CHORD DIAGRAMS, CONTACT-TOPOLOGICAL QUANTUM FIELD THEORY, AND CONTACT CATEGORIES

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Abstract

In this thesis we study some interesting mathematics arising at the intersection of the studies of contact topology and sutured Floer homology. Although this work was originally motivated by the study of contact elements in sutured Floer homology, we also obtain results in pure contact topology.

We consider contact elements in the sutured Floer homology of solid tori, as part of the (1+1)-dimensional topological quantum field theory defined by Honda–Kazez–Matić in [33]. We find that the $\mathbb{Z}_2$ sutured Floer homology of solid tori with longitudinal sutures forms a "categorification of Pascal's triangle", a triangle of vector spaces. Contact structures on solid tori with longitudinal sutures correspond bijectively to chord diagrams, which are sets of disjoint properly embedded arcs in the disc; these may in turn be identified with contact elements. The contact elements form distinguished subsets of the vector spaces in the categorified Pascal's triangle, of order given by the Narayana numbers. We find natural "creation and annihilation operators" which allow us to define a QFT-type basis of each SFH vector space, consisting of contact elements. We show that sutured Floer homology in this case reduces to the combinatorics of chord diagrams. We prove that contact elements are in bijective correspondence with comparable pairs of basis elements with respect to a certain partial order, and in a natural and explicit way. We also prove numerous results about the structure of contact elements and investigate various algebraic structures which arise.

Our main theorem, describing how contact elements lie in sutured Floer homology, has a purely combinatorial interpretation, as a statement about chords on discs subject to a certain surgery and a single addition relation. The algebraic and combinatorial structures which naturally arise in this description have intrinsic contact-topological meaning.

In particular, the QFT-type basis of sutured Floer homology, and its partial order, have a natural interpretation in pure contact topology, related to the contact category of a disc: the partial order enables us to tell when the sutured solid cylinder obtained by "stacking" two chord diagrams has a tight contact structure. This leads us to extend Honda's notion of contact category to a "bounded" contact category, containing chord diagrams and contact structures which occur within a given contact solid cylinder. We compute this bounded contact category in certain cases. Moreover, the decomposition of a contact element into basis elements naturally gives a triple of contact structures on solid cylinders which we regard as a type of "distinguished triangle" in the contact category. We also use the algebraic structures arising among contact elements to extend the notion of contact category to a 2-category.

Acknowledgments

I would like to thank Yasha Eliashberg, Steve Kerckhoff, and everyone else with whom I have had interesting mathematical discussions during my time at Stanford. I would also like to thank my family for their ongoing support. I have found Stanford to be a stimulating and supportive academic environment in which to perform research in an extraordinarily difficult area of science, and I am deeply appreciative of all those who have contributed to this atmosphere, including all mathematics students and faculty with whom I have interacted.

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Chapter 1

Introduction

We begin by surveying our results and how they fit into previously existing work. Since much of our work is simply about the combinatorics of arranging chords on circles, we begin in section 1.1 starting from a completely elementary perspective. We then summarise our results, as they relate to the study of contact elements in sutured Floer homology (section 1.2); and then, as they relate to pure contact topology and the study of contact categories (section 1.3). We also make some remarks about future directions and questions (section 1.4) and some notes about the structure of this thesis (section 1.5).

1.1 Fun with chord diagrams

The main results of this thesis can be described as elementary combinatorial results about chord diagrams, which have applications to contact topology and sutured Floer homology.

Definition 1.1.1 (Chord diagram) *A chord diagram Γ is a set of disjoint properly embedded arcs (chords) in a disc D^2, considered up to homotopy relative to endpoints.*

Consider a chord diagram with n chords; it has $2n$ marked points on the boundary of the disc, connected in pairs by disjoint chords. We declare one of those marked

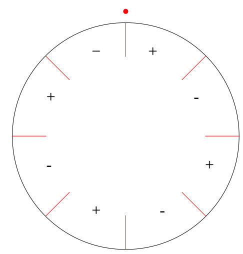


Figure 1.1: Base point and sign of regions.

points on the boundary a base point; rotating a chord diagram will generally give a distinct chord diagram.

(Most rigorously, a chord diagram of n chords is an embedding of pairs

$$\Big([0,1] \times \{1,\ldots,n\}, \{0,1\} \times \{1,\ldots,n\}\Big) \longrightarrow \left(D^2, \{p_1,\ldots,p_{2n}\}\right),$$

where $p_1, \ldots, p_{2n}$ are $2n$ distinct distinguished points on ∂D, and p_1 is the base point; embeddings are considered up to relative homotopy and pre-composition by permutations of $\{1,\ldots,n\}$. We identify Γ with the image of this map.)

The chords of a chord diagram divide the disc D into regions, which we alternately label as positive or negative. The labelling is induced from a labelling on the arcs of ∂D^2 between marked points; we declare that the arc immediately clockwise of the base point is positive, and the arc immediately anticlockwise is negative. See figure 1.1.

Remark 1.1.2 (Denoting base point) *The base point will always be denoted by a solid red dot in our diagrams.*

Definition 1.1.3 (Euler class of chord diagram) *The* (relative) euler class e *of a chord diagram* Γ *is the sum of the signs of the regions of* $D - \Gamma$.

That is, a + region counts as +1 and a − region counts as −1. It's not difficult to see that e has opposite parity to n, and $|e| \leq n - 1$.

We consider a certain vector space generated by chord diagrams.

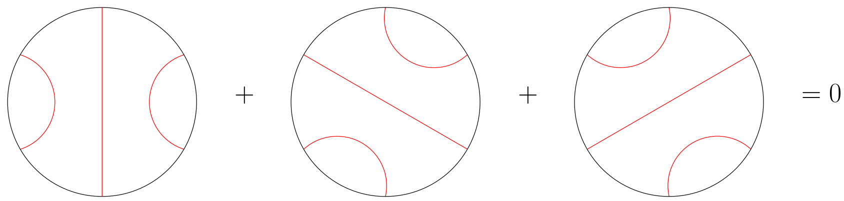


Figure 1.2: The bypass relation.

Definition 1.1.4 (Combinatorial *SFH*) *The $\mathbb{Z}_2$-vector space generated by chord diagrams of n chords and euler class e, subject to the* bypass relation *in figure 1.2, is called $SFH_{comb}(T, n, e)$. The $\mathbb{Z}_2$-vector space generated by all chord diagrams of n chords, subject to the same relation, is called $SFH_{comb}(T, n)$.*

We will show that these combinatorial objects are isomorphic to $SFH(T, n, e)$ and $SFH(T, n)$, which are objects defined by counting certain holomorphic curves in certain almost complex manifolds, in due course. The letters SFH stand for "sutured Floer homology".

The bypass relation means that if we have three chord diagrams $\Gamma_1, \Gamma_2, \Gamma_3$ which are all identical, except in a sub-disc $D' \subset D$, on which each of $\Gamma_1, \Gamma_2, \Gamma_3$ contains three arcs, respectively in the three arrangements shown in figure 1.2, then we consider them to sum to zero.

The terminology "bypass" comes from contact geometry. A bypass is a "physical" contact-geometric object, that is, a concrete contact 3-manifold with boundary. We shall make the contact geometry clear as we go on, but the idea of "bypasses" here can be considered purely as a type of surgery on a chord diagram, which we call a "bypass move".

Definition 1.1.5 (Arc of attachment) *An* arc of attachment*, or* attaching arc *in a chord diagram Γ is an embedded arc which intersects the chords of Γ at precisely three points, namely, its two endpoints, and one interior point.*

We consider attaching arcs equivalent if they are homotopic through attaching arcs. (Most rigorously, an attaching arc is an embedding of pairs

$$\left([0,1], \left\{ 0, \frac{1}{2}, 1 \right\} \right) \longrightarrow (\text{Int}\, D, \text{Int}\, \Gamma)$$

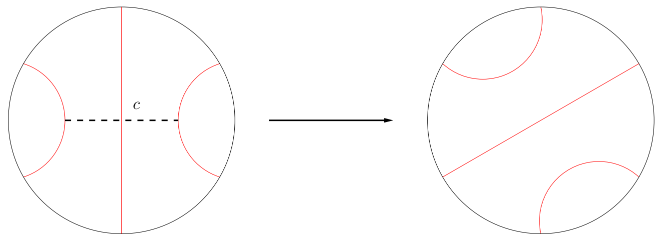


Figure 1.3: Upwards bypass move.

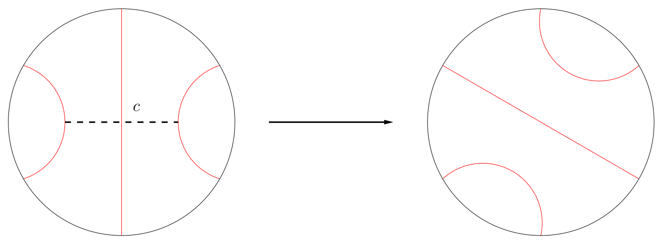


Figure 1.4: Downwards bypass move.

up to homotopy and pre-composition by reflection of $[0, 1]$. We identify the attaching arc with the image of this map.)

A bypass move is something done along an arc of attachment, and it may be done upwards or downwards.

Definition 1.1.6 (Bypass moves) *Let c be an attaching arc in a chord diagram Γ.*

(i) The upwards bypass move Up_c *along c on Γ consists of removing a small disc neighbourhood of c and replacing it with another disc with chords as shown in figure 1.3.*

(ii) The downwards bypass move Down_c *along c on Γ consists also of removing a small neighbourhood of c, but now replacing it as shown in figure 1.4.*

We see that chord diagrams related by bypass moves naturally come in triples, and such triples are defined to sum to 0 in SFH_{comb}. In particular, in $\Gamma' = \mathrm{Up}_c \Gamma$, there is an attaching arc c' such that $\mathrm{Down}_{c'} \Gamma' = \Gamma$, and in $\Gamma'' = \mathrm{Down}_c \Gamma$, there is an attaching arc c'' such that $\mathrm{Up}_{c''} \Gamma'' = \Gamma$. We can think of bypass moves as performing a local 60° rotation on part of a chord diagram; see figure 1.5. Since a local 180°

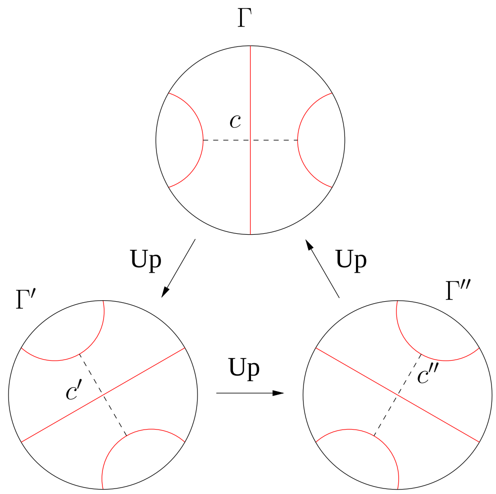


Figure 1.5: Bypass triple.

rotation gives the identity, "three bypass moves is the identity". This observation, as we will see, is the source of much interesting algebraic and categorical structure.

Definition 1.1.7 (Bypass triple) *Three chord diagrams* Γ, Γ', Γ'' *form a* bypass triple *if there exists an attaching arc* c *on* Γ *such that*

$$\Gamma' = \mathrm{Up}_c\ \Gamma, \quad \Gamma'' = \mathrm{Down}_c\ \Gamma.$$

The above observation makes it clear that the existence of such an attaching arc on Γ is equivalent to existence of such arcs on Γ' or Γ''. If two distinct chord diagrams are related by a bypass move, then there is a unique third chord diagram forming a bypass triple.

In general a bypass move on a chord diagram need not produce a chord diagram; it may produce a closed loop. A diagram with a closed loop is considered to be zero in SFH_{comb}. In this case, the effect of the bypass move in the opposite direction leaves the chord diagram unchanged; the bypass relation still holds and is of the form $x + x + 0 = 0$.

The main combinatorial result of this thesis gives a nice basis for each vector space $SFH_{comb}(T, n, e)$, and shows that when chord diagrams are decomposed into a sum of basis elements, this decomposition has certain nice properties. There will be a partial

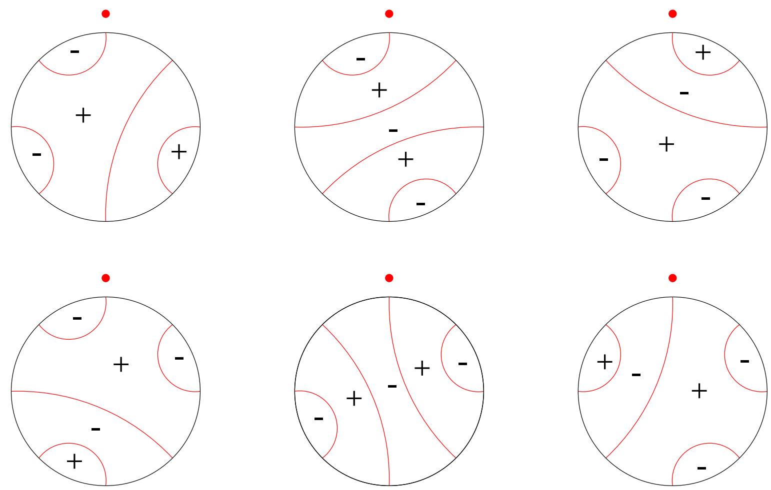


Figure 1.6: Chord diagrams in $SFH(T, 4, -1)$.

order on this basis, and chord diagrams will correspond bijectively with pairs of basis elements which are comparable with respect to this partial order.

For instance, consider $SFH_{comb}(T, 4, -1)$. This vector space is spanned by the 6 chord diagrams which have 4 chords and relative euler class -1: see figure 1.6.

We will show, and it was essentially known previously in [33], that

$$SFH_{comb}(T, 4, -1) = \mathbb{Z}_2^3.$$

Our basis will consist of the three chord diagrams in the top row of figure 1.6, labelled with the words, respectively:

$$- - +, \quad - + -, \quad + - -.$$

Note that they are labelled by words w on $\{-, +\}$ containing 2 minus signs and 1 plus sign. The number of such words is $\binom{3}{1}$; in general, the basis for $SFH_{comb}(T, n+1, e)$ will be labelled by words on $\{-, +\}$ of length n whose symbols sum to e, and the number of such words is $\binom{n}{k}$, where $k = (n+e)/2$. We will write v_w to denote the basis element labelled by w.

On this set of words, there is a partial order defined by "all minus signs move right (or stay where they are)". (In this simple case, it is actually a total order; but this will not be true for words of longer length. For instance, in $SFH_{comb}(T, 5, 0)$ we find the words $- + + -$ and $+ - - +$, which are not comparable.) Thus,

$$- - + \preceq - + - \preceq + - -.$$

In terms of our basis, the 6 chord diagrams (arranged as in figure 1.6) are

$$\begin{array}{ccc} v_{--+} & v_{-+-} & v_{+--} \\ v_{--+} + v_{-+-} & v_{--+} + v_{+--} & v_{-+-} + v_{+--}. \end{array}$$

Moreover, there are six pairs of words w_1, w_2 which are *comparable* with respect to $\preceq$, namely three "doubles"

$$(- - +, - - +), \quad (- + -, - + -), \quad (+ - -, + - -)$$

and three less trivial pairs

$$(- - +, - + -), \quad (- - +, + - -), \quad (- + -, + - -).$$

And in fact, for each pair, there is precisely one chord diagram having that pair as its first and last basis element.

That is, there is a bijection

$$\{\text{Chord diagrams}\} \leftrightarrow \{\text{Comparable pairs of words}\}$$

given by taking a chord diagram to the first and last basis elements in its basis decomposition.

This is a general fact, and our main theorem. Moreover, this bijection, and its inverse, can be described explicitly. That is, given a chord diagram, we can algorithmically extract its first and last basis elements, and they are comparable. Conversely, given two comparable words, we can algorithmically produce the chord diagram for which those words give its first and last basis elements. We will also say more about the set of basis chord diagrams that occur in a given chord diagram; as well as relationships between the various vector spaces $SFH_{comb}(T, n, e)$.

An information-theoretic note from this result is that a chord diagram of 4 chords can be encoded in 6 bits, with the redundancy that the first 3 bits form a word lesser than the second 3 bits, with respect to $\preceq$. In general, a chord diagram of $n+1$ chords can be encoded in $2n$ bits, with a similar redundancy.

This particular example, with 4 chords and $e = -1$, is actually the essence of Honda's octahedral axiom (see [21], also section 3.1.6 below).

While this result is largely combinatorics, the motivation, notation, and applications come from the theory of sutured Floer homology, with its connections to topological quantum field theory and contact topology.

1.2 Contact elements in SFH of solid tori

The original motivation for this work was to understand in detail the contact elements in the sutured Floer homology of a very simple sutured manifold, namely a solid torus with longitudinal sutures. We now give an overview of this aspect of our results.

1.2.1 Sutured Floer homology and contact structures

We will review the theory of sutured Floer homology more fully in section 2.2. For the purposes of introduction, it is sufficient to note four facts about sutured Floer homology.

First, sutured Floer homology theory *associates to certain sutured 3-manifolds* $(M, \mathbf{\Gamma})$ *a* $\mathbb{Z}_2$*-vector space* $SFH(M, \mathbf{\Gamma})$. For present introductory purposes, a sutured manifold can be thought of as a 3-manifold M with boundary, with some disjoint oriented simple closed curves $\mathbf{\Gamma}$ drawn on the boundary ∂M, dividing ∂M into alternating positive and negative regions. The sutured manifolds for which SFH is defined are called *balanced*.

Remark 1.2.1 (Notation: the letter $\mathbf{\Gamma}$) *This letter is used to denote chord diagrams, and also to denote sutures on sutured manifolds. This is not unusual, because* *in the present context, both arise as dividing sets on convex surfaces in contact manifolds; and dividing sets are often denoted by* Γ*. However, to avoid confusion, for now we shall use the letter* Γ *to denote chord diagrams; to denote sutures, we shall use the boldface* $\mathbf{\Gamma}$*.*

For present purposes, we consider our sutured manifold to be a solid torus, with $2n$ parallel longitudinal sutures, denoted (T, n). Its sutured Floer homology is known to be $\mathbb{Z}_2^{2^{n-1}}$ (see [33]).

Second, sutured Floer homology *associates to a contact structure* ξ *on* $(M, \mathbf{\Gamma})$ *an element* $c(\xi) \in SFH(-M, -\mathbf{\Gamma})$. Here the minus signs refer to reversed orientation. When we refer to a contact structure on a sutured 3-manifold $(M, \mathbf{\Gamma})$, we require it to be compatible with the sutures $\mathbf{\Gamma}$, in the sense that the boundary ∂M is convex with dividing set Γ, and the positive/negative regions of ∂M as a convex surface agree with the positive/negative regions arising from the sutures. The contact element satisfies $c(\xi) = 0$ if ξ is overtwisted; if $c(\xi) \neq 0$ then ξ is tight. We will review the notions of tight and overtwisted, and other relevant contact geometry, more fully in section 2.1 and chapter 4.

In our case of a solid torus with longitudinal sutures, a tight contact structure can be described by examining the dividing set on a convex meridional disc, which is a chord diagram of n chords. The tight contact structures on (T, n), up to isotopy rel boundary, are in bijective correspondence with chord diagrams of n chords (see [22], but note [24], also [23, 25, 17, 18]; we also prove this as part of our study of bypasses, as proposition 4.2.11). That is, there is exactly one tight contact structure, up to isotopy rel boundary, for each such chord diagram. And, in a notationally-executed blatant cover-up of the unpleasant reversals of orientation and extra minus signs, we will write $SFH(T, n)$ to denote the SFH of the appropriately orientation-reversed manifold. The orientation reversal is never an issue in the following, so hopefully the abuse of notation will not cause too much confusion. We still have $SFH(T, n) = \mathbb{Z}_2^{2^{n-1}}$.

In any case, the upshot is that:

We may regard chord diagrams of n *chords as contact elements in* $SFH(T, n)$*.*

Third, sutured Floer homology $SFH(M, \mathbf{\Gamma})$ *splits as a direct sum.* This direct sum is over spin-c structures on $(M, \mathbf{\Gamma})$.

In our simple case of (T, n), for which $SFH(T, n+1) = \mathbb{Z}_2^{2^n}$, this sum over spin-c structures corresponds to a row of Pascal's triangle.

Theorem 1.2.2 (Honda–Kazez–Matić [33], Juhász [37]) *$SFH(T, n+1) = \mathbb{Z}_2^{2^n}$ and splits as a direct sum over spin-c structures*

$$SFH(T, n+1) = \mathbb{Z}_2^{\binom{n}{0}} \oplus \cdots \oplus \mathbb{Z}_2^{\binom{n}{n}}.$$

If ξ is a contact structure on the sutured manifold $(T, n+1)$ with relative euler class e, then its contact element $c(\xi)$ lies in the summand $\mathbb{Z}_2^{\binom{n}{k}}$, where $k = (e+n)/2$.

The *relative euler class* of a contact structure (evaluated on a meridional disc, which generates $H^2(T, \partial T)$) is precisely the relative euler class of the corresponding chord diagram. With $n+1$ chords, this is an integer e of the same parity as n, and $-n \leq e \leq n$. The $n+1$ possible values of e correspond precisely to the $n+1$ direct summands above. We denote by $SFH(T, n+1, e)$ the summand containing the contact elements of relative euler class e.

Fourth, *an inclusion of sutured manifolds induces a map on SFH* [33]. More precisely, an inclusion of sutured manifolds $(M', \mathbf{\Gamma}') \hookrightarrow (M, \mathbf{\Gamma})$ together with a contact structure ξ'' on $(M - M', \mathbf{\Gamma} \cup \mathbf{\Gamma}')$ determines a map on SFH. This map takes the contact element $c(\xi')$ of a contact structure ξ' on $(M', \mathbf{\Gamma}')$ to the contact element $c(\xi' \cup \xi'')$ of the contact structure $\xi' \cup \xi''$ on $(M, \mathbf{\Gamma})$. This is a property of the type found in topological quantum field theories, and we can call it *TQFT-inclusion.* We will use this principle to describe our basis for SFH, among other things.

The above indicates (but does not explain) the origin of the letters "SFH" in the definition of the combinatorial vector space in section 1.1.

The original motivation of this work was to answer the question:

How do contact elements lie in sutured Floer homology?

A first proposition in this direction is that all tight contact elements are distinct in $SFH(T, n)$. This was known to Honda–Kazez–Matić in [33]; we will prove it again.

Proposition 1.2.3 (Contact elements distinct) *Distinct tight contact structures (up to isotopy) on (T, n), or equivalently, distinct chord diagrams, give distinct contact elements of $SFH(T, n)$.*

Recalling the bijection between chord diagrams and tight contact structures on (T, n), this means that chord diagrams of n chords may be identified with contact elements in $SFH(T, n)$.

A second proposition is that the meaning of addition in SFH corresponds in a precise sense to bypass moves on chord diagrams. This was probably known to the authors of [33], although the whole of this result was not made explicit. The set of contact elements in $SFH(T, n, e)$ is not a subgroup under addition, but the extent to which it is closed under addition is described by bypass moves.

Proposition 1.2.4 (Addition means bypass moves) *Suppose a, b are contact elements in $SFH(T, n, e)$. Then $a+b$ is a contact element if and only if a, b are related by a bypass move. If so, then $a + b$ is the third element of their bypass triple.*

(Note here we are identifying chord diagrams with contact elements, and we will continue this abuse of notation throughout.)

The combinatorial version of sutured Floer homology described in section 1.1 seems to have been known in [33], although it was not made explicit; it also appears to be the origin of Honda's "contact category" [21]. In any case, the bypass relation alone does not show that SFH is the combinatorial object described in section 1.1. But it is; in some sense the "only" relation between contact elements is the bypass relation.

Proposition 1.2.5 (SFH is combinatorial) *There is an isomorphism*

$$SFH_{comb}(T, n, e) \stackrel{\cong}{\longrightarrow} SFH(T, n, e).$$

This isomorphism takes a chord diagram to the contact element of the tight contact structure on (T, n) with that chord diagram as its dividing set on a meridional disc.

1.2.2 Categorification of Pascal's triangle

If we consider all the sutured Floer homology groups $SFH(T,n)$ and their decompositions into direct sums of $SFH(T,n+1,e)$, over all possible n and e, we can arrange these in a triangle. (Recall $-n \leq e \leq n$ and $e \equiv n \bmod 2$.)

$$\begin{array}{ccccccccc} & & & & SFH(T,1,0) & & & & \\ & & SFH(T,2,-1) & & \oplus & & SFH(T,2,1) & & \\ SFH(T,3,-2) & & \oplus & & SFH(T,3,0) & & \oplus & & SFH(T,3,2) \end{array}$$

These vector spaces are isomorphic respectively to the following "categorification of Pascal's triangle".

$$\begin{array}{ccccccc} & & & \mathbb{Z}_2^{\binom{0}{0}} & & & \\ & & \mathbb{Z}_2^{\binom{1}{0}} & \oplus & \mathbb{Z}_2^{\binom{1}{1}} & & \\ & \mathbb{Z}_2^{\binom{2}{0}} & \oplus & \mathbb{Z}_2^{\binom{2}{1}} & \oplus & \mathbb{Z}_2^{\binom{2}{2}} & \\ \mathbb{Z}_2^{\binom{3}{0}} & \oplus & \mathbb{Z}_2^{\binom{3}{1}} & \oplus & \mathbb{Z}_2^{\binom{3}{2}} & \oplus & \mathbb{Z}_2^{\binom{3}{3}} \end{array}$$

We will see that there are various maps between these vector spaces. There are maps denoted

$$\begin{array}{cccc} B_-, B_+ : & SFH(T,n) & \longrightarrow & SFH(T,n+1) \\ & \wr\| & & \wr\| \\ & \mathbb{Z}_2^{2^{n-1}} & \longrightarrow & \mathbb{Z}_2^{2^n} \end{array}$$

which we call *creation* maps. They are defined by the picture in figure 1.7 of "creating a chord and adding a $\pm$ outermost region near the base point". They take chord diagrams to chord diagrams, i.e. contact elements to contact elements.

In the combinatorial definition of SFH, it is clear that they are linear maps. The fact that they are linear in bona fide sutured Floer homology comes from the TQFT-inclusion property [33], as we will see in chapter 3; we will define the operators more precisely in section 3.1.

The maps B_-, B_+ respectively subtract or add 1 to the relative euler class of the chord diagram / contact structure; so that, restricting to particular summands,

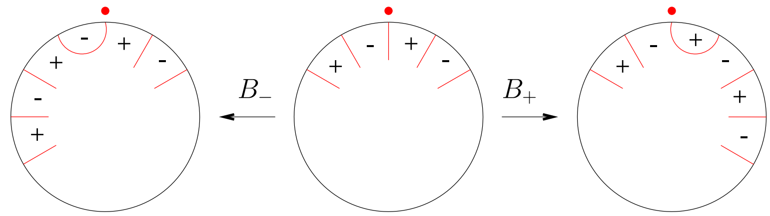


Figure 1.7: Creation maps $B_\pm$.

B_-, B_+ respectively define maps

$$\begin{array}{ccc} SFH(T,n,e) & \stackrel{B_-}{\longrightarrow} & SFH(T,n+1,e-1) \\ \mathbb{Z}_2^{\binom{n-1}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}} \end{array}$$

and

$$\begin{array}{ccc} SFH(T,n,e) & \stackrel{B_+}{\longrightarrow} & SFH(T,n+1,e+1) \\ \mathbb{Z}_2^{\binom{n-1}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k+1}} \end{array}$$

where $k = (n+e-1)/2$. (Strictly speaking, a $B_\pm$ is defined on each $SFH(T,n,e)$ or $SFH(T,n)$, but we denote them all by $B_\pm$; alternatively, $B_\pm$ may be considered to act on the direct sum of all the $SFH(T,n)$.)

These two maps "categorify the Pascal recursion".

Proposition 1.2.6 (Categorification of Pascal recursion) *There are maps*

$$B_\pm : SFH(T,n,e) \longrightarrow SFH(T,n+1,e\pm 1)$$

which correspond to "creating" a chord as in figure 1.7 above. These are injective linear maps and

$$SFH(T,n+1,e) = B_+\left(SFH(T,n,e-1)\right) \oplus B_-\left(SFH(T,n,e+1)\right).$$

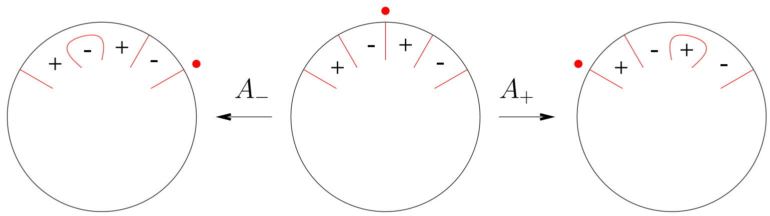


Figure 1.8: Annihilation maps $A_\pm$.

Similarly, there are two maps

$$\begin{array}{rccc} A_-, A_+ : & SFH(T, n+1) & \longrightarrow & SFH(T, n) \\ & \wr\| & & \wr\| \\ & \mathbb{Z}_2^{2^n} & \longrightarrow & \mathbb{Z}_2^{2^{n-1}} \end{array}$$

which we may call *annihilation* maps, defined by "closing off an outermost $\pm$ region near the base point". See figure 1.8.

Again, this is clearly linear in the combinatorial version of SFH; it is also linear as an application of the TQFT-property of SFH; further details in chapter 3.

The maps A_+, A_- respectively add or subtract 1 to the relative euler class of the chord diagram / contact structure; so that, restricting to these summands, again, we have

$$\begin{array}{ccc} SFH(T, n+1, e) & \xrightarrow{A_-} & SFH(T, n, e-1) \\ \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n-1}{k-1}} \end{array}$$

and

$$\begin{array}{ccc} SFH(T, n+1, e) & \xrightarrow{A_+} & SFH(T, n, e+1) \\ \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n-1}{k}} \end{array}$$

where $k = (n+e)/2$.

The creation and annihilation operators satisfy some relations.

Proposition 1.2.7 (Annihilation operators) *There are maps*

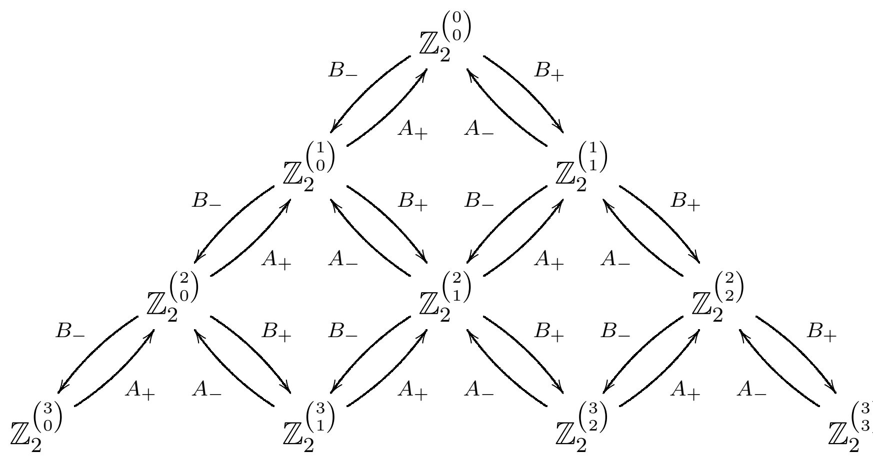
$$A_\pm : SFH(T, n+1, e) \longrightarrow SFH(T, n, e \pm 1)$$

which correspond to "annihilating" a chord in a chord diagram as in figure 1.8 above.

These are surjective and satisfy

$$A_+ \circ B_- = A_- \circ B_+ = 1 \quad \textit{and} \quad A_+ \circ B_+ = A_- \circ B_- = 0.$$

Thus, the $A_\pm, B_\pm$ operators give a categorification of Pascal's triangle, in the sense of the following diagram:



(The effect of a composition $B_\pm \circ A_\pm$ is also easily understood, with the basis described in the next section.)

1.2.3 Basis, words, orderings, and quantum field theory

Denote by $v_\emptyset$ the nonzero element of $SFH(T,1) = \mathbb{Z}_2$ (the "vacuum"), which corresponds to the unique chord diagram with 1 chord (lemma 3.1.1). Then in

$$SFH(T, n+1, e) \cong \mathbb{Z}_2^{\binom{n}{k}}$$

there are $\binom{n}{k}$ contact elements of the form

$$B_\pm \, B_\pm \, \cdots \, B_\pm \, v_\emptyset$$

where there are n_+ of the B_+'s, and n_- of the B_-'s, satisfying $k = n_+$, $n = n_+ + n_-$ and $e = n_+ - n_-$.

Denote by $W(n_-, n_+)$ the set of all words on $\{-,+\}$ of length $n = n_+ + n_-$, with n_- minus signs and n_+ plus signs; equivalently, which sum to $e = n_+ - n_-$. For every word $w \in W(n_-, n_+)$ there is a corresponding element $v_w = B_w v_\emptyset$ in $SFH(T, n+1, e)$. Here B_w denotes the string of B_+'s and B_-'s corresponding to w. Each $v_w \in SFH(T, n+1, e)$ is a contact element, corresponding to a chord diagram Γ_w of $n+1$ chords and relative euler class e.

Remark 1.2.8 (Conventions for variables) *Unless mentioned otherwise, we will assume that the variables n_-, n_+, n, e, k are related so that $SFH(T, n+1, e) = \mathbb{Z}_2^{\binom{n}{k}}$ contains the contact elements v_w with $w \in W(n_-, n_+)$. That is, they are related by*

$$k = (e+n)/2, \quad e = 2k - n, \quad n_+ = k, \quad n = n_+ + n_-, \quad e = n_+ - n_-.$$

The set $W(n_-, n_+)$ has some orderings.

Definition 1.2.9 (Lexicographic ordering) *There is a total order on $W(n_-, n_+)$ obtained from regarding $-$ as coming before $+$ in the dictionary. This also induces a total order on the elements $v_w \in SFH(T, n+1, e)$ and the chord diagrams Γ_w.*

We will usually read words from left to right, but we note that reading words from right to left also gives a total lexicographic order.

Definition 1.2.10 (Partial ordering $\preceq$) *There is a partial order $\preceq$ on $W(n_-, n_+)$ defined by: $w_1 \preceq w_2$ if and only if, for all $i = 1, \ldots, n_-$, the i'th $-$ sign in w_1 occurs to the left of (or in the same position as) the i'th $-$ sign in w_2. This also induces a partial order, also denoted $\preceq$, on the elements $v_w \in SFH(T, n+1, e)$ and the chord diagrams Γ_w.*

Thus the partial order $\preceq$ essentially says "all minus signs move right (or stay where they are)". It is clear that this is a partial order, and a sub-order of the lexicographic total order. It is equivalent to "all $+$ signs move left (or stay where they are)".

Note that $|W(n_-, n_+)| = \binom{n}{n_+} = \binom{n}{k}$, so that there are as many contact elements $v_w \in SFH(T, n+1, e)$ as the dimension of $SFH(T, n+1, e)$. Even better:

Proposition 1.2.11 (QFT basis) *The set of v_w, for $w \in W(n_-, n_+)$, forms a basis for $SFH(T, n+1, e)$.*

The analogy, of course, is with operators for the creation and annihilation of particles in quantum field theory. We think of particles with charge (spin?) ± 1, and consider $SFH(T, n+1, e)$ as the space generated by n-particle states of charge e. Each chord diagram with $n+1$ chords and relative euler class e becomes an "n-particle state of charge e"; the chord diagram with 1 chord, "the vacuum". We think of B_+ as "creating a charge $+1$ particle" and B_- as "creating a charge -1 particle"; and similarly, we think of A_+ as "annihilating a charge -1 particle" and A_- "annihilating a charge $+1$ particle". The bypass relation can be thought of as saying "the superposition of two bypass-related states is the third state in their triple".

The fact that the v_w form a basis says that "the space of n-particle states has a basis obtained by applying creation operators to the vacuum". This is usual in quantum field theory. However, for bosons, creation operators commute; for fermions, they anti-commute; in our case, there is no commutation relation whatsoever, and any applications of creation operators in different orderings are independent. Perhaps our particles have "irrational spin", then. Or, ours are "free particles", where "free" is understood in the sense of "free group".

1.2.4 Catalan and Narayana numbers

Since one of our goals is to understand how contact elements / chord diagrams lie in SFH, a simple first question is: *How many contact elements are there in $SFH(T, n)$?*

The number of distinct chord diagrams of n chords is given by the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, \ldots.$$

The Catalan numbers are classical combinatorial objects and have been studied extensively for centuries; they are prolific in mathematics. For instance, the number of ways of arranging n pairs of brackets meaningfully is C_n; it is not difficult to see that such bracketings are in bijective correspondence with chord diagrams of n chords.

The Catalan numbers can also be defined recursively by $C_0 = 1$, $C_1 = 1$ and then

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0.$$

The number of distinct chord diagrams with n chords is C_n; and by proposition 1.2.3 above each chord diagram gives a distinct contact element; thus the contact elements form a distinguished subset of size C_n in $SFH(T, n) \cong \mathbb{Z}_2^{2^{n-1}}$, which has $2^{2^{n-1}}$ elements.

We can then refine our question, splitting according to relative euler class: *How many contact elements are there in $SFH(T, n+1, e)$?*

Let this number be C_{n+1}^e. So there are C_{n+1}^e chord diagrams with $n+1$ chords and relative euler class e; and e is an integer of the same parity as n satisfying $-n \leq e \leq n$. It will also be useful to define $C_{n+1,k} = C_{n+1}^{2k-n} = C_{n+1}^e$, following our convention in remark 1.2.8; so that k is an integer, $0 \leq k \leq n$.

From counting chord diagrams of various relative euler classes, we have

$$C_{n+1} = C_{n+1,0} + C_{n+1,1} + \cdots + C_{n+1,n} = C_{n+1}^{-n} + C_{n+1}^{-n+2} + \cdots + C_{n+1}^{n}.$$

The numbers C_{n+1}^e form a triangle, which is known as the *Catalan triangle*. Its entries are known as the *Narayana numbers*.

$$\begin{array}{ccccccccc} &&&& C_1^0 &&&& \\ &&& C_2^{-1} && C_2^1 &&& \\ && C_3^{-2} && C_3^0 && C_3^2 && \\ & C_4^{-3} && C_4^{-1} && C_4^1 && C_4^3 & \\ C_5^{-4} && C_5^{-2} && C_5^0 && C_5^2 && C_5^4 \end{array} = \begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 3 && 1 && \\ & 1 && 6 && 6 && 1 & \\ 1 && 10 && 20 && 10 && 1 \end{array}$$

The Narayana numbers are usually given as $N_{n,k} = C_{n,k-1}$; we have shifted them for our purposes. They have an explicit formula, although we shall not use it:

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

or

$$C_{n+1}^{e} = C_{n+1,k} = N_{n+1,k+1} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}.$$

There is a substantial literature on the Narayana numbers (e.g. [1, 3, 4, 6, 11, 20, 35, 40, 45, 46, 47]); we will restate some of their properties.

Proposition 1.2.12 (Narayana numbers) *The Narayana numbers give the number of chord diagrams C_{n+1}^e in $SFH(T, n+1, e)$, and satisfy the following relations:*

(i) $C_{n+1} = C_{n+1,0} + C_{n+1,1} + \cdots + C_{n+1,n}$.

(ii)

$$C_{n+1,k} = C_{n,k} + C_{n,k-1} + \sum_{\substack{n_1+n_2=n \\ k_1+k_2=k-1}} C_{n_1,k_1}\, C_{n_2,k_2},$$

or equivalently,

$$C_{n+1}^{e} = C_{n}^{e-1} + C_{n}^{e+1} + \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C_{n_1}^{e_1}\, C_{n_2}^{e_2}.$$

One can think of this recursion as a slightly more complicated version of Pascal's triangle, incorporating the Catalan recursion. In fact, we will also show that there is a "categorification" of this recursion also, in a certain tenuous sense.

Proposition 1.2.13 (Categorification of Catalan recursion) *There is an operator*

$$M : SFH(T, n_1, e_1) \otimes SFH(T, n_2, e_2) \longrightarrow SFH(T, n_1 + n_2 + 1, e_1 + e_2)$$

which, applied to contact elements $c(\xi_1) \otimes c(\xi_2)$, gives the contact element obtained by "merging" the corresponding chord diagrams. The operator M reduces to a creation *operator $B_\pm$ (a "degenerate merge" regarding $SFH(T,0)$ as trivial) in the case $n_1 = 0$ or $n_2 = 0$. Every contact element in $SFH(T, n+1, e)$ can then be written uniquely as $M(c(\xi_1), c(\xi_2))$, where $c(\xi_i)$ is a contact element in $SFH(T, n_i, e_i)$, and n_i, e_i satisfy $n_1 + n_2 = n$ and $e_1 + e_2 = e$ (possibly $n_i = 0$). That is,*

$$\left\{ \begin{array}{c} \text{Contact el'ts in} \\ SFH(T, n+1, e) \end{array} \right\} = \bigsqcup_{\substack{n_1+n_2=n \\ e_1+e_2=e}} M \left(\left\{ \begin{array}{c} \text{Contact el'ts in} \\ SFH(T, n_1, e_1) \end{array} \right\}, \left\{ \begin{array}{c} \text{Contact el'ts in} \\ SFH(T, n_2, e_2) \end{array} \right\} \right)$$

In particular,

$$SFH(T, n+1, e) = \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} M(SFH(T, n_1, e_1), SFH(T, n_2, e_2)).$$

Note this is by no means a direct sum; this is simply a statement about surjectivity of M.

We also have a crucial enumerative result for our main theorem: a bijection between comparable pairs and chord diagrams (section 3.2). Recall that the partial order $\preceq$ of $W(n_-, n_+)$ indexes the basis elements of $SFH(T, n+1, e)$.

Proposition 1.2.14 (Number of comparable pairs) *The number of pairs w_0, w_1 in $W(n_-, n_+)$ with $w_0 \preceq w_1$ is C_{n+1}^e.*

1.2.5 Contact elements and comparable pairs

Our main theorems flesh out the purely enumerative bijection of proposition 1.2.14, giving an explicit bijection between contact elements and comparable pairs of words. A little more precisely, a general contact element is determined by decomposing it in terms of basis elements and looking at the first and last basis elements among them.

Alternatively, we can think of every state as a morphism from a first state to a last state. This is one origin of our categorical perspective.

Theorem 1.2.15 (Min, max basis elements) *Consider a contact element*

$$v \in SFH(T, n+1, e).$$

Writing v as a sum of basis vectors v_w, where $w \in W(n_-, n_+)$, there is a lexicographically first v_{w_-} and last v_{w_+} basis vector amongst them. Then for every basis vector v_w occurring in the sum, $w_- \preceq w \preceq w_+$. In particular, $w_- \preceq w_+$.

Theorem 1.2.16 (Contact elements and comparable pairs) *The map*

$$\Phi : \left\{ \begin{array}{c} \textit{Contact} \\ \textit{elements in} \\ SFH(T, n+1, e) \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Chord diagrams} \\ \textit{with } n+1 \textit{ chords,} \\ \textit{euler class } e \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Comparable pairs} \\ \textit{of words } w_1 \preceq w_2 \\ \textit{in } W(n_-, n_+) \end{array} \right\}$$

given by taking a contact element v, and sending it to (w_-, w_+), where v_{w_-}, v_{w_+} are respectively the first and last basis vectors in the basis decomposition of v, is a bijection.

That is, given any comparable pair $w_1 \preceq w_2$, there is precisely one contact element which, when written as a sum of basis elements, has v_{w_1} as its first and v_{w_2} as its last.

We will denote the unique contact element with first basis element v_{w_-} and last basis element v_{w_+} by $[w_-, w_+]$ or $[v_{w_-}, v_{w_+}]$ or $[\Gamma_{w_-}, \Gamma_{w_+}]$, and throughout we will abuse notation, often identifying contact elements with chord diagrams and basis contact elements with words; hopefully this will not cause too much confusion.

