (Creation/annihilation relations)

$$A_+ \circ B_- = A_- \circ B_+ = 1$$
$$A_+ \circ B_+ = A_- \circ B_- = 0$$

Proposition (Categorification of Pascal recursion)

$$SFH(T, n+1, e) = B_+ SFH(T, n, e-1) \oplus B_- SFH(T, n, e+1)$$

I.e.:

$$\mathbb{Z}_2^{\binom{n+1}{k}} = B_+ \mathbb{Z}_2^{\binom{n}{k-1}} \oplus B_- \mathbb{Z}_2^{\binom{n}{k}}$$

QFT analogy

$SFH(T, n+1, e)$ = "n-particle states of charge e"

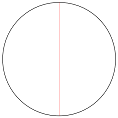


Figure: "The vacuum" $v_\emptyset \in SFH(T, 1, 0) = \mathbb{Z}_2$.

"Basis: apply creation operators to the vacuum"

$W(n_-, n_+) = \{\text{Words on } \{-,+\}, n_- - \text{signs}, n_+ + \text{signs}\}$

For $w \in W(n_-, n_+)$, form $v_w \in SFH(T, n+1, e)$.
$(n = n_- + n_+, e = n_+ - n_-.)$

QFT analogy

$$SFH(T, n+1, e) = \text{“}n\text{-particle states of charge } e\text{”}$$

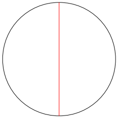


Figure: “The vacuum” $v_\emptyset \in SFH(T, 1, 0) = \mathbb{Z}_2$.

“Basis: apply creation operators to the vacuum”

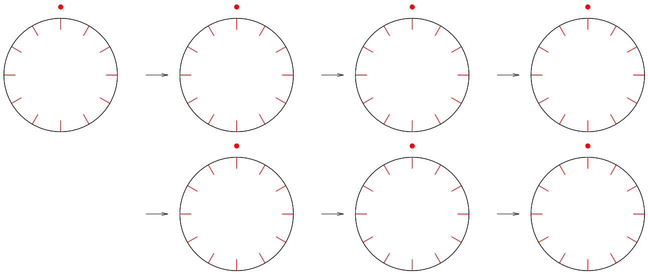
$$W(n_-, n_+) = \{\text{Words on } \{-, +\}, n_- \; - \text{ signs}, n_+ \; + \text{ signs}\}$$

For $w \in W(n_-, n_+)$, form $v_w \in SFH(T, n+1, e)$.
$(n = n_- + n_+, e = n_+ - n_-.)$

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



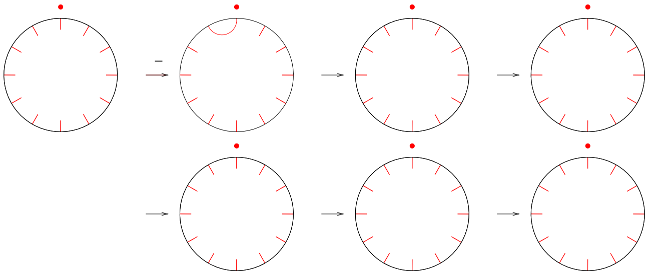
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



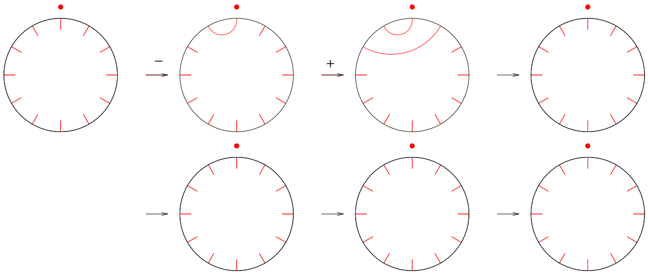
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



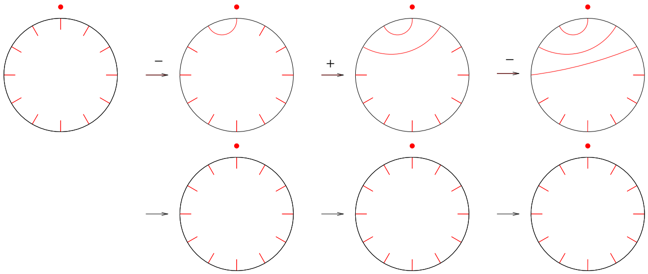
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



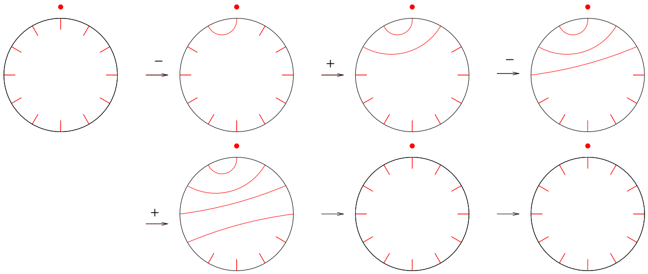
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



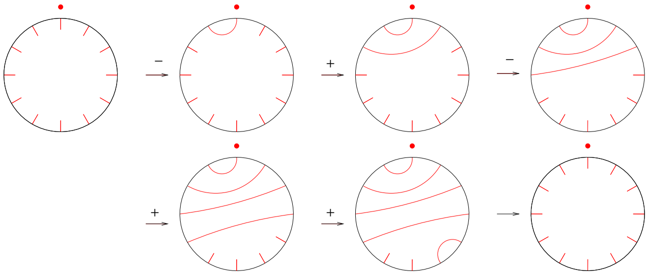
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



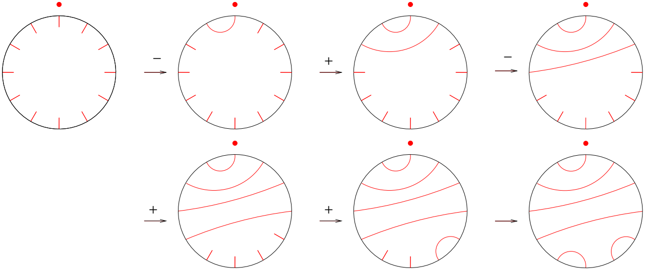
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

“QFT basis”

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



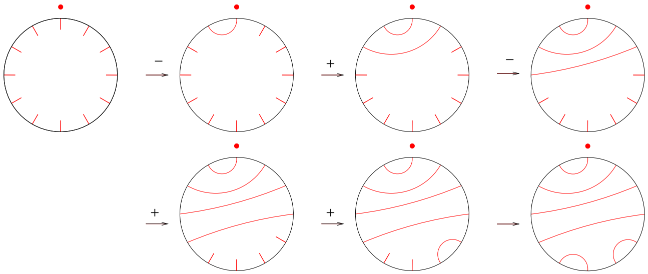
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

"QFT basis"

E.g.

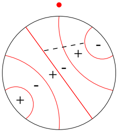
$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



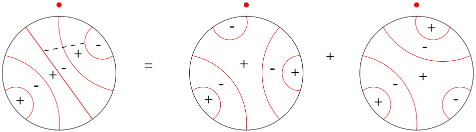
Proposition

The v_w, $w \in W(n_-, n_+)$, form a basis for $SFH(T, n+1, e)$.