1.2.6 Moves on chord diagrams and words

The proofs of the main theorems are by explicit construction. Given $w_- \preceq w_+$, we show how to construct a chord diagram whose decomposition has v_{w_-} as its first and v_{w_+} as its last element. Then by the enumerative bijection of proposition 1.2.14, this is shown to be a bijection.

We will build up a method for performing bypass moves on basis chord diagrams, in order to turn any Γ_{w_1} into Γ_{w_2}, whenever $w_1 \preceq w_2$, by upwards bypass moves. And conversely, we will show how to turn Γ_{w_2} into Γ_{w_1} by downwards bypass moves. This method will be explicitly analogous to certain combinatorial "word-processing" moves on the corresponding words, consisting of moving certain blocks of $-$ signs past certain blocks of $+$ signs.

Multiple bypass moves will be performed on *bypass systems*.

Definition 1.2.17 (Bypass system) *A* bypass system *is a finite set of disjoint arcs of attachment.*

We will build up enough machinery to construct bypass moves to take us from any w_1 to w_2, for $w_1 \preceq w_2$. This will give us a *bypass system of the comparable pair* $w_1 \preceq w_2$.

Then, we will show that performing all these bypass moves *in the opposite direction*, gives us a chord diagram whose decomposition has w_1, w_2 as first and last elements.

Proposition 1.2.18 (Bypass system of a comparable pair) *Suppose* $\Gamma_1 \preceq \Gamma_2$ *are basis chord diagrams.*

(i) *On* Γ_1*, there exists a bypass system* $FBS(\Gamma_1, \Gamma_2)$ *such that performing* upwards *bypass moves on it gives* Γ_2*.*

(ii) *On* Γ_2*, there exists a bypass system* $BBS(\Gamma_1, \Gamma_2)$ *such that performing* downwards *bypass moves on it gives* Γ_1*.*

Proposition 1.2.19 (Bypass system of pair, opposite direction) *Performing* downwards *bypass moves on* $FBS(\Gamma_1, \Gamma_2)$ *or* upwards *bypass moves on* $BBS(\Gamma_1, \Gamma_2)$ *gives a chord diagram* Γ *such that in its basis decomposition,* Γ_1 *and* Γ_2 *appear, and for every basis element* Γ_w *in this decomposition,* $\Gamma_1 \preceq \Gamma_w \preceq \Gamma_2$*. That is,* Γ_1 *is a total minimum and* Γ_2 *a total maximum, with respect to* $\preceq$*, among all the basis elements occurring in the decomposition.*

(Again we are identifying chord diagrams with contact elements.)

The proofs of these propositions are based on correspondences between the following notions, which we will define in due course.

$$
\left\{ \begin{array}{c} \text{elementary move} \\ \text{on a word} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{bypass move} \\ \text{on an attaching arc} \end{array} \right\}
$$
$$
\left\{ \begin{array}{c} \text{generalised elementary} \\ \text{move on a word} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{bypass moves on the} \\ \text{bypass system of a} \\ \text{generalised attaching arc} \end{array} \right\}
$$
$$
\left\{ \begin{array}{c} \text{nicely ordered sequence of} \\ \text{generalised elementary moves} \\ \text{on a word} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{bypass moves on the} \\ \text{bypass system of a} \\ \text{nicely ordered sequence of} \\ \text{generalised attaching arcs} \end{array} \right\}.
$$

The final correspondence is strong enough to give the constructions in the above two propositions, explicitly; which in turn give the main theorems.

1.2.7 Contact elements are tangled

Having shown that contact elements are determined by the first and last basis elements in their decomposition, a natural question arises: what are the other basis elements between the ones between the first and last? We can give some answers about what lies in between.

A first property, from theorem 1.2.15, is that every other basis element lies between the first and last, with respect to $\preceq$.

Second, we can prove results about *the number of basis elements* in a contact element. For a basis element, this answer is clear: one — itself. In any other case, we will show that the answer is even.

Proposition 1.2.20 (Size of basis decomposition) *Every chord diagram which is not a basis element has an even number of basis elements in its decomposition.*

Third, we can show that basis elements occurring in the decomposition of a contact element are "tangled up", in some sense. We have said that the first and last basis elements of a contact element Γ are comparable to all others in the decomposition. We show that no other basis element has this property.

Theorem 1.2.21 (Not much comparability) *Suppose v_w occurs in the basis decomposition of $v = [v_{w_-}, v_{w_+}]$ and is comparable, with respect to $\preceq$, with every other basis element occurring in the decomposition. Then $w = w_-$ or w_+.*

In a sense, therefore, we cannot "untangle" the partial ordering among the basis elements in the decomposition of a contact element.

More generally, we will show that for any basis element v_w in the decomposition of v, other than $v_{w_\pm}$, the number of basis elements $v_{w'}$ of Γ such that $w' \preceq w$ is even; and the number of $v_{w'}$ with $w \preceq w'$ is even also (proposition 7.3.11). This implies the above theorem.

Fourth, the presence of $\pm$ symbols in certain positions in both w_- and w_+ implies the presence of certain symbols in similar positions in all w occurring in the decomposition of $[v_{w_-}, v_{w_+}]$. In fact, such symbols also tell us about the corresponding chord diagram:

- (Lemma 7.3.2) A chord diagram $\Gamma = [\Gamma_-, \Gamma_+]$ has an outermost region at the base point, if and only if the words for Γ_-, Γ_+ begin with the same symbol, if and only if all basis elements of Γ have words which begin with the same symbol.

- (Lemmas 7.3.3-7.3.7) Similarly, for various locations on the disc, a chord diagram has an outermost region at that location, if and only if the words for Γ_-, Γ_+ both possess a certain property (ending with the same symbol; having the j'th $-$ sign not the first in its block; etc.), if and only if each basis element of Γ has the same property.

Fifth, and finally, we note it is possible to give an algorithm to write down the basis decomposition of any $[v_{w_-}, v_{w_+}]$ with $w_- \preceq w_+$. However, this basically just replicates the construction of bypass systems in the construction of the chord diagram, in combinatorial language (or writes a computer program to manipulate chord diagrams!). We make some remarks along these lines in section 7.3.4.

1.2.8 Computation by rotation

None of the above gives a good way to *compute* all contact elements (not just basis elements). One good way to enumerate them is to use the fact that we may always rotate a chord diagram until there is an outermost region adjacent to the base point, and then it lies in the image of $B_\pm$. Such a rotation gives a linear operator in sutured Floer homology, and we will give a recursive formula (proposition 8.1.1) for this operator, as well as describing it explicitly (proposition 8.1.2). There is interesting combinatorics in the matrix of this operator; we wonder if it has other applications.

1.2.9 Simplicial structure

We will also show that there is a simplicial structure on the SFH vector spaces forming the various diagonals of Pascal's triangle. We note that our creation and annihilation operators $A_\pm, B_\pm$ were defined at a particular point, namely the base point, but there are $2n$ marked points on the boundary of the disc. Choosing other points gives more creation and annihilation operators, which, as it turns out, obey the same relations as face and degeneracy maps in simplicial structures. The associated boundary maps make the categorified Pascal's triangle into a double chain complex.

Proposition 1.2.22 (Simplicial structure) *On each diagonal of Pascal's triangle, there are face and degeneracy maps giving it a simplicial structure, with boundary maps making each diagonal into a chain complex with trivial homology, and the whole triangle into a double complex.*

1.3 Contact categories, stacking

Our investigations of the sutured Floer homology of the solid torus, arguably the simplest nontrivial sutured 3-manifold, lead us to develop some considerable algebraic and combinatorial structure: chord diagrams, creation and annihilation operators, bypass moves, QFT-type basis, and partial order; as related by the various results of the previous section.

We now find that much of this structure has direct contact-geometric meaning; it has applications independent of sutured Floer homology. Moreover, some of these notions are contact-geometric to begin with, like "bypass moves". We find that these algebraic and combinatorial structures are intimately related to contact structures on solid cylinders $D \times I$, by regarding the bypass moves of the preceding constructions as actual bypass attachments. This leads us to consider the "contact category" of Honda [21], and various extensions and generalisations of it. Indeed, we seem to be led in the direction of a "categorification of contact geometry".

1.3.1 Contact "cobordisms" and stackability

Bypass moves arise from the contact-geometric construction of attaching a *bypass*. A bypass is a thickened half-disc with a particular contact structure; we will discuss contact geometry preliminaries in section 2.1, and we will analyse bypasses and bypass attachments in some detail in section 4.1. Bypasses are very interesting objects because they can be considered both as elementary building blocks of contact manifolds, and also as half of an overtwisted disc, which "spoils" a contact manifold. Overtwisted contact geometry is "trivial" in the sense that it reduces to homotopy theory [7], and most of the interest in contact topology lies in tight (i.e. non-overtwisted) contact structures.

Attachment of bypasses on a disc gives a cylinder $D^2 \times I$ with distinct dividing sets on $D^2 \times \{0\}$ and $D^2 \times \{1\}$, leading to a construction we call *stacking*. Given two chord diagrams Γ_0, Γ_1, we form a sutured solid cylinder $\mathcal{M}(\Gamma_0, \Gamma_1)$. This is a 3-manifold with boundary (and corners) $D^2 \times I$, with sutures on the bottom $D \times \{0\}$ and top $D \times \{1\}$ being Γ_0 and Γ_1 respectively. We can ask whether there is a tight contact structure on this sutured manifold: if so, we say Γ_1 is *stackable* on Γ_0. We then think of the contact structure on the solid cylinder as a "cobordism" between the two convex discs given by the chord diagrams Γ_0, Γ_1 (even though the boundary of the cylinder consists of more than $D \times \{0, 1\}$). More details will be given in section 4.2.

The question of whether Γ_1 is stackable on Γ_0 is a linear question in SFH.

Proposition 1.3.1 (Stackability map) *There is a linear map*

$$m : SFH(T,n) \otimes SFH(T,n) \longrightarrow \mathbb{Z}_2$$

which takes pairs of contact elements, corresponding to pairs of chord diagrams Γ_0, Γ_1, *to* 0 *or* 1 *respectively as* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is overtwisted or tight.*

Thus, the linear map m is the answer, as a boolean function, to the question "is Γ_1 stackable on Γ_0?" Moreover, the summands $SFH(T,n,e)$ of $SFH(T,n)$ are "orthogonal" with respect to this question:

Proposition 1.3.2 (Relative euler class orthogonality) *Let* Γ_0 *and* Γ_1 *be chord diagrams with* n *chords. If* Γ_0, Γ_1 *have distinct relative euler class then* $m(\Gamma_0, \Gamma_1) = 0$.

We can actually give a complete description of m. Whether $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight is intimately related to the partial order $\preceq$; in fact, on basis chord diagrams Γ_w for $w \in W(n_-, n_+)$, the answer *is* $\preceq$ (regarded as a boolean function).

Proposition 1.3.3 (Contact interpretation of $\preceq$) $\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ *is tight if and only if* $w_0 \preceq w_1$.

Then we can use this to obtain a result for general chord diagrams.

Proposition 1.3.4 (General stackability) *Let* Γ_0 *and* Γ_1 *be two chord diagrams of* n *chords with relative euler class* e. *Then* Γ_1 *is stackable on* Γ_0 *(i.e.* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight) if and only if the cardinality of the set*

$$\left\{ (w_0, w_1) \; : \; \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \textit{ occurs in the decomposition of } \Gamma_i \end{array} \right\}$$

is odd.

We will show various other properties of m and $\mathcal{M}$:

(i) (Lemma 4.2.4) $m(\Gamma, \Gamma) = 1$, i.e. $\mathcal{M}(\Gamma, \Gamma)$ is tight.

(ii) (Lemma 4.2.5) If Γ_0, Γ_1 respectively have outermost chords γ_0, γ_1 in the same position, then $m(\Gamma_0, \Gamma_1) = m(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1)$.

(iii) (Lemma 4.2.6) If Γ_0, Γ_1 are related by a bypass move then, when placed in the right order, $m(\Gamma_0, \Gamma_1)$ is tight. (The order is made explicit in lemma 4.2.6.)

The last lemma relates stackability to bypass moves. In fact, if Γ_1 can be obtained from Γ_0 by attaching bypasses on top of Γ_0, then we have a construction of $\mathcal{M}(\Gamma_0, \Gamma_1)$, along with a contact structure on the sutured solid cylinder. Under certain conditions, one can prove that this contact structure is tight: such questions were considered in [29], and the results of that paper, including the important notion of *pinwheels*, are applied here.

We will see that bypass triples naturally give rise to triples of tight contact cobordisms (lemma 4.2.6). Further, we will see that our explicit construction of bypass moves from Γ_{w_1} to Γ_{w_2}, via a bypass system $FBS(w_1, w_2)$ for any $w_1 \preceq w_2$, when considered as a set of actual contact-geometric bypass attachments, gives a tight contact structure on $\mathcal{M}(\Gamma_{w_1}, \Gamma_{w_2})$. More generally (lemma 7.2.2), for a given chord diagram $\Gamma = [\Gamma_-, \Gamma_+]$, there are tight contact structures on $\mathcal{M}(\Gamma, \Gamma_-)$, $\mathcal{M}(\Gamma_+, \Gamma)$ and $\mathcal{M}(\Gamma_-, \Gamma_+)$ obtained by attaching bypasses along bypass systems $FBS(\Gamma_-, \Gamma_+)$ and $BBS(\Gamma_-, \Gamma_+)$. This is a generalisation of the notion of bypass triple; we will have more to say about various possible generalisations of bypass triples as we proceed.

1.3.2 Contact categories

The question of which dividing sets are stackable on which others arises naturally in the notion of *contact category* defined by Honda [21]. Honda shows that this category possesses certain properties similar to those of a triangulated category, and behaves functorially with respect to SFH. For a given surface Σ, the contact category $\mathcal{C}(\Sigma)$ has objects corresponding to dividing sets on Σ, and morphisms corresponding to contact structures on $\Sigma \times I$ with dividing sets specified on $\Sigma \times \{0,1\}$ (as we need it, a rigorous definition is given in 4.2.15). A nontrivial (tight) morphism $\Gamma_0 \longrightarrow \Gamma_1$ precisely means that Γ_1 is stackable on Γ_0. Our map m then precisely describes the nontrivial morphisms in the contact category of a disc.

We can go further, or rather, narrower. We can start from a given cobordism

$\mathcal{M}(\Gamma_0, \Gamma_1)$ with tight contact structure, and ask what chord diagrams occur as dividing sets of discs *inside* this cobordism (definition 4.2.8). We can give a criterion for when a chord diagram Γ exists (lemma 4.2.9). Using this, we obtain easily, for instance, that the only chord diagram existing in $\mathcal{M}(\Gamma, \Gamma)$ is Γ itself (lemma 4.2.10).

This leads us to define the notion of *bounded contact category* (definition 4.2.18) $\mathcal{C}^b(\Gamma_0, \Gamma_1)$. The idea is that it is the "subcategory of the contact category of the disc which is contained in $\mathcal{M}(\Gamma_0, \Gamma_1)$", or the "subcategory of the contact category bounded by Γ_0 and Γ_1". Its objects are those dividing sets Γ which occur in a tight $\mathcal{M}(\Gamma_0, \Gamma_1)$, and its morphisms are those cobordisms $\mathcal{M}(\Gamma, \Gamma')$ which occur in $\mathcal{M}(\Gamma_0, \Gamma_1)$. We prove (lemma 4.2.19) that this $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ is indeed a category.

Therefore, lemma 4.2.10, that the only chord diagram existing in $\mathcal{M}(\Gamma, \Gamma)$ is Γ, says that the bounded contact category $\mathcal{C}^b(\Gamma, \Gamma)$ is trivial.

Any partially ordered set can be considered as a category; hence the set of words $W(n_-, n_+)$ with the partial order $\preceq$ can be considered as a category. Conversely, it's easy to specify under what conditions a given category can be considered as a partially ordered set (lemma 4.2.20). We can then prove the following.

Proposition 1.3.5 *The bounded contact category $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ is partially ordered.*

In fact, suppose a tight contact structure on $\mathcal{M}(\Gamma_0, \Gamma_1)$ is obtained by attaching bypasses to Γ_0 along a bypass system c. Then "each successive bypass attachment creates another morphism". The set of subsets of c (i.e. the power set $\mathcal{P}(c)$ of the finite set of attaching arcs) can be considered a partially ordered set under inclusion, and hence a category. Then we obtain a covariant functor Up from $\mathcal{P}(c)$ to $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ by performing bypass attachments along subsets of c; and similarly we can obtain a contravariant Down functor (lemma 4.2.21).

Contact categories have structures resembling "exact triangles" and "cones", analogously to a triangulated category. Bypass triples resemble exact triangles: the composition of two morphisms in the triangle is overtwisted, since a bypass is half an overtwisted disc. Two bypass-related chord diagrams determine a third one (their sum in SFH), which can be regarded as the cone of the morphism between them.

When $\mathcal{M}(\Gamma_0, \Gamma_1)$ can be obtained by multiple bypass attachments along a bypass system (an *elementary cobordism*), we have generalised versions of bypass triples, exact triangles and cones. This includes triples $(\Gamma, \Gamma_-, \Gamma_+)$ where $\Gamma = [\Gamma_-, \Gamma_+]$. However, we believe these notions are still in an unsatisfactory state; they leave us discontented. In particular, not every cobordism is elementary (lemma 4.2.14), so there is no general notion of "cone" or "exact triangle" for a morphism.

We discuss these and related issues in sections 4.2.5, 4.2.10, 7.2.2 and 7.2.3.

1.3.3 Computation of bounded contact categories

Using our interpretation of the partial order $\preceq$ as describing stackability, we can compute some bounded contact categories.

First, we can compute $\mathcal{C}^b(\Gamma_{w_0}, \Gamma_{w_1})$ for any basis chord diagrams $\Gamma_{w_0}, \Gamma_{w_1}$ corresponding to words $w_0, w_1 \in W(n_-, n_+)$; for a tight cobordism, we assume $w_0 \preceq w_1$.

Definition 1.3.6 (Partially ordered set $\boldsymbol{W(w_0, w_1)}$) *Given two words w_0, w_1 in $W(n_-, n_+)$ which are comparable, $w_0 \preceq w_1$, let*

$$W(w_0, w_1) = \{w \in W(n_-, n_+) \ : \ w_0 \preceq w \preceq w_1\}$$

i.e. the subset of $W(n_-, n_+)$ bounded below by w_0 and above by w_1. We endow $W(w_0, w_1)$ with the partial order inherited from $W(n_-, n_+)$.

As $W(w_0, w_1)$ is a partially ordered set, it may be considered as a category. The result is that this is precisely the bounded contact category.

Proposition 1.3.7 (Bounded contact category of basis cobordism) *For words $w_0 \preceq w_1$ in $W(n_-, n_+)$ corresponding to basis chord diagrams $\Gamma_{w_0}, \Gamma_{w_1}$,*

$$\mathcal{C}^b\left(\Gamma_{w_0}, \Gamma_{w_1}\right) \cong W(w_0, w_1).$$

The word $w \in W(w_0, w_1)$ corresponds to the basis chord diagram Γ_w.

That is, the chord diagrams occurring in $\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ are precisely the basis chord diagrams Γ_w with $w_0 \preceq w \preceq w_1$; and convex discs in the cobordism with dividing sets Γ_w, $\Gamma_{w'}$ can be separated, Γ_w below $\Gamma_{w'}$, if and only if $w \preceq w'$.

If we take w_0 and w_1 to be the total minimum and maximum in $W(n_-, n_+)$, i.e. $(-)^{n_-}(+)^{n_+}$ and $(+)^{n_+}(-)^{n_-}$ respectively, then the bounded contact category is precisely $W(n_-, n_+)$; the chord diagrams occurring are precisely all the basis chord diagrams. This leads us to consider a sort of "universal cobordism".

Definition 1.3.8 (The "universal cobordism") *We denote the cobordism*

$$\mathcal{M}\left(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma_{(+)^{n_+}(-)^{n_-}}\right)$$

by $\mathcal{U}(n_-, n_+)$ and call it the "universal cobordism".

We can denote its bounded contact category by

$$\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right) = \mathcal{C}^b\left(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma_{(+)^{n_+}(-)^{n_-}}\right).$$

Then as a special case of the preceding proposition we have:

Proposition 1.3.9 (Bounded contact category of universal cobordism) *For any n_-, n_+, there is an isomorphism of categories*

$$\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right) \cong W(n_-, n_+).$$

The word $w \in W(n_-, n_+)$ corresponds to the basis chord diagram Γ_w.

We may therefore regard $\mathcal{U}(n_-, n_+)$ as a "geometric realisation" of the category $W(n_-, n_+)$, in a moral (not technical) sense; and similarly, each $\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ with $w_0 \preceq w_1$ as "realizing" the sub-category $W(w_0, w_1)$.

Although in a sense $\mathcal{U}(n_-, n_+)$ has the "most complicated" bounded contact category among cobordisms between basis chord diagrams with given $n_\pm$, it is just a bypass cobordism. Indeed, $\Gamma_{(+)^{n_+}(-)^{n_-}}$ can be obtained from $\Gamma_{(-)^{n_-}(+)^{n_+}}$ by a single bypass attachment (see section 6.1.4). In effect, the computation of $\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ tells us what "bypasses exist inside the bypass". The presence of extra chords near the attaching arc allows for extra "intermediate" bypasses.

As it turns out, the universal cobordism actually describes the "bypasses inside *any* bypass". We can compute the bounded contact category of *any* bypass cobordism, i.e. any cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ obtained from attaching a single bypass above Γ_0. Here Γ_0, Γ_1 need not be basis chord diagrams. The answer is isomorphic to the bounded contact category of some universal cobordism. In fact, if we take the "largest universal cobordism" that can be embedded into the chord diagram around the bypass attachment, the bounded contact category of the cobordism is isomorphic to that of the corresponding universal cobordism. The presence of other chords makes no difference to the bounded contact category. The precise statement (theorem 6.2.4) and proof is given in section 6.2.2.

1.3.4 Categorical meaning of main theorems

Our main theorems can be interpreted in this language of contact categories. This largely amounts to saying the same thing with fancier words, but may still be of interest.

These theorems show that for $w_- \preceq w_+$, we have a bypass system $FBS(w_-, w_+)$ such that $\text{Up}_{FBS(w_-,w_+)} \Gamma_{w_-} = \Gamma_{w_+}$ and $\text{Down}_{FBS(w_-,w_+)} \Gamma_{w_-} = \Gamma = [\Gamma_{w_-}, \Gamma_{w_+}]$. In terms of the bounded contact category, the pair $w_- \preceq w_+$ corresponds to a morphism $\Gamma_{w_-} \longrightarrow \Gamma_{w_+}$ in the bounded contact category of the universal cobordism $\mathcal{U}(n_-, n_+)$, representing an embedded cobordism $\mathcal{M}(\Gamma_{w_-}, \Gamma_{w_+})$ in $\mathcal{U}(n_-, n_+)$. Morphisms between basis chord diagrams are precisely the morphisms of the bounded contact category of the universal cobordism.

Moreover, attaching bypasses along $FBS(w_-, w_+)$ actually gives the tight contact structure on $\mathcal{M}(\Gamma_{w_-}, \Gamma_{w_+})$ (lemma 7.2.2); so it is *elementary*.

Proposition 1.3.10 (Tight basis cobordisms elementary) *Let* Γ_0 *and* Γ_1 *be basis chord diagrams, and suppose* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight. Then* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is elementary.*

We may therefore consider $\Gamma, \Gamma_{w_-}, \Gamma_{w_+}$ as a "generalised bypass triple" or "exact triangle". In fact, $\Gamma = [\Gamma_{w_-}, \Gamma_{w_+}]$ can be regarded as the "cone", the third element in an exact triangle arising from the morphism $\Gamma_{w_-} \longrightarrow \Gamma_{w_+}$ between basis chord diagrams.

Hence chord diagrams of n chords and euler class e are in bijective correspondence with the morphisms of the bounded contact category $\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ of the universal cylinder, and may be regarded as their cones. We make this precise in proposition 7.2.3.

1.3.5 A contact 2-category

We can also consider generalisations of Honda's contact category in another direction.

The abstract nonsense interpretation of our main theorem in proposition 7.2.3 says that in our simple case (i.e. a disc), the objects of the contact category can themselves be viewed as morphisms. For chord diagrams are described by pairs $w_0 \preceq w_1$, and as we have seen, a partial order is a category. In this spirit, we can obtain a *contact 2-category* $\mathcal{C}(n+1, e)$: the objects of Honda's contact category can be regarded themselves as its 1-morphisms, and the morphisms of that category become 2-morphisms, or "morphisms between morphisms".

Proposition 1.3.11 (Contact 2-category) *There is a 2-category $\mathcal{C}(n+1, e)$ such that:*

(i) its objects are words $w \in W(n_-, n_+)$, or equivalently basis chord diagrams Γ_w;

(ii) its 1-morphisms are chord diagrams of $n+1$ chords with relative euler class e;

(iii) its 2-morphisms $\Gamma_0 \to \Gamma_1$ are contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$, with (vertical) composition given by stacking contact structures.

Note that $\mathcal{C}(n+1, e) \cong \mathcal{C}^b(\mathcal{U}(n_-, n_+)) \cong W(n_-, n_+)$ as a 1-category; so $\mathcal{C}(n+1, e)$ can be regarded as a "2-category" structure on $W(n_-, n_+)$ or $\mathcal{U}(n_-, n_+)$. It may even be that considering the situation over all values of n and e, the contact category becomes a 3-category.

1.4 Questions and directions

The results of these thesis lead to many further questions, in several different directions. Here we present some thoughts.

From the perspective of sutured Floer homology, we have given a basis for the SFH of certain sutured solid tori, and a description of the contact elements. These are the solid tori with longitudinal sutures. Even if we confine our attention to solid tori, we can ask: what occurs for sutures of different slope? If the situation is at all similar to the longitudinal case, then there should be interesting combinatorics, even in the case of only 2 sutures, since the number of contact structures on a solid torus with 2 boundary sutures of slope p/q is related to the continued fraction decomposition of p/q (see [22]).

Sticking with longitudinal sutures, our solid tori are of the form $D^2 \times S^1$, with sutures of the form $F \times S^1$, F a finite set. Sutured manifolds of this type — $\Sigma \times S^1$ with sutures $F \times S^1$ for $F \subset \partial\Sigma$ finite — form the dimensionally reduced $(1+1)$-dimensional TQFT described in [33], and which we discuss in section 8.3.1. From this perspective, our results describe some aspects of this TQFT for a disc. We can ask how our results extend to more complicated surfaces: we make some remarks in this direction in section 8.3.4, and note how our computations apply immediately to other surfaces, but there is certainly more work to be done here.

We are working only with SFH with $\mathbb{Z}_2$ coefficients. Our work clearly carries over to $\mathbb{Z}$ coefficients. We must note however that with $\mathbb{Z}$ coefficients, contact elements are only defined up to sign ([33]; see also [30], [43], [31]). What are the details in the $\mathbb{Z}$-coefficient case, and do they lead to any new consequences? What about twisted coefficients?

We show in section 8.3.2 that the structures arising in the SFH of our solid tori lead to a notion of contact 2-category. Does this extend further, for instance to a 3-category? The two types of morphisms in the 2-category correspond to "two different directions" in a contact 3-manifold; can a 3-category incorporate the three dimensions of the manifold? Does this extend in any nice way to more complicated surfaces than discs? Can we improve the TQFT structure? We make some remarks on the possibilities in section 8.3.3.

From the perspective of contact topology, we have introduced a notion of "bounded contact category" and computed it in some simple cases. We have also noted that in principle there is an algorithm to compute it in any given case. Is there a simpler description? Our computations for cobordisms of basis chord diagrams or single bypass cobordisms give the bounded contact category as a partially ordered category of words on two symbols. More complicated cobordisms will have bounded contact categories which are in some sense “amalgamated products” of these; what can we say about them?

We have considered only contact categories on discs; of course we would like to know about higher genus surfaces, which we expect to be more complicated. Moreover, we have used the word “cobordism” to describe topologically trivial “cobordisms”, indeed not even really cobordisms. Can anything be said for more general cobordisms?

Our categorical structures are somewhat rudimentary, although they do seem to be of interest to contact topology. Can we refine or redefine these structures so as to have more pleasing algebraic properties? Are there more general notions of exact triangles, cones, kernels, or other concepts, than those we discuss in sections 4.2.10 and 7.2.3? For instance, does our “snake lemma” (lemma 7.2.4) have more than coincidental significance?

We have computed an operator on SFH for rotation of a chord diagram; can we use this for any pure contact-topology applications?

We have some interesting results about the contact manifolds given by attaching bypasses to convex surfaces; even though not every contact structure arises from a bypass system, can we describe contact structures purely from some sort of decorated surface? Is there then some form of “contact Reidemeister moves”, related to manipulations of bypasses, that allow us to better understand contact geometry?

1.5 Structure of this thesis

This introductory chapter gives an overview of our results, in a narrative order; results relating to contact elements in sutured Floer homology are described separately from results about contact categories and cobordisms. However, the body of this thesis presents results in a more logical ordering, which is a little different, proving results about contact elements in SFH, and results about contact categories, in parallel; see figure 1.9.

We begin in chapter 2 by giving some preliminaries on contact geometry (section 2.1) and on sutured Floer homology (section 2.2). We briefly review some fundamentals of contact geometry, including convex surfaces and edge-rounding (section 2.1.2), and contact structures on balls and solid tori (section 2.1.3). We recall the definition of sutured Floer homology (section 2.2.1), its splitting over spin-c structures (section 2.2.2), and briefly review contact elements and topological quantum field theory properties (section 2.2.3).

In chapter 3, we take our first steps, in sutured Floer homology (section 3.1) and in enumerative combinatorics (section 3.2). We build up our picture of SFH vector spaces categorifying Pascal's triangle: beginning from the vacuum (section 3.1.1), we introduce creation and annihilation operators (section 3.1.2), show the distinctness of contact elements (section 3.1.3), the bypass relation and meaning of addition (section 3.1.4), we establish our basis and related results (section 3.1.5), and remark on the octahedral axiom (section 3.1.6). Then we turn to enumerative combinatorics, proving a $TQFT$-version of the Narayana recursion (section 3.2.1), and the relationship of Narayana numbers to the partial order $\preceq$ (section 3.2.2).

In chapter 4, we return to contact geometry with a thorough study of bypasses (section 4.1) and contact categories and "cobordisms" (section 4.2). In section 4.1 we describe the contact geometry of a bypass (section 4.1.1); we consider when bypasses can be found in a given manifold (section 4.1.2), including in particular our usual case of 3-balls (section 4.1.3). We explain how topologically trivial contact structures can be built out of bypasses (section 4.1.4), and consider multiple bypasses and pinwheels (section 4.1.5). In section 4.2, we introduce the notion of stackability (section 4.2.1) and establish basic properties of the stackability map m (section 4.2.2). We then consider bypass cobordisms (section 4.2.3) and give a combinatorial criterion for the existence of a chord diagram in a cobordism (section 4.2.4). We generalise to consider elementary cobordisms (section 4.2.5), before formally introducing the contact category (section 4.2.6) and its bounded variant (section 4.2.7), which we show is a partial order (section 4.2.8). Having defined all these categories, we note some functorial properties (section 4.2.9) and other categorical structures such as distinguished triangles (section 4.2.10).

After this abstract nonsense, we turn in chapter 5 to consider concrete basis chord diagrams. We show how to construct them from given words (section 5.1) — in fact, two different ways (sections 5.1.2 and 5.1.3) — how to decompose contact elements in this basis (section 5.2), and then prove the relationship between the partial order on these basis elements and stackability (section 5.3), including our general result on stackability (section 5.3.4).

With basis chord diagrams thus understood, in chapter 6 we consider bypass systems on them. Section 6.1 describes in detail the possible bypass moves on basis diagrams, giving a bypass system taking Γ_{w_0} to Γ_{w_1}, for any $w_0 \preceq w_1$; and then in section 6.2 we use these results to compute certain bounded contact categories explicitly. As noted in the introduction, we build up machinery giving correspondences between combinatorial moves on words and bypass moves on chord diagrams, of increasingly general type. After illustrating our methods with some examples (section 6.1.1), we formalise them. We first consider elementary moves on words and bypass moves on attaching arcs, and establish a correspondence between them (sections 6.1.2–6.1.4). After some general remarks about bypass systems (section 6.1.5), we consider generalised elementary moves on words, generalised attaching arcs and their bypass systems, and establish a correspondence between them (sections 6.1.6–6.1.8). Even more generally, we then consider multiple generalised attaching arcs (sections 6.1.9–6.1.11), and multiple generalised elementary moves (section 6.1.12). For any comparable pair of words, we define a sequence of generalised elementary moves (section 6.1.13) and a bypass system; and establish a correspondence between them (sections 6.1.14–6.1.15). Having completed all this, without too much more effort we may compute the bounded contact category of a basis cobordism (section 6.2.1); and with some more effort, we may compute the bounded contact category of a bypass cobordism (section 6.2.2).

Chapter 7 then turns to a study of contact elements. We complete the proof of our main theorems (section 7.1) and then give various consequences and properties of contact elements (sections 7.2–7.3). We discuss how the main theorem describes a generalised bypass triple with contact and categorical implications (section 7.2.2), and we give a categorical restatement of our main theorems (section 7.2.3). We then establish properties of contact elements such as the number of elements in a decomposition (section 7.3.1), symbolic interpretation of outermost regions (section 7.3.2), various algebraic and ordering relations within contact elements (section 7.3.3); we then remark on more general questions (section 7.3.4).

Finally, chapter 8 contains numerous additional considerations. In section 8.1 we introduce the rotation operator, and compute it, recursively (section 8.1.2) and then explicitly (section 8.1.3). In section 8.2 we give the simplicial structure on the various diagonals of the categorified Pascal's triangle, and make the categorified Pascal's triangle into a double complex. And we close with section 8.3, in which we introduce our contact 2-category and make some remarks about improving it, and extending our results beyond discs.

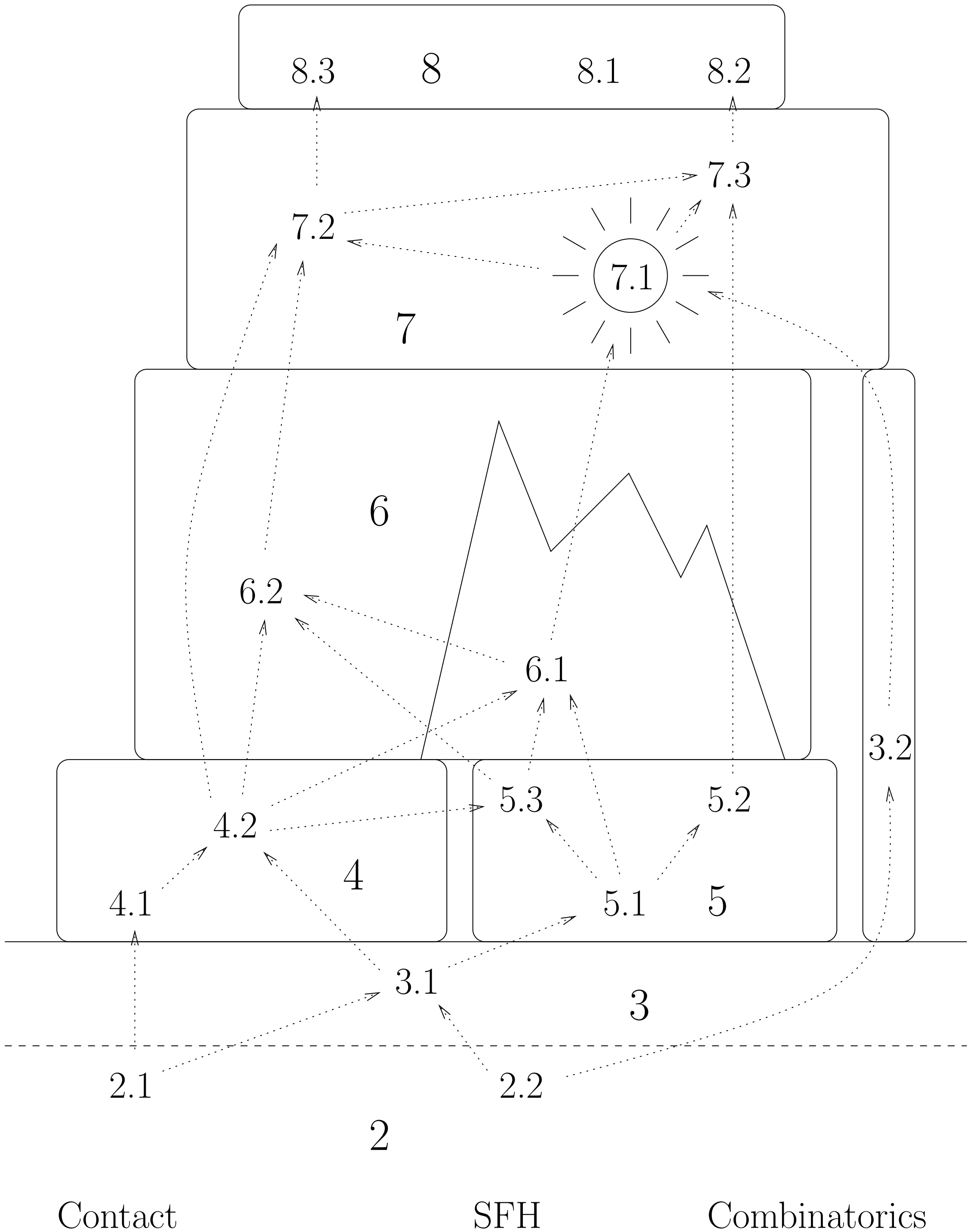


Figure 1.9: Approximate dependency and flavour of chapters and sections.

Chapter 2

Preliminaries

2.1 Contact geometry preliminaries

We recall some basic facts about 3-dimensional contact geometry. For general introductions to the subject, the reader is referred to any of [9, 26, 27, 38].

2.1.1 Fundamental facts

A *contact structure* ξ on a 3-manifold M is a totally non-integrable 2-plane field. A contact structure can always be described locally as the kernel of a 1-form α; the non-integrability condition then becomes $\alpha \wedge d\alpha \neq 0$.

A curve in M everywhere tangent to ξ is called *legendrian*. A simple closed legendrian curve C bounding a surface Σ has a *Thurston-Bennequin number* $tb(C)$, which is given by how many times ξ rotates along C, relative to the trivialisation of the tangent bundle of M along C given by Σ.

Contact structures naturally diverge into two types. *Overtwisted* contact structures are those which contain an *overtwisted disc*. An overtwisted disc is an embedded disc bounded by a legendrian curve with Thurston–Bennequin number 0. The classification of such contact structures is homotopy-theoretic [7]. A non-overtwisted contact structure is called *tight*. The classification of tight contact structures is much more subtle: see, e.g., [16, 17, 18, 22, 23].

An embedded surface Σ in a contact manifold (M, ξ) has a *characteristic foliation*, which is the singular foliation given by $T\Sigma \cap \xi$. The characteristic foliation on a surface determines the germ of the contact structure nearby [15].

2.1.2 Convex surfaces

A fundamental notion for understanding contact structures on 3-manifolds is that of *convex surfaces*: for a general reference the reader is referred to [15]. A convex surface is an embedded surface Σ in a contact manifold (M, ξ) for which there exists a contact vector field X transverse to Σ. (A contact vector field is a vector field whose flow preserves ξ.) If a convex surface has boundary, we require it to be legendrian.

Convex surfaces are *generic*. In particular, every closed embedded surface Σ in M is C^∞ close to a convex surface; if Σ has boundary, we may need to make a C^0 perturbation near the boundary.

The *dividing set* Γ of a convex surface is the subset of Σ where $X \in \xi$; we can think of this as where "ξ is vertical". The dividing set of a convex surface is a properly embedded 1-manifold. If we have a 1-form α for our contact structure, then Γ divides Σ into positive and negative regions R_+, R_- where $\alpha(X) > 0$ or $\alpha(X) < 0$ respectively. The dividing set "divides" the characteristic foliation in the sense that this foliation can be directed by a vector field which dilates an area form on R_+ and contracts it on R_-. In fact, given a dividing set Γ, *any* characteristic foliation on Σ which is divided by Γ can be taken to any other by a C^0-small isotopy of Σ in M. Thus, in some sense, the dividing set is what fundamentally describes the contact structure near the surface.

As a special case, certain curves C on a convex surface Σ can be realised as legendrian curves: this is the *legendrian realisation principle* (see [22]). Cutting Σ along $\Gamma \cup C$, we obtain several components; if each component has boundary which intersects Γ, then C can be legendrian realised.

It's easy to determine tightness near a convex surface: a convex surface $\Sigma \neq S^2$ has a tight neighbourhood if and only if its dividing set has no contractible components; and a convex S^2 has a tight neighbourhood if and only if its dividing set has a single component [15, 22].

For a 3-manifold with boundary M, we can assume the boundary ∂M is convex, and then the convexity gives M the structure of a *sutured manifold*. We can define a *sutured manifold* $(M, \mathbf{\Gamma})$ (following [36]) as a compact oriented 3-manifold with boundary M, with a set $\mathbf{\Gamma} \subset \partial M$ of disjoint annuli and tori on the boundary. The annuli in $\mathbf{\Gamma}$ have oriented core curves called *sutures*. Removing the set $\mathbf{\Gamma}$ from ∂M breaks $\partial M - \mathbf{\Gamma}$ into connected components, which are oriented so that their boundaries agree with the sutures; in particular, orientations alternate on ∂M as we cross a suture. The components of $\partial M - \mathbf{\Gamma}$ are given a sign, *positive* or *negative*, respectively as the normal vector defined by their orientation enters or exits M. The positive and negative components are denoted $R_+(\mathbf{\Gamma})$ and $R_-(\mathbf{\Gamma})$ respectively.

For a contact 3-manifold with convex boundary, a neighbourhood of the dividing set on ∂M forms a set of annuli, the curves of the dividing set form sutures, and the positive and negative regions of the convex surface can be taken as the positive and negative regions of the sutured manifold. For our purposes we may abuse notation and regard $\mathbf{\Gamma}$ as the dividing set. Hence, given a sutured manifold $(M, \mathbf{\Gamma})$, we will say that ξ is a *contact structure on the sutured manifold* $(M, \mathbf{\Gamma})$ if ξ is a contact structure such that ∂M is convex with dividing set $\mathbf{\Gamma}$ and appropriate positive and negative regions.

In the following, we will often consider two convex surfaces Σ_1, Σ_2 with dividing sets Γ_1, Γ_2 which meet along a common boundary C, forming a "corner". We require this common boundary to be legendrian. If so, the dividing sets Γ_1, Γ_2 must "interleave" along C, as shown on the left of figure 2.1. The number of intersections $|\Gamma_1 \cap C| = |\Gamma_2 \cap C|$ is precisely half $|tb(C)|$. We may round the corner to obtain a smooth surface. The dividing curves then behave as shown on the right of figure 2.1; we may think of the rule as "down and to the right; up and to the left".