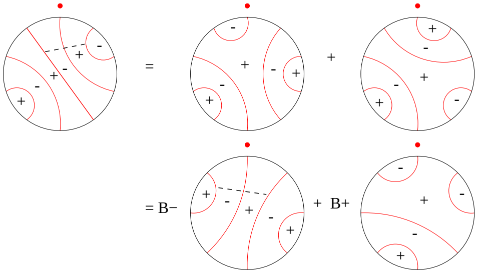
Basis decomposition



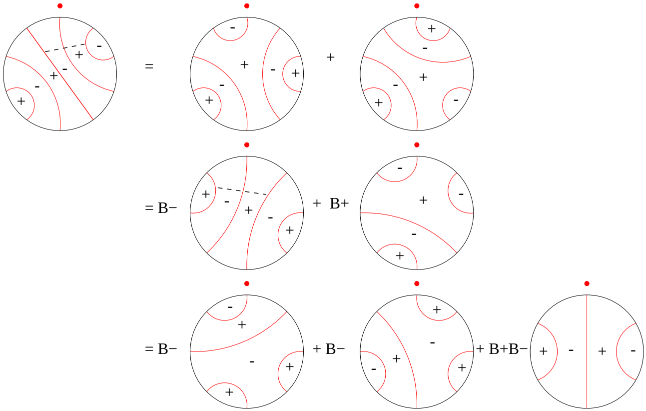
Basis decomposition



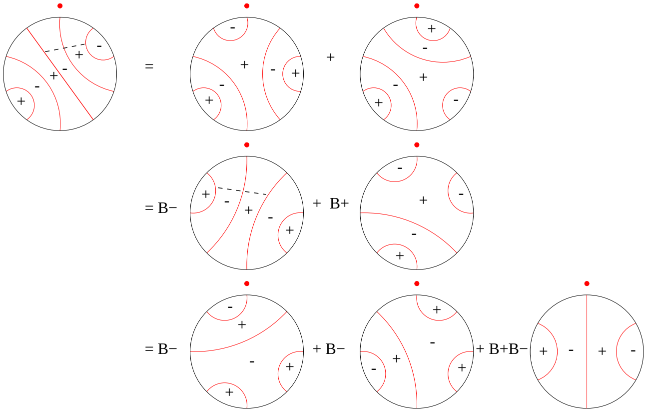
Basis decomposition



Basis decomposition

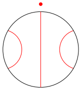


Basis decomposition



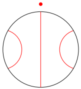
$$= B_-B_-B_+B_+v_\emptyset + B_-B_+B_+B_-v_\emptyset + B_+B_-(B_-B_+v_\emptyset + B_+B_-v_\emptyset)$$
$$= v_{--++} + v_{-++-} + v_{+--+} + v_{+-+-}$$

Examples of basis decomposition

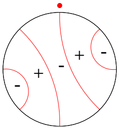


$$\begin{matrix} - & + \\ + & - \end{matrix}$$

Examples of basis decomposition

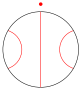


$-$	$+$
$+$	$-$

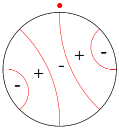


$-$	$-$	$+$
$+$	$-$	$-$

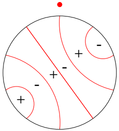
Examples of basis decomposition



−	+
+	−

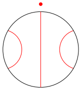


−	−	+
+	−	−

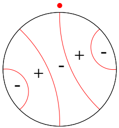


−	−	+	+
−	+	+	−
+	−	−	+
+	−	+	−

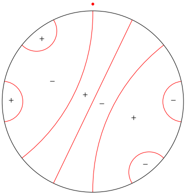
Examples of basis decomposition

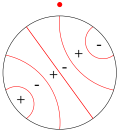


−	+
+	−



−	−	+
+	−	−





−	−	+	+
−	+	+	−
+	−	−	+
+	−	+	−

−	+	−	+	−	+
−	+	−	+	+	−
−	+	+	−	−	+
−	+	+	−	+	−
+	−	−	+	−	+
+	−	−	+	+	−
+	−	+	−	−	+
+	−	+	+	−	−
+	+	−	−	+	−
+	+	−	+	−	−

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Orderings on $W(n_-, n_+)$

- Lexicographic ordering: Total order.
- Partial order $\preceq$: "All minus signs move right (or stay where they are)."

E.g.

$$--++ \preceq +-+-$$

but

$$-++-, \quad +--+ \text{ not comparable.}$$

Theorem

Write a contact element v as a sum of basis vectors

$$v = \sum_w v_w, \quad w \in W(n_-, n_+).$$

Let w_-, w_+ be (lex.) first and last words occurring. Then for all words w in decomposition, $w_- \preceq w \preceq w_+$.

Orderings on $W(n_-, n_+)$

- Lexicographic ordering: Total order.
- Partial order $\preceq$: "All minus signs move right (or stay where they are)."

E.g.

$$- - ++ \preceq + - +-$$

but

$$- + +-, \quad + - - + \text{ not comparable.}$$

Theorem

Write a contact element v as a sum of basis vectors

$$v = \sum_w v_w, \quad w \in W(n_-, n_+).$$

Let w_-, w_+ be (lex.) first and last words occurring. Then for all words w in decomposition, $w_- \preceq w \preceq w_+$.

Orderings on $W(n_-, n_+)$

- Lexicographic ordering: Total order.
- Partial order $\preceq$: "All minus signs move right (or stay where they are)."

E.g.

$$- - + + \preceq + - + -$$

but

$$- + + -, \quad + - - + \text{ not comparable.}$$

Theorem

Write a contact element v as a sum of basis vectors

$$v = \sum_w v_w, \quad w \in W(n_-, n_+).$$

Let w_-, w_+ be (lex.) first and last words occurring. Then for all words w in decomposition, $w_- \preceq w \preceq w_+$.

Chord diagram = comparable pair

Now have

$$\Phi : \{\text{Contact elements}\} \longrightarrow \{\text{Comparable pairs of words}\}$$
$$v = \sum_w v_w \mapsto (w_-, w_+)$$

Proposition

These sets have the same cardinality.
I.e. # comparable pairs of words = C_n^e.

Theorem

Φ is a bijection.

I.e. for any $w_- \preceq w_+$ $\exists!$ contact element with v_{w_-} first, v_{w_+} last.

Chord diagram = comparable pair

Now have

$$\Phi : \{\text{Contact elements}\} \longrightarrow \{\text{Comparable pairs of words}\}$$

$$v = \sum_w v_w \mapsto (w_-, w_+)$$

Proposition

These sets have the same cardinality.
I.e. $\#$ comparable pairs of words = C_n^e.

Theorem

Φ is a bijection.

I.e. for any $w_- \preceq w_+$ $\exists!$ contact element with v_{w_-} first, v_{w_+} last.

Chord diagram = comparable pair

Now have

$$\Phi : \{\text{Contact elements}\} \longrightarrow \{\text{Comparable pairs of words}\}$$

$$v = \sum_w v_w \mapsto (w_-, w_+)$$

Proposition

These sets have the same cardinality.
I.e. $\#$ comparable pairs of words = C_n^e.

Theorem

Φ is a bijection.

I.e. for any $w_- \preceq w_+$ $\exists!$ contact element with v_{w_-} first, v_{w_+} last.

Other properties of contact elements

Notation $v = [w_-, w_+]$.

Proposition

The number of terms in the basis decomposition of a contact element v is

$$\begin{cases} 1 & \text{if } v \text{ is a basis element.} \\ \text{even} & \text{otherwise.} \end{cases}$$

Theorem (Not much comparability)

Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.
Suppose w is comparable with every other element in the decomposition.
Then $w = w_-$ or w_+.

Other properties of contact elements

Notation $v = [w_-, w_+]$.

Proposition

The number of terms in the basis decomposition of a contact element v is

$$\begin{cases} 1 & \text{if } v \text{ is a basis element.} \\ \text{even} & \text{otherwise.} \end{cases}$$

Theorem (Not much comparability)

Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.
Suppose w is comparable with every other element in the decomposition.
Then $w = w_-$ or w_+.

Other properties of contact elements

Notation $v = [w_-, w_+]$.

Proposition

The number of terms in the basis decomposition of a contact element v is

$$\begin{cases} 1 & \text{if } v \text{ is a basis element.} \\ \text{even} & \text{otherwise.} \end{cases}$$

Theorem (Not much comparability)

Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.
Suppose w is comparable with every other element in the decomposition.
Then $w = w_-$ or w_+.

Other properties of contact elements

Notation $v = [w_-, w_+]$.

Proposition

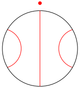
The number of terms in the basis decomposition of a contact element v is

$$\begin{cases} 1 & \text{if } v \text{ is a basis element.} \\ \text{even} & \text{otherwise.} \end{cases}$$

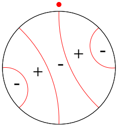
Theorem (Not much comparability)

Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.
Suppose w is comparable with every other element in the decomposition.
Then $w = w_-$ or w_+.