2.1.3 Contact structures on balls and solid tori

The classification of tight contact structures on a ball is simple. We have already seen that on a convex boundary S^2, the contact structure in a neighbourhood is tight

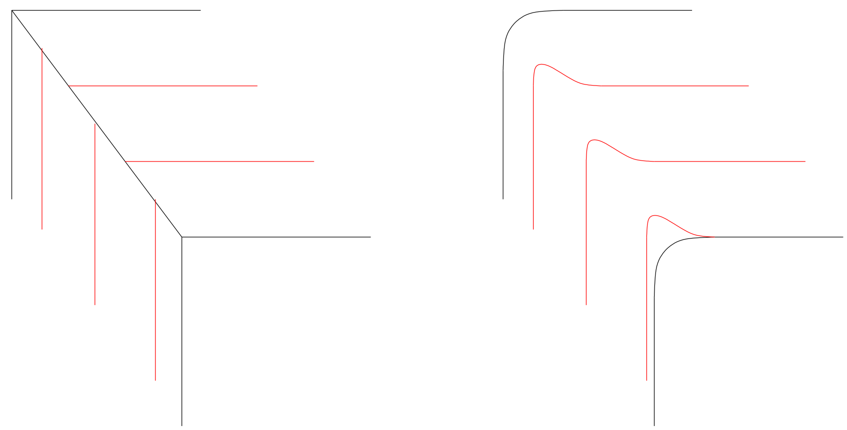


Figure 2.1: Edge rounding of convex surfaces.

if and only if the dividing set Γ is connected. It is a theorem of Eliashberg that in this case, the space of tight contact structures, fixed on the boundary sphere, is connected; hence, up to isotopy, there is a unique tight contact structure [8].

We next turn to tight contact structures ξ on (T, n), i.e. the solid torus with $2n$ longitudinal sutures. We can cut along a convex meridional disc D with legendrian boundary, and obtain a 3-ball. Then on D, the dividing set Γ_D is a properly embedded 1-manifold; if it has closed components then D has an overtwisted neighbourhood; hence Γ_D is a chord diagram. As it turns out, no matter what choice we take for our convex D, we obtain the same chord diagram: this follows from [23] or [25]; we will also prove it as proposition 4.2.11 as a corollary of our study of bypasses. (This is not the case when the sutures are not longitudes: see e.g. [22].)

Thus, the contact structure ξ determines the chord diagram Γ_D. Conversely, any chord diagram Γ_D on D determines an S^1-invariant contact structure ξ on the torus $D \times S^1 = T$; one can show this is tight (again see [23], [25] or proposition 4.2.11). Hence there is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Chord diagrams} \\ \text{with } n \text{ chords} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Tight contact structures} \\ \text{on } (T, n) \text{ up to isotopy} \end{array} \right\}.$$

Moreover, the dividing set Γ_D cuts D into regions with signs; in specifying the sutured manifold (T, n), we also specify positive and negative regions on the boundary torus;

this determines which regions of $D - \Gamma_D$ are positive and negative, as in section 1.1. The relative euler class $e(\xi)$ of ξ takes the class of the meridional disc to the number obtained by summing the euler characteristics of these regions with sign. Since the regions are all discs, we simply add the number of positive regions and subtract the number of negative regions, giving the "relative euler class" described in 1.1.

2.2 Sutured Floer homology preliminaries

2.2.1 Brief overview of SFH

The theory of sutured Floer homology was introduced by Juhász in [36]. It is an extension of Heegaard Floer homology theory, developed in [41, 42, 43, 44], for manifolds with boundary. We mention a few basic results of this theory, and refer to those papers for details and proofs.

Sutured Floer homology is an invariant of a *balanced sutured manifold*. A sutured manifold $(M, \mathbf{\Gamma})$ is *balanced* if it satisfies the following conditions: M has no closed components; $\chi(R_+(\mathbf{\Gamma})) = \chi(R_-(\mathbf{\Gamma}))$; and every boundary component of M has an annular suture. (In particular, there are no toric sutures.)

As in Heegaard Floer homology, we consider a Heegaard decomposition of our manifold. A *sutured Heegaard diagram* is a compact oriented surface with boundary Σ, with some simple closed curves α_i and β_i drawn on it. The α_i are pairwise disjoint, and the β_i are pairwise disjoint.

From a sutured Heegaard diagram, one constructs a sutured manifold by taking $\Sigma \times I$ and gluing 2-handles to $\alpha_i \times \{0\}$ and $\beta_i \times \{1\}$. This is our 3-manifold with boundary M. The sutures are then defined by the annuli $\mathbf{\Gamma} = \partial\Sigma \times I$, with oriented core curves $\partial\Sigma \times \{1/2\}$. The balanced condition means that the following conditions hold: $|\alpha| = |\beta| = d$, where α (resp. β) is the set of α_i (resp. β_i); every component of $\Sigma \backslash (\cup \alpha_i)$ contains a boundary component of Σ; and every component of $\Sigma \backslash (\cup \beta_i)$ contains a boundary component of Σ. Every balanced sutured manifold has a sutured Heegaard diagram satisfying these conditions.

Again as in Heegaard Floer homology, we consider a symmetric product $\text{Sym}^d(\Sigma)$, which is Σ^d/S_d, where S_d is the symmetric group acting by permutation of coordinates. If Σ has a complex structure, then so also does this $2d$-manifold. There are two totally real tori

$$\mathbb{T}_\alpha = (\alpha_1 \times \cdots \times \alpha_d)/S_d \quad \text{and} \quad \mathbb{T}_\beta = (\beta_1 \times \cdots \times \beta_d)/S_d \quad \text{in } \mathrm{Sym}^d(\Sigma).$$

These tori $\mathbb{T}_\alpha, \mathbb{T}_\beta$ will generally intersect in a finite set of points, corresponding to d-tuples of intersection points of α_i and β_i, one on each α_i and one on each β_i.

Given a sutured Heegaard diagram (Σ, α, β), we consider $CF(\Sigma, \alpha, \beta)$, a chain complex generated by the finite set of points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. To define a differential ∂ on this complex, we consider holomorphic curves in $\mathrm{Sym}^d(\Sigma)$ with certain boundary conditions, given below.

First, we consider a more elementary notion. A *Whitney disc* from $\mathbf{x} \in \mathbb{T}_\alpha$ to $\mathbf{y} \in \mathbb{T}_\beta$ is a map u from the unit disc $D \subset \mathbb{C}$ into $\mathrm{Sym}^d(\Sigma)$ satisfying the following boundary conditions: $u(-i) = \mathbf{x}$; $u(i) = \mathbf{y}$; u takes the "left side" of ∂D (i.e. with real part ≤ 0) into $\mathbb{T}_\alpha$; and u takes the "right side" of ∂D into $\mathbb{T}_\beta$. The set of homotopy classes of such discs is denoted $\pi_2(\mathbf{x}, \mathbf{y})$. We consider holomorphic discs which are Whitney discs.

We consider how our Whitney discs intersect the various regions of $\Sigma \backslash (\bigcup \alpha_i \cup \bigcup \beta_i)$. There is a well-defined intersection number with each such region, depending only on the homotopy class of a Whitney disc. We label the regions of $\Sigma \backslash (\bigcup \alpha_i \cup \bigcup \beta_i)$ as $D_1, \ldots, D_m$, and call linear combinations of the D_i *domains*. To measure the intersection number with a region, we take a random point z_i in each D_i; then a Whitney disc has a well-defined intersection number $n_{z_i}(u)$ with each z_i. This is the algebraic intersection number of u with $\{z_i\} \times \mathrm{Sym}^{d-1}(\Sigma)$, and it depends only on the homotopy class of u and the component D_i in which z_i lies. The *domain of* u is then $D(u) = \sum n_{z_i}(u) D_i$; since it depends only on the homotopy class of u, we may speak of $D(\phi)$, where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. When we have a Whitney disc connecting $\mathbf{x}$ to $\mathbf{y}$, its domain D has boundary which (algebraically) runs from $\mathbf{x}$ to $\mathbf{y}$ along every α_i and β_i: that is, $\partial(\partial D \cap \alpha_i) = (\mathbf{x} \cap \alpha_i) - (\mathbf{y} \cap \alpha_i)$ and $\partial(\partial D \cap \beta_i) = (\mathbf{x} \cap \beta_i) - (\mathbf{y} \cap \beta_i)$. A domain D satisfying these two equalities is called a *domain connecting* $\mathbf{x}$ *to* $\mathbf{y}$; the set of such domains is called $D(\mathbf{x}, \mathbf{y})$.

For a domain D connecting $\mathbf{x}$ to $\mathbf{y}$, we may define $\mathcal{M}(D)$ to be the moduli space of holomorphic Whitney discs with domain D. Our differential will count index-1 families of such discs. Since there is a holomorphic $\mathbb{R}$-action on D preserving $-i$ and i, we may take a quotient of an index-1 family of curves by this action and obtain $\widehat{\mathcal{M}}(D)$. Our differential ∂ on $CF(\Sigma, \alpha, \beta)$ is then defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{D \in D(\mathbf{x},\mathbf{y}) \\ \text{Index}=1}} \# \widehat{\mathcal{M}}(D) \, \mathbf{y}.$$

Our sutured Heegaard diagram is also required to be *admissible*. An admissible diagram is one for which every periodic domain has positive and negative coefficients; a domain is *periodic* if its boundary is a sum of closed curves α_i and β_i (i.e. for each α_i and each β_i, each arc is covered an equal number of times). One can show that any balanced sutured Heegaard diagram is isotopic to an admissible one. Admissibility guarantees that the set of positive domains (i.e. having all coefficients positive) connecting $\mathbf{x}$ to $\mathbf{y}$ is finite: since holomorphic discs have positive domains by positivity of intersection, this means that the above sum is finite.

One can show, using Gromov compactness, that $\partial^2 = 0$. The homology of the complex is called *sutured Floer homology* $SFH(M, \mathbf{\Gamma})$; one can show that it is invariant of the choice of admissible balanced sutured Heegaard diagram.

2.2.2 Spin-c structures

We will now briefly explain how SFH splits as a direct sum over spin-c structures.

For our purposes, again following [36], we can regard a *spin-c structure* on $(M, \mathbf{\Gamma})$ as a vector field, satisfying certain boundary conditions, up to homotopy relative to the boundary in the complement of a ball.

More precisely, we require a vector field on $(M, \mathbf{\Gamma})$ to point out of M along $R_+(\mathbf{\Gamma})$, in along $R_-(\mathbf{\Gamma})$, and along the annuli $\mathbf{\Gamma}$ to be tangent to ∂M, as the gradient of the height function $S^1 \times I \longrightarrow I$. We will say such a vector field is *compatible with* $\mathbf{\Gamma}$. Two vector fields v, w on M, compatible with $\mathbf{\Gamma}$, are said to be *homologous* if there exists an open ball B in the interior of M such that v, w are homotopic in $M - B$. A *spin-c structure* on $(M, \mathbf{\Gamma})$ is a homology class of vector fields on M compatible with $\mathbf{\Gamma}$. We call the set of such homology classes $\text{Spin}^c(M, \mathbf{\Gamma})$.

Any spin-c structure $\mathfrak{s} \in \text{Spin}^c(M, \mathbf{\Gamma})$ has a *first chern class* $c_1(\mathfrak{s})$, defined as follows. Take a vector field v representing $\mathfrak{s}$, and form a perpendicular 2-plane field $v^\perp$. Then $c_1(\mathfrak{s}) = c_1(v^\perp) \in H^2(M; \mathbb{Z})$. Note that $c_1(\mathfrak{s})$ cannot be any element of $H^2(M; \mathbb{Z})$; since v is compatible with $\mathbf{\Gamma}$, $c_1(v^\perp)$ restricts to a particular homology class in $H^2(\partial M; \mathbb{Z})$. Since c_1 only depends on the homotopy class of $v^\perp$ over a 2-skeleton, altering $v^\perp$ inside a ball has no effect; so $c_1(\mathfrak{s})$ is well defined. Indeed, we see that $\text{Spin}^c(M, \mathbf{\Gamma})$ is an affine space over $H^2(M, \partial M; \mathbb{Z})$.

Each generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of $CF(\Sigma, \alpha, \beta)$ gives a spin-c structure $\mathfrak{s}(\mathbf{x})$ as follows. The sutured Heegaard diagram gives a Morse function $f : M \longrightarrow \mathbb{R}$ for which Σ is a level set and the α, β curves are intersections of stable and unstable manifolds with this level set. Moreover, f is easily chosen so that $\text{grad}\, f$ is compatible with $\mathbf{\Gamma}$. The point $\mathbf{x}$ is a d-tuple of intersection points of α_i and β_i curves, one on each curve; and each such intersection point corresponds to a trajectory $\gamma_{\mathbf{x}}$ between an index-1 and index-2 critical point of f. Thus $\mathbf{x}$ gives a set of pairwise disjoint trajectories of $\text{grad}\, f$ connecting all the index one and index two critical points of f in pairs. We may modify $\text{grad}\, f$ on a neighbourhood of each of these trajectories to give a nowhere zero vector field v. Then $\mathfrak{s}(\mathbf{x})$ is the spin-c structure represented by v.

Given two points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we may join the corresponding trajectories as $\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}$ to obtain a collection of oriented simple closed curves in M, which we denote $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(M; \mathbb{Z})$. We can then prove that $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(M; \mathbb{Z})$ is Poincaré dual to $\mathfrak{s}(\mathbf{x}) - \mathfrak{s}(\mathbf{y})$ (since $\text{Spin}^c(M, \mathbf{\Gamma})$ is affine over $H^2(M, \partial M; \mathbb{Z})$). Note that $\epsilon(\mathbf{x}, \mathbf{y})$ can be homotoped to lie entirely on the α and β curves, and hence corresponds to some curves in $\mathbb{T}_\alpha \cup \mathbb{T}_\beta \subset \text{Sym}^d(\Sigma)$. We note that $H_1(\Sigma; \mathbb{Z}) = H_1(M; \mathbb{Z})$ under inclusion, and we may regard $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(\Sigma; \mathbb{Z})$ also.

Now, if $\mathfrak{s}(\mathbf{x}) \neq \mathfrak{s}(\mathbf{y})$ then $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0 \in H_1(\Sigma; \mathbb{Z})$; this curve is not a boundary. But any Whitney disc connecting $\mathbf{x}$ to $\mathbf{y}$ must give such a boundary; so there are no Whitney discs in this case, and in particular, no holomorphic Whitney discs.

Thus, the differential ∂ on $CF(\Sigma, \alpha, \beta)$ takes $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

having the same spin-c structure. Hence SFH splits as a sum over spin-c structures:

$$SFH(M, \mathbf{\Gamma}) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M, \mathbf{\Gamma})} SFH(M, \mathbf{\Gamma}, \mathfrak{s}).$$

2.2.3 Contact elements and TQFT

We now briefly explain how a contact structure ξ on $(M, \mathbf{\Gamma})$ gives rise to a *contact element* or *contact class* in $SFH(-M, -\mathbf{\Gamma})$. If we use $\mathbb{Z}$-coefficients, there is a (± 1) ambiguity; but with $\mathbb{Z}_2$ coefficients the contact element is a well-defined single element. Here we follow the definition of Honda–Kazez–Matić in [30], and we only consider $\mathbb{Z}_2$ coefficients. The contact class in this case is an extension of the definition of a contact class in the Heegaard Floer homology of a closed manifold, as defined in [43] and reformulated in [31].

The central concept in this definition is that of a *partial open book decomposition* $(S, R_+(\mathbf{\Gamma}), h)$ of a sutured manifold $(M, \mathbf{\Gamma})$. Here S is a surface with boundary, $R_+(\mathbf{\Gamma})$ is a subsurface of S, and h is a *partial monodromy map* $S - \overline{R_+(\mathbf{\Gamma})} \longrightarrow S$. We first consider $S \times [-1, 1]/\sim$, thickening S and taking the quotient by the relation $\sim$, which identifies all boundary points $(x, t) \sim (x, t')$ for $x \in \partial S$ and $t, t' \in [-1, 1]$: this is the "binding" of the open book.

The manifold M is then given by gluing $(x, 1)$ to $(h(x), -1)$, for $x \in S - \overline{R_+(\mathbf{\Gamma})}$. This manifold has boundary consisting of $R_+(\mathbf{\Gamma}) \times \{1\}$, which becomes $R_+(\mathbf{\Gamma})$ in the sutured manifold; also $(S - \mathrm{Im}\,(h)) \times \{-1\}$, which becomes $R_-(\mathbf{\Gamma})$ in the sutured manifold; and their boundaries $\partial(R_+(\mathbf{\Gamma})) \times \{1\}$, $\partial(S - \mathrm{Im}\,(h)) \times \{-1\}$, which have been glued together, form the sutures.

Now, we recall Giroux's theorem [19] (see also [10]) that isotopy classes of contact structures on a (closed) 3-manifold correspond precisely to open book decompositions modulo positive stabilisation. In their paper [30], Honda–Kazez–Matić extend this result to sutured manifolds and partial open books.

Thus, given a contact structure on $(M, \mathbf{\Gamma})$, we take a corresponding partial open book decomposition. The partial open book decomposition then gives rise to a sutured Heegaard splitting along a surface Σ, obtained by gluing together two "opposite pages" of the partial open book. (In this partial open book, however, two opposite pages will not usually be homeomorphic surfaces.) We take a *basis* for $(S, R_+(\mathbf{\Gamma}))$, which is a set of pairwise disjoint properly embedded arcs a_i in $S - \overline{R_+(\mathbf{\Gamma})}$, so that after cutting S along the a_i, the surface deformation retracts to $\overline{R_+(\mathbf{\Gamma})}$. From such a basis, it is possible to obtain some α and β curves on our sutured Heegaard surface, giving rise to a balanced sutured Heegaard diagram. For α and β curves obtained by their method, there is a canonical pairing between α_i and β_i curves, and there is a canonical intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ of each pair near the boundary of S.

In this construction, it turns out that $\partial\Sigma$ and $\partial(R_+(\mathbf{\Gamma}))$, which are equal as sets, have opposite orientation. Thus we end up with $\mathbf{x} \in CF(\Sigma, \beta, \alpha)$, rather than $CF(\Sigma, \alpha, \beta)$.

It is then shown that this point $\mathbf{x}$ satisfies $\partial\mathbf{x} = 0$, so we may consider $\mathbf{x}$ as an element of SFH. However, because of the orientation issue, we must take $\mathbf{x}$ in $SFH(-M, -\mathbf{\Gamma})$. For a different choice of partial open book decomposition or basis curves, this element behaves naturally under corresponding isomorphisms of SFH. Thus we may speak of *the contact element* $c(\xi) \in SFH(-M, -\mathbf{\Gamma})$.

The contact class is known to satisfy various properties, also noted in [30]: for instance, $c(\xi) = 0$ when ξ is overtwisted, or when the partial monodromy h is not "right-veering" (see [32, 34]).

In [33], Honda–Kazez–Matić proved that SFH has some of the properties of a topological quantum field theory. We give a $\mathbb{Z}_2$ version.

Theorem 2.2.1 (Honda–Kazez–Matić [33]) *Let $(M', \mathbf{\Gamma}')$ be a sutured submanifold of $(M, \mathbf{\Gamma})$ lying in Int M, and let ξ be a contact structure on $(M - Int(M'), \mathbf{\Gamma} \cup \mathbf{\Gamma}')$. Let $M - Int(M')$ have m components which are isolated, i.e. components which do not intersect ∂M. Then ξ induces a natural map*

$$\Phi_\xi : SFH(-M', -\mathbf{\Gamma}') \longrightarrow SFH(-M, -\mathbf{\Gamma}) \otimes V^m,$$

where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \widehat{HF}(S^1 \times S^2)$. This map has the property that for any contact *structure* ξ' *on* $(M', \mathbf{\Gamma}')$,

$$\Phi_\xi \left(c(\xi') \right) = c(\xi' \cup \xi) \otimes x^{\otimes m},$$

where x *is the contact class of the standard tight contact structure on* $S^1 \times S^2$.

2.2.4 Solid tori

Juhász [37] and Honda–Kazez–Matić [30, 33] have proved theorems to calculate SFH when two sutured manifolds are glued together in certain ways. One immediate corollary, given in [33], is a computation of SFH for solid tori with longitudinal sutures. We have seen in the introduction that

$$SFH(T, n) = \mathbb{Z}_2^{2^{n-1}}$$

and the direct sum over spin-c structures is

$$SFH(T, n+1) = \mathbb{Z}_2^{\binom{n}{0}} \oplus \mathbb{Z}_2^{\binom{n}{1}} \oplus \cdots \oplus \mathbb{Z}_2^{\binom{n}{n}} = \bigoplus_e SFH(T, n+1, e)$$

The relative euler class of the chord diagram, or contact structure, corresponds precisely to these summands [30, 33, 37]. Recall that contact structures on (T, n) are in bijective correspondence with chord diagrams of n chords; and the relative euler class of the contact structure is the relative euler class of the chord diagram. Thus, a chord diagram Γ with n chords and euler class e gives rise to an element of $SFH(T, n, e)$. This is theorem 1.2.2 from section 1.2.1 in our overview.

Chapter 3

First steps

3.1 First observations in SFH

3.1.1 The vacuum

Let us begin by considering the case of $(T,1)$, the solid torus with one pair of longitudinal sutures. We have $SFH(T,1) = SFH(T,1,0) = \mathbb{Z}_2$. Contact elements in $SFH(T,1)$ arise from contact structures which correspond to chord diagrams with 1 chord. There are not many of these! See figure 3.1. This chord diagram gives the unique tight contact structure on this sutured manifold; and it is a standard neighbourhood of a closed legendrian curve.

Such a contact manifold can be contact embedded in the standard contact S^3; or even in a slightly smaller sutured manifold $S^3 - B^3 = B^3$ with one suture. It is generally true that $\widehat{HF}(M) = SFH(M - B^3, \mathbf{\Gamma})$ where $\mathbf{\Gamma}$ is a single curve on the sphere. In either of these cases, the contact element for the standard contact structure

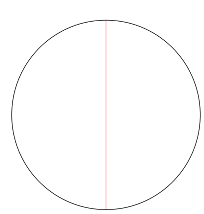


Figure 3.1: Chord diagram with 1 chord: the vacuum.

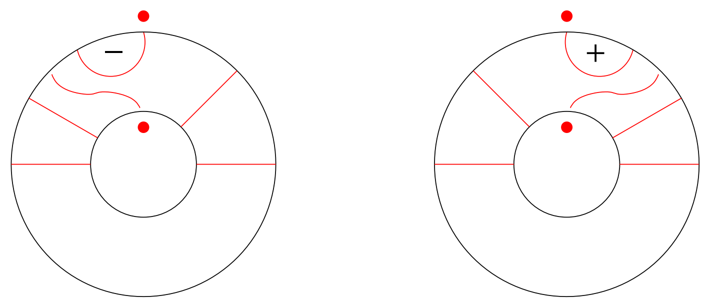


Figure 3.2: Inclusion of sutured manifolds for $B_\pm$.

is nonzero; by Stein fillability of S^3, for instance. By TQFT-inclusion, we have the following lemma.

Lemma 3.1.1 *The contact element of the unique tight contact structure on* $(T, 1)$ *is the nonzero element* $v_\emptyset \in \mathbb{Z}_2 = SFH(T, 1)$*.* ∎

We call this contact element $v_\emptyset$ *the vacuum*, in our quantum field theory interpretation. The vacuum state in quantum field theory is not zero.

3.1.2 Creation and annihilation

We have defined orientation and sign conventions on chord diagrams in section 1.1. We now define creation and annihilation operators properly.

We consider an embedding $(T, n) \hookrightarrow (T, n+1)$, together with a contact structure on $(T, n+1) - (T, n)$, in order to use TQFT-inclusion. Such an embedding is given by embedding a disc inside a larger disc, all times S^1. On the intermediate manifold $(T, n+1) - (T, n)$, which is an annulus times S^1 with $2n+2$ longitudinal sutures "on the outside" and $2n$ longitudinal sutures "on the inside", we can specify an S^1-invariant contact structure by drawing a dividing set on the annulus, which is assumed to be convex. We use the two dividing sets depicted in figure 3.2, respectively for positive and negative creation operators. Note we must mark base points; recall these are denoted by a solid red dot.

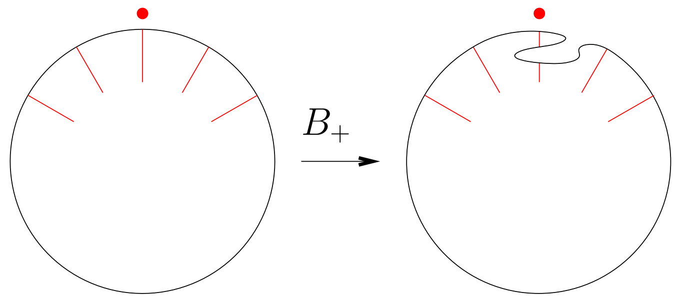


Figure 3.3: "Creating by annihilating".

Definition 3.1.2 (Creation operators) *The* creation operators *are the maps*

$$B_-, B_+ : SFH(T, n) \longrightarrow SFH(T, n+1)$$

given by TQFT-inclusion, from $(T, n) \hookrightarrow (T, n+1)$ *together with the contact structures on* $(T, n+1) - (T, n)$ *described by the dividing sets given in figure 3.2.*

Given a contact structure on (T, n), described by a chord diagram Γ of n chords, applying $B_\pm$ to its contact element gives the contact element of the contact structure described by the chord diagram with $(n + 1)$ chords, adding a chord enclosing an outermost $\pm$ region near the base point, as described in the introduction.

Moreover, if Γ has relative euler class e, then after applying $B_\pm$ to its contact element, we have the contact element arising from a chord diagram with euler class $e \pm 1$. So $B_\pm$ takes contact elements lying in $SFH(T, n, e)$ to contact elements lying in $SFH(T, n + 1, e \pm 1)$.

As an aside, note that B_+ and B_- "create" an extra chord by the "creation" of an extra piece on our solid torus. But they can also be viewed as "creating" an extra chord by "annihilating" part of the manifold as shown in figure 3.3. This is a direct proof that the associated inclusion of contact manifolds takes tight contact structures to tight contact structures.

Similarly, we can define annihilation maps corresponding to a similar inclusion of sutured manifolds $(T, n + 1) \hookrightarrow (T, n)$, with certain dividing sets specified on an annulus in $(T, n) - (T, n + 1)$.

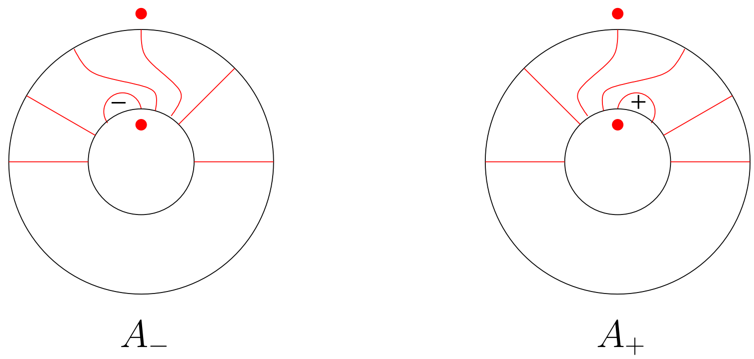


Figure 3.4: Inclusion of sutured manifolds for $A_\pm$.

Definition 3.1.3 (Annihilation operators) *The* annihilation operators *are the maps*

$$A_+, A_- : SFH(T, n+1) \longrightarrow SFH(T, n)$$

given by TQFT-inclusion, from the inclusion $(T, n+1) \hookrightarrow (T, n)$ *together with the contact structures on* $(T, n) - (T, n+1)$ *described by the dividing sets given in figure 3.4.*

It's clear that the effect of $A_\pm$ on contact elements corresponds to the effect on chord diagrams described in the introduction. And $A_\pm$ takes contact elements lying in $SFH(T, n+1, e)$ to contact elements lying in $SFH(T, n, e \pm 1)$.

Note that if A_+ is applied to a contact element arising from a chord diagram with an outermost positive region at the base point, then we obtain a contact element not arising from a chord diagram, but from a diagram with a closed loop. The corresponding contact structure is overtwisted, and the contact element is zero.

(We note parenthetically that A_+ and A_- "annihilate" a chord by the "creation" of an extra piece on our solid torus. While we can "create by creating" and "create by annihilating", we can only "annihilate by creating" — we cannot "annihilate by annihilating".)

Note that there are actually "creation" and "annihilation" operators which we can consider, not just near the base point, but at any specific location. We will refer specifically to some of these later; for now we note that they exist.

It's also now clear that the creation and annihilation effects have the relations

$$A_+ \circ B_- = A_- \circ B_+ = 1 \quad \text{and} \quad A_+ \circ B_+ = A_- \circ B_- = 0,$$

when applied to contact elements. Just place these figures of annuli together. Restricting to each summand $SFH(T, n+1, e)$, we have now proved proposition 1.2.7.

3.1.3 Nontriviality and uniqueness

We can now see that contact classes are distinct and nonzero; this argument also appeared in Honda–Kazez–Matić [33].

Lemma 3.1.4 *Any tight contact structure ξ on (T, n), corresponding to a chord diagram Γ, has nonzero contact element $c(\xi)$.*

Proof For any such contact element $c(\xi)$, corresponding to a chord diagram Γ, at least one of the annihilation operators A_+ or A_- reduces it to the contact element of a chord diagram with fewer chords (i.e. at most one of these can create a closed loop). By repeatedly applying annihilation operators in this way we may reduce the chord diagram to one chord, i.e. the vacuum $v_\emptyset \neq 0$. The composition of these annihilation operators is a linear map which takes the $c(\xi)$ to $v_\emptyset \neq 0$. Hence $c(\xi) \neq 0$. ∎

The following argument also appeared in [33]: this is proposition 1.2.3.

Proposition (Contact elements distinct) *Distinct tight contact structures (up to isotopy) on (T, n), or equivalently, distinct chord diagrams, give distinct contact elements of $SFH(T, n)$.*

Proof Let Γ_1, Γ_2 be two distinct dividing sets of n chords. There is a sequence of annihilation operators which reduces the contact element of Γ_1 to the vacuum state but which, when applied to the contact element of Γ_2, at some point creates a closed curve in the corresponding dividing set. These annihilation operators might not be applied in the positions of A_+, A_-, but may be at other positions; we have noted that there is nothing special about annihilating at the base point. The composition

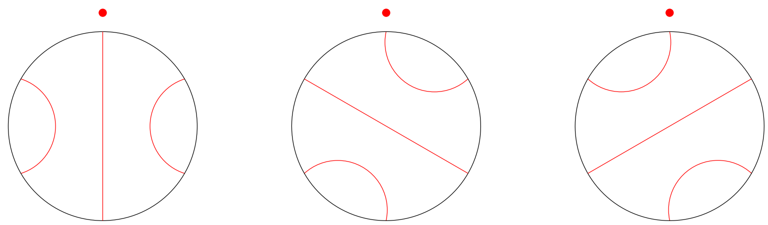


Figure 3.5: Chord diagrams in $SFH(T,3,0)$.

of these operators takes the contact element of Γ_1 to $v_\emptyset$ but takes that of Γ_2 to 0; hence they cannot be equal. ∎

This establishes a bijective correspondence:

$$\left\{ \begin{array}{c} \text{Tight contact} \\ \text{structures on } (T,n) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Chord diagrams} \\ \text{of } n \text{ chords} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Nonzero contact} \\ \text{elements in } SFH(T,n) \end{array} \right\},$$

and also on the corresponding refinements by relative euler class.

Remark 3.1.5 (Lax notation) *We will often denote by* Γ *a chord diagram, or its corresponding contact element, and drop the notation* $c(\xi)$*. The meaning should be clear.*

3.1.4 Bypasses and addition

The simplest case for which there is more than one chord diagram is 3 chords and $e = 0$. Since $SFH(T,3,0) = \mathbb{Z}_2^2$ and $C_3^0 = 3$, there are 3 chord diagrams giving 3 distinct elements of $\mathbb{Z}_2^2$: see figure 3.5.

These three dividing sets are related by bypass moves, as described in section 1.1. They form the simplest possible nontrivial bypass triple.

The 3 nonzero elements of $\mathbb{Z}_2^2$ have the property that the sum of any two of them is equal to the third; or equivalently in mod 2 arithmetic, the sum of all three is zero. Hence the 3 contact elements have the same property. Thus in this case, “bypasses do mean addition”. The result in general follows from TQFT-inclusion.

Proposition 3.1.6 *Suppose that three chord diagrams $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple. Then $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$.*

Conversely, suppose three chord diagrams $\Gamma_1, \Gamma_2, \Gamma_3$ satisfy $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. Then $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple.

Proof Suppose $\Gamma_1, \Gamma_2, \Gamma_3$ form a bypass triple in $SFH(T, n, e)$. Then they are obtained by taking the three chord diagrams of 3 chords in figure 3.5, and adding an annulus to the outside of their discs containing the same diagram. This gives an inclusion of one solid torus inside another, with a contact structure specified in the intermediate region, and hence by TQFT-inclusion we obtain a linear map

$$\mathbb{Z}_2^2 \cong SFH(T, 3, 0) \longrightarrow SFH(T, n, e).$$

Since the sum of the three contact classes in $SFH(T, 3, 0)$ is zero, after applying this linear map, the sum of $\Gamma_1, \Gamma_2, \Gamma_3$ is zero also.

For the converse: proof by induction on the number of chords n. For $n = 3$ it is clear. Note that if three chord diagrams sum to zero then they all have the same relative euler class. Furthermore, since all contact elements of chord diagrams are nonzero, if the Γ_i sum to zero then they are all distinct.

We use the following fact: given any two distinct chord diagrams with the same number of chords and relative euler class, there exists an annihilation operator, annihilating at some location (possibly not at the base point), that creates no closed curves on either. If annihilating at every position creates a closed curve on at least one of the diagrams, then the two chord diagrams consist entirely of outermost chords, enclosing all positive regions on one chord diagram, and enclosing all negative regions on the other. Thus the chord diagrams have distinct relative euler class, a contradiction.

Applying this to Γ_1 and Γ_2, we find an annihilation operator A which reduces the number of chords by 1, and such that $A\Gamma_1, A\Gamma_2$ are nonzero. If $A\Gamma_1$ and $A\Gamma_2$ are distinct then from linearity of A and $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ we find that $A\Gamma_3$ is also nonzero; and thus we have reduced to a smaller case and are done by induction. If $A\Gamma_1$ and $A\Gamma_2$ are equal, then we have the situation that an annihilation operator takes two distinct dividing sets and outputs the same dividing set. It follows that the situation

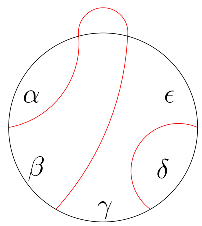


Figure 3.6: Distinct Γ_1, Γ_2 for which $A\Gamma_1 = A\Gamma_2$. Here $\alpha, \beta, \gamma, \delta, \epsilon$ denote that the two diagrams contain identical chords in these regions.

must be as in figure 3.6; and hence Γ_1, Γ_2 are related by a bypass. Then $\Gamma_3 = \Gamma_1 + \Gamma_2$ (by the first part of the proposition) is the third diagram in their bypass triple. ■

This proposition is simply a reformulation of proposition 1.2.4, which is now also proved. So, the set of contact elements does not form an additive subgroup; but the extent to which it is closed under addition precisely describes the existence of bypasses.

3.1.5 The basis

We now show that the elements v_w, for $w \in W(n_-, n_+)$, form a basis. Recall (section 1.2.3) v_w is obtained from applying $B_\pm$ to $v_\emptyset \in SFH(T, 1, 0)$ repeatedly, according to the word w. We prove proposition 1.2.11:

Proposition (QFT basis) *The set of v_w, for $w \in W(n_-, n_+)$, forms a basis for $SFH(T, n+1, e)$.*

Proof First we show the v_w are linearly independent. For this suppose that some $v_{w_1} + \cdots + v_{w_j} = 0$. Then, to this sum apply a sequence of annihilation operators which undoes the creation operators in the definition of w_1. The composition A of these operators takes e_{w_1} to the vacuum $v_\emptyset \neq 0$ and every other v_{w_i} to 0; hence

$$A(v_{w_1} + \cdots + v_{w_j}) = v_\emptyset = 0,$$

which is a contradiction.

The number of v_w is the number of $w \in W(n_-, n_+)$, which is $\binom{n}{k}$, which is the dimension of $SFH(T, n+1, e)$. Hence they form a basis. ■

We can now prove proposition 1.2.6; it only remains to prove that the B_-, B_+ are injective and form the "categorification of the Pascal recursion":

$$SFH(T, n+1, e) = B_+ SFH(T, n, e-1) \oplus B_- SFH(T, n, e+1).$$

PROOF The basis of $SFH(T, n+1, e)$ consists of elements v_w. If w begins with a $+$, $w = +w'$, then $v_w = B_+ v_{w'} \in B_+ SFH(T, n, e-1)$. If w begins with a $-$, $w = -w'$, then $v_w = B_- v_{w'} \in B_- SFH(T, n, e+1)$. This proves the recursion, and injectivity is clear. ■

Given any chord diagram / contact element, there are simple algorithms to determine its decomposition as a sum of basis elements. These will be described in detail in section 5.2. Essentially, there is either an outermost region at the base point, or there is not. If there is an outermost region, then we can factor out a $B_\pm$ and reduce to a smaller chord diagram. If there is no such outermost region, then we can perform upwards and downwards bypass moves near the base point to write our chord diagram as a sum of two other chord diagrams, each of which contains an outermost region at the base point. We can proceed in this way until we reach the vacuum; and at this point we have our decomposition. We illustrate with an example in figure 3.7.

We can now prove proposition 1.2.5.

Proposition (SFH is combinatorial) *There is an isomorphism*

$$SFH_{comb}(T, n, e) \stackrel{\cong}{\longrightarrow} SFH(T, n, e).$$

This isomorphism takes a chord diagram to the contact element of the tight contact structure on (T, n) with that chord diagram as its dividing set on a meridional disc.

PROOF Recall that $SFH_{comb}(T, n, e)$ is defined as the $\mathbb{Z}_2$-vector space freely generated by appropriate chord diagrams, say $\mathbb{Z}_2\langle V \rangle$, modulo bypass relations; let the subspace of $\mathbb{Z}_2\langle V \rangle$ generated by bypass relations be $\mathcal{B}$. There is certainly a linear vector space

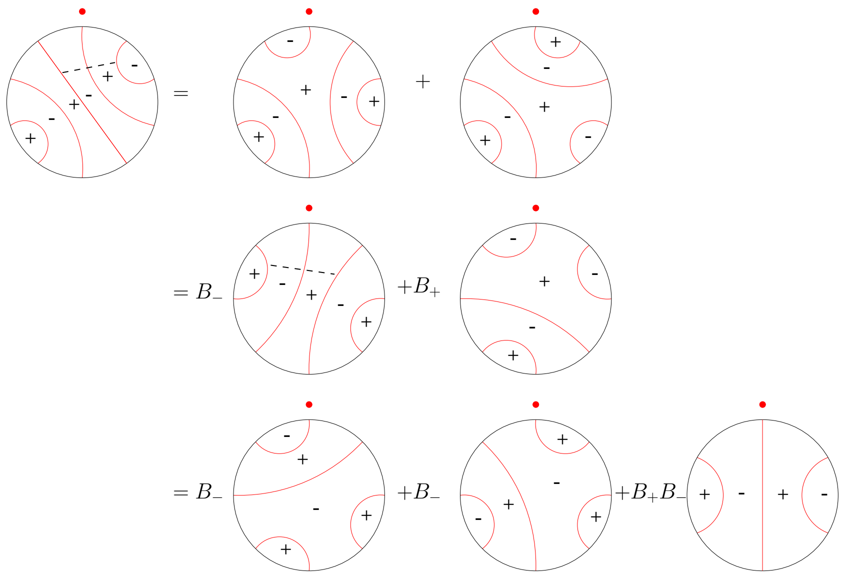


Figure 3.7: Decomposition into basis elements. From here we have $B_- B_- B_+ B_+ v_\emptyset$, $B_- B_+ B_+ B_- v_\emptyset$, $B_+ B_- B_- B_+ v_\emptyset$ and $B_+ B_- B_+ B_- v_\emptyset$, hence $v_{--++} + v_{-++-} + v_{+--+} + v_{+-+-}$.

map $\mathbb{Z}_2\langle V \rangle \longrightarrow SFH(T,n,e)$, taking chord diagrams to the corresponding contact elements in SFH. Moreover, this map takes $\mathcal{B} \mapsto \{0\}$, and so descends to a map $\phi : SFH_{comb} \longrightarrow SFH$. Since SFH is generated by chord diagrams, ϕ is surjective. Now in $SFH_{comb}(T,n,e)$, the chord diagrams Γ_w for $w \in W(n_-, n_+)$ form a spanning set: every chord diagram can be decomposed, using the bypass relation, as a sum of Γ_w. Thus the dimension of SFH_{comb} is $\leq \binom{n}{k}$, while the dimension of SFH is $\binom{n}{k}$. However ϕ is surjective, so the dimension of SFH must actually be $\binom{n}{k}$ and ϕ must be an isomorphism. ∎

3.1.6 The octahedral axiom

We now pause briefly to return to the example of $SFH(T,4,-1) = \mathbb{Z}_2^3$ discussed in section 1.1. Recall we have 6 chord diagrams, 3 basis elements v_{--+}, v_{-+-} and v_{+--}, and the 6 contact elements are

$$\begin{array}{ccc} v_{--+} & v_{-+-} & v_{+--} \\ v_{--+} + v_{-+-} & v_{--+} + v_{+--} & v_{-+-} + v_{+--}. \end{array}$$

Thus the 6 contact elements in $\mathbb{Z}_2^3$ consist of the 3 basis elements and all sums of them in pairs. We may consider the elements of $\mathbb{Z}_2^3$ as the vertices of a cube with coordinates (x, y, z), with planes through the origin corresponding to 2-dimensional subspaces and lines through the origin corresponding to 1-dimensional subspaces. We see that each 2-dimensional subspace generated by two basis elements (with equations $x = 0$, $y = 0$ and $z = 0$) contains 3 contact elements which form a bypass triple; and also the subspace $x + y + z = 0$. Thus, the 6 vertices of the cube which are contact elements contain between them 4 triangles which are bypass triples. We can thus take these 6 vertices and arrange them as an octahedron; then 4 of the 8 faces are exact. This is the arrangement which appears in the octahedral axiom of Honda [21].

One can think of every $SFH(T, n, e)$ as containing contact elements which describe some higher-order version of the octahedral axiom.

3.2 Enumerative combinatorics

In this section we collect some enumerative results that we will need later.

3.2.1 Catalan, Narayana, and merging

Recall the Catalan numbers are given recursively by $C_0 = 1$, $C_1 = 1$ and

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0.$$

We will define our (shifted) Narayana numbers $C_{n+1,k} = C_{n+1}^e$ recursively by $C_1^0 = 1$ and

$$C_{n+1}^e = C_n^{e-1} + C_n^{e+1} + \sum_{\substack{n_1+n_2=n \\ e_1+e_2=e}} C_{n_1}^e \, C_{n_2}^e$$

We will show that the number of chord diagrams with n chords satisfies the Catalan recursion, and the number of chord diagrams with n chords and relative euler class e satisfies the Narayana recursion. The initial values are clearly right.

One easy way to see this recursion is a *merging operation*. Given two chord diagrams Γ_1, Γ_2 with n_1, n_2 chords and relative euler classes e_1, e_2, we can combine

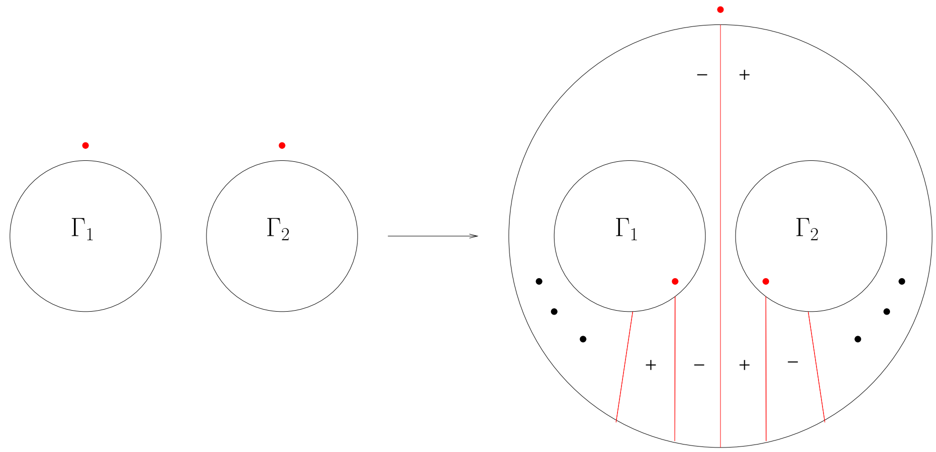


Figure 3.8: Merging operation.

them into a larger chord diagram with $n_1 + n_2 + 1$ chords, and relative euler class $e_1 + e_2$, as shown in figure 3.8.