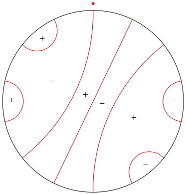
Examples of basis decomposition



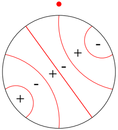
−	+
+	−



−	−	+
+	−	−



−	+	−	+	−	+
−	+	−	+	+	−
−	+	+	−	−	+
−	+	+	−	+	−
+	−	−	+	−	+
+	−	−	+	+	−
+	−	+	−	−	+
+	−	+	+	−	−
+	+	−	−	+	−
+	+	−	+	−	−



−	−	+	+
−	+	+	−
+	−	−	+
+	−	+	−

Summary of results

- Distinct chord diagrams/contact structures give distinct contact elements.
- Contact elements not a subgroup, but “addition means bypasses”.
- Can give a basis for each $SFH(T, n, e)$ consisting of chord diagrams / contact elements.
- There is a partial order $\preceq$ on each basis.
- Chord diagrams / contact structures correspond precisely to comparable pairs of basis elements.

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Stacking construction

Given Γ_0, Γ_1 chord diagrams, consider $\mathcal{M}(\Gamma_0, \Gamma_1)$:

- sutured solid cylinder $D \times I$
- Γ_i sutures along $D \times \{i\}$
- Vertical interleaving sutures along $\partial D \times I$.

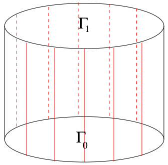


Figure: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

$\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight if it admits a tight contact structure.
I.e. after rounding corners, sutures form single component.

Stacking construction

Given Γ_0, Γ_1 chord diagrams, consider $\mathcal{M}(\Gamma_0, \Gamma_1)$:

- sutured solid cylinder $D \times I$
- Γ_i sutures along $D \times \{i\}$
- Vertical interleaving sutures along $\partial D \times I$.

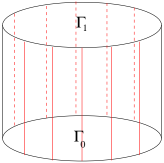


Figure: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

$\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight if it admits a tight contact structure.
I.e. after rounding corners, sutures form single component.

Stacking construction

Given Γ_0, Γ_1 chord diagrams, consider $\mathcal{M}(\Gamma_0, \Gamma_1)$:

- sutured solid cylinder $D \times I$
- Γ_i sutures along $D \times \{i\}$
- Vertical interleaving sutures along $\partial D \times I$.

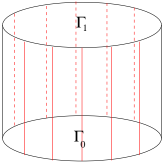


Figure: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

$\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight if it admits a tight contact structure.
I.e. after rounding corners, sutures form single component.

Stackability constructions

Proposition (Stackability map)

There is a linear map

$$m : SFH(T,n,e) \otimes SFH(T,n,e) \longrightarrow \mathbb{Z}_2$$

taking (Γ_0, Γ_1) *to* 1 *if* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight, and* 0 *if overtwisted.*

Proposition (Contact interpretation of $\preceq$)

$\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ *is tight iff* $w_0 \preceq w_1$.

Proposition (General stackability)

Γ_0, Γ_1 *chord diagrams, n chords, euler class e.*

$$\mathcal{M}(\Gamma_0, \Gamma_1) \text{ is tight} \Leftrightarrow \# \left\{ (w_0, w_1) : \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in } \Gamma_i \end{array} \right\} \text{ is odd.}$$

Stackability constructions

Proposition (Stackability map)

There is a linear map

$$m : SFH(T,n,e) \otimes SFH(T,n,e) \longrightarrow \mathbb{Z}_2$$

taking (Γ_0, Γ_1) *to* 1 *if* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight, and* 0 *if overtwisted.*

Proposition (Contact interpretation of $\preceq$)

$\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ *is tight iff* $w_0 \preceq w_1$.

Proposition (General stackability)

Γ_0, Γ_1 *chord diagrams, n chords, euler class e.*

$$\mathcal{M}(\Gamma_0, \Gamma_1) \text{ is tight} \Leftrightarrow \# \left\{ (w_0, w_1) : \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in } \Gamma_i \end{array} \right\} \text{ is odd.}$$

Stackability constructions

Proposition (Stackability map)

There is a linear map

$$m : SFH(T,n,e) \otimes SFH(T,n,e) \longrightarrow \mathbb{Z}_2$$

taking (Γ_0, Γ_1) *to* 1 *if* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight, and* 0 *if overtwisted.*

Proposition (Contact interpretation of $\preceq$)

$\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ *is tight iff* $w_0 \preceq w_1$.

Proposition (General stackability)

Γ_0, Γ_1 *chord diagrams, n chords, euler class e.*

$$\mathcal{M}(\Gamma_0, \Gamma_1) \text{ is tight} \Leftrightarrow \# \left\{ (w_0, w_1) : \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in } \Gamma_i \end{array} \right\} \text{ is odd.}$$

Other stackability properties

- $m(\Gamma, \Gamma) = 1$.
- Suppose Γ_0, Γ_1 have an outermost chord γ in the same position.
  Then $m(\Gamma_0, \Gamma_1) = m(\Gamma_0 - \gamma, \Gamma_1 - \gamma)$.
- Γ_0, Γ_1 related by bypass move (in correct order).
  Then $m(\Gamma_0, \Gamma_1) = 1$.

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Contact category

Honda (unpublished...)

Σ surface.

Contact category $\mathcal{C}(\Sigma)$

Objects:

- Dividing sets Γ on Σ (= Contact structures near Σ)

Morphisms $\Gamma_0 \longrightarrow \Gamma_1$:

- Contact structures on $\Sigma \times I$ with $\Gamma_{\Sigma \times \{i\}} = \Gamma_i$.

Properties:

- Behaves functorially w.r.t. SFH.
- Obeys some of the axioms of a triangulated category:
  - Distinguished triangles = bypass triples
  - Octahedral axiom $\sim$ 6 contact elements in $SFH(T,4,1) \cong \mathbb{Z}_2^3$.

Contact category
Honda (unpublished...)

Σ surface.

Contact category $\mathcal{C}(\Sigma)$

Objects:

- Dividing sets Γ on Σ (= Contact structures near Σ)

Morphisms $\Gamma_0 \longrightarrow \Gamma_1$:

- Contact structures on $\Sigma \times I$ with $\Gamma_{\Sigma \times \{i\}} = \Gamma_i$.

Properties:

- Behaves functorially w.r.t. *SFH*.
- Obeys some of the axioms of a triangulated category:
  - Distinguished triangles = bypass triples
  - Octahedral axiom $\sim$ 6 contact elements in $SFH(T, 4, 1) \cong \mathbb{Z}_2^3$.

Contact category

Honda (unpublished...)

Σ surface.

Contact category $\mathcal{C}(\Sigma)$

Objects:

- Dividing sets Γ on Σ (= Contact structures near Σ)

Morphisms $\Gamma_0 \longrightarrow \Gamma_1$:

- Contact structures on $\Sigma \times I$ with $\Gamma_{\Sigma \times \{i\}} = \Gamma_i$.

Properties:

- Behaves functorially w.r.t. *SFH*.
- Obeys some of the axioms of a triangulated category:
  - Distinguished triangles = bypass triples
  - Octahedral axiom $\sim$ 6 contact elements in $SFH(T, 4, 1) \cong \mathbb{Z}_2^3$.

Contact category
Honda (unpublished...)

Σ surface.

Contact category $\mathcal{C}(\Sigma)$

Objects:

- Dividing sets Γ on Σ (= Contact structures near Σ)

Morphisms $\Gamma_0 \longrightarrow \Gamma_1$:

- Contact structures on $\Sigma \times I$ with $\Gamma_{\Sigma \times \{i\}} = \Gamma_i$.

Properties:

- Behaves functorially w.r.t. *SFH*.
- Obeys some of the axioms of a triangulated category:
  - Distinguished triangles = bypass triples
  - Octahedral axiom $\sim$ 6 contact elements in $SFH(T, 4, 1) \cong \mathbb{Z}_2^3$.

A 2-category

On D^2 have $\mathcal{C}(D^2, n, e)$
(restrict to chord diagrams, n chords, euler class e):

- Objects in $\mathcal{C}(D^2, n, e)$ (= chord diagrams) given by partial order $[w_-, w_+]$.
- A partial order is a category.