Note the specification of base points. The merging operation is also defined when $n_1 = 0$ or $n_2 = 0$; in this case the chord diagram added on one side is null; and the operation reduces to the effect of B_+ or B_-. It's easy to see that any given chord diagram can be expressed as the merge of two smaller (possibly null) chord diagrams, and in precisely one way.

Counting the number of chord diagrams of n chords without regard to relative euler class gives the Catalan recursion, where each term $C_k C_{n-k}$ counts the number of merged chord diagrams with k chords in the left diagram and $n - k$ in the right. Doing the same thing but keeping track of relative euler class, we obtain the Narayana recursion. And it is now clear from this interpretation that

$$C_n = \sum_e C_n^e.$$

We have now proved proposition 1.2.12.

We further note that the "merging" operation precisely describes an inclusion of sutured manifolds — in this case, $(T, n_1) \sqcup (T, n_2) \to (T, n_1 + n_2 + 1)$ — together with a contact structure on the intermediate manifold $(T, n_1 + n_2 + 1) - ((T, n_1) \sqcup (T, n_2))$. Thus TQFT-inclusion applies. Note $SFH((T, n_1) \sqcup (T, n_2)) =$

$SFH(T, n_1) \otimes SFH(T, n_2)$.

Definition 3.2.1 (Merge operator) *The inclusion of sutured manifolds, and contact structure on the intermediate manifold, described by the merging of two chord diagrams as above, gives a linear map*

$$M : SFH(T, n_1) \otimes SFH(T, n_2) \longrightarrow SFH(T, n_1 + n_2 + 1)$$

which restricts to a map

$$M : SFH(T, n_1, e_1) \otimes SFH(T, n_2, e_2) \longrightarrow SFH(T, n_1 + n_2 + 1, e_1 + e_2)$$

on each summand.

When n_1 or n_2 is 0, the definition of M naturally extends as a creation operator.

Since every contact element in $SFH(T, n, e)$ lies in the image of M, applied to contact elements, and contact elements span $SFH(T, n, e)$, we have proved proposition 1.2.13.

3.2.2 Counting comparable pairs

This is proposition 1.2.14.

Proposition *The number of pairs w_0, w_1 in $W(n_-, n_+)$ with $w_0 \preceq w_1$ is C^e_{n+1}.*

In order to prove this, we give a "baseball" interpretation of the partial order $\preceq$. Given a word $w \in W(n_-, n_+)$, call the m'th symbol from the left the m'th *innings*. Call the sum of the first m symbols the *score after m innings*. Then, for $w_0, w_1 \in W(n_-, n_+)$, the relation $w_0 \preceq w_1$ means precisely that after every innings, w_1 has a score higher than (or equal to) w_0.

(Note, this is a low-scoring version of baseball: in every innings, each team scores ± 1 run. If is also a fixed version of baseball: since $w_0, w_1 \in W(n_-, n_+)$, the scores at the end of the game, after all n innings, are equal. In any case, an innings where the lead changes from one team to the other is precisely the case when the corresponding words are not comparable; comparable words are uninteresting as spectator sport, unfortunately. Two words are comparable if and only if they describe a low-scoring, fixed, and uninteresting baseball game.)

Proof (of proposition 1.2.14) First, there is a bijection between pairs of comparable words of length n with k plus signs, and monotone increasing functions

$$f : \{1, \ldots, n+1\} \longrightarrow \{1, \ldots, n+1\},$$

i.e. $f : [n+1] \longrightarrow [n+1]$, satisfying $f(i) \leq i$ for all i and taking $k+1$ distinct values. The bijection is described as follows. Given a pair of comparable words $w_0 \preceq w_1$, we know that for all j, the j'th + sign in w_0 is to the right of the j'th + sign in w_1. Insert a + at the start of w_0 and w_1 to obtain w_0', w_1', so these are now words of length $n+1$ with $k+1$ plus signs. For $i \in \{1, \ldots, n+1\}$, let the number of + signs up to and including the i'th symbol of w_0 be $j(i)$; then define $f(i)$ to be the position of the $j(i)$'th + sign in w_1.

Conversely, given such a function, we can easily reconstruct the words w_0, w_1. The positions of the + signs in w_1' are precisely the values of f. And the positions of the + signs in w_0' are precisely those i for which $f(i)$ jumps, $f(i) > f(i-1)$.

The number of such functions $f : [n+1] \longrightarrow [n+1]$ with $k+1$ distinct values is well known to be $N_{n+1,k+1} = C_{n+1,k} = C^e_{n+1}$.

To see this, we can show that $F_{n,k}$, the number of increasing $f : [n] \longrightarrow [n]$ with $f(i) \leq i$ taking k values, satisfies the Narayana recursion; clearly $F_{n,k} = N_{n,k}$ for small values. Clearly any such function has the fixed point $f(1) = 1$. The number with no other fixed points is $F_{n-1,k}$. The number with a fixed point $f(2) = 2$ is $F_{n-1,k-1}$. Otherwise let j be the least fixed point ≥ 2. We can then "break the function into two" at that fixed point, and the number of such functions is given by $F_{j-2,k_1} F_{n-j+1,k_2}$ over the possible k_1, k_2 where $k_1 + k_2 = k$. ∎

Chapter 4

Contact considerations: Combinatorial, categorical, cobordisms

4.1 On Bypasses

Having covered some contact geometry preliminaries in section 2.1, we now move on to consider bypasses in some detail. Bypasses can be considered as the "smallest building blocks" of contact structures; the contact structures we consider can be constructed entirely out of bypasses. On the other hand, a bypass is "half an overtwisted disc".

Hence, the smallest step in contact geometry is half way to oblivion. Such is the precariousness of all tight contact life.

4.1.1 The bypass manifold

Our bypass moves come from actual contact geometric objects called *bypasses*. A bypass is essentially half an overtwisted disc. Recall that an overtwisted disc is a disc bounded by a legendrian curve with Thurston-Bennequin number 0. A convex overtwisted disc D has dividing set consisting of a single closed loop. A thickened convex overtwisted disc $D \times I$ has dividing set on its boundary consisting of 3 closed

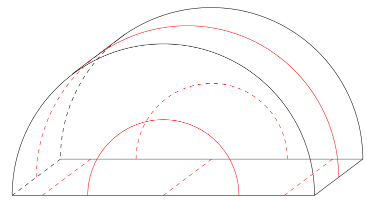


Figure 4.1: Bypass with convex boundary.

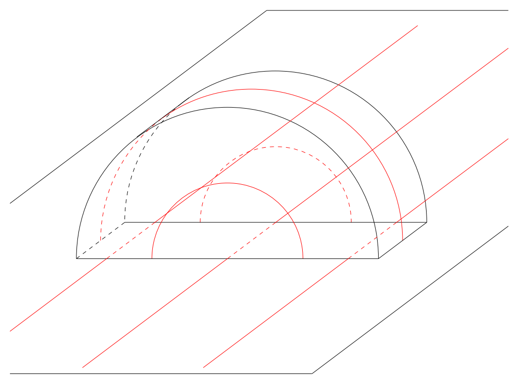


Figure 4.2: Attaching a bypass.

loops: one on each of $D \times \{0\}$, $D \times \{1\}$, and another loop in $\partial D \times I$. A bypass is then half of this object, where we slice through a diameter of D, times I; we consider the "sliced" part of the bypass to be the "base". On this slice, the dividing set consists of three arcs of the form $\{\cdot\} \times I$; see figure 4.1.

Since the contact structure near a surface is (up to C^0 isotopy) determined by the dividing set, we may attach a bypass to any convex surface along an attaching arc, above (resp. below): see figure 4.2. Rounding (figure 4.3) and flattening (figure 4.4 then gives the surgery on dividing sets we have called an upwards (resp. downwards) bypass move.

Clearly, then, adding two bypasses, above and below a convex surface, along the same attaching arc, attaches an overtwisted disc — hence gives an overtwisted contact structure.

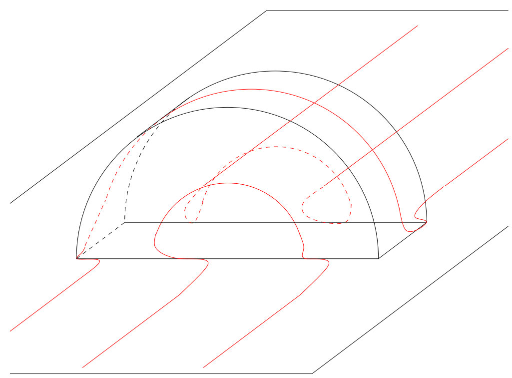


Figure 4.3: Rounding a bypass attachment.

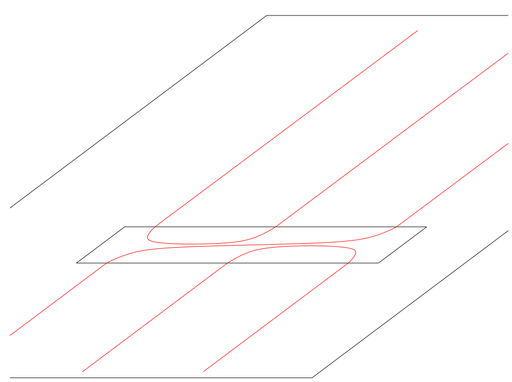


Figure 4.4: Effect of bypass attachment.

4.1.2 When do bypasses exist?

In the following, we often consider the question of whether a bypass exists in a given contact manifold. More precisely, suppose we have a contact 3-manifold (M, ξ) with convex boundary, and we would like to know whether there exists a bypass inside M along an attaching arc c on ∂M. There are some situations in which there is an easy answer to this question: and this will be enough for our purposes.

(i) If ξ is tight, and performing the bypass attachment along c, inwards into M, would result in a convex surface with an overtwisted neighbourhood (easily detected by looking at the dividing set, see section 2.1), then no such bypass exists.

(ii) If performing the bypass attachment along c, inwards into M, would result in a convex surface with dividing set isotopic to the original ∂M, then the bypass exists. This principle — "trivial bypasses always exist" — has been mischievously named the "right to life" principle by Honda–Kazez–Matić [25, 28].

(iii) If a bypass exists in a certain location, then a bypass necessarily exists at other locations too: this principle is called "bypass rotation" in [29]. The idea behind bypass rotation is that, after attaching a bypass along one attaching arc, other arcs of attachment may become trivial, and hence bypasses along them exist by the right-to-life principle. In particular, consider figure 4.5 below: if there exists a bypass above the (solid) arc in the right of the figure, then there also exists a bypass along the (dotted) arc in the left of the figure. The slogan for an upwards bypass is: "bypasses to my left are redundant after me". Conversely, for downwards bypasses, "bypasses to my right are redundant after me".

4.1.3 Bypasses on a tight 3-ball

We will be interested in bypasses along an attaching arc on a tight 3-ball with convex boundary. We can ask two questions:

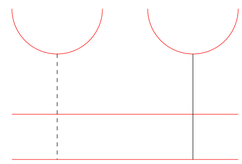


Figure 4.5: Bypass rotation.

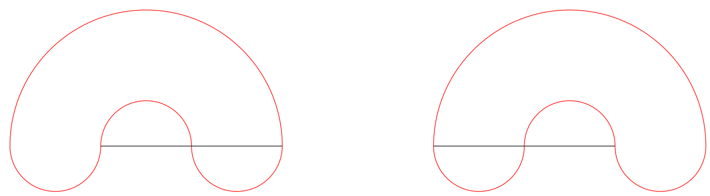


Figure 4.6: Possible attaching arcs on a tight ∂B^3. We view this as looking at ∂B^3 from the outside.

(i) After adding a bypass to the outside of the ball, does it remain tight?

(ii) (As in the previous subsection.) Does a bypass exist along our attaching arc, inside the manifold?

Topologically, an arc of attachment may be arranged in only two ways on a tight S^2 boundary: see figure 4.6.

In the first case, the answer to the first question is "yes" and the second question "no". To see this, we note that adding a bypass to the outside of the ball has no change on the effect of the topology of the dividing set; it is still connected. Moreover, the contact structure on the enlarged ball, obtained by adding the bypass to the pre-existing tight contact structure on the ball, is again tight — this follows from the right-to-life principle. The second answer is "no", since if such a bypass existed, removing it would lead to a disconnected dividing set, contradicting the tightness of the contact structure.

In the second case we obtain precisely the opposite answers, and for similar reasons: "no" and "yes" respectively.

In the first case, we call the arc of attachment *outer*, and in the second case *inner*.

Since a solid cylinder is a 3-ball, the above applies to any attaching arc on a tight contact $D \times I$.

4.1.4 Bypasses are building blocks

We now examine how bypasses are "elementary building blocks" for contact structures. In particular, we show how a tight solid cylinder can be constructed out of bypasses. The proof is in essence a version of Honda's *imbalance principle* (see [22]), although there are complications arising from the corners and boundary.

Lemma 4.1.1 (Cobordisms constructed out of bypasses) *Suppose that on the cylinder $D \times I$ there is a tight contact structure ξ with dividing sets Γ_0, Γ_1 on $D \times \{0\}$, $D \times \{1\}$ and with vertical dividing set along $\partial D \times I$. Then $(D \times I, \xi)$ is contactomorphic to the thickened convex surface $D \times \{0\}$ with some finite set of bypass attachments.*

PROOF Obviously Γ_0, Γ_1 must have the same number n of chords. Our proof is by induction on n. For $n = 1$ or 2 the tightness of $D \times I$ implies that $\Gamma_0 = \Gamma_1$; if $\Gamma_0 \neq \Gamma_1$ then it is easy to see after edge rounding that we have an overtwisted 3-ball.

Suppose now that Γ_0, Γ_1 have a common outermost chord γ enclosing an outermost region R. Then we consider another arc δ in D running close and parallel to γ, enclosing it and the outermost region R. We can legendrian realise δ and (possibly after perturbing) consider the convex surface $\delta \times I$. After some edge rounding, we can make $\partial(\delta \times I)$ a legendrian curve, which intersects the dividing set on $\partial(D \times I)$ in two points. Since the contact structure is tight, there is only one possible dividing set on $\delta \times I$. Indeed, cutting along $\delta \times I$ we obtain two solid cylinders, both of which must be tight, and both of which (after re-sharpening corners) have vertical sutures on $\partial D \times I$. One of these contains the same dividing set γ on the top and bottom, and hence is contactomorphic to an I-invariant contact structure. The other cylinder has dividing sets on both ends with $n - 1$ chords; hence by induction is obtained by attaching bypasses to the base. Thus the original cylinder is constructed by bypass attachments.

Now suppose that Γ_1 has an outermost chord γ which does not occur in Γ_0. Let its endpoints, labelled clockwise, be p and q, and let the next marked point clockwise on D be r. Then on Γ_0, there is no outermost chord joining p and q (by assumption), nor joining q and r (which after edge rounding would give a closed component along with γ of the dividing set). Thus the two chord diagrams must appear as shown in

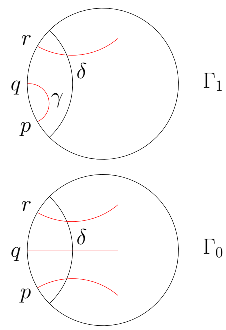


Figure 4.7: Arc δ on Γ_0, Γ_1.

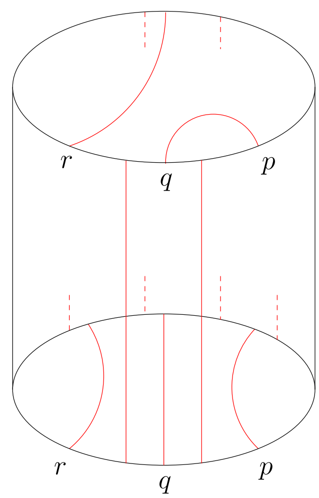


Figure 4.8: Dividing set on cylinder obtained by cutting along $\delta \times I$. Here $\delta \times I$ forms the back of the picture; its dividing set is to be determined but its boundary points are shown.

figure 4.7, and we may take an arc δ on D as shown, which intersects Γ_0 in 3 points and Γ_1 in 1 point.

After perturbing if necessary, we consider a convex $\delta \times I$; we will determine the dividing set on $\delta \times I$. By the interleaving property of dividing sets, the dividing set on $\delta \times I$ has six boundary points (3 from Γ_0, 1 from Γ_1, and 2 from the vertical sutures after rounding), hence contains 3 arcs. Cutting the cylinder along $\delta \times I$ gives two smaller cylinders. The smaller cylinder containing $\{p, q, r\} \times I$ has boundary dividing set as shown in figure 4.8. Since this is tight, we see there is only one possible dividing set on $\delta \times I$, as shown in figure 4.9.

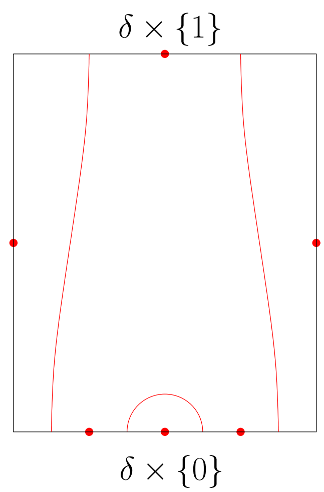


Figure 4.9: Dividing set on $\delta \times I$. The red dots show interleaving intersections with the dividing set $\partial(D \times I)$.

Hence there is a bypass above Γ_0 along a sub-arc of δ. Attaching this bypass to Γ_0 gives a dividing set with an outermost chord in the same position as Γ_1. Hence, after removing a layer containing this bypass attachment, we have reduced to the previous case and we are done. ■

4.1.5 Bypass systems and pinwheels

We now increase the level of difficulty and consider attaching several bypasses to a surface S, along a bypass system (recall definition 1.2.17).

Obviously, as in the previous section, we are interested in taking a convex disc and adding bypasses above a bypass system. The question of when the resulting contact manifold is tight has been completely answered by Honda–Kazez–Matić [29]; the same paper has more general results also. The key indicator is an object known as a *pinwheel*.

Definition 4.1.2 (Pinwheel) *An (upwards)* pinwheel *is an embedded polygonal region P on a convex surface satisfying the following conditions.*

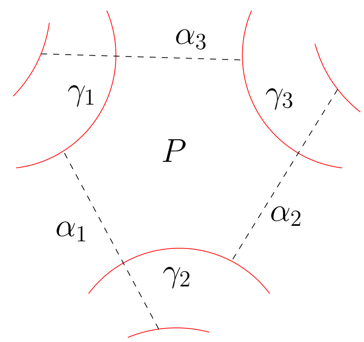


Figure 4.10: An (upwards) pinwheel.

(i) The boundary of P consists of $2k$ $(k \geq 1)$ consecutive sides

$$\gamma_1, \alpha_1, \gamma_2, \alpha_2, \ldots, \gamma_k, \alpha_k,$$

labelled anticlockwise, where γ_i is an arc on a chord of the dividing set Γ, and α_i is half of an arc of attachment c_i.

(ii) For each i, c_i extends beyond α_i in the direction shown in figure 4.10, and does not again intersect P.

It's clear enough that attaching bypasses above a surface along the attaching arcs of a pinwheel results in an overtwisted contact structure. The result of Honda–Kazez–Matić is that the converse is true as well: if there is no pinwheel, then the result is tight.

Theorem 4.1.3 (Honda–Kazez–Matić [29]) *Let D be a convex disc with legendrian boundary and c a bypass system. The contact manifold obtained by attaching bypasses above a standard neighbourhood of D along c is tight if and only if there are no pinwheels in D.* ∎

A similar result obviously holds for attaching bypasses downwards along a convex disc; the orientation of the pinwheels is reversed. Hence we speak of *upwards pinwheels* and *downwards pinwheels*.

4.2 Contact cobordisms

The precariousness of tight contact life, with every bypass move half way to the oblivion of overtwistedness, leads to interesting combinatorial and categorical relationships. However, not all of these are (yet) as nice as one might hope.

Although this section purports to be about "contact cobordisms", the only "cobordisms" we consider are manifolds $D \times I$, which we consider as "cobordisms" between $D \times \{0\}$ and $D \times \{1\}$. Moreover, we consider only vertical sutures $F \times I \subset \partial D \times I$, where $F \subset \partial D$ is finite. This situation is difficult enough for our purposes; and clearly some of the following applies in far greater generality.

4.2.1 Stackability

We now formalise a construction we have already seen in the foregoing.

Suppose we have two chord diagrams Γ_0, Γ_1, both with the same number of chords n, and with marked base points. Then we consider the cylinder $D \times I$, where D is a disc and $I = [0, 1]$. Its boundary is $(D \times \{0\}) \cup (\partial D \times I) \cup (D \times \{1\})$. We now draw some sutures on this boundary. We draw the chord diagram Γ_0 on $D \times \{0\}$, and Γ_1 on $D \times \{1\}$. We do this so that the marked points are aligned at points $\{p_i\} \times \{0, 1\}$, and the base points are aligned at points $\{p_0\} \times \{0, 1\}$. We then choose $2n$ points $\{q_i\}$ on ∂D, evenly spaced between successive p_i; and we draw the $2n$ curves $\{q_i\} \times [0, 1]$ on $\partial D \times I$.

We think of the $[0, 1]$ factor as giving the "vertical" direction: the positive direction is up and the negative direction is down.

Thus $D \times I$ can now be considered as a sutured 3-ball (with corners). It can also be considered as a 3-ball with corners and a contact structure specified near the boundary. The "corners" $\partial D \times \{0, 1\}$ can be made Legendrian, and the two surfaces intersecting along these corners have interleaving dividing sets as required. Hence we may round the corners and obtain a 3-ball B, with dividing set still denoted Γ in abusive notation. If Γ on this rounded ball is connected, then we can take the unique tight contact structure (up to isotopy rel boundary) on B with boundary dividing set Γ. If Γ is disconnected, then any contact structure on the ball with this boundary

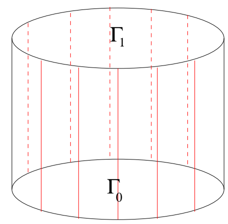


Figure 4.11: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

dividing set is overtwisted. Such contact structures can be considered equally as structures on a rounded ball, or the cylinder $D \times I$.

The manifold so obtained from Γ_0, Γ_1 is really a sutured manifold, but we can call it "tight" or "overtwisted".

Definition 4.2.1 ($\mathcal{M}(\Gamma_0, \Gamma_1)$) *Given two chord diagrams Γ_0, Γ_1 with n chords, the sutured manifold $\mathcal{M}(\Gamma_0, \Gamma_1)$ is $D \times I$ with sutures*

$$(\Gamma_0 \times \{0\}) \cup \left(\left(\bigcup \{q_i\} \right) \times I \right) \cup (\Gamma_1 \times \{1\}) .$$

See figure 4.11.

Definition 4.2.2 (Tight/overtwisted $\mathcal{M}(\Gamma_0, \Gamma_1)$) *We say $\mathcal{M}(\Gamma_0, \Gamma_1)$ is* tight *if it admits a tight contact structure, i.e. if after rounding corners, the sutures of $\mathcal{M}(\Gamma_0, \Gamma_1)$ are connected. Otherwise we say $\mathcal{M}(\Gamma_0, \Gamma_1)$ is* overtwisted.

Definition 4.2.3 (Stackable) *We say that a chord diagram Γ_1 is* stackable *on another chord diagram Γ_0 if $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight.*

We can now define a map m from the TQFT-property of SFH. Suppose we take the boundary of $\mathcal{M}(\Gamma_0, \Gamma_1)$, which after rounding we consider an S^2, and remove a small neighbourhood of a point on one of the vertical curves running between Γ_0 and Γ_1, to obtain a (large) disc with sutures / dividing set. Then, taking a product of this disc with S^1, with S^1-invariant contact structure from the dividing sets, we can regard this construction as arising from an inclusion of two solid tori (each with $2n$

longitudinal boundary sutures) into a single solid torus (with 2 longitudinal boundary sutures), i.e.

$$(T, n) \cup (T, n) \hookrightarrow (T, 1),$$

and with a specified contact structure on the intermediate (pants) $\times S^1$. Hence there is a map

$$m : SFH(T, n) \otimes SFH(T, n) \longrightarrow SFH(T, 1) = \mathbb{Z}_2.$$

(Note that on our $\partial B^3 = S^2$, the two discs with chord diagrams Γ_0, Γ_1 are oriented; one agrees with the orientation of S^2, the other does not. Thus, one of the $SFH(T, n)$ factors above should have reversed orientation. This is in addition to the orientation issues discussed in section 1.2.1! Nonetheless, by precomposing by the a linear map on the orientation-reversed $SFH(T, n)$ factor, such a map m still exists.)

Two chord diagrams Γ_0, Γ_1 give contact elements in $SFH(T, n)$ and hence give a contact element in $SFH(T, 1)$. This is an overtwisted contact structure, in the case that $\mathcal{M}(\Gamma_0, \Gamma_1)$ has disconnected sutures, and hence gives contact element 0. Otherwise, it is the unique tight contact structure on $(T, 1)$, in the case that $\mathcal{M}(\Gamma_0, \Gamma_1)$ has connected sutures, i.e. is tight, and then gives the contact element 1. Clearly this restricts to maps on various summands $SFH(T, n, e)$.

That is, $m(\Gamma_0, \Gamma_1) \in \mathbb{Z}_2$ is 0 or 1, respectively as $\mathcal{M}(\Gamma_0, \Gamma_1)$ is overtwisted or tight. In fact, the map m could also be defined purely combinatorially with this as the definition, using the combinatorial version of SFH. We have therefore proved proposition 1.3.1:

Proposition (Stackability map) *There is a linear map*

$$m : SFH(T, n) \otimes SFH(T, n) \longrightarrow \mathbb{Z}_2$$

which takes pairs of contact elements, corresponding to pairs of chord diagrams Γ_0, Γ_1, to 0 or 1 respectively as $\mathcal{M}(\Gamma_0, \Gamma_1)$ is overtwisted or tight. ■

Obviously m also restricts to summands to give maps

$$m : SFH(T, n, e_1) \otimes SFH(T, n, e_2) \longrightarrow \mathbb{Z}_2.$$

This function is easily calculated, simply by rounding the dividing set on $\mathcal{M}$ and counting components. We next give some properties of m.

4.2.2 First properties of $\mathcal{M}$ and m

Lemma 4.2.4 ($\mathcal{M}(\Gamma, \Gamma)$ tight; m "positive definite") *For any chord diagram Γ, $\mathcal{M}(\Gamma, \Gamma)$ is tight. That is, $m(\Gamma, \Gamma) = 1$.*

We give two proofs.

Proof (# 1) Since a chord diagram has no closed curves, there is a tight I-invariant contact structure on $D^2 \times I$ with convex boundary and dividing set identical to the sutures of $\mathcal{M}(\Gamma, \Gamma)$. ∎

We can alternatively prove the result by looking at the sutures on $\mathcal{M}(\Gamma, \Gamma)$ in more detail, and showing explicitly that after rounding, they are connected. This proof may be less elegant, but it is similar in spirit to some subsequent proofs (e.g. section 5.3.3 proving proposition 1.3.3), and leads to a useful lemma.

Proof (# 2) Proof by induction on $|\Gamma|$, the number of chords in the chord diagram. For $|\Gamma| = 1$, there is only one chord diagram possible on the disc; and the dividing set on $\partial \mathcal{M}(\Gamma, \Gamma)$, after rounding corners, is obviously connected.

Now consider a general Γ. Let γ be an outermost chord of Γ. Thus we may consider $\Gamma - \gamma$ to be a chord diagram with $|\Gamma| - 1$ components. We reduce the case of $\mathcal{M}(\Gamma, \Gamma)$ to the case of $M(\Gamma - \gamma, \Gamma - \gamma)$, proving the result by induction.

Note that $\gamma \times \{1\}$ has two endpoints; and after rounding corners, one of these is connected to an endpoint of $\gamma \times \{0\}$. Thus on the rounded ball we have a connected arc c, part of the dividing set, of the form $c = c_1 \cup (\gamma \times \{1\}) \cup c_2 \cup (\gamma \times \{0\}) \cup c_3$, where the c_i are (rounded versions of) arcs $q_i \times [0, 1]$ of $\mathcal{M}(\Gamma, \Gamma)$. But now folding corners in a slightly different way, we can perform a "finger move" on the dividing set and see that this is equivalent to the dividing set of $\mathcal{M}(\Gamma - \gamma, \Gamma - \gamma)$; where all of c becomes one of the $q_i \times [0, 1]$ arcs. See figure 4.12. ∎

The argument of this proof is useful in its own right, so we record it.

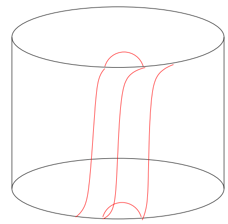


Figure 4.12: $\mathcal{M}(\Gamma, \Gamma)$ with some edge-rounding.

Lemma 4.2.5 (Cancelling outermost chords) *Suppose each of Γ_0 and Γ_1 has an outermost chord γ_0, γ_1 in the same position. Then*

$$\mathcal{M}(\Gamma_0, \Gamma_1) \quad \textit{is tight iff} \quad \mathcal{M}(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1) \quad \textit{is tight.}$$

That is, $m(\Gamma_0, \Gamma_1) = m(\Gamma_0 - \gamma_0, \Gamma_1 - \gamma_1)$. ■

We prove now that m is trivial when Γ_0, Γ_1 have distinct euler classes. We can therefore say that the splitting

$$SFH(T, n) = \bigoplus_e SFH(T, n, e)$$

is orthogonal with respect to the bilinear form m. This is proposition 1.3.2:

Proposition (Relative euler class orthogonality) *Let Γ_0 and Γ_1 be chord diagrams with n chords. If Γ_0, Γ_1 have distinct relative euler class then $m(\Gamma_0, \Gamma_1) = 0$.*

Proof Suppose $m(\Gamma_0, \Gamma_1) = 1$, i.e. $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight, and after rounding corners, the suture on the boundary S^2 of the rounded cobordism has a single component. This dividing set cuts S^2 into two discs D_+ and D_-, one positive and one negative. The orientation of S^2 as the boundary of $\mathcal{M}(\Gamma, \Gamma)$ agrees with the orientation on Γ_1 and disagrees with the orientation on Γ_0.

We can take a polygonal decomposition of S^2 where each face lies in D_+ or D_-; we can take one with $4n$ vertices (coming from each of the intersection points of Γ_i with ∂D), with $8n$ edges (n from Γ_0, n from Γ_1, $2n$ from $\partial D \times \{0\}$, $2n$ from $\partial D \times \{1\}$, and $2n$ from $\partial D \times I$), and with $4n+2$ faces ($n+1$ from $D \times \{0\}$, $n+1$ from $D \times \{1\}$, and $2n$

from $\partial D \times I$). This polygonal decomposition restricts to a polygonal decomposition of D_+ and D_-, with precisely the same vertices, and the same number of edges. The decompositions of D_+, D_- both have $4n$ vertices and $6n$ edges (n from Γ_0, n from Γ_1, n from $\partial D \times \{0\}$, n from $\partial D \times \{1\}$, and $2n$ from $\partial D \times (0,1)$). Let $f_{i,\pm}$ be the number of positive or negative faces in Γ_i. Then

$$0 = \chi(D_+) - \chi(D_-) = f_{1,+} + f_{0,-} - f_{1,-} - f_{0,+} = e(\Gamma_1) - e(\Gamma_0)$$

as required. ■

In fact, the argument of the above proof shows that, restricting to summands, m becomes

$$m : SFH(T, n, e_1) \otimes SFH(T, n, e_2) \longrightarrow SFH(T, 1, e_1 - e_2)$$

and $SFH(T, 1, e_1 - e_2)$ is nontrivial only when $e_1 = e_2$.

We now see m is bilinear, satisfies the "positive definite" condition $m(\Gamma, \Gamma) = 1$, and satisfies a natural orthogonality relation. In this way it behaves something like a metric. However, m is not symmetric; nor is m antisymmetric; and in fact, for a pair of chord diagrams (Γ_0, Γ_1), even with the same euler class, any one of the pairs

$$(m(\Gamma_0, \Gamma_1), m(\Gamma_1, \Gamma_0)) = (0,0),\ (0,1),\ (1,1)$$

is possible. The pair $(1,1)$ is possible even when $\Gamma_0 \neq \Gamma_1$.

For example, taking the three pairs shown in figure 4.13 for (Γ_0, Γ_1) give these three pairs for $(m(\Gamma_0, \Gamma_1), m(\Gamma_1, \Gamma_0))$.

4.2.3 Bypass cobordisms and bypass triples

We now consider bypasses, as in section 4.1, in the context of cobordisms $\mathcal{M}(\Gamma_0, \Gamma_1)$.

The simplest nontrivial contact cobordism is a *bypass cobordism* obtained by taking a disc D, with dividing set Γ, and attaching a bypass along an arc c. We have seen (lemma 4.1.1) that every tight contact cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ can be decomposed

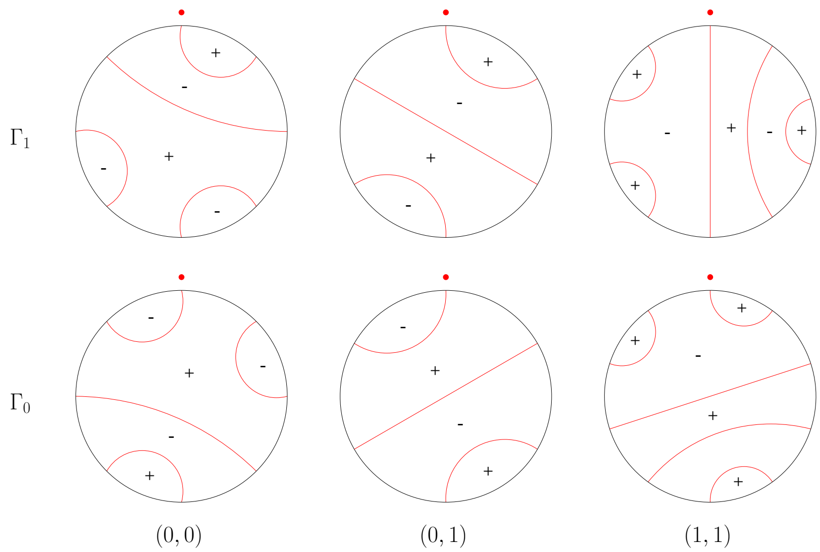


Figure 4.13: Pairs (Γ_0, Γ_1) giving various values for $(m(\Gamma_0, \Gamma_1), m(\Gamma_1, \Gamma_0))$.

into bypass cobordisms.

As long as we attach the bypass along a nontrivial attaching arc c (i.e. c intersects three different components of Γ), there can be no pinwheel, and hence we obtain a tight contact structure on $D \times I$, where there is dividing set Γ on $D \times \{0\}$, $\text{Up}_c(\Gamma)$ on $D \times \{1\}$, and vertical dividing curves on $\partial D \times I$. That is, $\mathcal{M}(\Gamma, \text{Up}_c(\Gamma))$ is tight.

More generally, as we have seen (definition 1.1.7), bypass-related chord diagrams naturally come in triples $\Gamma, \text{Up}_c(\Gamma)$ and $\text{Down}_c(\Gamma)$. Their tightness or overtwistedness is given by the following lemma.

Lemma 4.2.6 *Let c be a nontrivial arc of attachment on a chord diagram Γ. Then*

(i) all of $\mathcal{M}(\Gamma, \text{Up}_c(\Gamma))$, $\mathcal{M}(\text{Up}_c(\Gamma), \text{Down}_c(\Gamma))$ and $\mathcal{M}(\text{Down}_c(\Gamma), \Gamma)$ are tight, with tight contact structure given by a single bypass attachment,

$$m(\Gamma, \text{Up}_c(\Gamma)) = m(\text{Up}_c(\Gamma), \text{Down}_c(\Gamma)) = m(\text{Down}_c(\Gamma), \Gamma) = 1;$$

(ii) all of $\mathcal{M}(\Gamma, \text{Down}_c(\Gamma))$, $\mathcal{M}(\text{Down}_c(\Gamma), \text{Up}_c(\Gamma))$, $\mathcal{M}(\text{Up}_c(\Gamma), \Gamma)$ are overtwisted,

$$m(\Gamma, \text{Down}_c(\Gamma)) = m(\text{Down}_c(\Gamma), \text{Up}_c(\Gamma)) = m(\text{Up}_c(\Gamma), \Gamma) = 0.$$

PROOF We just saw that $\mathcal{M}(\Gamma, \text{Up}_c(\Gamma))$ is tight with contact structure given by a single bypass attachment. Recalling figure 1.5 and the symmetry between the chord diagrams of a bypass triple, part (i) follows; and part (ii) follows from showing $\mathcal{M}(\Gamma, \text{Down}_c(\Gamma))$ is overtwisted.

If Γ has only 3 chords, then there is only one possible arrangement (up to rotation of the cylinder) for $\mathcal{M}(\Gamma, \text{Down}_c(\Gamma))$; the result is then true by inspection. If there are more than three chords, then we see that there must be some outermost chord γ which is disjoint from c; hence it appears in all of Γ, $\text{Up}_c(\Gamma)$ and $\text{Down}_c(\Gamma)$. Applying lemma 4.2.5, our manifold is tight if and only if the same is true for $\Gamma - \gamma$, $\text{Up}_c(\Gamma) - \gamma$ and $\text{Down}_c(\Gamma) - \gamma$. Repeatedly applying lemma 4.2.5 we reduce to the case of 3 chords. ■

This lemma has a more purely contact interpretation. Recall that $\mathcal{M}(\Gamma, \Gamma)$ is a tight contact 3-ball.

Lemma 4.2.7 *Every nontrivial arc of attachment c on $\Gamma \times \{0\}$ or $\Gamma \times \{1\}$ in $\mathcal{M}(\Gamma, \Gamma)$ is outer.* ■

The notion of an *outer* attaching arc on a ball was defined in section 4.1.3.

A weaker statement can be proved purely algebraically: if Γ_0, Γ_1 are bypass-related then precisely one of $\mathcal{M}(\Gamma_0, \Gamma_1)$, $\mathcal{M}(\Gamma_1, \Gamma_0)$ is tight. This statement is equivalent to

$$m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0) = 1.$$

For the third bypass in their triple is $\Gamma_0 + \Gamma_1$, and from lemma 4.2.4 we have $m(\Gamma_0, \Gamma_0) = m(\Gamma_1, \Gamma_1) = m(\Gamma_0 + \Gamma_1, \Gamma_0 + \Gamma_1) = 1$. So

$$\begin{aligned} 1 &= m(\Gamma_0 + \Gamma_1, \Gamma_0 + \Gamma_1) \\ &= m(\Gamma_0, \Gamma_0) + m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0) + m(\Gamma_1, \Gamma_1) \\ &= m(\Gamma_0, \Gamma_1) + m(\Gamma_1, \Gamma_0). \end{aligned}$$

4.2.4 What bypasses and chord diagrams exist in a cobordism?

Putting together the above considerations, it is now straightforward to describe what bypasses and chord diagrams exist in a cobordism, in an algorithmic way.

For a tight $\mathcal{M}(\Gamma_0, \Gamma_1)$, which is a 3-ball, as discussed in section 4.1.3 above, for any arc of attachment c, there exists a bypass inside $\mathcal{M}(\Gamma_0, \Gamma_1)$ along c if and only if c is inner. Equivalently, for c on $D \times \{0\}$, a bypass exists inside $\mathcal{M}(\Gamma_0, \Gamma_1)$ along c if and only if

$$m(\mathrm{Up}_c(\Gamma_0), \Gamma_1) = 1.$$

We also know from lemma 4.1.1 above that any cobordism is constructed from bypass attachments. Thus, we can determine what chord diagrams "exist inside a cobordism". By this, precisely, we mean the following:

Definition 4.2.8 (Existence of chord diagram in cobordism) *Let Γ be a chord diagram and $\mathcal{M}(\Gamma_0, \Gamma_1)$ a tight cobordism, where Γ_0, Γ_1 contain n chords. The chord diagram Γ is said to* exist *or* occur *in $\mathcal{M}(\Gamma_0, \Gamma_1)$ if there exists an embedded convex surface $D' \subset \mathcal{M}(\Gamma_0, \Gamma_1) = D \times I$ such that:*

(i) the boundary $\partial D'$ lies on $\partial D \times I$ and intersects the dividing set of $\partial \mathcal{M}(\Gamma_0, \Gamma_1)$ in precisely $2n$ points (i.e. as efficiently as possible), and hence inherits a base point from Γ_0 (or Γ_1);

(ii) with the base point so inherited, D' has dividing set Γ.

Thus, a chord diagram Γ exists in $\mathcal{M}(\Gamma_0, \Gamma_1)$ if and only if there is a sequence of inner bypasses which we successively "dig out", until we find Γ as a boundary of a smaller "excavated" manifold.

Lemma 4.2.9 (Criterion for existence of Γ in a cobordism) *A chord diagram Γ exists in $\mathcal{M}(\Gamma_0, \Gamma_1)$ if and only if there exists a sequence of chord diagrams*

$$\Gamma_0 = G_0,\ G_1,\ \ldots,\ G_k = \Gamma$$

and attaching arcs $c_0, \ldots, c_{k-1}$, with c_i on G_i, such that:

(i) for $i = 0, \ldots, k-1$*, we have* $G_{i+1} = \mathrm{Up}_{c_i} G_i$.

(ii) c_i *is inner on* $\mathcal{M}(G_i, \Gamma_1)$*, or equivalently,* $m(G_{i+1}, \Gamma_1) = 1$. ∎

There is of course a similar result "excavating" from Γ_1 rather than Γ_0.

Asking the question for the cobordism $\mathcal{M}(\Gamma, \Gamma)$, recall that by lemma 4.2.7 all arcs of attachment are outer. Hence there are no inner arcs of attachment; hence no bypasses inside; and by the above result, no other chord diagrams existing inside.

Lemma 4.2.10 *The only chord diagram existing in* $\mathcal{M}(\Gamma, \Gamma)$*, with the unique tight contact structure, is* Γ. ∎

This lemma actually permits us to prove a classification result for tight contact structures on solid tori with longitudinal bypasses. We cited this without proof in section 2.1.3 above, but our methods now permit us to prove it directly (in the spirit of [25]). Although it is a detour, it illustrates the use of these methods.

Proposition 4.2.11 *Tight contact structures up to isotopy on the solid torus* $D^2 \times S^1$ *with boundary dividing set* $F \times S^1$*,* $F \subset \partial D^2$ *a finite set,* $|F| = 2n$*, are in bijective correspondence with chord diagrams of* n *chords.*

Proof First suppose we have a tight contact structure on this solid torus. Take a convex meridional disc D intersecting the boundary dividing set in precisely $2n$ points; its dividing set is some chord diagram Γ of n chords. Cutting along D gives $\mathcal{M}(\Gamma, \Gamma)$ with a tight contact structure, hence isotopic to a standard neighbourhood of the convex disc D.

If we take any other convex meridional disc D' intersecting the boundary dividing set in $2n$ points, then after taking (if necessary) a finite cover of the solid torus, we may consider D' disjoint from D. Cutting along D we still obtain a (thicker!) $\mathcal{M}(\Gamma, \Gamma)$ with tight contact structure, a (thicker!) standard convex neighbourhood of D. Hence the chord diagram on D' exists in $\mathcal{M}(\Gamma, \Gamma)$, and by the above result is Γ. So there is a well-defined map from isotopy classes of tight contact structures to chord diagrams.