Contact 2-category $\mathcal{C}(n+1, e)$

- Objects = words in $W(n_-, n_+)$ = basis chord diagrams
- 1-morphisms = $\{$partial order $\preceq\}$ = chord diagrams
- 2-morphisms = contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$.

Proposition

This is a 2-category.

A 2-category

On D^2 have $\mathcal{C}(D^2, n, e)$
(restrict to chord diagrams, n chords, euler class e):

- Objects in $\mathcal{C}(D^2, n, e)$ (= chord diagrams) given by partial order $[w_-, w_+]$.
- A partial order is a category.

Contact 2-category $\mathcal{C}(n+1, e)$

- Objects = words in $W(n_-, n_+)$ = basis chord diagrams
- 1-morphisms = $\{$partial order $\preceq\}$ = chord diagrams
- 2-morphisms = contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$.

Proposition

This is a 2-category.

A 2-category

On D^2 have $\mathcal{C}(D^2, n, e)$
(restrict to chord diagrams, n chords, euler class e):

- Objects in $\mathcal{C}(D^2, n, e)$ (= chord diagrams) given by partial order $[w_-, w_+]$.
- A partial order is a category.

Contact 2-category $\mathcal{C}(n+1, e)$

- Objects = words in $W(n_-, n_+)$ = basis chord diagrams
- 1-morphisms = $\{$partial order $\preceq\}$ = chord diagrams
- 2-morphisms = contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$.

Proposition

This is a 2-category.

Outline

1. Background
   - Sutured Floer homology, contact elements, TQFT
   - Solid tori, Catalan, Narayana
2. Contact elements in $SFH(T)$
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
   - Partial order, main theorem
3. Contact geometry applications
   - Stackability
   - Contact 2-category
4. Idea of proof of main theorems
   - Comparable pairs and bypass systems

An explicit construction

Prove correspondence

$$\left\{ \text{ Chord diagrams } \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Comparable pairs} \\ \text{of words} \end{array} \right\}$$

Essential idea:

Given $w_1 \preceq w_2$, construct a chord diagram Γ whose decomposition has w_1 first and w_2 last.

- Along the way, show that every other word w in the decomposition has $w_1 \preceq w \preceq w_2$.
- Elementary combinatorics gives $\# \{\text{pairs } (w_1 \preceq w_2)\} = C_n^e$.
- Done.

An explicit construction

Prove correspondence

$$\left\{ \text{ Chord diagrams } \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Comparable pairs} \\ \text{of words} \end{array} \right\}$$

Essential idea:

Given $w_1 \preceq w_2$, construct a chord diagram Γ whose decomposition has w_1 first and w_2 last.

- Along the way, show that every other word w in the decomposition has $w_1 \preceq w \preceq w_2$.
- Elementary combinatorics gives $\# \{\text{pairs } (w_1 \preceq w_2)\} = C_n^e$.
- Done.

An explicit construction

Prove correspondence

$$\left\{ \text{ Chord diagrams } \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Comparable pairs} \\ \text{of words} \end{array} \right\}$$

Essential idea:

Given $w_1 \preceq w_2$, construct a chord diagram Γ whose decomposition has w_1 first and w_2 last.

- Along the way, show that every other word w in the decomposition has $w_1 \preceq w \preceq w_2$.
- Elementary combinatorics gives $\# \{\text{pairs } (w_1 \preceq w_2)\} = C_n^e$.
- Done.

Bypass systems

Take $\Gamma_{w_1}, \Gamma_{w_2}$ basis chord diagrams, $w_1 \preceq w_2$.

Proposition

1. *On Γ_{w_1} there exists a bypass system $FBS(\Gamma_{w_1}, \Gamma_{w_2})$ such that performing upwards bypass moves along it gives Γ_{w_2}.*
2. *On Γ_{w_2} there exists a bypass system $BBS(\Gamma_{w_1}, \Gamma_{w_2})$ such that performing downwards bypass moves gives Γ_{w_1}.*

Proposition

Performing either:

1. *downwards bypass moves on Γ_{w_1} along $FBS(\Gamma_{w_1}, \Gamma_{w_2})$, or*
2. *upwards bypass moves on Γ_{w_2} along $BBS(\Gamma_{w_1}, \Gamma_{w_2})$*

gives a chord diagram containing w_1, w_2 in decomposition and:

- *for all words w in the decomposition, $w_1 \preceq w \preceq w_2$.*

Bypass systems

Take $\Gamma_{w_1}, \Gamma_{w_2}$ basis chord diagrams, $w_1 \preceq w_2$.

Proposition

1. *On Γ_{w_1} there exists a bypass system $FBS(\Gamma_{w_1}, \Gamma_{w_2})$ such that performing upwards bypass moves along it gives Γ_{w_2}.*
2. *On Γ_{w_2} there exists a bypass system $BBS(\Gamma_{w_1}, \Gamma_{w_2})$ such that performing downwards bypass moves gives Γ_{w_1}.*

Proposition

Performing either:

1. *downwards bypass moves on Γ_{w_1} along $FBS(\Gamma_{w_1}, \Gamma_{w_2})$, or*
2. *upwards bypass moves on Γ_{w_2} along $BBS(\Gamma_{w_1}, \Gamma_{w_2})$*

gives a chord diagram containing w_1, w_2 in decomposition and:

- *for all words w in the decomposition, $w_1 \preceq w \preceq w_2$.*

Proof by increasingly difficult example

Easy level

"Elementary move" on word = Bypass move on "attaching arc".

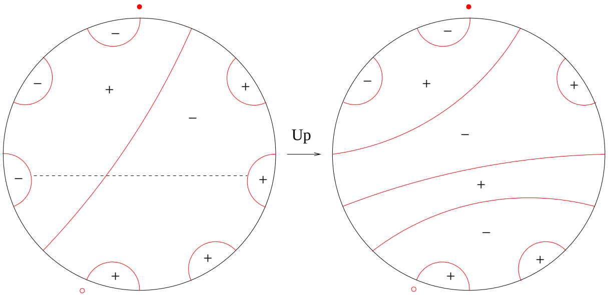


Figure: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$.

Proof by incresingly difficult example

Medium level

$$\left\{ \begin{array}{c} \text{“Generalized elementary} \\ \text{move” on word} \end{array} \right\} = \left\{ \begin{array}{c} \text{Bypass moves on} \\ \text{“generalized attaching arc”} \end{array} \right\}$$

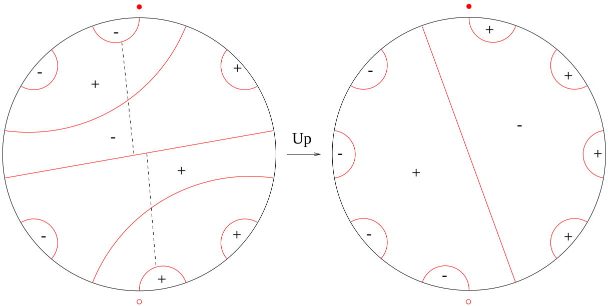


Figure: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$.

Proof by increasingly difficult example

Hard level

$$\left\{ \begin{array}{c} \text{"Nicely ordered sequence"} \\ \text{of "generalized elementary} \\ \text{moves" on word} \end{array} \right\} = \left\{ \begin{array}{c} \text{Bypass moves on} \\ \text{"well placed sequence" of} \\ \text{"generalized attaching arcs"} \end{array} \right\}$$

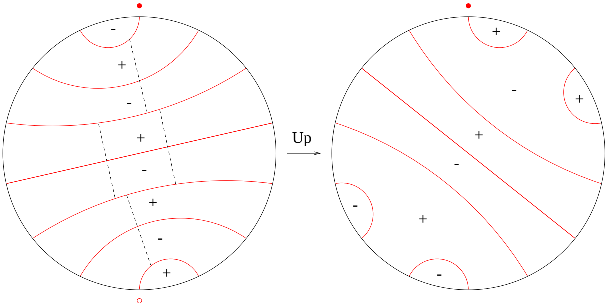


Figure: Upwards moves from Γ_{-+-+-+} to Γ_{++-+--}.