Conversely, given a chord diagram Γ, we take the tight contact cobordism $\mathcal{M}(\Gamma, \Gamma)$ and glue the ends together. This gives an S^1-invariant contact structure on the solid torus $D \times S^1$, with desired boundary dividing set, and a convex meridional disc with dividing set Γ. To see that it is tight, suppose that there were an overtwisted disc D'. After taking a finite cover if necessary, we may assume D' disjoint from D. Then, cutting along D, we have an overtwisted disc in the tight (invariant $D \times I$) $\mathcal{M}(\Gamma, \Gamma)$, a contradiction. ■

4.2.5 Elementary cobordisms and generalised bypass triples

Another simple type of cobordism is one obtained by attaching bypasses; not just one bypass, but along a bypass system.

Definition 4.2.12 (Elementary cobordism) *A tight cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ such that the tight contact structure can be constructed by attaching bypasses above $D \times \{0\}$ along a bypass system c on Γ_0 is called an* elementary cobordism*.*

Obviously a bypass cobordism is a elementary cobordism; every tight cobordism can be decomposed into elementary cobordisms.

We can think of the multiple bypasses in an elementary cobordism as a "generalised bypass". However, unlike single bypass attachments, the result of attaching the bypasses need not be tight; and the result of attaching bypasses even along a bypass system containing only nontrivial attaching arcs may contain closed curves. The tightness of elementary cobordisms, and the existence of closed curves, is determined by the existence or non-existence of pinwheels, by section 4.1.5 above. In following chapters, we will find a large class of elementary cobordisms which contain no pinwheels, so that the bypass attachments lead to tight contact structures.

In any case, an elementary cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ with bypass system c_0 on Γ_0 naturally gives rise to a triple of diagrams

$$\Gamma_0, \quad \Gamma_1 = \text{Up}_c(\Gamma), \quad \Gamma_{-1} = \text{Down}_c(\Gamma)$$

which we can think of as a "generalised bypass triple". Here we have extended the notation Up_c and Down_c to bypass systems in the obvious way. In view of figure 1.5 and the symmetry of bypass moves, which can be regarded as "local 60° rotations", we note that there is also a corresponding bypass system c_1 on Γ_1 on which upwards and downwards bypass moves give Γ_{-1}, Γ_0 respectively; and similarly there is a corresponding bypass system on Γ_{-1}. However, Γ_{-1} need not be a chord diagram: for instance, a trivial attaching arc in c may lead to a closed loop in Γ_{-1}.

A generalised bypass triple will not usually sum to zero; the relationship is more complicated, and we will investigate some aspects of this relationship in detail in the following. For now, we note that if we expand, say, every downwards bypass move as a sum of the null and upwards bypass moves, we obtain a sum with 2^k terms, where c contains k arcs; that is, a sum over all subsets of c, including the empty and full subsets.

Lemma 4.2.13 (Expanding down over up) *For any bypass system c on Γ,*

$$\text{Down}_c(\Gamma) = \sum_{c' \subseteq c} \text{Up}_{c'}(\Gamma).$$

Similarly,

$$\text{Up}_c(\Gamma) = \sum_{c' \subseteq c} \text{Down}_{c'}(\Gamma).$$

■

We might ask whether every tight cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ is elementary. However, any such optimism is crushed by the following example.

Lemma 4.2.14 (Not every cobordism is elementary) *With Γ_0, Γ_1 as shown in figure 4.14, the cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight but not elementary.*

Proof It's easy to verify, rounding corners, that $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight. Suppose the cobordism were elementary, so that $\Gamma_1 = \text{Up}_c \Gamma_0$ for some bypass system c on Γ_0. On Γ_0 there are only three nontrivial attaching arcs, namely α, β, γ as shown in figure 4.14. It's easy to verify that $\mathcal{M}(\text{Up}_\alpha \Gamma_0, \Gamma_1)$ and $\mathcal{M}(\text{Up}_\gamma \Gamma_0, \Gamma_1)$ are both overtwisted, while $\mathcal{M}(\text{Up}_\beta \Gamma_0, \Gamma_1)$ is tight. Hence c can only consist of copies of β and trivial attaching arcs; hence $\text{Up}_c \Gamma_0 = \text{Up}_\beta \Gamma_0$. However $\text{Up}_\beta \Gamma_0 \neq \Gamma_1$. ■

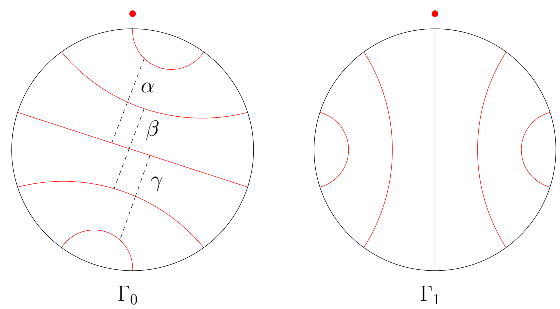


Figure 4.14: Non-elementary cobordism.

4.2.6 The contact category

Category-theoretic notions have been introduced into contact geometry by Honda [21]. Given a surface Σ and a finite set of points F on $\partial\Sigma$, there is a contact category $\mathcal{C}(\Sigma, F)$.

Definition 4.2.15 (Contact category) *The category $\mathcal{C}(\Sigma, F)$ is defined as follows.*

- *The* objects *are isotopy classes of tight dividing sets Γ on Σ with $\partial\Gamma = F$.*
- *The* morphisms $\Gamma_0 \longrightarrow \Gamma_1$ *are:*
  - *(i) isotopy classes of tight contact structures on $\Sigma \times I$, with boundary dividing set Γ_0 on $\Sigma \times \{0\}$, Γ_1 on $\Sigma \times \{1\}$, and a vertical dividing set on $\partial\Sigma \times I$; and*
  - *(ii) a single morphism, denoted $*$, referring to overtwisted structures on $\Sigma \times I$ with the same boundary conditions. (We think of this as the zero morphism.)*
- Composition *of morphisms is given by gluing cobordisms.*

Honda has showed that this category obeys many of the properties of a *triangulated category*. (For a general reference on triangulated categories, see e.g. [14].) In particular, there are distinguished triangles, arising from bypass additions, and these obey an octahedral axiom [21]. The TQFT-properties of SFH imply that it also behaves functorially with respect to SFH.

In our case $\Sigma = D^2$, F contains $2n$ points; we can denote this category by $\mathcal{C}(D^2, n)$. Our work already can computed $\mathcal{C}(D^2, n)$ algorithmically from chord diagrams. The objects are chord diagrams of n chords, and the morphisms $\Gamma_0 \longrightarrow \Gamma_1$ are contact structures on $\mathcal{M}(\Gamma_0, \Gamma_1)$. If $m(\Gamma_0, \Gamma_1) = 0$ then there is only the overtwisted morphism $* : \Gamma_0 \longrightarrow \Gamma_1$. If $m(\Gamma_0, \Gamma_1) = 1$ then there is precisely one other morphism, the unique (up to isotopy) tight contact structure on $\mathcal{M}(\Gamma_0, \Gamma_1)$. Composition of two morphisms $\Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2$ is $*$ if either of the two composed morphisms are. If both composed morphisms are tight, then the composition is the tight contact structure if $m(\Gamma_0, \Gamma_2) = 1$ and Γ_1 exists in $\mathcal{M}(\Gamma_0, \Gamma_2)$; otherwise it is the overtwisted morphism.

By relative euler class orthogonality (proposition 1.3.2), a nontrivial morphism $\Gamma_0 \longrightarrow \Gamma_1$ exists only when Γ_0, Γ_1 have the same relative euler class e. Hence we define $\mathcal{C}(D^2, n, e)$ to be the full subcategory of $\mathcal{C}(D^2, n)$ on those objects which are chord diagrams of relative euler class e.

4.2.7 The bounded contact category

A natural way to restrict the contact category is to consider only those objects which can be embedded in a given tight cobordism. This leads us to the following notion of *bounded contact category*, tenuously analogous to the concept of a bounded category. We start from a given tight cobordism $\Gamma_0 \stackrel{\xi}{\rightarrow} \Gamma_1$. We would like to consider a category where

- the objects are those dividing sets which occur in the given cobordism (see definition 4.2.8); and
- the morphisms are those "cobordisms which exist" in the given cobordism.

Note that with this definition, there are no overtwisted morphisms.

The morphisms are defined precisely as follows. This sounds rather technical but the intuitive meaning is clear: a contact cobordism which embeds in another, compatible with the notions of "up" and "down" in the cobordism.

Definition 4.2.16 (Existence of cobordism inside cobordism) *Let* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *be a cobordism with a contact structure* ξ. *We say that the cobordism* $\mathcal{M}(\Gamma, \Gamma')$ *with contact structure* ξ' exists *or* occurs *in* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *if:*

- Γ, Γ' *both exist in* $\mathcal{M}(\Gamma_0, \Gamma_1)$*;*
- Γ*,* Γ' *can be taken to lie on discs* D, D' *in* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *such that* D, D' *are disjoint and* D *is below* D' *(i.e. cutting along* D *gives two sutured cylinders* $\mathcal{M}(\Gamma_0, \Gamma)$ *and* $\mathcal{M}(\Gamma, \Gamma_1)$*, and the disc* D' *lies in* $\mathcal{M}(\Gamma, \Gamma_1)$*); hence* $\mathcal{M}(\Gamma, \Gamma')$ *embeds in* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *in a way compatible with dividing sets of* ξ*; and*
- *the restriction of* ξ *to this embedded* $\mathcal{M}(\Gamma, \Gamma')$ *is isotopic to* ξ'*.*

Note that if ξ is the unique tight contact structure on $\mathcal{M}(\Gamma_0, \Gamma_1)$, then ξ' must also be tight, and hence the unique tight contact structure on $\mathcal{M}(\Gamma, \Gamma')$. We therefore may just say that $\mathcal{M}(\Gamma, \Gamma')$ occurs in the tight $\mathcal{M}(\Gamma_0, \Gamma_1)$.

It might seem at first glance that this clearly forms a category, but there is an issue with compositions of morphisms. Suppose we have two morphisms $\Gamma \to \Gamma'$ and $\Gamma' \to \Gamma''$, so that there are cobordisms $\mathcal{M}(\Gamma, \Gamma')$ and $\mathcal{M}(\Gamma', \Gamma'')$ occurring inside $\mathcal{M}(\Gamma_0, \Gamma_1)$ (with contact structures agreeing up to isotopy). In both of these sub-cobordisms, there are discs D_1, D_2 with dividing set Γ', but there is no reason why we should be able to glue them together inside $\mathcal{M}(\Gamma_0, \Gamma_1)$.

However, we can avoid this problem — indeed avoid geometry altogether — and use the notion of "gluing cobordisms" in the following lemma.

Lemma 4.2.17 ("Gluing cobordisms") *Suppose that there are two cobordisms*

$$\mathcal{M}(\Gamma, \Gamma') \quad \textit{and} \quad \mathcal{M}(\Gamma', \Gamma'')$$

which occur in the tight cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$*. Then a tight cobordism* $\mathcal{M}(\Gamma, \Gamma'')$ *also occurs in* $\mathcal{M}(\Gamma_0, \Gamma_1)$*; and in turn the chord diagram* Γ' *occurs in this* $\mathcal{M}(\Gamma, \Gamma'')$*.*

Proof This is virtually immediate from the criterion (lemma 4.2.9) for the existence of chord diagrams in a cobordism. Above our first cobordism $\mathcal{M}(\Gamma, \Gamma')$ occurring in $\mathcal{M}(\Gamma_0, \Gamma_1)$ we have a cobordism $\mathcal{M}(\Gamma', \Gamma_1)$. But the fact that the cobordism $\mathcal{M}(\Gamma', \Gamma'')$

exists in $\mathcal{M}(\Gamma_0, \Gamma_1)$ means that Γ'' exists in any tight cobordism $\mathcal{M}(\Gamma', \Gamma_1)$. Hence above our first cobordism $\mathcal{M}(\Gamma, \Gamma')$ is a cobordism $\mathcal{M}(\Gamma', \Gamma'')$; putting these together yields a cobordism $\mathcal{M}(\Gamma, \Gamma'')$ occurring in $\mathcal{M}(\Gamma_0, \Gamma_1)$, and containing a disc with dividing set being chord diagram Γ'. ■

This lemma says that a composition $\Gamma \longrightarrow \Gamma' \longrightarrow \Gamma''$ is always uniquely defined and always tight. We can therefore make the following definition.

Definition 4.2.18 (Bounded contact category) *Let $\mathcal{M}(\Gamma_0, \Gamma_1)$ be a tight cobordism. The* bounded contact category $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ *is defined as follows.*

- *The* objects *are the chord diagrams Γ which exist in the tight $\mathcal{M}(\Gamma_0, \Gamma_1)$. (See definition 4.2.8.)*
- *There is one* morphism $\Gamma \longrightarrow \Gamma'$ *whenever the tight cobordism $\mathcal{M}(\Gamma, \Gamma')$ exists in the tight $\mathcal{M}(\Gamma_0, \Gamma_1)$. (See definition 4.2.16.)*
- Composition of morphisms *is given by gluing cobordisms as in lemma 4.2.17: $\Gamma \longrightarrow \Gamma' \longrightarrow \Gamma''$ composes to the unique $\Gamma \longrightarrow \Gamma''$.*

Thus, between any two objects the number of morphisms will be either zero or one.

Note that we have eschewed geometry from this definition; perhaps there is a more geometric definition. But lemma 4.2.17 shows that the composition $\Gamma \to \Gamma' \to \Gamma''$ gives a morphism $\Gamma \longrightarrow \Gamma''$, which contains Γ', and which has contact structure the union of the two original morphisms; so it is a natural contact-geometric form of composition.

Lemma 4.2.19 *For any tight $\mathcal{M}(\Gamma_0, \Gamma_1)$, $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ is a category.* ■

Clearly the notion of "bounding" cobordisms in this way can be nested. So, if the tight $\mathcal{M}(\Gamma, \Gamma')$ exists in the tight $\mathcal{M}(\Gamma_0, \Gamma_1)$, then we get a covariant fully faithful functor

$$\mathcal{C}^b(\Gamma, \Gamma') \longrightarrow \mathcal{C}^b(\Gamma_0, \Gamma_1),$$

injective on objects and morphisms. In particular, $\mathcal{C}^b(\Gamma, \Gamma')$ is isomorphic to the full sub-category on the image of its objects in $\mathcal{C}^b(\Gamma_0, \Gamma_1)$.

4.2.8 The bounded contact category is partially ordered

We now show that the bounded contact category of a tight contact cobordism has the structure of a partial order.

We begin by noting that a partially ordered set can be considered as a category: the objects are the elements of the set, and for each pair (x, y) of elements, there is one morphism $x \to y$ between them if they are related by the partial order, $x \preceq y$, and no morphisms between them otherwise. It's clear that the axioms of a partial ordering imply that the axioms of a category are satisfied. We can call this the *category of a partially ordered set.*

On the other hand, we may ask when a given category arises from a partial ordering. Considering the axioms of a partial ordering, and the construction of the previous paragraph, the following lemma is clear.

Lemma 4.2.20 (When a category is a partial order) *Let $\mathcal{C}$ be a category. Suppose that $\mathcal{C}$ satisfies the following conditions.*

(i) For every pair A, B of objects of $\mathcal{C}$, there is at most one morphism $A \to B$.

(ii) If there are morphisms $A \to B$ and $B \to A$, then $A = B$.

Define the relation on objects of $\mathcal{C}$: $A \preceq B$ if and only if there is a morphism $A \to B$.

Then $\preceq$ is a partial order on $Ob(\mathcal{C})$, and $\mathcal{C}$ is the category of this partially ordered set. ■

Having made this observation, we have proposition 1.3.5:

Proposition *The bounded contact category $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ is partially ordered.*

Proof We only need to verify the conditions of lemma 4.2.20 above. The first is true by definition.

For the second, suppose we have two objects Γ, Γ' of $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ and morphisms $\Gamma \to \Gamma' \to \Gamma$. After composing, we obtain a sub-cobordism $\mathcal{M}(\Gamma, \Gamma)$ of $\mathcal{M}(\Gamma_0, \Gamma_1)$ with the inherited tight contact structure, in which the chord diagram Γ' exists. But this contradicts lemma 4.2.10 above, unless $\Gamma = \Gamma'$. ■

As a result, we may write $\preceq$ for the partial order on $\mathcal{C}^b(\Gamma_0, \Gamma_1)$. Note that for any $\Gamma \preceq \Gamma'$ in this partial order, we must have $m(\Gamma, \Gamma') = 1$. As for the converse, we do not know: if Γ, Γ' are objects in $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ and $m(\Gamma, \Gamma') = 1$, is $\Gamma \preceq \Gamma'$?

We actually have a little more than a mere partial ordering on $\mathcal{C}^b(\Gamma_0, \Gamma_1)$. This ordering has a unique minimal element Γ_0, and a unique maximal element Γ_1. Category-theoretically, $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ has an initial object Γ_0 and a final object Γ_1.

We may think of $\mathcal{M}(\Gamma_0, \Gamma_1)$, with its unique tight contact structure, as a "geometric realisation" — in a moral sense, not the technical sense — of the bounded contact category $\mathcal{C}^b(\Gamma_0, \Gamma_1)$. Slicing this cylinder (generically, i.e. along convex surfaces) geometrically realises the objects; and the pieces into which the cobordism is cut by these slices realise the morphisms.

4.2.9 Functorial properties of elementary cobordisms

Elementary cobordisms, being obtained simply by attaching bypasses along a bypass system, possess various nice categorical properties.

Let $\mathcal{M}(\Gamma_0, \Gamma_1)$ be a tight elementary cobordism obtained by attaching bypasses to the convex disc $D \times \{0\}$, with dividing set Γ_0, along the bypass system $c = \{c_1, \ldots, c_k\}$. Let $\mathcal{P}(c)$ denote the power set of c, i.e. the set of all subsets of c. Clearly $\mathcal{P}(c)$ is partially ordered under inclusion, and hence can be considered as a category. Note that attaching bypasses along any subset c' of c gives an object $\text{Up}_{c'}(\Gamma_0)$ of $\mathcal{C}^b(\Gamma_0, \Gamma_1)$, and a morphism from the initial object $\Gamma_0 \to \text{Up}_{c'}(\Gamma_0)$ arising from the sub-cobordism $\mathcal{M}(\Gamma_0, \text{Up}_{c'}(\Gamma_0))$ of $\mathcal{M}(\Gamma_0, \Gamma_1)$.

Moreover, if we have two subsets $c' \subset c''$ of c, then the convex surface arising from attaching bypasses along c' can be taken to lie entirely below the surface arising from attaching bypasses along c''. Thus the two objects $\text{Up}_{c'}(\Gamma_0)$, $\text{Up}_{c''}(\Gamma_0)$ are related by a morphism, or by the partial order. Phrasing this in fancier language, and applying the same reasoning to downwards bypasses from Γ_1, gives the following result.

Lemma 4.2.21 (Up and down functors) *Let $\mathcal{M}(\Gamma_0, \Gamma_1)$ be tight and elementary, arising from attaching bypasses above Γ_0 along a bypass system c. Let $\mathcal{P}(c)$ denote* *the power set of c, considered as a category. Then there is a covariant functor*

$$\begin{aligned} \mathrm{Up}_c \ : \ \mathcal{P}(c) &\longrightarrow \mathcal{C}^b(\Gamma_0, \Gamma_1) \\ c' &\mapsto \mathrm{Up}_{c'}(\Gamma_0) \\ (c' \subseteq c'') &\mapsto (\mathrm{Up}_{c'}(\Gamma_0) \to \mathrm{Up}_{c''}(\Gamma_0)) \, . \end{aligned}$$

Similarly, the tight elementary $\mathcal{M}(\Gamma_0, \Gamma_1)$ can be considered to arise from bypass attachments below Γ_1 along a bypass system d, and there is a contravariant functor

$$\begin{aligned} \mathrm{Down}_d \ : \ \mathcal{P}(d) &\longrightarrow \mathcal{C}^b(\Gamma_0, \Gamma_1) \\ d' &\mapsto \mathrm{Down}_{d'}(\Gamma_1) \\ (d' \subseteq d'') &\mapsto (\mathrm{Down}_{d''}(\Gamma_1) \to \mathrm{Down}_{d'}(\Gamma_1)) \, . \end{aligned}$$

■

Note the functors are well-defined on morphisms since, in all the categories concerned, there is at most one morphism between any two objects. Also note the assumption of tightness implies that c contains no upwards pinwheels and d contains no downwards pinwheels.

This functor need not be injective or surjective on objects, and usually is neither. For the functor to be non-injective indicates that some arc of attachment of c is trivial, or *becomes* trivial after bypass attachments along other arcs. For the functor to be non-surjective indicates that not every dividing set existing in $\mathcal{M}(\Gamma_0, \Gamma_1)$ arises from bypass attachments along the given arcs.

4.2.10 Other categorical structures

Suppose we have a generalised bypass triple Γ, $\Gamma' = \mathrm{Up}_c \Gamma$, $\Gamma'' = \mathrm{Down}_c \Gamma$, where c is a bypass system without any pinwheels (upwards or downwards). We have corresponding bypass systems on c', c'' on Γ', Γ'' and successive upwards bypass attachments give a triangle of morphisms in $\mathcal{C}(D, n)$.

$$\begin{array}{ccc} & \Gamma & \\ \nearrow & & \searrow \\ \mathrm{Down}_c(\Gamma) & \longleftarrow & \mathrm{Up}_c \, \Gamma \end{array}$$

The composition of any two morphisms in this triangle is an overtwisted contact structure, in fact containing an overtwisted disc along every attaching arc of the intermediate dividing set, since each bypass is half an overtwisted disc.

We might think of such a "generalised bypass triple" as a "distinguished triangle" or "exact triangle". Honda notes that bypass triples can be considered analogous to the distinguished triangles of a triangulated category. One of the ways in which the category fails to be triangulated is that not every morphism extends to a distinguished triangle. The axioms of a triangulated category require that every morphism extends to a distinguished triangle — the third element in the triangle being the *cone* of the morphism.

With our generalised notion of distinguished triangle, every elementary cobordism arising from a bypass system without (up or down) pinwheels includes into a distinguished triangle; however the cone depends on the choice of bypass system, which is unsatisfactory. (We will see in chapter 6 and discuss in section 7.2.3 certain somewhat "canonical" bypass systems, but these only exist for basis chord diagrams.) Elementary cobordisms form a much larger class than bypass cobordisms, but by lemma 4.2.14 not all the tight cobordisms. In any case we may consider $\mathrm{Down}_c(\Gamma)$ to be "a cone" of the morphism $\Gamma \longrightarrow \mathrm{Up}_c(\Gamma)$: "The cone of Up_c is Down_c".

We might also think of "Down as the kernel/cokernel of Up". The existence of kernels or cokernels would make a contact category something like an abelian category. One natural notion of the kernel of a cobordism $\mathcal{M}(\Gamma_0, \Gamma_1)$ might be a "minimal" cobordism $\mathcal{M}(\Gamma_K, \Gamma_0)$ such that gluing the tight $\mathcal{M}(\Gamma_K, \Gamma_0)$ to the tight $\mathcal{M}(\Gamma_0, \Gamma_1)$ gives an overtwisted contact structure; and a cokernel might be the corresponding object glued at the other end of $\mathcal{M}(\Gamma_0, \Gamma_1)$. However, this notion has several problems. For instance, there may be many bypasses which exist upwards within $\mathcal{M}(\Gamma_0, \Gamma_1)$ along various attaching arcs c_i on Γ_0. For each c_i, there is a tight cobordism $\mathcal{M}(\text{Down}_{c_i} \Gamma_0, \Gamma_0)$ such that the composition $\text{Down}_{c_i} \Gamma_0 \longrightarrow \Gamma_0 \longrightarrow \Gamma_1$ is zero; these are all "minimal" in the sense that they are bypass cobordisms. And the set of such c_i need not be disjoint: there may be bypass attachments along attaching arcs which intersect. The notion of a kernel is therefore problematic; we remain discontented.

Chapter 5

The basis of contact elements

We now examine *basis chord diagrams* in detail. We first consider (section 5.1) how to construct them, and (section 5.2) how to decompose in terms of them. Then (section 5.3) we consider the partial order $\preceq$ and its relation to stackability of basis chord diagrams.

5.1 Construction of basis elements

5.1.1 An example

Consider the basis element v_{-+-++}. Suppose we want to draw the corresponding chord diagram.

One way to proceed is to note that by definition $v_{-+-++} = B_-(v_{+-++})$. Hence there is an outermost chord which encloses a negative region and which lies immediately to the "left" of the base point; one of its endpoints is the base point. If we were to consider removing this outermost chord, including its endpoints (including the base point), and placing a new "temporary" base point to its "left" (i.e. "jumping over" the location of the previous outermost chord), we should then have v_{+-++}. But $v_{+-++} = B_+(v_{-++})$, hence v_{+-++} has an outermost chord at the base point enclosing a positive region; this also tells us the location of a chord on v_{-+-++}.

We can then repeat. Since $v_{-++} = B_-(v_{++})$, there must be an outermost chord at the "temporary" base point in this new chord diagram, enclosing a negative region; and hence we may locate a chord on v_{-+-++}. Similarly, we can locate the remaining chords, until we reduce to the vacuum diagram $v_\emptyset$. See figure 5.1.

5.1.2 The base point construction algorithm

The above can be formalised into an algorithm to construct the chord diagram of a basis element v_w. The algorithm starts from the base point; hence the name.

Algorithm 5.1.1 (Base point construction algorithm) *Let w be a word in the symbols $+$ and $-$ of length n. Consider a disc with $2n+2$ points marked on the boundary, and one of those marked points called the* base point. *Proceed through the word from left to right, and at each stage draw a chord and move to a new, temporary base point as follows. Once there is a chord ending at a marked point, that marked point is called* used.

(i) If the symbol is $-$, draw a chord from the current temporary base point to the next unused marked point anticlockwise ("left" in the diagram) from it. After drawing this chord, move the temporary base point to the next unused marked point in the anticlockwise direction ("left" in the diagram). (I.e., immediately anticlockwise of the new chord.)

(ii) If the symbol is $+$, draw a chord from the current temporary base point to the next unused marked point clockwise ("right") from it. After drawing this chord, move the temporary base point to the next unused marked point in the clockwise direction. (I.e., immediately clockwise of the new chord.)

This constructs n chords connecting $2n$ marked points. Finally, connect the remaining two marked points with a chord. The base point returns to its initial position, which is the "permanent" base point.

The stages in the construction of v_{-+-++} are depicted in figure 5.1.

The words "left" and "right" to describe directions around a circle may seem horrendously bad as terminology, only being appropriate near the base point, only

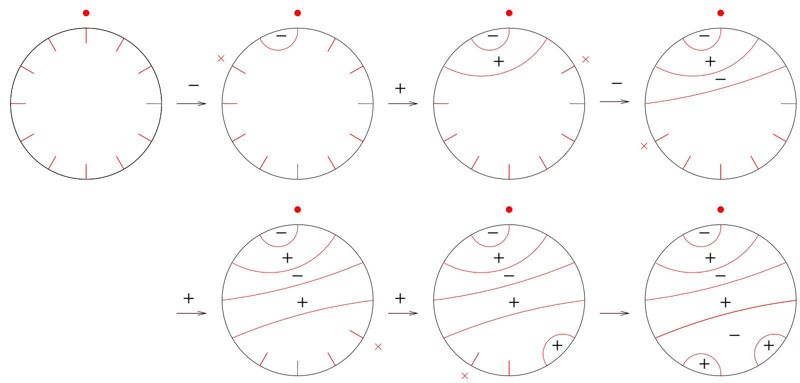


Figure 5.1: Construction of v_{-+-++}. The temporary base point at each stage is denoted by a red cross.

when the base point is drawn at the top of the circle as in our diagrams, and eventually conflicting around the other side of the circle! But we will see there is a reason for it, and it does make some sense.

That this produces v_w may appear rather obvious, but since the details will be needed later we provide them. In fact, we will need to describe the mechanics of this algorithm in gruesome detail.

Proposition 5.1.2 (Base point algorithm works) *The algorithm 5.1.1 is well-defined; in particular, at every stage of this algorithm, the chord described can be made disjoint from all previously drawn chords; and uniquely up to homotopy rel endpoints within the disc minus the previously drawn chords.*

Moreover, it actually produces the chord diagram Γ_w.

First, a little notation.

Definition 5.1.3 (Labels of marked points) *We label the $2n+2$ marked points with integers modulo $2n+2$. The base point is labelled 0 and the numbering proceeds clockwise. (So the marked point immediately "right" / clockwise of the base point 0 is the point 1.)*

Remark 5.1.4 (Labelling convention) *Marked points will always be labelled with respect to the (permanent) base point. In the various steps of the base point algorithm, as the "temporary base point" moves, the numbering of marked points does not change.*

Note that with this choice of labelling, a chord connecting two consecutive points $(2j-1, 2j)$ encloses a *negative* region; while an chord connecting two consecutive points $(2j, 2j+1)$ encloses a *positive* region.

Definition 5.1.5 (Discrete interval) *A* discrete interval *of marked points* $[a, b]$ *on the circle is a set of marked points of the form* $\{a, a+1, \ldots, b\}$.

Definition 5.1.6 (Substring) *A* substring *of a word/string* w *is a set of adjacent symbols of* w.

So, for instance, $-+$ is a substring of $--++--++$ but $---+$ is not.

Definition 5.1.7 (Block) *Maximal substrings of identical symbols are called* blocks.

Definition 5.1.8 (Leading and following symbols) *A symbol in* w *which is the first in its block (read left to right) we shall call a* leading *symbol. Non-leading symbols are called* following.

Now we can describe the mechanics of the algorithm precisely. We can locate how the temporary base point moves at each step of the algorithm; and where the chord is drawn at every step. As it turns out, the *odd*-numbered marked points serve as useful indicators of where we are up to in the base point algorithm. We consider the base point construction algorithm for a word $w \in W(n_-, n_+)$, describing a basis element of $SFH(T, n+1, e)$, with our usual notation conventions.

Lemma 5.1.9 (Mechanics of base point construction algorithm)

(i) Consider the stage of the base point algorithm which processes the i'th $-$ *sign* $(1 \leq i \leq n_-)$ *in* w. *Let* i_+ *be the number of* $+$ *symbols processed up to this point. At this stage:*

(a) A chord is drawn with endpoints:

$$\begin{array}{rl} (1-2i, 2i_+) & \text{if the present (i.e. } i\text{'th) } - \text{ sign is leading} \\ (1-2i, 2-2i) & \text{if the present } - \text{ sign is following} \end{array}$$

In particular, the chord has an endpoint at the odd-numbered marked point $1 - 2i$.

(b) *The temporary base point then moves to the marked point* $-2i$.

(c) *The set of used marked points is the discrete interval* $[1 - 2i, 2i_+]$.

(ii) *Consider the stage of the base point algorithm which processes the* j*'th* $+$ *sign* $(1 \leq j \leq n_+)$ *in* w. *Let* j_- *be the number of* $-$ *signs processed up to this point. At this stage:*

(a) *A chord is drawn with endpoints:*

$$(-2j_-, 2j - 1) \text{if the present (i.e. } j\text{'th) + sign is leading}$$
$$(2j - 2, 2j - 1) \text{if the present + sign is following}$$

In particular, the chord has an endpoint at the odd-numbered marked point $2j - 1$.

(b) *The temporary base point then moves to the marked point* $2j$.

(c) *The set of used marked points is the discrete interval* $[-2j_-, 2j - 1]$.

Proof (of proposition 5.1.2 & lemma 5.1.9) Proof by induction on the number of symbols processed in w. We consider processing $+$ signs; $-$ signs are obviously similar. The result is clear as we process the first symbol in w. A leading $+$ sign will "switch sides" and connect a "negatively labelled", or "left", or anticlockwise-of-the-base point marked point to a "positively labelled", "right" or clockwise-of-the-base point marked point. A following $+$ sign will give a chord enclosing a positive outermost interval clockwise from the temporary base point.

At each stage before termination, the set of used marked points is then as given by 5.1.9, as is the temporary base point. Hence the next chord can always be drawn, and uniquely up to homotopy in the disc (minus the previous chords) rel endpoints. At the final stage before termination, the discrete interval of used marked points consists of all but two of the marked points; hence the remaining two must be adjacent, and connecting them is possible (and uniquely up to homotopy in the same way).

That the algorithm produces v_w is easily seen by induction on the length of the word. For words of length 1 (or even 0) it is clear. Let $w = sw'$ where s is a symbol (i.e. $+$ or $-$) and w' is a word one symbol shorter than w. Then by induction the algorithm applied to w' produces $v_{w'}$; and it's also clear that the algorithm for w produces $B_s(v_{w'})$ as claimed. ∎

Lemma 5.1.10 (Existence of root point) *The position of the final chord drawn in the base point construction algorithm only depends on n, e (or equivalently n_-, n_+) and the final symbol of w. The final chord encloses an outermost region of sign given by that final symbol, and has an endpoint at the point numbered*

$$2n_+ + 1 = -2n_- - 1 = e + n + 1 = e - (n + 1)$$

with respect to the base point.

Proof After having processed all symbols, from the previous lemma, the used marked points form the discrete interval

$$\begin{array}{rl} [-2n_-, 2n_+ - 1] & \text{if the final symbol is a } + \text{ sign} \\ [1 - 2n_-, 2n_+] & \text{if the final symbol is a } - \text{ sign} \end{array}$$

Hence if w ends in a $+$, then the final chord connects the two remaining unused points

$$(2n_+, 2n_+ + 1) = (-2n_- - 2, -2n_- - 1)$$

and hence encloses an outermost positive region; while if w ends in a $-$, then the final chord connects

$$(2n_+ + 1, 2n_+ + 2) = (-2n_- - 1, -2n_-)$$

and encloses an outermost negative region. ∎

Definition 5.1.11 (Root point) *The marked point numbered*

$$2n_+ + 1 = -2n_- - 1 = e \pm (n + 1)$$

with respect to the base point is called the root point*.*

Remark 5.1.12 (Denoting root point) *In our diagrams, the root point will be denoted by a hollow red dot.*

We now see that our terminology of "left" and "right" is not so horrendous after all. For as we perform the base point algorithm, all chord additions are done so that the discrete interval of used marked points never crosses the root point; and hence talking about "left" and "right" of the base point (i.e. anticlockwise and clockwise) never ceases to make sense. We can then define the terminology properly.

Definition 5.1.13 (Left/west & right/east) *The marked points forming the discrete interval* $[e-n-1,0]$ *are called the* left side *or* westside *of the circle. The marked points forming the discrete interval* $[0,e+n+1]$ *are called the* right side *or* eastside*.*

Thus, to move left from the base point is to move anticlockwise; but to move left from the root point is to move clockwise. We can now use this algorithm to number chords and regions in the chord diagram Γ_w corresponding to v_w.

Definition 5.1.14 (Base-± numbering of chords and regions)

(i) The chord created in the base point construction algorithm by processing the i'th $-$ sign of w is called the base-i'th $-$ chord*. It encloses a $-$ region, which is also a region of the completed chord diagram Γ_w, which we call the* base-i'th $-$ region*.*

(ii) The chord created in the base point construction algorithm by processing the j'th $+$ sign of w is called the base-j'th $+$ chord*. It encloses a $+$ region, which is also a region of the completed chord diagram Γ_w, which we call the* base-j'th $+$ region*.*

Note that every chord has a base-± numbering, except the final one constructed in our algorithm, i.e. the chord at the root. And every region has a base-± numbering, except the two regions adjacent to the root point, of which there is one positive and one negative.

This consideration makes it explicit how the relative euler class of the chord diagram Γ_w is $e = n_+ - n_-$; every $-$ sign creates a $-$ region, and every $+$ sign creates a $+$ region, in the algorithm; with two regions left at the end, of opposite sign.

5.1.3 The root point construction algorithm

The above algorithm takes a word w and constructs the corresponding chord diagram, starting from the base point, reading w from left to right. But equally, we can construct the chord diagram from the root point, reading w from right to left. In some sense this is more natural, since the word denotes a composition of the creation operators $B_\pm$, and compositions of functions are applied from right to left.

The algorithm is basically identical; except that, because we proceed from the "bottom" of our chord diagram to the "top", clockwise and anticlockwise are reversed. However, if we draw our diagrams as we do, with the base point at the top and root point at the bottom, then "left" and "right" are *not* reversed.

Algorithm 5.1.15 (Root point construction algorithm) *We process a word w in $W(n_-, n_+)$, from right to left. Begin with a disc with $2n+2$ marked points on its boundary, and one of those points called the* root point*. At each stage draw a chord and move to a new, temporary root point as follows.*

(i) For a $-$ symbol, draw a chord from the current root point to the next unused marked point clockwise*/left from it. After drawing this chord, move the temporary root point to the next unused marked point in the* clockwise*/left direction. (I.e., immediately clockwise of the new chord.)*

(ii) For a $+$ symbol, draw a chord from the current base point to the next unused marked point anticlockwise*/right from it. After drawing this chord, move the base point to the next unused marked point in the anticlockwise/right direction. (I.e., immediately anticlockwise of the new chord.)*

This constructs n chords connecting $2n$ marked points. Finally, connect the remaining two marked points with a chord. The root point now moves back to its initial, permanent position.

We can easily prove that there are similar results to proposition 5.1.2 and lemma 5.1.9 for this algorithm: it works, and actually constructs Γ_w.

We also have root-numberings of chords and regions, which we will need later, in analogy to our base-numberings.

Definition 5.1.16 (Root-± numbering of chords and regions)

(i) The chord created in the root point construction algorithm by processing the i'th $-$ sign of w (from the left*) is called the* root-i'th $-$ chord*. It encloses a $-$ region, which is also a region of the completed chord diagram Γ_w, which we call the* root-i'th $-$ region*.*

(ii) The chord created in the base point construction algorithm by processing the j'th $+$ sign of w (from the left*) is called the* root-j'th $+$ chord*. It encloses a $+$ region, which is also a region of the completed chord diagram Γ_w, which we call the* root-j'th $+$ region*.*

Note especially that the root point construction algorithm processes w from *right to left*: but when we speak of the root-i'th $\pm$ region we are reading w from *left to right*. This is confusing, but makes subsequent considerations easier.

Also note that every chord has a root-$\pm$ numbering, except the chord at the base point. And every region has a root-$\pm$ numbering, except the two regions adjacent to the base point, of which there is one positive and one negative.

Thus, *every chord* has some numbering, whether from the base or the root.

5.2 Decomposition into basis elements

We now examine the decomposition of a chord diagram as a sum of basis chord diagrams. We give two algorithms which compute this decomposition. (Of course, they give the same result!) One proceeds from the base point, and the other from the root point.

We saw in section 3.1.5 that there is a natural way to expand out a given chord diagram as a sum of basis elements. We now formalise this procedure. The procedure may seem obvious, from the example, but will actually give us interesting information about the elements that occur in the decomposition of a chord diagram.

Given a diagram Γ with base point and root point identified, we will successively obtain sets of chord diagrams

$$\{\Gamma\} = \Upsilon_0 \rightsquigarrow \Upsilon_1 \rightsquigarrow \cdots \rightsquigarrow \Upsilon_n$$

where Υ_k is the set of all diagrams obtained at the k'th stage, and obtained from decomposing the diagrams in Υ_{k-1}. The final set Υ_n will contain precisely the elements of the basis decomposition of Γ. In particular, $|\Upsilon_k| \geq |\Upsilon_{k-1}|$ and

$$\Gamma = \sum_{\Gamma' \in \Upsilon_0} \Gamma' = \sum_{\Gamma' \in \Upsilon_1} \Gamma' = \cdots = \sum_{\Gamma' \in \Upsilon_n} \Gamma'.$$

To make this precise, note that the first k steps of the base point construction algorithm depend only on the k leftmost symbols of the word w; and the first k steps of the root point construction algorithm depend only on the k rightmost symbols of w.

Definition 5.2.1 (Partial chord diagrams) *Let w be a word of length $k \leq n$.*

(i) The partial chord diagram for $w\cdot$ *is a disc with $2n+2$ marked points on the boundary, including a base and root point, and the first k chords drawn in processing any word of length n beginning with w in the base point construction algorithm.*

(ii) The partial chord diagram for $\cdot w$ *is a disc with $2n+2$ marked points on the boundary, including a base and root point, and the first k chords drawn in processing any word of length n ending in w in the root point construction algorithm.*

The dots in $w\cdot$ and $\cdot w$ describe "where the rest of the word goes". If we cut our disc along the chords of a partial chord diagram, all the unused marked points lie in a single component; we call this the *unused disc*.

We will label the elements of Υ_k as $\Gamma_{w\cdot}$ (resp. $\Gamma_{\cdot w}$), where w varies over words of length k. The chord diagram $\Gamma_{w\cdot}$ (resp. $\Gamma_{\cdot w}$) will be the sum of all basis elements of Γ whose words begin (resp. end) with w, and it will contain the partial chord diagram for $w\cdot$ (resp. $\cdot w$).

We may think of Γ itself as corresponding to the empty word, $\Gamma = \Gamma_{\emptyset\cdot} = \Gamma_{\cdot\emptyset}$.

Algorithm 5.2.2 (Base point decomposition algorithm) *Begin with*

$$\Upsilon_0 = \{\Gamma\} = \{\Gamma_{\emptyset\cdot}\}.$$

At the k'th step, we take Υ_{k-1}, and for each element $\Gamma_{w\cdot}$ of Υ_{k-1}, corresponding to the word w of length $k-1$, we do the following.

(i) If there exists a word $w' = w+$ or $w-$ such that $\Gamma_{w\cdot}$ contains the partial chord diagram for $w'\cdot$, then we place $\Gamma_{w\cdot}$ in Υ_k and name it $\Gamma_{w'\cdot}$.

(ii) Otherwise, there is no such word. Hence neither of the two chords added in the k'th stage of the base point construction algorithm for the words $w\pm$ lie in $\Gamma_{w\cdot}$. Equivalently, we consider the location of the temporary base point after $k-1$ stages of the base point construction algorithm for w; then, on the unused disc of Γ_w, there is no outermost chord at the temporary base point.

We then consider an arc of attachment which runs close to the boundary of the unused disc, which is centred on the chord emanating from the temporary base point (as shifted after $k-1$ stages of the base point construction algorithm for w), and which has its two ends on the two chords emanating from the marked points adjacent to the temporary base point on the unused disc. We perform the two possible bypass moves, obtaining two distinct chord diagrams. One of these contains the partial chord diagram for $w-\cdot$, and the other contains the partial chord diagram for $w+\cdot$. We label them $\Gamma_{w-\cdot}$ and $\Gamma_{w+\cdot}$ and place them in Υ_k.

This constructs Υ_k from Υ_{k-1}.

It's clear from the algorithm that the Υ_k have the desired properties. Precisely, we have the following.

Lemma 5.2.3 *For each k, the elements of Υ_k obtained in the base point decomposition algorithm can be grouped in some fashion so as to be summable, and they sum to Γ. The basis decomposition of $\Gamma_{w\cdot}$ contains all the basis elements of Γ whose words begin with w, and contains the partial chord diagram for $w\cdot$. The elements of Υ_n are basis elements and are precisely those occurring in the decomposition of Γ.* ■

In fact, to see how to sum the elements of Υ_k, bracket them exactly according to how they came from Υ_{k-1}; the decomposition process actually gives us a directed binary tree of chord diagrams, equivalent to a bracketing. To see that Υ_n consists of basis elements, note that its elements are all partial chord diagrams for words of length n; this leaves only one possible place for the remaining chord.

We may apply the same idea from the root point rather than the base point, to $\Gamma_{\cdot w}$ rather than $\Gamma_{w\cdot}$.

Algorithm 5.2.4 (Root point decomposition algorithm) *Begin with*

$$\Upsilon_0 = \{\Gamma\} = \{\Gamma_{\cdot\emptyset}\}.$$

At the k'th step, we take Υ_{k-1}, and for each element $\Gamma_{\cdot w}$ of Υ_{k-1}, corresponding to the word w of length $k-1$, we do the following.

(i) If there exists a word $w' = -w$ or $+w$ such that $\Gamma_{\cdot w}$ contains the partial chord diagram for $\cdot w'$, then we place $\Gamma_{\cdot w}$ in Υ_k and name it $\Gamma_{\cdot w'}$.

(ii) Otherwise, there is no such word. Hence neither of the two chords added in the k'th stage of the root point construction algorithm for the words $\pm w$ lie in $\Gamma_{\cdot w}$. Equivalently, we consider the location of the temporary root point after $k-1$ stages of the root point construction algorithm for w; then, on the unused disc of $\Gamma_{\cdot w}$, there is no outermost chord at the temporary root point.

We then consider an arc of attachment which runs close to the boundary of the unused disc, which is centred on the chord emanating from the temporary root point (as shifted after $k-1$ stages of the root point construction algorithm for w), and which has its two ends on the two chords emanating from the marked *points adjacent to the temporary root point on the unused disc. We perform the two possible bypass moves, obtaining two distinct chord diagrams. One of these contains the partial chord diagram for $\cdot - w$, and the other contains the partial chord diagram for $\cdot + w$. We label them $\Gamma_{\cdot - w}$ and $\Gamma_{\cdot + w}$ and place them in Υ_k.*

This constructs Υ_k from Υ_{k-1}.

Lemma 5.2.5 *For each k, the elements of Υ_k obtained in the root point decomposition algorithm can be grouped in some fashion so as to be summable, and they sum to Γ. The basis decomposition of $\Gamma_{\cdot w}$ contains all the basis elements of Γ whose words end with w, and contains the partial chord diagram for $\cdot w$. The elements of Υ_n are basis elements and are precisely those occurring in the decomposition of Γ.* ∎

5.3 Contact interpretation of the partial order $\preceq$

We now consider $\mathcal{M}(\Gamma_0, \Gamma_1)$, where each Γ_i is a *basis* chord diagram, and ask when it is tight, i.e. when Γ_1 is stackable on Γ_0. We have seen in proposition 1.3.2 that if Γ_0, Γ_1 have distinct relative euler classes then $\mathcal{M}(\Gamma_0, \Gamma_1)$ is overtwisted. So the chord diagrams Γ_0, Γ_1 correspond to words w_0, w_1 in the same $W(n_-, n_+)$.

Write $\mathcal{M}(w_0, w_1) = \mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$. In this section, we will prove proposition 1.3.3:

Proposition *$\mathcal{M}(w_0, w_1)$ is tight if and only if $w_0 \preceq w_1$.*

5.3.1 Easy direction

Lemma 5.3.1 *If w_0 does not precede w_1 with respect to $\preceq$, then $\mathcal{M}(w_0, w_1)$ is overtwisted.*

Proof By the "baseball interpretation", there is some point in the game, playing innings from left to right, where team 0 takes the lead. Hence there is a point in the game, at the m'th innings, where team 0 moves precisely one step ahead. That is, there is some m such that in w_0, there are i minus signs and j plus signs up to the m'th position, but in w_1 there are $i+1$ minus signs and $j-1$ plus signs up to the

m'th position. Moreover, since team 0 just took the lead, the m'th symbol in w_0 is a $+$, while in w_1 the m'th symbol is a $-$.

By lemma 5.1.9 then, after the m'th stage of the base point algorithm, in Γ_0 the discrete interval of used marked points is $[-2i, 2j-1]$, while in Γ_1 the discrete interval of used marked points is $[1-2(i+1), 2(j-1)] = [-2i-1, 2j-2]$. After rounding corners, the chords with endpoints in these intervals precisely match and will form closed curves on the rounded $\mathcal{M}(w_0, w_1)$. Since the m'th stage is not the final stage, the rounded $\mathcal{M}(w_0, w_1)$ has several components of sutures. Thus $\mathcal{M}(w_0, w_1)$ is overtwisted. ■

5.3.2 Preliminary cases

Lemma 5.3.2 *If $\mathcal{M}(w_0, w_1)$ is overtwisted, then a separate component of the dividing set can be observed in constructing the basis chord diagrams Γ_0, Γ_1 with the base point construction algorithm, before the final step.*

Proof We know that, after rounding, we have a system of at least 2 connected curves on S^2 and, by the argument of proposition 1.3.2 (proved in section 4.2.2), the total euler class is 0; hence there are at least 3 components. Thus there is some component γ that intersects neither the root point on Γ_0 nor the root point on Γ_1. On $\mathcal{M}(w_0, w_1)$, γ contains some of the chords on Γ_0, and some on Γ_1, but no chords with endpoints at either root point. Thus, a separate component γ can be observed at some stage of the base point algorithm before the final step. ■

Lemma 5.3.3 *The proposition for w_0, w_1 beginning with the same symbol, $w_0 = sw_0'$, $w_1 = sw_1'$, where $s \in \{+, -\}$, reduces to the proposition for w_0', w_1', i.e. shorter words.*

Proof We note that, by lemma 4.2.5, cancelling outermost chords, $\mathcal{M}(w_0, w_1)$ is contactomorphic to $\mathcal{M}(w_0', w_1')$, through rounding and re-folding. And clearly $w_0 \preceq w_1$ iff $w_0' \preceq w_1'$. ■

5.3.3 Proof of proposition

We now suppose that $w_0 \preceq w_1$, and show that $\mathcal{M}(w_0, w_1)$ is tight.

We prove this by induction on the length n of the words w_0 and w_1. It is clearly true by inspection for words of length 1 and 2; now assume it is true for all lengths less than n, and consider words w_0, w_1 of length n.

By lemma 5.3.2, we know that if $\mathcal{M}(w_0, w_1)$ is overtwisted, we will see a closed loop before the final stage of the base point construction algorithm.

We will show by induction on m, that no closed loop appears at the m'th stage of the base point construction algorithm, before the final step. By lemma 5.3.3 and lemma 5.3.1, we can assume w_0 begins with a $-$ and w_1 begins with a $+$; so no closed loop appears at the first stage; the result is true for $m = 1$. At the m'th stage of the algorithm, let $[a_m, b_m]$ denote the discrete interval of used marked points on Γ_0 and $[c_m, d_m]$ on Γ_1. The hypothesis $w_0 \preceq w_1$ means that for all m, $a_m - 1 < c_m$ and (equivalently) $b_m - 1 < d_m$.

So now suppose that there is no closed loop at any stage before m, but a closed loop appears at stage m. At the previous $(m-1)$'th stage, we had discrete intervals of used marked points $[a_{m-1}, b_{m-1}]$ and $[c_{m-1}, d_{m-1}]$. Let us examine what can happen at the m'th stage.

On Γ_0, there are three possible positions for the chord added at the m'th step of the algorithm. These three possibilities connect the pairs of marked points

$$(a_{m-1} - 2, a_{m-1} - 1)\,, \quad (a_{m-1} - 1, b_{m-1} + 1)\,, \quad \text{or} \quad (b_{m-1} + 1, b_{m-1} + 2)\,.$$

Similarly, on Γ_1 there are three possible positions for the new chord. At least one of these new chords must form part of the new closed loop; else it would have appeared earlier. We will assume that the new chord on Γ_0 is part of the new closed loop; the case where the new chord lies on Γ_1 is similar.

Let this new chord on Γ_0, added at the m'th stage, be γ_m. Note γ_m cannot include the marked point a_m, since a_m on $D \times \{0\}$ connects to $a_m - 1$ on $D \times \{1\}$, and $a_m - 1 < c_m$, so this is left of all used points of Γ_1 at this stage, and cannot form a closed loop. Thus, γ_m is $(b_{m-1} + 1, b_{m-1} + 2) = (b_m - 1, b_m)$, and it forms part of a closed loop.

We see that γ_m encloses an outermost region on the eastside of Γ_0, hence a positive

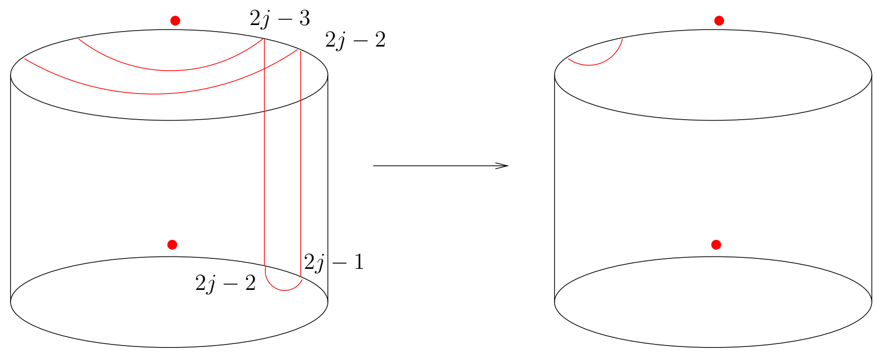


Figure 5.2: Finger move on $\mathcal{M}(\Gamma_0, \Gamma_1)$.

outermost region. Hence it is constructed by processing a following + sign in w_0. (If w_0 begins with a +, this also creates a positive outermost region, but we have dealt with the case $m = 1$.) Let this be the j'th + sign in w, so using lemma 5.1.9, $(b_m - 1, b_m) = (2j-2, 2j-1)$ Thus $w_0 = u + +v$, where u (possibly empty) contains $j-2$ plus signs, and has length $m-2$.

Now, the endpoints of γ_m on Γ_0 connect to the two marked points

$$\{b_m - 2, b_m - 1\} = \{2j-3, 2j-2\} \quad \text{on } \Gamma_1.$$

We have $d_m > b_m - 1$, so these are not the rightmost points among the used points on Γ_1, at this m'th stage. However, the closed loop we have just created cannot involve any of the points right of $2j-2 = b_m - 1$ on Γ_1, since these points connect to marked points right of b_m on Γ_0, which have not been used yet.

Thus, the chord emanating from $2j-2$ on Γ_1 must go to the westside, enclosing a $-$ region, and must be created by processing a leading $-$ symbol in w_1. And the chord emanating from $2j-3$ on Γ_1, by lemma 5.1.9, is created by processing the $(j-1)$'th + sign in w_1. Thus $w_1 = y + -z$, where y (possibly empty) contains $j-2$ plus signs.

Now, rounding the ball and refolding, we may perform a "finger move", pushing the whole new chord γ_m off $D \times \{0\}$ and up to $D \times \{1\}$, which has the effect of removing γ_m from $D \times \{0\}$, and closing off the marked points labelled $2j-3, 2j-2$ on $D \times \{1\}$. See figure 5.2.

The chord diagram on $D \times \{0\}$ then reduces to the chord diagram for $w_0' = u + v$, deleting the $(j-1)$'th $+$ sign from w_0. The chord diagram on $D \times \{1\}$ reduces to the chord diagram for $w_1' = y - z$, also deleting the $(j-1)$'th $+$ sign. Thus the situation reduces to $\mathcal{M}(w_0', w_1')$ for two smaller words obtained from deleting the $(j-1)$'th $+$ signs from both w_0 and w_1; since we deleted the same numbered $+$ signs, $w_0' \preceq w_1'$. But we know that the proposition is true for all smaller length words, so $\mathcal{M}(w_0', w_1')$ is tight; so there cannot be any closed loop, and we have a contradiction.

Thus, adding the m'th chord in the base point construction algorithm for Γ_0 and Γ_1, we never see a closed loop. By induction, at every stage before the end of the algorithm, we never see a closed loop. Hence there are no closed loops, and $\mathcal{M}(w_0, w_1)$ is tight.

This concludes the proof of proposition 1.3.3.

5.3.4 Which chord diagrams are stackable?

Notice that proposition 1.3.3 gives the value of m on all basis elements. This defines m completely, and m describes stackability. Thus, we can give an answer to the general question: given two chord diagrams Γ_0 and Γ_1, is Γ_1 stackable on Γ_0? For we can simply expand out $m(\Gamma_0, \Gamma_1) = 1$ as a sum over basis elements. This proves proposition 1.3.4.

Proposition (General stackability) *Let Γ_0 and Γ_1 be two chord diagrams with n chords and relative euler class e. Then Γ_1 is stackable on Γ_0 (i.e. $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight) if and only if the cardinality of the set*

$$\left\{ (w_0, w_1) \ : \ \begin{array}{c} w_0 \preceq w_1 \\ \Gamma_{w_i} \text{ occurs in the decomposition of } \Gamma_i \end{array} \right\}$$

is odd. ■

Chapter 6

Bypass systems on basis chord diagrams

In this chapter we will investigate performing bypass moves on basis chord diagrams, attaching bypasses along bypass systems. This chapter contains the main constructions which are at the core of this thesis.

In section 6.1, we will show, inter alia, that by performing bypass moves in a controlled way, we can go from a given basis chord diagram to many others — in particular, to any other basis chord diagram to which it is comparable under the partial order $\preceq$.

Then, in section 6.2, we will turn to more contact-categorical matters, and use these bypass systems to compute certain bounded contact categories.

6.1 Concrete combinatorial constructions

In this section we will prove proposition 1.2.18, constructing bypass systems that take Γ_1 to Γ_2 and vice versa, whenever $\Gamma_1 \preceq \Gamma_2$ are basis chord diagrams. And we will prove proposition 1.2.19, describing how performing bypass moves in the opposite direction along such bypass systems gives a chord diagram with a prescribed minimum and maximum in its basis decomposition. The construction will be explicit. As mentioned in section 1.2.6, our approach will be to develop a series of increasingly

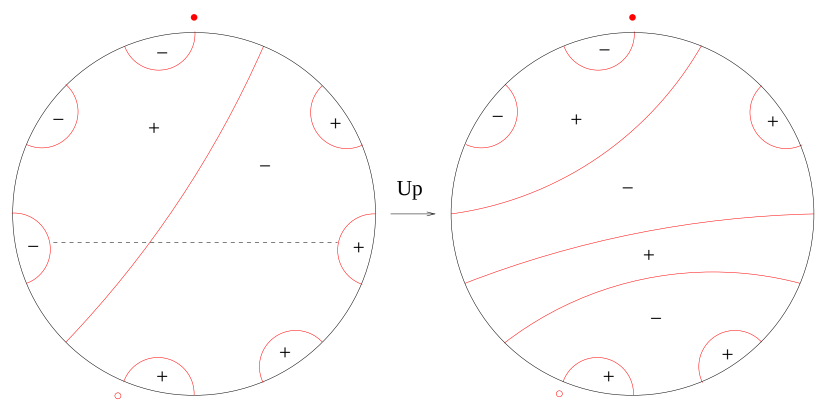


Figure 6.1: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$.

involved analogies between combinatorial manipulations on words $w \in W(n_-, n_+)$, and bypass systems on basis chord diagrams Γ_w. We start with single bypass moves, and proceed to general bypass systems. But first, we begin with some illustrative examples.

6.1.1 A menagerie of examples

These examples are of increasing difficulty. They illustrate various phenomena one observes when performing multiple bypass moves on a basis chord diagram.

First, we show how to go from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$. Here we "move the third $-$ sign past the first two $+$ signs". Moves like this are called *forwards elementary moves* and they are obtained by single upwards bypasses. See figure 6.1.

Next, we show how to go from the same starting diagram $\Gamma_{---++++}$ to $\Gamma_{-++--++}$, moving both the second and third $-$ sign past the first two $+$ signs. This is also a forwards elementary move, and is obtained from a single upwards bypass. In general, an elementary move consists of taking a string of contiguous $-$ symbols and moving them to the right, past an adjacent string of contiguous $+$ symbols. See figure 6.2.

This bypass move can be thought of as encoding the instruction "move the second $-$ sign past the first $+$ sign". If we think of the $-$ signs as remaining in order, then in this process, the third $-$ sign must be "brought along for the ride", past the first $+$ sign as well. Alternatively, if we "treat the two $-$ signs individually", and perform

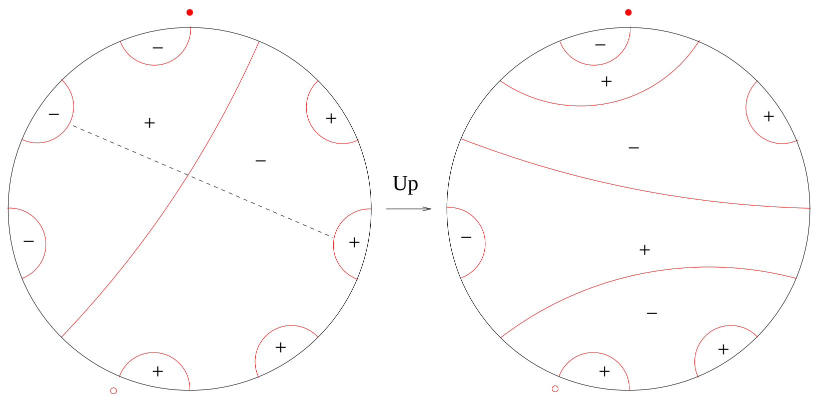


Figure 6.2: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{-++--++}$.

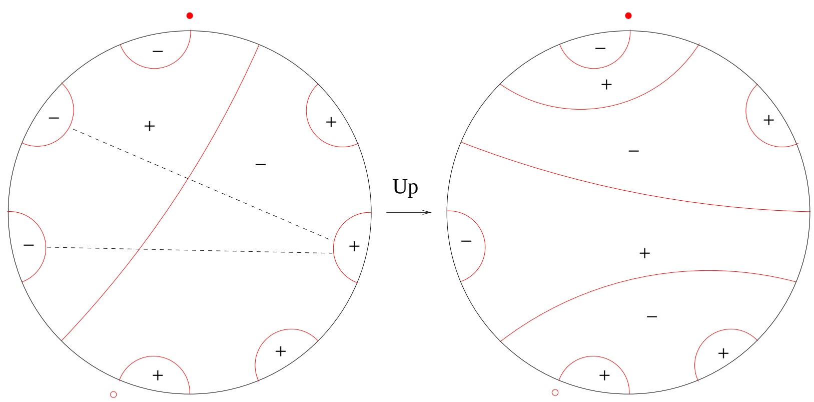


Figure 6.3: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{-++--++}$, another way.

one bypass move for each, respectively encoding the instruction to move them past the first $+$ sign, we obtain figure 6.3.

We see it gives the same result. This is an instance of the general phenomenon of "redundancy of bypasses" or "bypass rotation" (see section 4.1.2). See figure 6.4.

Next, we show how to go from $\Gamma_{--++--++}$ to $\Gamma_{++--++--}$. Here we "move the first and second $-$ signs past the first and second $+$ signs, and move the third and fourth $-$ signs past the third and fourth $+$ signs". There are two forwards elementary moves involved, but in some sense they do not interfere with each other; this is obtained by two upwards bypasses. See figure 6.5.

While the position of each of these bypass arcs, taken individually, might seem clear now from the foregoing, note that there are actually *two distinct* ways to place

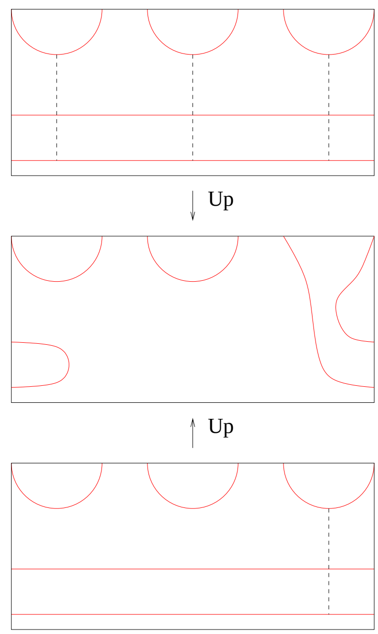


Figure 6.4: Redundancy of bypasses, or bypass rotation.

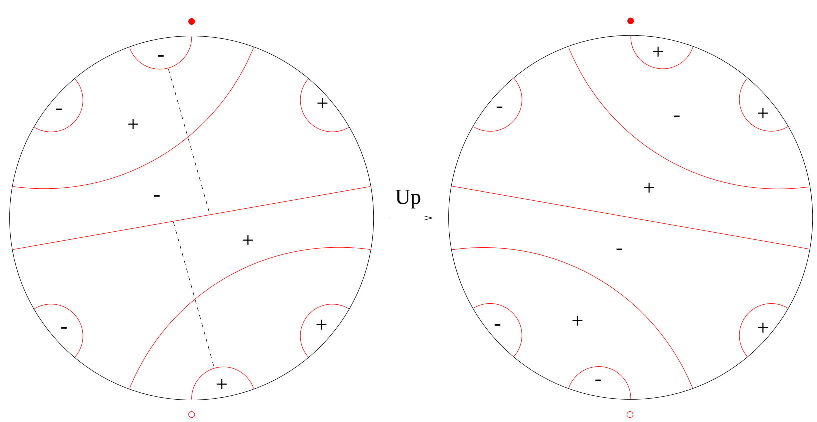


Figure 6.5: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++--++--}$.

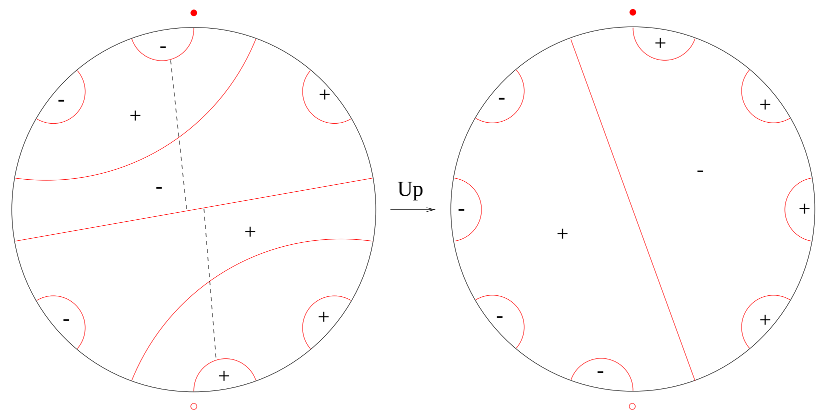


Figure 6.6: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$.

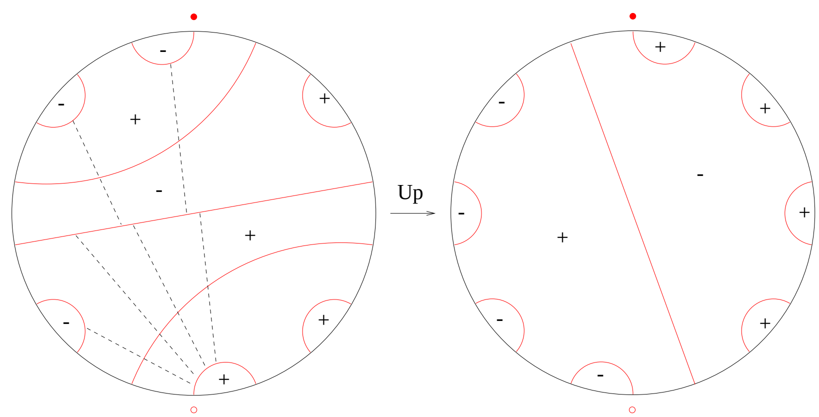


Figure 6.7: "Individual care" approach to $\Gamma_{--++--++} \to \Gamma_{++++----}$.

them relative to the other. If we consider these attaching arcs, placed in the other possible arrangement, we obtain a drastically different result: we go from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$. See figure 6.6.

Thus, the relative positioning of bypass arcs in this way corresponds to some sort of "carrying" or "compounding" phenomenon. Each arc itself moves some string of $-$ signs past some string of $+$ signs. But if two arcs are in this arrangement, those $-$ signs moved right by the first elementary move are then carried in the second also. Alternatively, the "treating each $-$ sign individually" approach here, from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$ requires six bypass arcs: 2 for the first $-$ sign, 2 for the second, 1 for the third, and 1 for the fourth. See figure 6.7.

In general, in the following, we will apply the "take individual care" approach, because it is easier to formalise, even though the sets of bypass moves so obtained often contain massive redundancy. This will lead to the notion of the *coarse bypass system of a pair* of comparable basis chord diagrams, which we will then refine to a "minimal" *bypass system of a pair.*

We will spend the rest of this section formalising all the above constructions.

6.1.2 Elementary moves on words

Given a word w, recall we group it into *blocks* of $+$ and $-$ symbols and hence may write

$$w = (-)^{a_1}(+)^{b_1} \cdots (-)^{a_k}(+)^{b_k}.$$

Possibly $k = 1$; possibly $a_1 = 0$; possibly $b_k = 0$; but every other a_i and b_i is nonzero. So w as written above has $2k$ blocks (or $2k - 1$ or $2k - 2$ blocks if $a_1 = 0$ or/and $b_k = 0$.)

We now make a combinatorial definition of moves on words of $+$ and $-$ symbols.

Definition 6.1.1 (Elementary moves on words) *Let w be a word in the symbols $\{+, -\}$.*

(i) A forwards elementary move *consists of taking a contiguous substring of w of the form $(-)^a(+)^b$ and replacing it with $(+)^b(-)^a$.*

(ii) A backwards elementary move *consists of taking a contiguous substring of w of the form $(+)^b(-)^a$ and replacing it with $(-)^a(+)^b$.*

Collectively we call these elementary moves*.*

The effect of an elementary move is therefore to "slide some $-$'s past some $+$'s". The forwards or backwards nature of the move corresponds to the $-$ signs moving forwards or backwards, as the word is read from left to right. Note that if w' is obtained from w by a forwards elementary move, then $w \preceq w'$; while if w' is obtained from w by a backwards elementary move, then $w' \preceq w$.

Lemma 6.1.2 (Number of elementary moves) *The word*

$$w = (-)^{a_1} (+)^{b_1} \cdots (-)^{a_k} (+)^{b_k}$$

has precisely

$$a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$$

nontrivial forwards elementary moves and

$$b_1 a_2 + b_2 a_3 + \cdots + b_{k-1} a_k$$

nontrivial backwards elementary moves, for a total of

$$a_1 b_1 + b_1 a_2 + a_2 b_2 + \cdots + b_{k-1} a_k + a_k b_k$$

elementary moves.

Proof For a nontrivial forwards move, we must choose a substring of $(-)^{a_i}(+)^{b_i}$ for some i of the form $(-)^A(+)^B$, and move them past each other. There are $a_i b_i$ such substrings. For a backwards move, we must choose a substring of $(+)^{b_i}(-)^{a_{i+1}}$ for some i of the form $(+)^B(-)^A$, and move them past each other. There are $b_i a_{i+1}$ such substrings. ∎

Definition 6.1.3 (Denoting elementary moves)

(i) The (i,j) forwards elementary move *$FE(i,j)$ moves the i'th $-$ sign (from the left), and all the minus signs to its right in the same block, to the position immediately to the right of the j'th $+$ sign (from the left).*

(ii) The (i,j) backwards elementary move *$BE(i,j)$ moves the j'th $+$ sign (from the left), and all the plus signs to its right in the same block, to the position immediately to the right of the i'th $-$ sign (from the left).*

Note that we must have $1 \leq i \leq n_-$ and $1 \leq j \leq n_+$ in this definition. But for i, j satisfying these inequalities, $FE(i,j)$ is not always defined; $FE(i,j)$ is only defined if the i'th $-$ sign is to the left of the j'th $+$ sign, and the j'th $+$ sign lies in block of $+$ symbols to the immediate right of the block with the i'th $-$ sign. For any pair (i, j), at most one of $FE(i, j)$ or $BE(i, j)$ is well-defined.

6.1.3 Anatomy of attaching arcs on basis chord diagrams

Let us define various types of attaching arcs.

Definition 6.1.4 (Types of attaching arcs) *Let c be an attaching arc on a chord diagram Γ.*

(i) If c intersects three distinct chords of Γ, c is nontrivial.

(ii) If c intersects less than three distinct chords of Γ, c is trivial.

- *(a) If c intersects precisely two distinct chords of Γ, c is* slightly trivial.
- *(b) If c intersects only one chord of Γ, c is* supertrivial.

For any trivial arc, performing a bypass move on it in one direction creates a closed curve, while performing a bypass move on it in the other direction leaves the chord diagram unchanged.

Definition 6.1.5 (Upwards, downwards trivial arcs) *A trivial attaching arc on a chord diagram on which*

(i) an upwards bypass move produces the same chord diagram, and a downwards bypass move creates a closed loop, is upwards.

(ii) a downwards bypass move produces the same chord diagram, and an upwards bypass move creates a closed loop, is downwards.

Finally, a supertrivial attaching arc c may come in two types. We consider traversing c from one end to the other; let the three intersection points of c with a chord γ, in some order along c, be p_1, p_2, p_3.

Definition 6.1.6 (Direct, indirect supertrivial arcs) *If the three intersection points p_1, p_2, p_3 lie in order along γ, c is* direct. *Otherwise, c is* indirect.

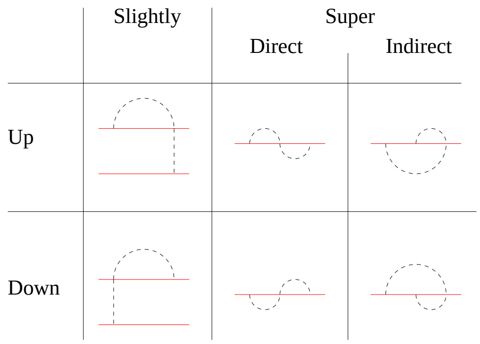


Figure 6.8: Types of trivial attaching arcs.

Our typology of trivial attaching arcs is depicted in figure 6.8.

We now give a complete description of nontrivial attaching arcs on basis chord diagrams. Writing a word w as

$$w = (-)^{a_1} (+)^{b_1} \cdots (-)^{a_k} (+)^{b_k},$$

the corresponding chord diagram Γ_w is as shown in figure 6.9.

Lemma 6.1.7 (Number of arcs of attachment) *There are precisely*

$$a_1 b_1 + b_1 a_2 + a_2 b_2 + \cdots + b_{k-1} a_k + a_k b_k$$

distinct possible nontrivial arcs of attachment on Γ_w*.*

(Recall arcs of attachment are equivalent if they are homotopic in the disc through attaching arcs.)

Proof The proof is based on the observation that the non-outermost chords neatly compartmentalise the disc into pieces.

We will give the proof when $a_1 \neq 0$ and $b_k \neq 0$; the cases where one or both of these are zero is similar. An arc of attachment intersects the chord diagram Γ_w in three points; for a nontrivial arc of attachment, the middle of these must lie on a non-outermost chord. There are precisely $2k-1$ non-outermost chords, corresponding to the $2k-1$ leading symbols in w (other than the very first symbol). Let these non-outermost chords be $c_1, d_1, c_2, d_2, \ldots, c_{k-1}, d_{k-1}, c_k$, respectively from base to root.

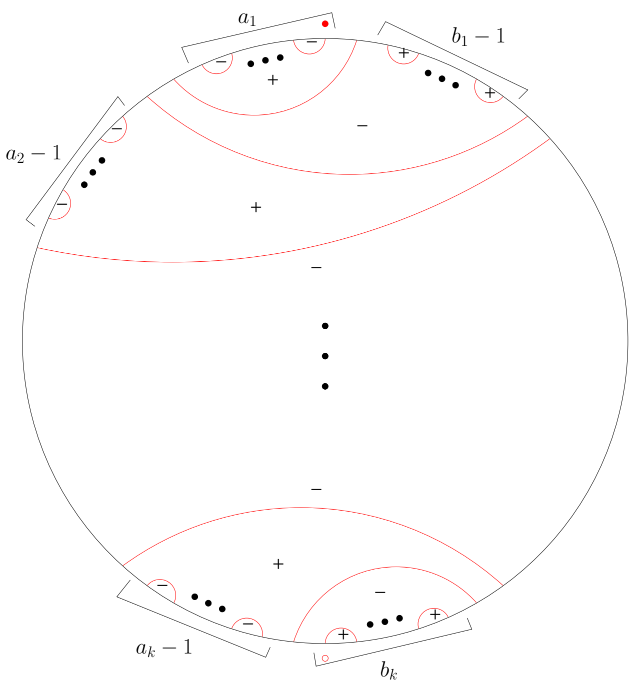


Figure 6.9: General basis chord diagram $(-)^{a_1} \cdots (+)^{b_k}$.

We count the number of nontrivial arcs of attachment with middle intersection point lying on each of these $2k-1$ chords.

Now c_i separates two regions; one of these (towards the base) has boundary with a_i other components of Γ_w; and the other (towards the root) has boundary with b_i other components of Γ_w. Thus there are $a_i b_i$ possible arcs of attachment centred on c_i.

Similarly, d_i separates two regions; one of these (towards the base) has boundary with b_i other components of Γ_w; the other (towards the root) has boundary with a_{i+1}. This gives $b_i a_{i+1}$ possible arcs of attachment centred on d_i. ∎

Suspiciously, the number of nontrivial elementary moves on w equals the number of nontrivial arcs of attachment on Γ_w. There is a nice bijection between them; to formalise this we need several definitions.

A small neighbourhood U of an attaching arc c in a chord diagram Γ is cut by Γ into 4 regions. Two of the components of $U - \Gamma$ intersect c, and two do not; and each component of $U - \Gamma$ lies in some component of $D - \Gamma$. If c is nontrivial, these 4 regions are distinct; if c is trivial, they are not all distinct. Hence we may make the following definition.

Definition 6.1.8 (Inner & Outer regions of attaching arc)

(i) The two components of $U - \Gamma$ which intersect c lie in components of $D - \Gamma$ which we call the inner regions *of c.*

(ii) The two regions of $U - \Gamma$ which do not intersect c lie in components of $D - \Gamma$ which we call the outer regions *of c.*

Clearly one outer region of c is positive and one is negative; similarly for the inner regions.

These regions all have distinct numberings, and so do the chords involved, motivating another definition.

Definition 6.1.9 (Prior, latter chords, regions) *Let c be a nontrivial or slightly trivial arc of attachment on a basis chord diagram. Consider the chords on which its endpoints lie.*

(i) The chord which was created first in the base point construction algorithm is the prior chord *of c.*

(ii) The chord created later is the latter chord *of c.*

Consider the outer regions of c.

(i) The outer region adjacent to its prior chord is the prior outer region *of c.*

(ii) The outer region adjacent to its latter chord is the latter outer region *of c.*

Note that this definition applies to any attaching arc except a supertrivial one.

Noting that the prior and latter outer regions are positive and negative in some order, we define a "direction" for non-supertrivial attaching arcs.

Definition 6.1.10 (Forwards and backwards arcs of attachment) *For a nontrivial arc of attachment:*

(i) it its prior outer region is negative (and latter outer region positive), it is called forwards.

(ii) if its prior outer region is positive (and latter outer region negative), it is called backwards.

For a slightly trivial arc of attachment:

(i) if its prior outer region is negative (and latter outer region positive), it is called quasi-forwards.

(ii) if its prior outer region is positive (and latter outer region negative), it is called quasi-backwards.

Now, for any nontrivial arc of attachment, the prior outer region is certainly not adjacent to the root point, and the latter outer region is not adjacent to the base point. Hence we may make the following definition, making use of base- and root-numbering of regions (definitions 5.1.14 and 5.1.16).

Definition 6.1.11 (Denoting forwards and backwards arcs of attachment)

(i) The nontrivial forwards attaching arc whose prior outer region is the base-i'th $-$ region and whose latter outer region is the root-j'th $+$ region is called the forwards (i, j) attaching arc $FA(i, j)$.

(ii) The nontrivial backwards attaching arc whose prior outer region is the base-j'th $+$ region and whose latter outer region is the root-i'th $-$ region is called the backwards (i, j) attaching arc $BA(i, j)$.

Note that these attaching arcs do not exist for all (i, j). The next lemma answers precisely when they do.

Lemma 6.1.12 (Existence of attaching arcs) *For a given word w, there is a forwards (resp. backwards) (i, j) attaching arc $FA(i, j)$ (resp. $BA(i, j)$) on Γ_w if and only if there is a forwards (resp. backwards) (i, j) elementary move $FE(i, j)$ (resp. $BE(i, j)$) on w.*

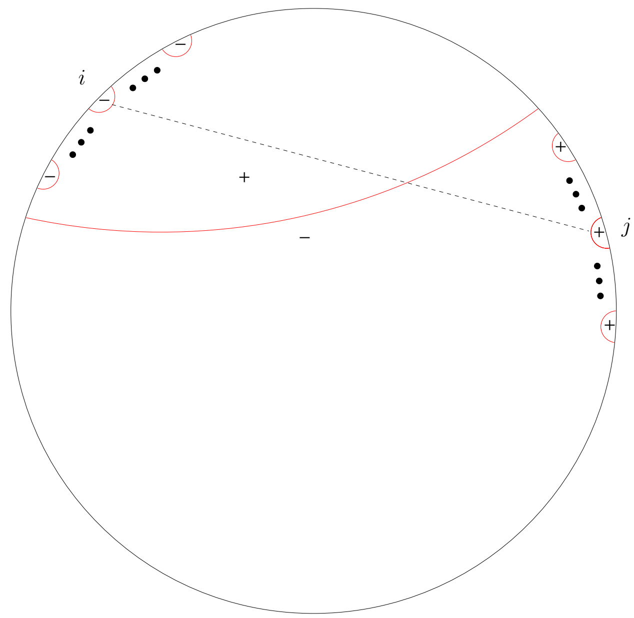


Figure 6.10: Forwards attaching arc $FA(i, j)$.

Proof We prove the forwards case; the backwards case is similar. Suppose there exists such an elementary move; so that the i'th $-$ sign and j'th $+$ sign appear as desired; within blocks of the form $(-)^a(+)^b$. Then, considering the base point construction algorithm, we see that the chord diagram for w contains an arrangement as shown in figure 6.10.

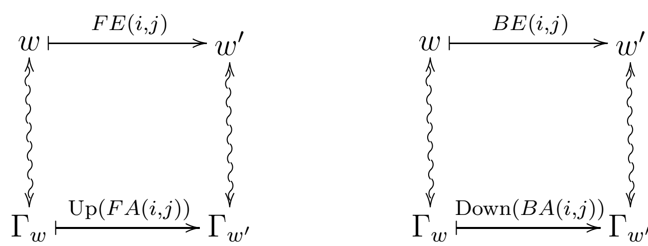
Thus there is a forwards or backwards (i, j) attaching arc, as desired. Conversely, any forwards or backwards (i, j) attaching arc comes in this arrangement, and hence there is a forwards or backwards (i, j) elementary move, as desired. ∎

Remark 6.1.13 (Forwards and backwards analogous) *Throughout this section we have constructions and lemmas which come in two varieties, one "forwards version" and one "backwards version". For the most part the backwards versions are entirely analogous to the forwards versions. To save space, we will often give arguments, and sometimes statements, for the forwards version only; but state the final results for both forwards and backwards versions.*

6.1.4 Single bypass moves and elementary moves

We now give the bijection between bypass moves on basis chord diagrams and elementary moves on words. This gives a complete combinatorial description of nontrivial bypass moves.

Lemma 6.1.14 (Bypass & elementary moves) *The chord diagram which is obtained from* Γ_w *by an upwards bypass move along* $FA(i,j)$ *(resp. downwards along* $BA(i,j)$*) is the basis chord diagram* $\Gamma_{w'}$*, where* $w' = FE(i,j)(w)$ *(resp.* $BE(i,j)(w)$*).*



In the other direction, a downwards bypass move along $FA(i,j)$ *(resp. upwards along* $BA(i,j)$*) gives* $\Gamma_w + \Gamma_{w'}$*.*

Proof Consider the (i,j) forwards elementary move $FE(i,j)$ on w and the forwards attaching arc $FA(i,j)$ on Γ_w. By lemma 6.1.12, one of these exists if and only if the other does. So the i'th $-$ sign and j'th $+$ sign in w occur in consecutive blocks, with the j'th $+$ sign in the block to the right of the block containing the i'th $-$ sign. In Γ_w, then, we have the situation depicted in figure 6.11, where we number the base-i'th $-$ region (and adjacent base $-$ regions) and the root-j'th $+$ region (and adjacent root $+$ regions). Let the last $-$ sign in the block with the i'th $-$ sign be the l'th $-$ sign in w (so $l \geq i$, possibly $l = i$), and let the first $+$ sign in the block with the j'th $+$ sign be numbered m (so $m \leq j$, possibly $m = j$).

An upwards bypass move along $FA(i,j)$ then has the effect shown. This has the effect of producing a basis chord diagram for the word w', where w' is obtained from w by swapping the string of i'th thru l'th $-$ signs with the string of m'th thru j'th $+$ signs, $(-)^{l-i+1}(+)^{j-m+1} \mapsto (+)^{j-m+1}(-)^{l-i+1}$. Thus w' is precisely the word obtained from w by moving the i'th $-$ sign (and all $-$ signs to the right of the i'th one, in the same block) past the j'th $+$ sign, i.e. by $FE(i,j)$.

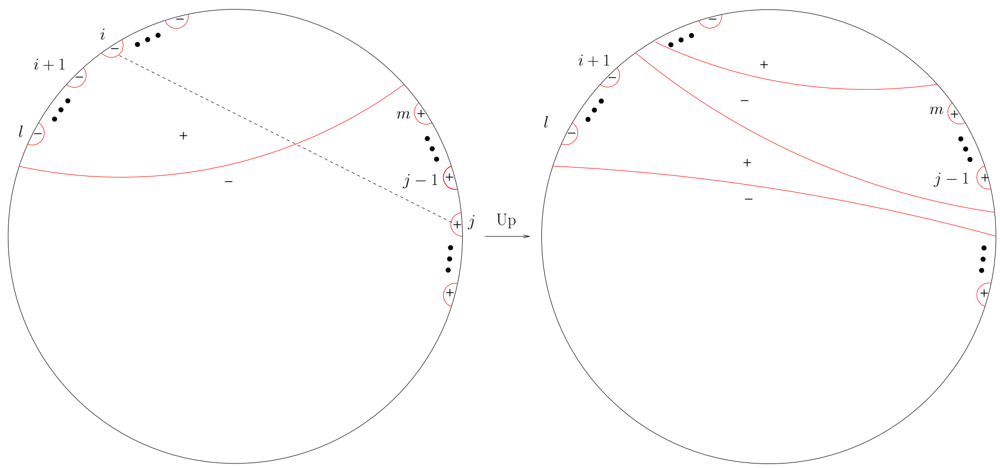


Figure 6.11: Effect of bypass move along $FA(i, j)$.

A similar argument works for backwards arcs of attachment and backwards elementary moves. The bypass relation then gives the final statement. ∎

In particular, performing a bypass move on a basis chord diagram gives either a basis diagram or a sum of two basis diagrams.

6.1.5 Stability of basis diagrams

We can now show how basis chord diagrams remain "stable" as we perform certain bypass moves on them. The idea is that, beginning with forwards attaching arcs and performing upwards bypass moves, we always remain within the class of basis chord diagrams. However, there are some technicalities, since phenomena like the following may occur:

- a bypass move on an attaching arc c_i may convert a nontrivial attaching arc c_j into a trivial one, or a slightly trivial c_j into a supertrivial one, or vice versa;
- a bypass move on a trivial attaching arc, while not changing the chord diagram, may change the locations of the other attaching arcs.

We must therefore take some care in the following lemma; it is for this reason that the anatomical terminology of "supertrivial", "quasi-forwards", "upwards trivial" and "direct supertrivial" has been introduced.

Lemma 6.1.15 *Suppose we have a bypass system $\{c_1, \ldots, c_m\}$ on a basis chord diagram, where each c_i is one of the following:*

(i) a nontrivial forwards attaching arc;

(ii) a slightly trivial, quasi-forwards, upwards attaching arc;

(iii) a supertrivial, direct, upwards attaching arc.

After performing an upwards bypass move along c_1, we still have a basis chord diagram, and each remaining attaching arc is of one of the above three types.

There is also a backwards version.

Proof By lemma 6.1.14 or (upwards) triviality of c_1, after performing the bypass move on c_1 we still have a basis chord diagram. It remains to show that each c_i other than c_1 remains one of the three specified types. We consider each of the possible $3 \times 3 = 9$ cases for c_1 and c_i.

(i) c_1 **nontrivial.**

(a) c_i **nontrivial.** The arc c_i intersects either $0, 1, 2$ or 3 of the same three chords as c_1. If it intersects none of the same chords, then their order of construction in the base point algorithm remains unchanged, so that c_i remains nontrivial and forwards. If c_i intersects one of the same chords as c_1, then the situation must be one of those shown in figure 6.12, with order of the chords as shown; c_i remains nontrivial and forwards.

If c_i intersects two of the same chords as c_1, then it either becomes slightly trivial, quasi-forwards, and upwards, as in figure 6.4; or the situation is as shown in figure 6.13, and c_i remains nontrivial and forwards.

If c_i intersects all three of the same chords as c_1, then it clearly becomes slightly trivial, upwards, and quasi-forwards.

Note that the cases in figures 6.12 and 6.13 come in pairs related by 180° rotations, with reversal of signs and numberings. For the rest of this argument, we only give one of each such pair of cases.

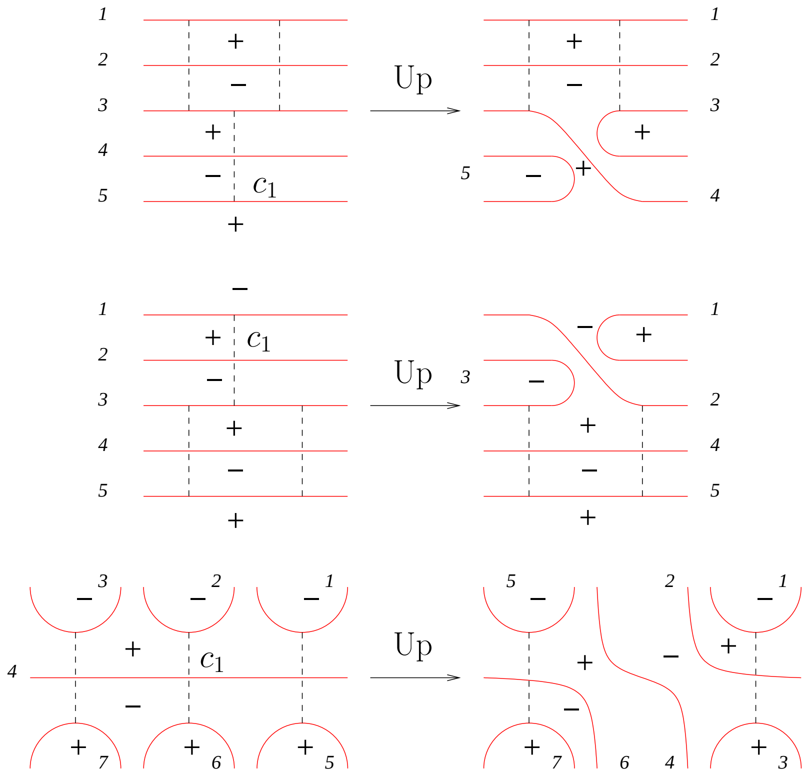


Figure 6.12: Case (i)(a), c_1, c_i with one common chord.

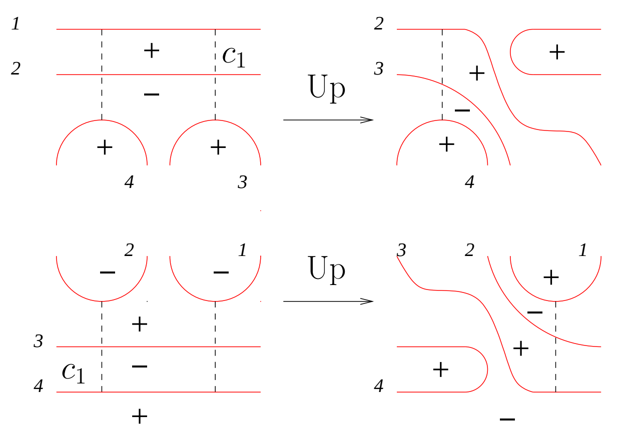


Figure 6.13: Case (i)(a), c_1, c_i with two common chords.

(b) c_i **slightly trivial.** By definition, c_i intersects two distinct chords of Γ. If neither of these chords intersects c_1, then c_i clearly remains slightly trivial, upwards, and quasi-forwards. If both of these chords intersect c_1, then the quasi-forwards and upwards conditions require the situation to be as in figure 6.14(a) (or a 180° rotated version of it), so that c_i becomes supertrivial direct upwards or remains slightly trivial upwards quasi-forwards. We may therefore assume that precisely one chord γ intersects both c_1 and c_i. If c_i only intersects γ once, then it's clear c_i remains slightly trivial upwards quasi-forwards. Thus we may assume c_i intersects γ twice. If the two intersections of c_i with γ lie on the same side of c_1, then it's clear c_i remains slightly trivial upwards quasi-forwards. If the two intersections of c_i with γ lie on opposite sides of c_1, then the upwards quasi-forwards conditions require that the situation is as in figure 6.14(b) (or a 180° rotation of it). Thus c_i becomes nontrivial forwards.

(c) c_i **supertrivial.** Here c_i only intersects one chord γ of Γ. If γ is disjoint from c_1, clearly c_i remains supertrivial upwards direct; we therefore assume c_1 intersects γ. If the intersection points of c_i with γ all lie on the same side of c_1, clearly c_i remains supertrivial upwards direct. If the intersection points of c_i with γ lie on both sides of c_1, then the upwards direct conditions require that the situation is as in figure 6.14(c) (or a 180° rotation of it); so c_i becomes slightly trivial upwards quasi-forwards.

(ii) c_1 **slightly trivial.** In this case c_1 intersects an chord of Γ twice; let γ be the arc of this chord lying between the intersection points. If c_i does not intersect γ, then after performing the upwards bypass move on c_1, the chord diagram is unchanged and the position of c_i is unchanged. We can therefore assume that c_i intersects γ.

(a) c_i **nontrivial.** The conditions that c_i is nontrivial, forwards, and intersects γ, require that the situation is as in figure 6.15(a) (or a 180°-reversed version). Together with the condition that c_1 is quasi-forwards, the ordering of chords must be as shown. We see that c_i remains nontrivial forwards,

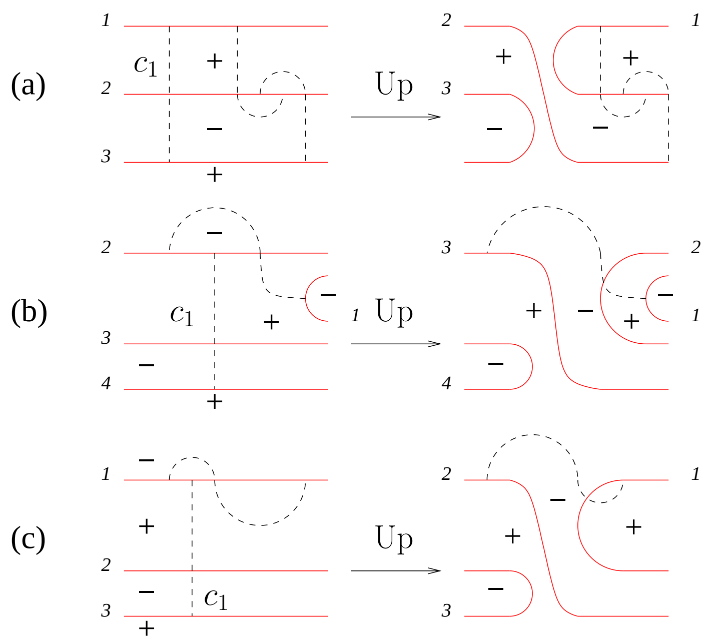


Figure 6.14: Cases (i)(b) and (i)(c).

although in a different position on the chord diagram.

(b) c_i **slightly trivial.** Clearly c_i intersects γ once or twice. Suppose c_i intersects γ twice. Then the condition that c_i is upwards quasi-forwards requires that the situation is one of those depicted in figure 6.15(b) (or a rotated version), with ordering of chords as shown. Thus c_i either remains slightly trivial upwards quasi-forwards, or becomes supertrivial upwards direct. Now suppose c_i intersects γ once. If c_i intersects both the same chords as c_1 this contradicts quasi-forwardness of c_i; thus the situation is as in figure 6.15(c) (or a rotated version). Thus c_i remains slightly trivial upwards quasi-forwards.

(c) c_i **supertrivial.** Here c_i may intersect γ 1, 2 or 3 times. If it intersects γ once, then the direct upwards condition requires that the situation is as in figure 6.15(d), so c_i becomes slightly trivial upwards quasi-forwards. The direct upwards condition precludes two intersections of c_i with γ. If c_i intersects γ 3 times, clearly c_i remains upwards direct supertrivial.

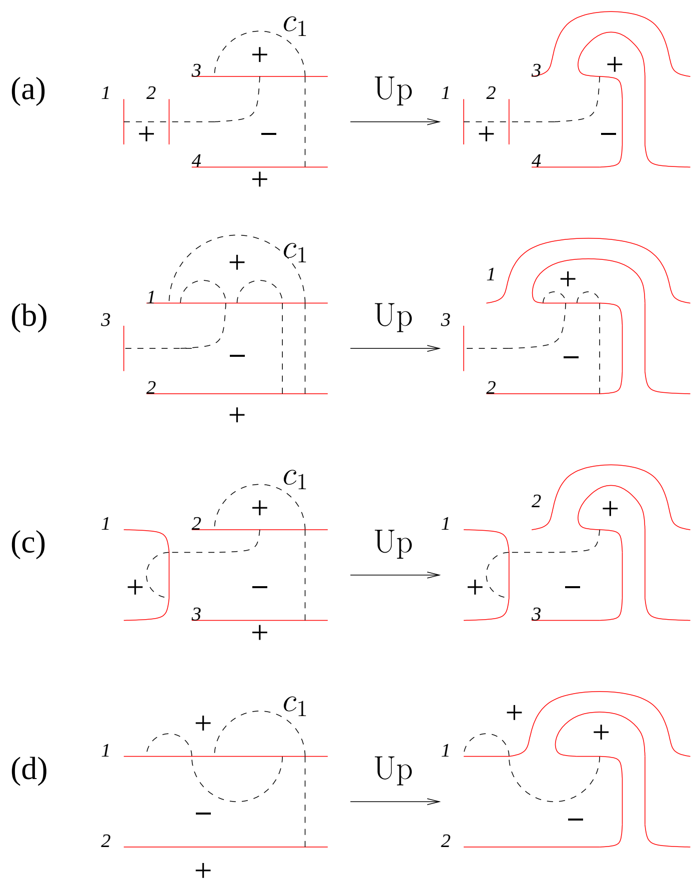


Figure 6.15: Case (ii).

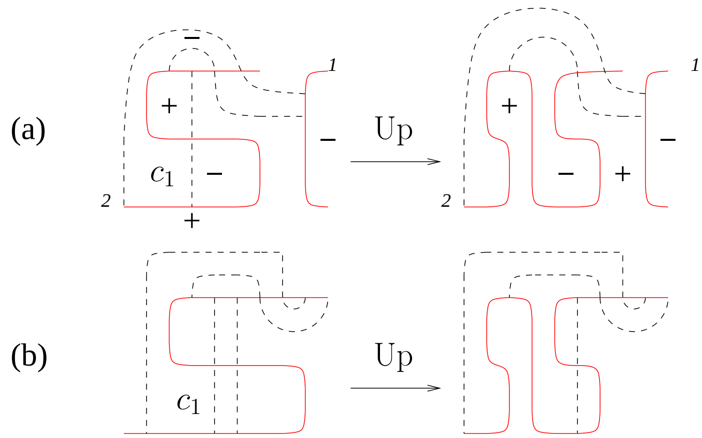


Figure 6.16: Case (iii).

(iii) c_1 **supertrivial.** In this case c_1 only intersects one chord γ of Γ. If c_i does not intersect γ, or intersects γ only once, then after performing the upwards bypass move on c_1, the chord diagram and the position of c_i is unchanged. We can therefore assume that c_i intersects γ at least twice; hence we only need consider c_i trivial.

(a) c_i **slightly trivial.** Clearly c_1 cuts γ into 4 arcs, and we are assuming c_i intersects γ twice. If the two intersection points of c_i with γ lie in the same component, clearly c_i remains slightly trivial upwards quasi-forwards. Thus we may assume c_i intersects γ in different arcs of γ, and the situation is as in figure 6.16(a) (or a rotated version). Hence c_i remains slightly trivial upwards quasi-forwards.

(b) c_i **supertrivial.** Again c_1 cuts γ into 4 arcs, and if c_i intersects γ all within the same arc, clearly c_i remains supertrivial direct upwards. Thus we assume c_i intersects γ in distinct arcs; the direct upwards condition requires that the situation is one of those depicted in figure 6.16(b) (or a rotated version). In every possibility c_i remains upwards direct supertrivial. ∎

This lemma now easily gives the following useful result.

Proposition 6.1.16 *The effect of performing upwards (resp. downwards) bypass moves on a basis chord diagram* Γ_{w_1} *along a bypass system consisting entirely of forwards (resp. backwards) attaching arcs is again a basis chord diagram* Γ_{w_2}*, with* $w_1 \preceq w_2$ *(resp.* $w_2 \preceq w_1$*).*

Proof Clearly the bypass system satisfies the hypotheses of lemma 6.1.15. Thus, after each bypass move, the bypass system still satisfies those hypotheses and the chord diagram remains a basis diagram. The bypass moves along trivial arcs have no effect on the chord diagram (although they may affect the bypass system); by lemma 6.1.14 each bypass move along a nontrivial forwards attaching arc produces another basis chord diagram, moving ahead in the partial order. ■

6.1.6 Generalised elementary moves on words

So far, we have defined *elementary moves* $FE(i,j)$ and $BE(i,j)$ on a word w. The forwards move $FE(i,j)$ moves the i'th $-$ sign (and all $-$ signs between it and the j'th $+$ sign) to the right, past the j'th $+$ sign: *provided that the j'th $+$ sign is in the block immediately to the right of the i'th $-$ sign.* This is a useful notion because it corresponds precisely to bypass moves on the chord diagram Γ_w. But we now generalise this notion, removing the somewhat artificial proviso in italics, to what we call *generalised elementary moves.*

Definition 6.1.17 (Generalised elementary move) *Let w be a word on $\{-,+\}$ with n_- $-$ signs and n_+ $+$ signs. Let $1 \leq i \leq n_-$ and $1 \leq j \leq n_+$.*

(i) If the i'th $-$ sign in w occurs to the left of the j'th $+$ sign, we define the forwards generalised elementary move $FE(i,j)$ *to take the i'th $-$ sign, and all $-$ signs between it and the j'th $+$ sign, and move them to a position immediately after the j'th $+$ sign.*

(ii) If the j'th $+$ sign in w occurs to the left of the i'th $-$ sign, we define the backwards generalised elementary move $BE(i,j)$ *to take the j'th $+$ sign, and all $+$ signs between it and the i'th $-$ sign, and move them to a position immediately after the i'th $-$ sign.*

It's clear that when the i'th $-$ sign and the j'th $+$ sign are in adjacent blocks, generalised elementary moves reduce to elementary moves. So we indeed have a generalisation, and we may use the same notation without contradiction.

To illustrate: if $w = - - - + + - ++$, then $FE(2,3)$ produces $- + + + - - - +$, and $BE(4,1)$ produces the word $- - - - + + ++$.

Since forwards generalised elementary moves move $-$ signs to the right, and backwards generalised elementary moves move $+$ signs to the right, the following lemma is clear.

Lemma 6.1.18 (Forwards moves move forward) *If w' can be obtained from w by a generalised forwards (resp. backwards) elementary move, then $w \preceq w'$ (resp. $w' \preceq w$).* ∎

Note that for any word $w \in W(n_-, n_+)$ and for any $1 \leq i \leq n_-$, $1 \leq j \leq n_+$, precisely one of $FE(i,j)$ or $BE(i,j)$ exists.

We have seen that a (forwards or backwards) elementary move on a word can be effected on a basis chord diagram by a single upwards bypass move along a (forwards or backwards) arc of attachment. It is also true that a (forwards or backwards) generalised elementary move can be effected by upwards bypass moves along (forwards or backwards) arcs of attachment — but more than one is required. We will now see how.

6.1.7 Generalised arcs of attachment

Recall that we defined forwards and backwards arcs of attachment $FA(i,j)$, $BA(i,j)$. The forwards arc $FA(i,j)$ connects the base-i'th $-$ region to the root-j'th $+$ region. The backwards arc $BA(i,j)$ connects the base-j'th $+$ region to the root-i'th $-$ region. We will now generalise this notion.

Definition 6.1.19 (Generalised arc of attachment) *A* generalised arc of attachment *in a chord diagram Γ is an arc which intersects Γ in an odd number of points, including both its endpoints. A generalised arc of attachment is* nontrivial *if all its intersection points with Γ lie on different components of Γ. Two generalised arcs of* *attachment are considered equivalent if they are homotopic through generalised arcs of attachment.*

It is clear that for any two chords in a chord diagram Γ, there is at most one nontrivial generalised arc of attachment between them, up to equivalence. As with (ungeneralised) arcs of attachment, we will usually implicitly consider generalised arcs of attachment up to equivalence, and speak of *the* generalised arc of attachment between two chords.

We have notions of prior and latter chords, prior and latter outer regions, and forwards and backwards, as for bona fide arcs of attachment. The two endpoints of a nontrivial generalised attaching arc c lie on two chords; the chord created first in the base point algorithm is its *prior chord*, the chord created later its *latter chord*. The complementary region of Γ not intersecting c but adjacent to its prior chord is its *prior outer region*; the region not intersecting c but adjacent to its latter chord is its *latter outer region*. Given a prior outer region and a latter outer region for c, it's clear that there is at most one nontrivial generalised attaching arc between them; hence we may speak of *the* generalised arc of attachment between the two regions. A generalised attaching arc with negative (resp. positive) prior outer region and positive (resp. negative) latter outer region is called *forwards* (resp. *backwards*).

The forwards generalised attaching arc whose prior outer region is the base-i'th $-$ region and whose latter outer region is the root-j'th $+$ region is called *the forwards* (i, j) *generalised attaching arc* $FA(i, j)$. The backwards generalised attaching arc whose prior outer region is the base-j'th $+$ region and whose latter outer region is the root-i'th $-$ region is called *the backwards* (i, j) *generalised attaching arc* $BA(i, j)$. Clearly this notation generalises the notation for attaching arcs, and clearly every nontrivial generalised attaching arc is of the form $FA(i, j)$ or $BA(i, j)$ for some (i, j).

In fact, for any pair (i, j), $1 \leq i \leq n_-$, $1 \leq j \leq n_+$ precisely one of $FA(i, j)$ or $BA(i, j)$ exists, as the next lemma makes clear.

Lemma 6.1.20 (Existence of generalised attaching arcs)

(i) There is an $FA(i, j)$ in Γ_w iff the i'th $-$ sign in w occurs before the j'th $+$ sign, iff there is an $FE(i, j)$ on w.

(ii) There is a $BA(i,j)$ in Γ_w iff the j'th $+$ sign in w occurs before the i'th $-$ sign, iff there is a $BE(i,j)$ on w.

Proof By definition, $FA(i,j)$ has prior outer region the base-i'th $-$ region, and latter outer region the root-j'th $+$ region; and by definition of prior and latter, and a little consideration of the relationship between the base and root algorithms, we see that if $FA(i,j)$ exists, then the i'th $-$ sign must occur before the j'th $+$ sign.

Conversely, suppose the i'th $-$ sign occurs before the j'th $+$ sign. Then in the base point construction algorithm, the i'th $-$ sign produces a chord γ_i, enclosing a negative region r_i. The j'th $+$ sign produces a chord γ', and the next symbol in w (or the final chord drawn in the algorithm) produces a chord γ_j; this chord is produced in the root point algorithm by the j'th $+$ sign, and encloses a positive region r_j in that algorithm. Since the base point algorithm produces γ_i before γ' before γ_j, it cannot be that r_i and r_j are adjacent. Therefore, there is a generalised attaching arc connecting γ_i to γ_j.

The proof is similar for backwards attaching arcs. ∎

Clearly, a generalised attaching arc is not something that we can perform a bypass move on. But from it, we can obtain a bypass system, and then perform bypass moves.

6.1.8 Bypass system of a generalised attaching arc

Since a generalised attaching arc intersects the chords of a chord diagram at an odd number of points, it may be broken into several bona fide attaching arcs which overlap only at endpoints. We can then perturb these endpoints in a specified way so that they become disjoint.

To make this precise, let c be a nontrivial generalised attaching arc in a basis chord diagram Γ_w. Let p be the intersection point of c with a chord γ of Γ_w, at an interior point of c. Then there is a "prior" and a "latter" direction along c from p, towards the endpoints of c on its prior and latter chords, respectively. Also, since c intersects distinct chords of Γ_w other than γ, in both directions from p, γ cannot be an outermost chord. Thus by our classification of chords in basis chord diagrams (lemma 5.1.9), γ runs from the westside to the eastside of Γ_w. Hence, from p, there is a well-defined "west" and "east" direction along γ.

Definition 6.1.21 (Bypass system of generalised arc) *Suppose c is a nontrivial generalised attaching arc in a basis chord diagram Γ_w which intersects Γ_w in $2m+1$ points. Then there is a unique way to split c into a series of attaching arcs $c_1, \ldots, c_m$, labelled from prior chord to latter chord, which intersect each other only at the endpoints. The* bypass system *of c is given as follows.*

(i) If c is a forwards generalised attaching arc, then $c_1, \ldots, c_m$ are forwards attaching arcs, and we perturb them as follows. At the intersection point p of c_i and c_{i+1} on a non-outermost chord γ of Γ_w, we move the endpoint of c_i slightly west of p along γ, and the endpoint of c_{i+1} slightly east of p along γ.

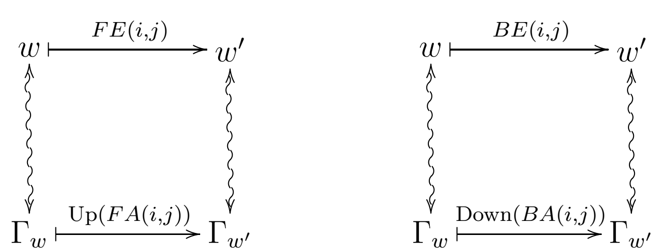
(ii) If c is a backwards generalised attaching arc, then $c_1, \ldots, c_m$ are backwards attaching arcs, and we perturb them as follows. At the intersection point p of c_i and c_{i+1} on a non-outermost chord γ of Γ_w, we move the endpoint of c_i slightly east of p along γ, and the endpoint of c_{i+1} slightly west of p along γ.

It's clear that this is indeed a bypass system. See figure 6.17 for an example. We now show that this edifice of definitions (and we have more to come!) is meaningful.

Lemma 6.1.22 (Generalised attaching arcs and elementary moves)

(i) Performing upwards bypass moves on Γ_w along the bypass system of $FA(i,j)$ gives $\Gamma_{w'}$, where $w' = FE(i,j)(w)$.

(ii) Performing downwards bypass moves on Γ_w along the bypass system of $BA(i,j)$ gives $\Gamma_{w'}$, where $w' = BE(i,j)(w)$.



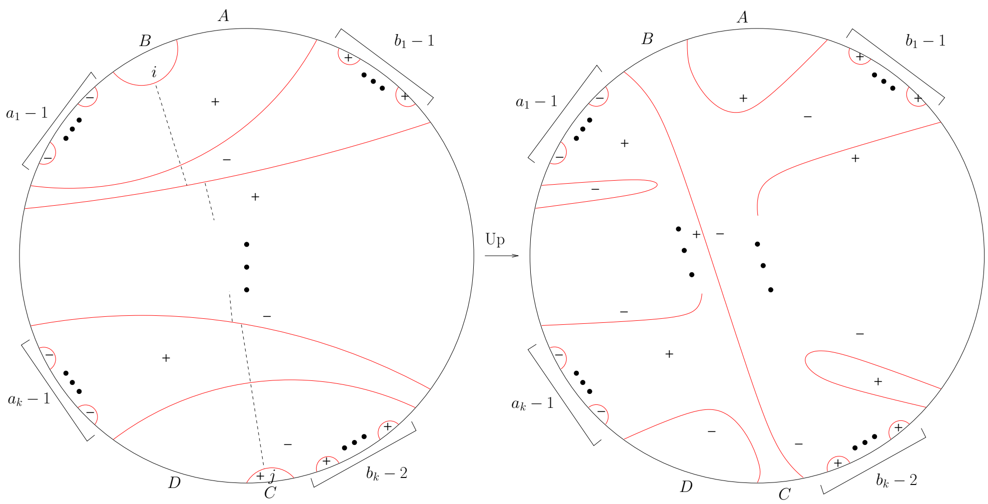


Figure 6.17: Effect of upwards bypass moves on a forwards generalised attaching arc.

Proof From lemma 6.1.20, $FA(i,j)$ exists precisely when $FE(i,j)$ does, i.e. when the i'th $-$ sign occurs before the j'th $+$ sign in w. First suppose that $j < n_+$, so there is a $(j+1)$'th $+$ sign in w. Let the substring of w between the i'th $-$ sign and the $(j+1)$'th $+$ sign be $(-)^{a_1}(+)^{b_1}\cdots(+)^{b_{k-1}}(-)^{a_k}(+)^{b_k}$. Then the situation appears as shown in figure 6.17. (Note that the chords constructed prior to the base-i'th $-$ chord lie in regions A or B accordingly as the i'th $-$ sign is following or leading; similarly, the chords constructed after the root-j'th $+$ chord lie in C or D accordingly as the j'th $+$ sign is the last in its block, or not.)

Performing upwards bypass moves along the arcs of attachment produces the result shown in figure 6.17, which corresponds to replacing the substring

$$(-)^{a_1}(+)^{b_1}\cdots(+)^{b_{k-1}}(-)^{a_k}(+)^{b_k}$$

of w with

$$(+)^{b_1+\cdots+b_k-1}(-)^{a_1+\cdots+a_k}(+).$$

That is, all the minus signs from the i'th $-$ sign, up to the j'th $+$ sign, have been moved to the immediate right of the j'th $+$ sign.

If $j = n_+$, the the picture is similar; the generalised arc of attachment has an endpoint on the chord created in the root point algorithm as the rightmost + sign in w is processed. The effect is to move all minus signs, from the i'th onwards, to the end of the word.

The backwards case is similar. ■

Loosely, the effect of performing upwards (resp. downwards) bypass moves along the bypass system of a forwards (resp. backwards) generalised attaching arc is to create one "long chord" running all along the generalised attaching arc, and "closing off" all the chords on either side of it.

6.1.9 Anatomy of multiple generalised arcs of attachment

We now consider taking several disjoint generalised arcs of attachment, and performing bypass moves along their bypass systems.

Note that for any two given forwards arcs of attachment $FA(i_1, j_1)$ and $FA(i_2, j_2)$, there is not always a unique way to place their bypass systems. For one thing, the two generalised arcs might intersect. Even if they do not intersect, it might be that having placed $FA(i_1, j_1)$, we can place $FA(i_2, j_2)$ on either side of it; and the results of performing bypass moves along the resulting bypass systems might be different.

Thus, when dealing with several generalised arcs of attachment, we need to specify precisely how they are placed. To this end, let us make some definitions. Note that a forwards generalised arc of attachment $FA(i, j)$, taken together with its prior and latter chords, splits the disc D into four regions; we now group these into "southwest" and "northeast" halves.

Definition 6.1.23 (Compass for generalised arc) *A forwards generalised arc of attachment $FA(i, j)$, together with its prior and latter chords, split the disc D into four pieces, proceeding clockwise around the disc:*

(i) The piece containing the prior outer region of $FA(i, j)$.

(ii) The piece which contains the marked points on the eastside immediately anti-clockwise/right of the latter chord of $FA(i, j)$

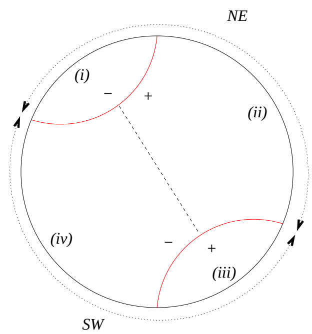


Figure 6.18: Compass points.

(iii) The piece containing the latter outer region of $FA(i, j)$.

(iv) The piece which contains the marked points on the westside immediately anti-clockwise/left of the prior chord of $FA(i, j)$.

Pieces (i) and (ii) are called the northeast *of $FA(i, j)$. Pieces (iii) and (iv) are called the* southwest *of $FA(i, j)$.*

See figure 6.18. Note that all chords and regions constructed in the base point construction algorithm prior to the base-i'th $-$ region lie in the northeast of $FA(i, j)$, and all chords and regions created after the root-j'th $+$ region lie to the southwest.

Similar definitions of "northwest" and "southeast" exist in the backwards case. The compass points are chosen with our westside/eastside in mind, thinking of the base point as the "north pole" and the root point as the "south pole".

6.1.10 Placing two generalised arcs of attachment

We now use these compass points to place multiple generalised arcs of attachment disjointly. We will consider the case of $FA(i_1, j_1)$ and $FA(i_2, j_2)$, where $i_1 < i_2$ and $j_1 \leq j_2$.

Recall that $FA(i_2, j_2)$ joins the base-i_2'th $-$ region to the root-j_2'th $+$ region. Now from the base point construction algorithm, since $i_1 < j_1$, the base-i_2'th $-$ region either lies

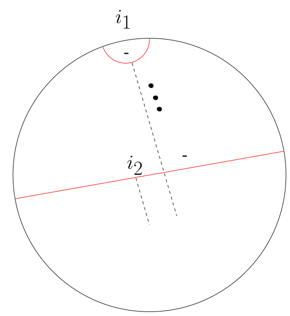


Figure 6.19: Placing two arcs: $i_1 < i_2$.

(i) entirely in the southwest of $FA(i_1, j_1)$; in this case there is no choice for the prior endpoint of $FA(i_2, j_2)$, up to equivalence; or

(ii) in both the southwest and northeast regions of $FA(i_1, j_1)$, and there is a choice: the prior endpoint of $FA(i_2, j_2)$ may be southwest or northeast of $FA(i_1, j_1)$.

If there is a choice, we choose southwest. See figure 6.19 showing case (ii).

Similarly, since $j_1 \leq j_2$, the root-j_2'th + region either lies

(i) entirely in the southwest of $FA(i_1, j_1)$, and identical to the root-j_1'th + region (i.e. $j_1 = j_2$); so there is a choice how to place the latter endpoint of $FA(i_2, j_2)$, which may be southwest or northeast of $FA(i_1, j_1)$ along the same latter chord;

(ii) entirely in the southwest of $FA(i_1, j_1)$ in region (iv) of figure 6.18; so there is no choice how to place the latter endpoint of $FA(i_2, j_2)$; or

(iii) entirely in the southwest of $FA(i_1, j_1)$ in region (iii) of figure 6.18, so $FA(i_2, j_2)$ might lie southwest or northeast of $FA(i_1, j_1)$.

If there is a choice, we always take the southwest choice. See figure 6.20 showing case (iii). Note that the conditions $i_1 < i_2$ and $j_1 \leq j_2$, with our "southwest choices", ensure that the arcs can be drawn disjoint from each other: the inequalities ensure that the prior and latter chords at least partially lie in the southwest of $FA(i_1, j_1)$.

So, with the above choices if necessary, we have a well-defined way to draw $FA(i_2, j_2)$ after $FA(i_1, j_1)$. It is "southwestmost".

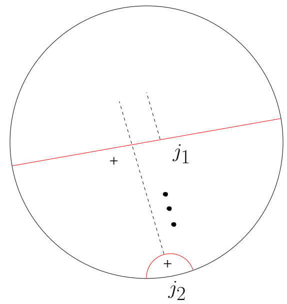


Figure 6.20: Placing two arcs: $j_1 < j_2$.

Lemma 6.1.24 (Placing two generalised arcs) *Let w be a word such that forwards generalised elementary moves $FE(i_1, j_1)$ and $FE(i_2, j_2)$ exist on w, where $i_1 < i_2$ and $j_1 \leq j_2$. There is one and only one way (up to equivalence) to draw $FA(i_1, j_1)$ and $FA(i_2, j_2)$ on Γ_w so that*

(i) $FA(i_1, j_1)$ and $FA(i_2, j_2)$ are disjoint;

(ii) for every chord of Γ_w which intersects both $FA(i_1, j_1)$ and $FA(i_2, j_2)$, the intersection point with $FA(i_2, j_2)$ lies southwest of $FA(i_1, j_1)$. ■

There is a similar result for backwards generalised attaching arcs also.

6.1.11 Placing multiple nicely ordered generalised arcs

We actually wish to consider multiple generalised arcs of attachment, which are "nicely ordered" in a similar way.

Definition 6.1.25 (Nicely ordered generalised arcs of attachment)

(i) A sequence of forwards generalised arcs of attachment

$$FA(i_1, j_1),\ FA(i_2, j_2), \ldots, FA(i_m, j_m)$$

on Γ_w is nicely ordered *if*

$$i_1 < i_2 < \cdots < i_m \quad \textit{and} \quad j_1 \leq j_2 \leq \cdots \leq j_m.$$

(ii) *A sequence of backwards generalised arcs of attachment*

$$BA(i_1, j_1),\ BA(i_2, j_2), \ldots, BA(i_m, j_m)$$

on Γ_w *is* nicely ordered *if*

$$j_m < j_{m-1} < \cdots < j_1 \quad \text{and} \quad i_m \leq i_{m-1} \leq \cdots \leq i_1.$$

Given a set of generalised attaching arcs, which are nicely ordered in this way, we now describe how to place them, giving a bypass system, following the ideas of section 6.1.10. We keep making the "southwest" choice any time we are faced with a choice, and then there is only one possibility for placing all the generalised arcs of attachment. For backwards arcs the result is similar, placing arcs from $BA(i_m, j_m)$ to $BA(i_1, j_1)$, always taking the "northwest" choice.

Lemma 6.1.26 (Arrangement of 3 nicely ordered arcs) *Let*

$$FA(i_1, j_1),\ FA(i_2, j_2),\ FA(i_3, j_3)$$

be a nicely ordered sequence of forwards generalised attaching arcs. Suppose that both $FA(i_1, j_1)$ *and* $FA(i_3, j_3)$ *intersect a chord of* Γ_w*. Then so does* $FA(i_2, j_2)$*.*

Proof In general, $FA(i, j)$ intersects precisely the following chords:

(i) The chord created by processing the i'th $-$ in w in the base point construction algorithm (or the symbol immediately preceding it, in the root point construction algorithm). This is the prior chord.

(ii) The chord created by processing the j'th $+$ in w in the root point construction algorithm (or the symbol immediately following it, in the base point construction algorithm). This is the latter chord.

(iii) The non-outermost chords created by processing "changes of sign" in w in between: more precisely, processing the leading $-$ and $+$ signs strictly after the

j'th $+$ sign and up to and including the i'th $-$ sign in the base point construction algorithm;

If both $FA(i_1, j_1)$ and $FA(i_3, j_3)$ intersect a chord γ, then in the base point construction algorithm, it is created by processing some symbol, which is one of:

(i) (from $FE(i_1, j_1)$) either the i_1'th $-$ sign, or a leading $-$ or $+$ sign after the i_1'th minus sign, up to and including the j_1'th $+$ sign, or the symbol immediately after the j_1'th $+$ sign;

(ii) (from $FE(i_3, j_3)$) either the i_3'th $-$ sign, or a leading $-$ or $+$ sign after the i_3'th minus sign, up to and including the j_3'th $+$ sign, or the symbol immediately after the j_3'th $+$ sign.

Combining these, since $i_1 < i_3$, and $j_1 \leq j_3$, γ must be created by a symbol which is either:

(i) the i_3'th $-$ sign, which is leading;

(ii) a leading $-$ or $+$ sign after the i_3'th $-$ sign, up to and including the j_1'th $+$ sign; or

(iii) the symbol immediately after the the j_3'th $+$ sign, and $j_1 = j_3$.

In every case, since $i_1 < i_2 < i_3$ and $j_1 \leq j_2 \leq j_3$, this chord γ also intersects $FA(i_2, j_2)$. ■

So now, suppose we have a nicely ordered sequence of forwards generalised arcs of attachment

$$FA(i_1, j_1),\ FA(i_2, j_2),\ \ldots,\ FA(i_m, j_m).$$

We know that we can place $FA(i_1, j_1)$, and then place $FA(i_2, j_2)$ "southwest" of it, as described in lemma 6.1.24. Then, we wish to place $FA(i_3, j_3)$. Let γ be a chord of Γ_w that intersects $FA(i_3, j_3)$. The previous lemma (6.1.26) governs how the forwards generalised arcs of attachment can intersect it. In particular, if either of $FA(i_1, j_1)$ or $FA(i_2, j_2)$ intersects γ, then $FA(i_2, j_2)$ certainly does. So if we require the intersection point of $FA(i_3, j_3)$ with γ to be southwest of $FA(i_2, j_2)$, then it is also southwest of $FA(i_1, j_1)$.

Proceeding inductively, we obtain the following lemma.

Lemma 6.1.27 (Placing multiple nicely ordered generalised arcs) *Let*

$$FA(i_1, j_1),\ FA(i_2, j_2),\ \ldots,\ FA(i_m, j_m)$$

be a nicely ordered sequence of forwards generalised arcs of attachment. They can be placed on Γ_w, so that:

(i) they are disjoint;

(ii) for any chord γ of Γ_w which nontrivially intersects at least one of these arcs, the set of $FA(i_k, j_k)$ intersecting γ is a discrete interval of k, of the form

$$FA(i_s, j_s),\ FA(i_{s+1}, j_{s+1}),\ \ldots,\ FA(i_t, j_t)$$

and moreover, for $u < v$, the intersection of γ with $FA(i_v, j_v)$ lies southwest of $FA(i_u, j_u)$;

(iii) none of the $FA(i_k, j_k)$, $k < m$, intersect the southwest region of $FA(i_m, j_m)$.

Moreover, there is only one way to place the arcs satisfying these conditions, up to equivalence (i.e. homotopy through generalised attaching arcs).

There is also a backwards version of this result.

PROOF For small m, we have proved the lemma. We show that we can inductively add a further $FA(i_m, j_m)$ to previously placed $FA(i_1, j_1), \ldots, FA(i_{m-1}, j_{m-1})$. We place $FA(i_m, j_m)$ to lie southwest of $FA(i_{m-1}, j_{m-1})$ as described in lemma 6.1.24. We note that, by inductive assumption (iii), there are no other arcs southwest of $FA(i_{m-1}, j_{m-1})$, and hence this specifies a unique way to place $FA(i_m, j_m)$; and disjointly, so (i) is true. Moreover, there are now no arcs southwest of $FA(i_m, j_m)$; so (iii) is true.

For every chord γ nontrivially intersecting one of these arcs, if it does not intersect $FA(i_m, j_m)$, then (ii) is true by inductive assumption. Otherwise, it intersects $FA(i_m, j_m)$, and the intersection point $FA(i_m, j_m)$ is southwest of all others; so by lemma 6.1.26 and inductive assumption, (ii) is again true. ■

We give a name to this construction.

Definition 6.1.28 (Bypass system of nicely ordered sequence) *Let*

$$FA(i_1, j_1),\ FA(i_2, j_2),\ \ldots,\ FA(i_m, j_m)$$

be a nicely ordered sequence of forwards generalised arcs of attachment. The bypass system of *this sequence is the bypass system obtained from placing these arcs as described in lemma 6.1.27.*

6.1.12 Nicely ordered sequences of generalised moves

We first extend the notion of "nicely ordered" to generalised elementary moves.

Definition 6.1.29 (Nicely ordered generalised elementary moves)

(i) A sequence of forwards generalised elementary moves

$$FE(i_1, j_1),\ FE(i_2, j_2), \ldots, FE(i_m, j_m)$$

on w is nicely ordered *if*

$$i_1 < i_2 < \cdots < i_m \quad \textit{and} \quad j_1 \leq j_2 \leq \cdots \leq j_m.$$

(ii) A sequence of backwards generalised elementary moves

$$BE(i_1, j_1),\ BE(i_2, j_2), \ldots, BE(i_m, j_m)$$

on w is nicely ordered *if*

$$j_m < j_{m_1} < \cdots < j_1 \quad \textit{and} \quad i_m \leq i_{m-1} \leq \cdots \leq i_1.$$

Nice ordering of generalised elementary moves is nice in the sense that it guarantees commutativity, as we shall now see. First, we have the following obvious lemma about redundancy.

Lemma 6.1.30 (Redundancy of generalised moves) *Suppose we have two well-defined forwards generalised elementary moves $FE(i_1, j)$ and $FE(i_2, j)$ on a word w, with $i_1 < i_2$. Then:*

(i) After applying $FE(i_1, j)$, then $FE(i_2, j)$ is no longer well-defined.

(ii) After applying $FE(i_2, j)$, then $FE(i_1, j)$ is still well-defined, and after applying it, the result is the same as simply applying $FE(i_1, j)$ alone:

$$FE(i_1, j)\ (w) = FE(i_1, j) \circ FE(i_2, j)\ (w) \qquad \blacksquare$$

In the case where $FE(i_2, j)$ is no longer well-defined, we may regard it as "the null move" and having trivial effect. Thus, we can extend the definition of $FE(i_2, j)$, to be the identity, where otherwise it is not defined. With this definition, we see that $FE(i_1, j)$ and $FE(i_2, j)$ commute, and their composition in either order is equal to $FE(i_1, j)$.

Lemma 6.1.31 *Let $FE(i_1, j_1)$ and $FE(i_2, j_2)$ be well-defined nontrivial forwards generalised elementary moves on a word w, with $i_1 < i_2$ and $j_1 < j_2$. Then:*

(i) After applying either of $FE(i_1, j_1)$ or $FE(i_2, j_2)$ to w, the other is still well-defined and nontrivial.

(ii) The effect of applying both $FE(i_1, j_1)$ and $FE(i_2, j_2)$ to w, in either order, is identical:

$$FE(i_1, j_1) \circ FE(i_2, j_2)\ (w) = FE(i_2, j_2) \circ FE(i_1, j_1)\ (w). \qquad \blacksquare$$

With the extended definition of $FE(i, j)$ and $BE(i, j)$ to be trivial when not otherwise defined, we obtain a general commutativity result:

Lemma 6.1.32 (General commutativity of generalised moves) *If $i_1 \leq i_2$ and $j_1 \leq j_2$, then $FE(i_1, j_1)$ and $FE(i_2, j_2)$ commute.* ■

Thus, for any nicely ordered set of generalised forwards elementary moves, they all commute; and hence we may speak of applying them to a word, without regard to their order. There are forwards and backwards versions.

6.1.13 Generalised elementary moves of a comparable pair

Suppose we have two comparable words $w_1, w_2 \in W(n_-, n_+)$ with $w_1 \preceq w_2$. Then, for every $1 \leq i \leq n_-$, the i'th $-$ sign in w_1 lies to the left of the i'th $-$ sign in w_2. That is, the i'th $-$ sign in w_1 has fewer $+$ signs to its left, than does the i'th $-$ sign in w_2. Suppose that the i'th $-$ sign has α_i $+$ signs to its left in w_1, and β_i $+$ signs to its left in w_2; so $\alpha_i \leq \beta_i$. So we have

$$\begin{array}{ccccccc} \alpha_1 & \leq & \alpha_2 & \leq & \cdots & \leq & \alpha_{n_-} \\ \wedge\!| & & \wedge\!| & & & & \wedge\!| \\ \beta_1 & \leq & \beta_2 & \leq & \cdots & \leq & \beta_{n_-} \end{array}$$

describing $w_1 \preceq w_2$.

Definition 6.1.33 (Elementary moves of comparable pair) *The* generalised forwards elementary moves of the pair $w_1 \preceq w_2$ *are*

$$FE(1, \beta_1),\ FE(2, \beta_2),\ \ldots,\ FE(n_-, \beta_{n_-}).$$

where β_i denotes the number of $+$ signs to the left of the i'th $-$ sign in w_2.

Recall that while $w_1 \preceq w_2$ means "all $-$ signs move right", some may not move at all; and if the i'th $-$ sign does not move, then the forwards generalised elementary move $FE(i, \beta_i)$ is trivial. Above we defined generalised elementary moves to be trivial when they are not otherwise defined. We could delete such moves from the sequence; but it is easier to "take individual care" of each symbol in this way.

Note this is a nicely ordered sequence, hence by lemma 6.1.32 the moves commute. The first move $FE(1, \beta_1)$ moves the first $-$ sign, and all $-$ signs between it and the

β_1'th + sign, to the immediate right of the β_1'th + sign. If $\beta_1 = \beta_2$, then $FE(2, \beta_2)$ is redundant; otherwise $\beta_1 < \beta_2$ and $FE(2, \beta_2)$ moves the second $-$ sign, and all the $-$ signs between it and the β_2'th + sign, to the immediate right of the β_2'th + sign. Continuing in this way, we see the result is

$$(+)^{\beta_1} \, (-) \, (+)^{\beta_2 - \beta_1} \, (-) \, \cdots \, (+)^{\beta_{n_-} - \beta_{n_- - 1}} \, (-) \, (+)^{n_+ - \beta_{n_-}}$$

which is w_2, by definition. We record this result.

Lemma 6.1.34 (Elementary moves between comparable words) *The result of applying the generalised forwards elementary moves of the pair $w_1 \preceq w_2$, to w_1, is w_2.* ■

There is also a backwards version. If we denote by δ_j the number of $-$ signs to the left of the j'th + sign in w_1. Then the *generalised backwards elementary moves of the pair $w_1 \preceq w_2$* are

$$BE(\delta_1, 1), \; BE(\delta_2, 2), \; \ldots, \; BE(\delta_{n_+}, n_+).$$

and the result of applying them to w_2 is w_1.

Thus we can go from w_1 to w_2 (and vice versa) by a well-defined nicely ordered sequence of generalised elementary moves. It now remains to show that this can be paralleled by bypass moves along the bypass system of a well-defined nicely ordered sequence of generalised arcs of attachment.

6.1.14 Bypass systems of nicely ordered sequences

We now consider in more detail the effect of performing bypass moves along the bypass system of a nicely ordered sequence of generalised attaching arcs. As we know from lemma 6.1.22, if we restrict to a single generalised attaching arc, then we obtain the basis chord diagram for the word obtained by performing a corresponding generalised elementary move.

We now see, in this subsection, that if we perform bypass moves along the bypass system of *multiple* generalised attaching arcs, when they are nicely ordered (as in definition 6.1.25) and then placed appropriately (as in definition 6.1.28), then we obtain the basis chord diagram for the word obtained by performing the corresponding composition of generalised elementary moves on the original word. More generally, all the lemmata of the previous section are paralleled by bypass systems of nicely ordered sequences of generalised attaching arcs.

In particular, in the previous section we proved (lemma 6.1.32) that generalised elementary moves in nicely ordered sequences commute — once we expand the definition a little to say that "when a generalised elementary move does not exist, it has trivial effect". A corresponding result is obvious for bypass moves on bypass systems: in a bypass system, the arcs of attachment are all disjoint, so the bypass moves on them obviously commute.

First, we consider redundancy.

Lemma 6.1.35 (Redundancy of generalised attaching arcs) *Let Γ_w be a basis chord diagram on which forwards generalised arcs of attachment $FA(i_1, j)$ and $FA(i_2, j)$ exist, with $i_1 < i_2$. Then:*

(i) After performing upwards bypass moves along the bypass system of $FA(i_1, j)$, then the bypass system of $FA(i_2, j)$ consists entirely of trivial bypasses. That is, performing upwards bypass moves along the bypass system $FA(i_2, j)$ has trivial effect.

(ii) After performing upwards bypass moves along the bypass system of $FA(i_2, j)$, then $FA(i_1, j)$ is still well-defined, and after applying it, the result is the same as simply applying $FE(i_1, j)$ alone:

$$\text{Up}(FA(i_1, j))\ (\Gamma_w) = \text{Up}(FA(i_1, j)) \circ \text{Up}(FA(i_2, j))\ (\Gamma_w)$$

Proof This is a proof by picture; see figure 6.21. ∎

We now consider this redundancy in more detail, so that it can be extended to the case of multiple attaching arcs.

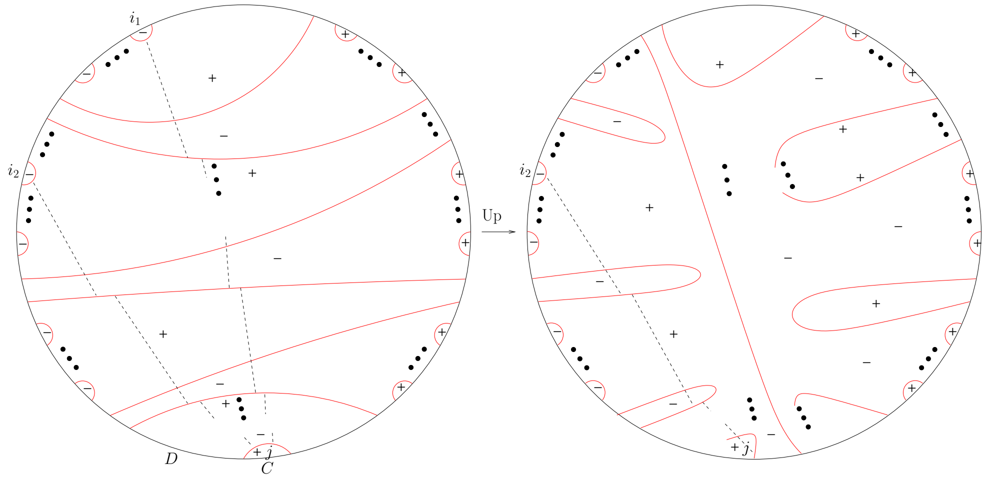


Figure 6.21: Redundancy with two generalised arcs.

Lemma 6.1.36 (Redundancy in two generalised arcs) *Suppose that $FA(i_1, j_1)$ and $FA(i_2, j_2)$ form a nicely ordered sequence of two forwards generalised attaching arcs on a basis chord diagram Γ_w. We write $FA_w(i_1, j_1), FA_w(i_2, j_2)$ to denote that they refer to Γ_w. After performing upwards bypass moves along $FA_w(i_1, j_1)$, we obtain a basis chord diagram $\Gamma_{w'}$, where $w' = FE(i_1, j_1)(w)$, by lemma 6.1.22.*

On $\Gamma_{w'}$, $FA_w(i_2, j_2)$ may no longer be a nontrivial generalised attaching arc; but it is equivalent *to the generalised attaching arc $FA_{w'}(i_2, j_2)$ on $\Gamma_{w'}$, in the following sense. If on $\Gamma_{w'}$ there is no forwards generalised arc $FA_{w'}(i_2, j_2)$, then $j_1 = j_2$ and the bypass system of $FA_w(i_2, j_2)$ consists entirely of trivial arcs of attachment. Otherwise:*

(i) The generalised attaching arc $FA_w(i_2, j_2)$ can be homotoped, rel endpoints, to $FA_{w'}(i_2, j_2)$.

(ii) This homotopy consists of a homotopy through generalised arcs, combined with finitely many (possibly none) local "pushing off" moves, of the sort depicted in figure 6.22.

(iii) Performing upwards bypass moves along the bypass system of $FA_w(i_2, j_2)$ or of $FA_{w'}(i_2, j_2)$ gives the same chord diagram.

Proof This is largely a proof by picture. Note that the chord created by processing the i_2'th $-$ sign in w or w' (or any word for that matter), in the base point algorithm,

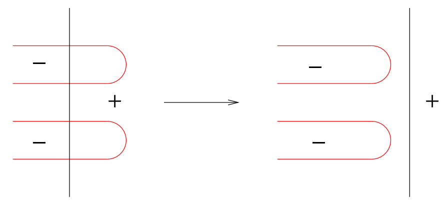


Figure 6.22: Pushing off trivial parts of a generalised attaching arc.

is the chord emanating from the marked point $1 - 2i_2$, by lemma 5.1.9. Thus, even after performing bypass moves along the bypass system of $FA_w(i_1, j_1)$, $FA_w(i_2, j_2)$ still has an endpoint on the chord created by processing the i_2'th $-$ sign in w in the base point construction algorithm; and similarly, it still has an endpoint on the chord created by processing the j_2'th $+$ sign in the root point construction algorithm. In particular, $FA_w(i_2, j_2)$ remains adjacent to the same marked points: it has the same "west end" for its prior chord and the same "east end" for its latter chord.

Recall (from lemma 6.1.20) that $FA(i, j)$ exists nontrivially iff $FE(i, j)$ does. If $FE(i_2, j_2)$ does not exist in w', but did exist in w, then we must have $j_1 = j_2$; otherwise the move $FE(i_1, j_1)$ would not move the i_2'th $-$ sign far enough, and $FE(i_2, j_2)$ would still have nontrivial effect. This is precisely the case when $FA(i_2, j_2)$ becomes redundant, as described above in lemma 6.1.35 and figure 6.21.

As we have seen, the effect of performing upwards bypass moves along the bypass system of a forwards generalised attaching arc is to create a "long chord", to close off outermost negative regions to the southwest, and to close off outermost positive regions to the northeast. Some of these outermost negative regions now have parts of $FA_w(i_2, j_2)$ inside them, and they are pushed off. After performing this homotopy, we certainly have $FA_{w'}(i_2, j_2)$.

As for the final claim, the effect of upwards bypass moves along the bypass systems before and after the homotopy are also best conveyed by picture; as in figure 6.21, there are many trivial bypasses, and by the principle expressed in figure 6.23, the effect is the same, after the "pushing off" homotopy of figure 6.22. ∎

We now consider a general nicely ordered sequence of forwards generalised attaching arcs.

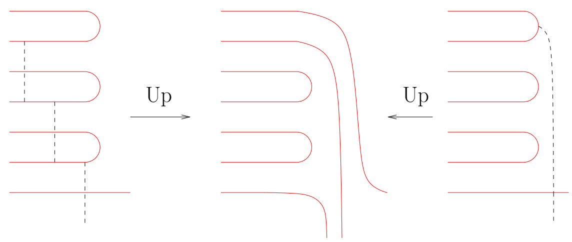


Figure 6.23: Pushing off makes no difference to effect of bypass moves.

Lemma 6.1.37 *Consider the bypass system of the nicely ordered sequence of forwards generalised attaching arcs on the basis chord diagram* Γ_w

$$FA_w(i_1, j_1),\ FA_w(i_2, j_2),\ \ldots,\ FA_w(i_m, j_m).$$

We use the subscript w *to denote that they refer to* Γ_w. *After performing upwards bypass moves along the bypass system of* $FA_w(i_1, j_1)$*, we obtain a basis chord diagram* $\Gamma_{w'}$*, where* $w' = FE(i_1, j_1)(w)$*, by lemma 6.1.22.*

On w'*, each of* $FA_w(i_2, j_2), \ldots, FA_w(i_m, j_m)$ *may no longer be a nontrivial generalised attaching arc. If the forwards generalised elementary move* $FE(i_k, j_k)$ *does not exist on* w' *then the bypass system of* $FA_w(i_k, j_k)$ *consists entirely of trivial arcs of attachment. But the other arcs are* equivalent *to the nontrivial generalised attaching arcs among* $FA_{w'}(i_2, j_2), \ldots, FA_{w'}(i_m, j_m)$ *on* $\Gamma_{w'}$*, in the following sense.*

(i) The nontrivial arcs among $FA_w(i_2, j_2), \ldots, FA_w(i_m, j_m)$ *can be simultaneously homotoped, rel endpoints, to* $FA_{w'}(i_2, j_2), \ldots, FA_{w'}(i_m, j_m)$*, placed "northeast to southwest" as described in definition 6.1.28.*

(ii) This homotopy consists of a homotopy through disjoint generalised attaching arcs, combined with finitely many (possibly none) local "pushing off" moves, possibly "pushing several arcs off several chords", of the sort depicted in figure 6.24.

(iii) Performing upwards bypass moves on $\Gamma_{w'}$ *along either bypass system has the same effect:* $FA_w(i_2, j_2), \ldots, FA_w(i_m, j_m)$ *or* $FA_{w'}(i_2, j_2), \ldots, FA_{w'}(i_m, j_m)$.

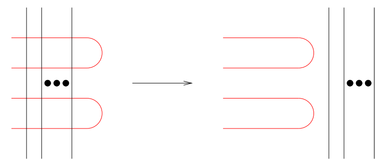


Figure 6.24: Pushing off trivial parts of multiple generalised attaching arcs.

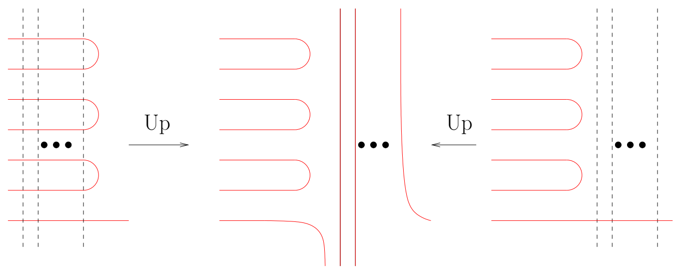


Figure 6.25: Pushing off several generalised attaching arcs makes no difference to effect of bypass moves.

PROOF This is again a proof by picture, except the pictures are a little more complicated than in the previous lemma. Again, the chords created by the processing the i_2'th $-$ sign in w or w' (or any word for that matter), in the base point algorithm, emanate from the same marked point $1-2i_2$, by lemma 5.1.9, so even after performing bypass moves along the bypass system of $FA_w(i_1, j_1)$, all the other $FA_w(i_k, j_k)$ have endpoints on the appropriate chords.

The statement about triviality of arcs follows as in the previous lemma.

The picture of the local homotopy is similar, as now the outermost regions closed off in performing bypass moves along the bypass system of $FA_w(i_1, j_1)$ may now have parts of several $FA_w(i_k, j_k)$ inside them, but they can all be pushed off simultaneously; and after performing this homotopy, we have all the $FA_{w'}(i_k, j_k)$.

As for the final claim, it is again best conveyed by picture. The general arrangement is shown in figure 6.25. ■

We now obtain a complete analogy between multiple generalised elementary moves and multiple generalised attaching arcs.

Lemma 6.1.38 (Multiple moves & bypass systems) *Suppose there is a nicely ordered sequence of forwards generalised elementary moves*

$$FE(i_1, j_1),\ FE(i_2, j_2),\ \ldots,\ FE(i_m, j_m)$$

on w*, and equivalently, a nicely ordered sequence of forwards generalised attaching arcs*

$$FA(i_1, j_1),\ FA(i_2, j_2),\ \ldots,\ FA(i_m, j_m)$$

on Γ_w*. If we perform upwards bypass moves along the bypass system of this nicely ordered sequence of forwards generalised attaching arcs, then we obtain* $\Gamma_{w'}$*, where* w' *is obtained from* w *by performing the above forwards generalised elementary moves.*

$$\begin{array}{ccc} w & \xmapsto{FE(i_1,j_1)\circ\cdots\circ FE(i_m,j_m)} & w' \\ \updownarrow & & \updownarrow \\ \Gamma_w & \xmapsto{\mathrm{Up}(FA(i_1,j_1),\ldots,FA(i_m,j_m))} & \Gamma_{w'} \end{array}$$

PROOF Proof by induction on m. For $m = 1$ it is true by lemma 6.1.22; now consider the case for general m. We know again from lemma 6.1.22 that performing the bypass moves of $FA(i_1, j_1)$, we obtain $\Gamma_{w''}$, where $w'' = FE(i_1, j_1)(w)$. Then, by the previous lemma, on $\Gamma_{w''}$, some of the arcs $FA_w(i_2, j_2)$, $\ldots$, $FA_w(i_m, j_m)$ are trivial (namely those with $j_k = j_1$), resulting in only trivial bypass moves; and the rest can be simultaneously homotoped to $FA_{w''}(i_2, j_2), \ldots, FA_{w''}(i_m, j_m)$, again placed properly "northeast to southwest", and in such a way that the effect of bypass moves along their bypass systems is unchanged. Then we are done by induction. ∎

The same all applies in a backwards version.

6.1.15 Bypass system of a comparable pair

We have now built so much superstructure that we can almost use it. Consider two comparable words $w_1 \preceq w_2$. We have a nicely ordered sequence of generalised elementary moves of the pair; now we define some generalised arcs and then a bypass system.

Definition 6.1.39 (Generalised arcs of comparable pair) *Consider two words $w_1, w_2 \in W(n_-, n_+)$ which are comparable, $w_1 \preceq w_2$. Let β_i denote the number of $+$ signs to the left of the i'th $-$ sign in w_2. Then the* nicely ordered sequence of forwards generalised attaching arcs of the pair $w_1 \preceq w_2$ *is*

$$FA(1, \beta_1),\ FA(2, \beta_2),\ \ldots,\ FA(n_-, \beta_{n_-}).$$

Similarly, we have a *nicely ordered sequence of backwards generalised attaching arcs of the pair* $w_1 \preceq w_2$, given by

$$BA(\delta_1, 1),\ BA(\delta_2, 2),\ \ldots,\ BA(\delta_{n_+}, n_+)$$

where δ_j denotes the number of $-$ signs to the left of the j'th $+$ sign in w_1.

As in definition 6.1.33, it's clear that these are nicely ordered sequences.

As noted for elementary moves, while "all $-$ signs move right", some may not move at all. If the i'th $-$ sign does not move, then we consider $FA(i, \beta_i)$ to be a null arc, with a null bypass system. For such $-$ signs at the start of the words, we might have $\beta_i = 0$. From all these generalised attaching arcs, we obtain a bypass system.

Definition 6.1.40 (Coarse bypass systems of comparable pair) *Let $w_1 \preceq w_2$ be comparable words.*

(i) The coarse forwards bypass system $CFBS(w_1, w_2)$ *of the pair $w_1 \preceq w_2$ is the bypass system of the nicely ordered sequence of forwards generalised attaching arcs of $w_1 \preceq w_2$.*

(ii) The coarse backwards bypass system $CBBS(w_1, w_2)$ *of the pair $w_1 \preceq w_2$ is the bypass system of the nicely ordered sequence of backwards generalised attaching arcs of $w_1 \preceq w_2$.*

We have called these bypass systems "coarse", because they may contain massive redundancy. We have taken our "individual care" approach, as discussed in our menagerie of examples.

Lemma 6.1.41 (Effect of coarse bypass systems)

(i) Performing upwards bypass moves on Γ_{w_1} *along* $CFBS(w_1, w_2)$ *gives* Γ_{w_2}*.*

(ii) Performing downwards bypass moves on Γ_{w_2} *along* $CBBS(w_1, w_2)$ *gives* Γ_{w_1}*.*

PROOF By lemma 6.1.34, the corresponding sequences of generalised elementary moves take w_1 to w_2 and vice versa. The corresponding effect on basis chord diagrams is now immediate from lemma 6.1.38. ∎

In fact, we can say a little more. Performing upwards bypass moves along the various attaching arcs of $CFBS(w_1, w_2)$ takes Γ_{w_1} to various basis chord diagrams, corresponding to words in $W(n_-, n_+)$, always moving forwards in the partial order $\preceq$. In categorical language, we have a covariant functor

$$\begin{aligned} \mathrm{Up}_{CFBS(w_1,w_2)} \; : \; \mathcal{P}\left(CFBS(w_1,w_2)\right) &\rightarrow W(n_-, n_+) \\ c' &\mapsto \text{word of basis diagram } \mathrm{Up}_{c'}(\Gamma_1) \end{aligned}$$

and a contravariant functor

$$\begin{aligned} \mathrm{Down}_{CBBS(w_1,w_2)} \; : \; \mathcal{P}\left(CBBS(w_1,w_2)\right) &\rightarrow W(n_-, n_+) \\ c' &\mapsto \text{word of basis diagram } \mathrm{Down}_{c'}(\Gamma_w). \end{aligned}$$

Here we think of the power set as partially ordered by $\subseteq$, and $W(n_-, n_+)$ partially ordered by $\preceq$; recall section 4.2.9. (In section 4.2.9 we considered a functor Up_c to a bounded contact category. Here $W(n_-, n_+)$ is also a bounded contact category, as we will see in section 6.2.1, proving proposition 1.3.9.)

Now $CFBS$ (resp. $CBBS$) is "coarse" in the sense that some proper subset may map to w_2 (resp. w_1) under this functor. We may take a *minimal subsystem* of the bypass system $CFBS(w_1, w_2)$ (resp. $CBBS(w_1, w_2)$). By this we mean a subset of these attaching arcs, such that

(i) this bypass system contains no trivial attaching arcs,

(ii) performing upwards bypass moves (resp. downwards bypass moves) along these attaching arcs gives Γ_{w_2} (resp. Γ_{w_1}), and

(iii) performing upwards bypass moves (resp. downwards bypass moves) along any proper subset of them does not give Γ_{w_2} (resp. Γ_{w_1}).

Note that condition (i) appears redundant, especially given condition (iii). It is in this case, since by definition $CFBS(w_1, w_2)$ contains no trivial attaching arcs. However, in general, performing the reverse of a "pushing off move" might well result in a bypass system satisfying (ii) and (iii) but not (i). Moreover, not every bypass system has a minimal sub-system; however, a bypass system with no trivial attaching arcs does have a minimal sub-system.

We say nothing about the uniqueness of this minimal bypass system, only that at least one exists. Hence, we use an indefinite article in the following definition.

Definition 6.1.42 (Bypass systems of a comparable pair) *Suppose $w_1 \preceq w_2$ are comparable words.*

(i) A forwards bypass system $FBS(w_1, w_2)$ *of the pair (w_1, w_2) is a minimal sub-system of the coarse forwards bypass system $CFBS(w_1, w_2)$.*

(ii) A backwards bypass system $BBS(w_1, w_2)$ *of the pair (w_1, w_2) is a minimal sub-system of the coarse backwards bypass system $CBBS(w_1, w_2)$.*

Now, at last, we prove propositions 1.2.18 and 1.2.19. Proposition 1.2.18 is now obvious from lemma 6.1.41 and the definition of a minimal bypass system.

And, we can now prove proposition 1.2.19; that performing bypass moves on these systems in the other direction gives a chord diagram whose basis decomposition has minimum Γ_{w_1} and maximum Γ_{w_2}, with respect to $\preceq$.

Proof (of proposition 1.2.19) We take a forwards bypass system $FBS(w_1, w_2)$; the case of $BBS(w_1, w_2)$ is similar. From proposition 1.2.18, we know that performing upwards bypass moves along this system on Γ_{w_1} gives Γ_{w_2}.

Now let Γ be the chord diagram obtained by performing downwards bypass moves on $FBS(w_1, w_2) = \{c_1, \ldots, c_m\}$ where the c_i are attaching arcs. We "expand down over ups" (lemma 4.2.13). Thus Γ is a sum of 2^m chord diagrams: for each of the 2^m subsets of $\{c_1, \ldots, c_m\}$, we have a sub-bypass system of $FBS(w_1, w_2)$, upon which we perform upwards bypass moves. Each of these 2^m chord diagrams is a basis chord diagram, by proposition 6.1.16. They are not necessarily distinct. But by minimality, each of the arcs of attachment is nontrivial, so the chord diagram Γ_{w_1} appears exactly once; and also by definition of minimality, Γ_{w_2} appears only after performing bypass attachments on the whole bypass system. For every other basis chord diagram Γ_w which appears, it is obtained from Γ_{w_1} by some sequence of upwards bypass moves along forwards attaching arcs; and then by attaching some more upwards bypass moves along forwards attaching arcs, we can obtain Γ_{w_2}; thus $w_1 \preceq w \preceq w_2$. ∎

Corollary 6.1.43 *For every pair $w_1 \preceq w_2$, there is a chord diagram such that, if we write it as a sum of basis chord diagrams, then Γ_{w_1} is lexicographically the first and Γ_{w_2} is lexicographically the last.* ∎

6.2 Contact categorical computations

In this section, we will compute the bounded contact category $\mathcal{C}^b(\Gamma_{w_0}, \Gamma_{w_1})$ for basis chord diagrams, and for bypass cobordisms.

6.2.1 Bounded contact category of the universal cobordism

We will now prove proposition 1.3.9, computing the bounded contact category of the universal cobordism, and then proposition 1.3.7, computing the bounded category of a general cobordism between basis elements. First, some lemmata.

Recall definition 4.2.8, what it means for a chord diagram to exist in a cobordism; and recall definition 1.3.8 of "universal cobordism".

Lemma 6.2.1 *If the chord diagram Γ exists in $\mathcal{U}(n_-, n_+)$, then $\Gamma = \Gamma_w$ for some word $w \in W(n_-, n_+)$.*

Proof (# 1, by combinatorial skiing) Such an object Γ must satisfy

$$m(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma) = 1 \quad \text{and} \quad m(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 1.$$

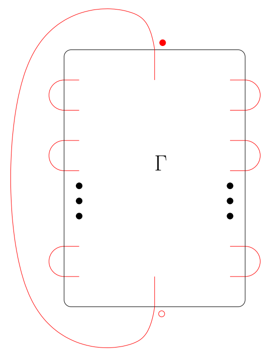


Figure 6.26: Chord diagram ski slope.

After edge rounding, either condition is equivalent to the condition that, when the chord diagram Γ is placed in figure 6.26, there must be one connected component of the closed curve so obtained. The disc is drawn as a rectangle, with the base point at the top and the root point at the bottom, and other points on the right or left.

We see that there can be no nesting of arcs on either side; thus

- the arcs from the base and root points must be outermost;
- every other arc must either be outermost on the left side, or outermost on the right side, or run from the left to right side.

Such a set of arcs joins together to form a "slalom course" (think of the top of the diagram as the top of a ski slope), successively rounding obstacles on left or right sides. The sequence in which the skier rounds obstacles on the left $(-)$ or right $(+)$ precisely gives the word for which the chord diagram is a basis element. ■

Proof (#2, by bypasses) If Γ exists in $\mathcal{U}(n_-, n_+)$, then by lemma 4.2.9 there must be a sequence of upwards bypasses from $\Gamma_{(-)^{n_-}(+)^{n_+}} = G_0$, through a sequence of chord diagrams $G_1, \ldots, G_k$ to $G_k = \Gamma$, where each G_i satisfies $m(G_i, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 1$. This last condition, after expanding out over the basis elements of G_i, simply counts the number of basis elements in the decomposition of G_k (every word is $\preceq (+)^{n_+}(-)^{n_-}$).

By the work of section 6.1.4 above, we know all possible upwards bypass moves on basis chord diagrams. An upwards bypass move along a forwards attaching arc gives another basis diagram, with word obtained by moving a string of $-$ signs past a string of + signs. An upwards bypass move along a backwards attaching arc gives a chord diagram which is a sum of two basis diagrams, namely the original basis diagram and one preceding it in the partial order. The former has one basis element in its decomposition (being a basis element!), while the latter has two. Hence the former satisfies the condition $m(\cdot, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 1$, and the latter does not.

Therefore, any bypass attachment upwards must move from a basis chord diagram to another basis chord diagram. Hence every chord diagram existing in $\mathcal{U}(n_-, n_+)$ is a basis diagram. ■

Out of this, we have a serendipitous corollary: proposition 1.2.20. We will also prove this proposition directly in section 7.3.1.

Proposition (Size of basis decomposition) *Every chord diagram which is not a basis element has an even number of basis elements in its decomposition.*

Proof For a non-basis chord diagram Γ, from proof #1 by skiing above, we must have $m(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 0$. We see this is equal to the number of words in the decomposition of Γ which precede $(+)^{n_+}(-)^{n_-}$. But every word in $W(n_-, n_+)$ precedes $(+)^{n_+}(-)^{n_-}$, so $m = 0$ says that Γ has an even number of basis elements in its decomposition. ■

Conversely, if we take the direct proof of this proposition in section 7.3.1, then we can obtain a third proof of lemma 6.2.1. For then we know a non-basis element Γ has an even number of basis elements in its decomposition, hence $m(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 0$, hence Γ cannot occur in $\mathcal{U}(n_-, n_+)$.

Lemma 6.2.2 *Every basis chord diagram Γ_w, $w \in W(n_-, n_+)$, exists in $\mathcal{U}(n_-, n_+)$.*

Proof Section 6.1 above shows how to construct a bypass system

$$FBS((-)^{n_-}(+)^{n_+}, w) \quad \text{on} \quad \Gamma_{(-)^{n_-}(+)^{n_+}}$$

such that performing upwards bypass attachments gives Γ_w. Each successive bypass attachment gives a basis diagram $\Gamma_{w'}$, where $w' \preceq w$, so $m(\Gamma_{w'}, \Gamma_{(+)^{n_+}(-)^{n_-}}) = 1$, hence the attaching arc is inner, hence $\Gamma_{w'}$ exists in $\mathcal{U}(n_-, n_+)$. ■

Lemma 6.2.3 *For any two words w, w' in $W(n_-, n_+)$ with $w \preceq w'$, the cobordism $\mathcal{M}(\Gamma_w, \Gamma_{w'})$ exists in $\mathcal{U}(n_-, n_+)$.*

PROOF We have seen that Γ_w exists in $\mathcal{U}(n_-, n_+)$. We now take the forwards bypass system $FBS(w, w')$ on Γ_w such that performing upwards bypass attachments gives $\Gamma_{w'}$. Again, each successive bypass attachment gives a basis chord diagram, which when paired with $\Gamma_{(+)^{n_+}(-)^{n_-}}$ via m gives 1, hence is inner, hence $\Gamma_{w'}$ exists in $\mathcal{U}(n_-, n_+)$ above Γ_w; and so $\mathcal{M}(\Gamma_w, \Gamma_{w'})$ exists in $\mathcal{U}(n_-, n_+)$. ■

Now we have proposition 1.3.9:

Proposition (Bounded contact category of universal cobordism) *For any n_-, n_+, there is an isomorphism of categories*

$$\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right) \cong W(n_-, n_+).$$

The word $w \in W(n_-, n_+)$ corresponds to the basis chord diagram Γ_w.

PROOF Lemma 6.2.1 above shows $\mathrm{Ob}\,\mathcal{C}^b(\mathcal{U}(n_-, n_+)) \subseteq W(n_-, n_+)$. Lemma 6.2.2 above shows $\mathrm{Ob}\,\mathcal{C}^b(\mathcal{U}(n_-, n_+)) \supseteq W(n_-, n_+)$. Lemma 6.2.3 above shows that the set of morphisms $\mathrm{Mor}\,\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ contains every pair related by the partial order $\preceq$ on $W(n_-, n_+)$. For any pair of words w, w' with $w \not\preceq w'$, $\mathcal{M}(\Gamma_w, \Gamma_{w'})$ is not tight, hence cannot exist in the tight $\mathcal{U}(n_-, n_+)$. Composition of morphisms in the bounded contact category is simply transitivity of the partial order $\preceq$. ■

And now proposition 1.3.7:

Proposition (Bounded contact category of basis cobordism) *For words $w_0 \preceq w_1$ in $W(n_-, n_+)$ corresponding to basis chord diagrams $\Gamma_{w_0}, \Gamma_{w_1}$,*

$$\mathcal{C}^b\left(\Gamma_{w_0}, \Gamma_{w_1}\right) \cong W(w_0, w_1).$$

The word $w \in W(w_0, w_1)$ corresponds to the basis chord diagram Γ_w.

PROOF The cobordism $\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ with tight contact structure exists in the universal cobordism $\mathcal{U}(n_-, n_+)$, as part of the preceding proposition. Hence its bounded contact category is the full sub-category on those objects Γ_w with $w_0 \preceq w \preceq w_1$. ■

6.2.2 Bounded contact category of a bypass cobordism

We now compute the bounded contact category of a bypass cobordism; that is, we compute $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ for any chord diagrams Γ_0, Γ_1 such that Γ_1 is obtained from an upwards bypass move on Γ_0. We prove that $\mathcal{C}^b(\Gamma_0, \Gamma_1) \cong W(n_-, n_+)$ for some n_-, n_+ that we will now describe; these n_-, n_+ are essentially largest possible so that $\mathcal{U}(n_-, n_+)$ embeds into the bypass attachment.

A general bypass attachment in a general chord diagram Γ can be considered as shown in figure 6.27. Let c be an attaching arc: as described in section 6.1.3, c has inner and outer $+$ and $-$ regions. The boundary of the inner $+$ region consists of: several arcs on the boundary of the disc; two chords of the dividing set which intersect c; and several other chords of Γ. Of these other chords of Γ bounding the inner $+$ region, let the number of those which lie anticlockwise of the outer $-$ region and clockwise of the inner $-$ region be $n_- - 1$. Similarly, the boundary of the inner $-$ region consists of: several arcs on the boundary of the disc; two chords of Γ which intersect c; and several other chords of Γ. Of these other chords, let the number which lie anticlockwise of the outer $+$ region and clockwise of the inner $+$ region be $n_+ - 1$.

Theorem 6.2.4 (Bounded contact category of bypass cobordism) *Let Γ_1 be obtained from Γ_0 by a single upwards bypass attachment, so that $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight. Let n_-, n_+ be defined as described above, and depicted in figure 6.27. Then*

$$\mathcal{C}^b(\Gamma_0, \Gamma_1) \cong W(n_-, n_+).$$

In figure 6.27, with chord diagram Γ_0 and attaching arc c, we have a region R which contains sub-arcs of $n_- + n_+ + 1$ chords from Γ_0: arcs of three chords intersecting c; and arcs of the $(n_- - 1) + (n_+ - 1)$ other chords described above bounding the inner regions of c. Note that, with base point as shown, R encloses precisely the chord diagram $\Gamma_{(-)^{n_-}(+)^{n_+}}$. Furthermore, the same region R on Γ_1 encloses a region which contains precisely the chord diagram $\Gamma_{(+)^{n_+}(-)^{n_-}}$. Thus we may think of $R \times I$ as an "embedded universal cobordism" $\mathcal{U}(n_-, n_+)$ in $\mathcal{M}(\Gamma_0, \Gamma_1)$ (however, we do not know that the vertical sutures also embed).

The point of the theorem is that the bounded contact category of $\mathcal{M}(\Gamma_0, \Gamma_1)$ is

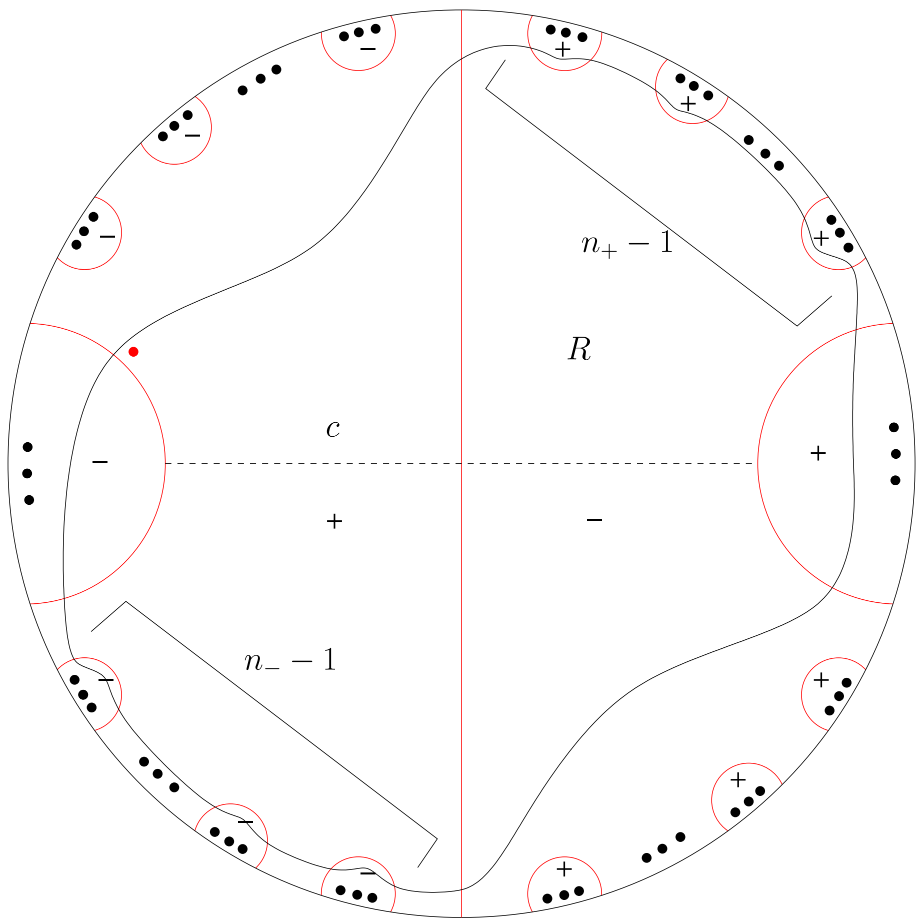


Figure 6.27: A general bypass attachment. Possible locations of other chords are denoted by black dots $(\ldots)$. The region R is enclosed by the black curve. Its base point is denoted by the solid red dot.

precisely that of $\mathcal{U}(n_-, n_+)$; all nontrivial bypasses attached in $\mathcal{M}(\Gamma_0, \Gamma_1)$ upwards from Γ_0 can be attached within R; and even as more bypasses are attached, they can be attached along R. Since chord diagrams exist in $\mathcal{M}(\Gamma_0, \Gamma_1)$ if and only if they can be reached by bypasses (lemma 4.2.9), the theorem is the consequence of the following two lemmas.

For a word $w \in W(n_-, n_+)$, let G_w denote the chord diagram which consists of taking Γ_0, and within the region R, replacing the chord diagram $\Gamma_{(-)^{n_-}(+)^{n_+}}$ with the chord diagram Γ_w.

Lemma 6.2.5 *Let $\mathcal{M}(\Gamma_0, \Gamma_1)$ be a bypass cobordism as above. For any word w in $W(n_-, n_+)$, the chord diagram G_w exists in $\mathcal{M}(\Gamma_0, \Gamma_1)$.*

Proof The work of section 6.1 gives a bypass system

$$FBS\left((-)^{n_-}(+)^{n_+}, w\right) \quad \text{on} \quad \Gamma_{(-)^{n_-}(+)^{n_+}}$$

upon which upwards bypass attachments give Γ_w. Moreover, by section 6.2.1, such bypasses also exist in the universal cobordism $\mathcal{U}(n_-, n_+)$.

With extra chords adjoined outside R, this gives a bypass system on Γ_0, which we denote $\{c_0, \ldots, c_{k-1}\}$, upon which upwards bypass moves give G_w.

As we successively perform these upwards bypass moves, let the diagrams obtained be $G_1, \ldots, G_k = G_w$; restricting to the region R, these give basis diagrams $D_1, \ldots, D_k = \Gamma_w$. We use lemma 4.2.5, that the tightness (or overtwistedness) of a cobordism is preserved upon removing common outermost chords on the upper and lower chord diagrams.

We have that bypasses exist along each c_i within $\mathcal{M}(D_i, \Gamma_{(+)^{n_+}(-)^{n_-}})$, so

$$m\left(D_{i+1}, \Gamma_{(+)^{n_+}(-)^{n_-}}\right) = 1.$$

Adding some extraneous arcs we have

$$m\left(G_{i+1}, \Gamma_1\right) = 1;$$

and hence these attaching arcs are inner at each stage; so bypasses exist along each c_i within $\mathcal{M}(G_i, \Gamma_1)$. Thus each chord diagram G_i exists in $\mathcal{M}(\Gamma_0, \Gamma_1)$; and hence so does $G_k = G_w$. ∎

Lemma 6.2.6 *Consider the chord diagram G_w for $w \in W(n_-, n_+)$, and the cobordism $\mathcal{M}(G_w, \Gamma_1)$ within $\mathcal{M}(\Gamma_0, \Gamma_1)$ as described above. Let c be a nontrivial attaching arc in G_w, such that a bypass exists upwards along c in the tight $\mathcal{M}(G_w, \Gamma_1)$. Then c is isotopic to an attaching arc lying entirely in the region R, and $\text{Up}_c\, G_w$ is of the form $G_{w'}$ for some word $w' \in W(n_-, n_+)$.*

Proof We consider all the possible locations of the nontrivial attaching arc c.

First suppose c is isotopic (through attaching arcs on G_w) to an attaching arc lying entirely in R. Then c can be taken as an attaching arc in Γ_w. If it is a forwards attaching arc, then it leads to a basis chord diagram $\Gamma_{w'}$ within R, or the chord diagram $G_{w'}$ in $\mathcal{M}(\Gamma_0, \Gamma_1)$, and by the previous lemma exists. If it is a backwards attaching arc, its existence is determined by $m(\text{Up}_c\, G_w, \Gamma_1)$, which by lemma 4.2.5 (cancelling corresponding outermost chords) is equivalent to $m(\text{Up}_c\, \Gamma_w, \Gamma_{(+)^{n_+}(-)^{n_-}})$, which is 0 since $\text{Up}_c\, \Gamma_w$ will be a sum of two basis elements. Hence the bypass does not exist.

We may therefore assume that c is not isotopic (in G_w) to an attaching arc in R; hence it intersects chords of G_w not entering R. Since c is nontrivial, c intersects three distinct chords of G_w. Moreover, c must have at least one endpoint on a chord which does not enter R (if both endpoints can be isotoped into R then so can the middle intersection point).

We now consider the arrangement of dividing curves on the whole boundary S^2 of $\mathcal{M}(G_w, \Gamma_1)$. We can regard this S^2 as consisting of four regions: two discs arising from the region R (containing Γ_w and $\Gamma_{(+)^{n_+}(-)^{n_-}}$ as dividing sets respectively) separated by two annuli containing identical dividing sets; although in rounding corners, the two identical dividing sets on the annuli meet each other relatively shifted by one marked point. Taking the general picture of a bypass attachment on Γ_0 depicted in figure 6.27, we can draw the dividing set on this S^2 by drawing Γ_1 "on the outside" of that diagram. We obtain figure 6.28, in which the four concentric regions are respectively (from inside to out): the disc with dividing set Γ_w; the two annuli with identical (but relatively shifted) dividing sets; and then the disc with $\Gamma_{(+)^{n_+}(-)^{n_-}}$ (although in "flipping" this disc to draw it in our diagram, the signs of regions are reversed).

Suppose now that the middle intersection point of c with G_w lies on a chord not entering R; hence half of c can be isotoped to lie entirely outside R. We now observe from the arrangement of 6.28, arising from the clockwise rotation (as depicted in the diagram) of Γ_1 relative to G_w, that we may slide an endpoint of c along the dividing set on the sphere S^2, until it approaches the middle intersection point of c, and the result is as depicted in figure 6.29. Hence performing an upwards bypass move along c would create a dividing set with multiple components; so it cannot exist inside the tight $\mathcal{M}(G_w, \Gamma_1) \subset \mathcal{M}(\Gamma_0, \Gamma_1)$. Thus no such bypass exists.

We may now assume that the middle intersection point of c with G_w lies on a chord entering R, hence which can be considered a chord of Γ_w. The two endpoints of c may:

(i) both lie on chords outside R, or

(ii) only one endpoint may lie on a chord which enters R.

These two cases are depicted in figure 6.30.

Consider again figure 6.28, which shows the arrangement of chords on G_w, including the base and root points of Γ_w. We see that if c exits R and then intersects a chord of G_w outside R, then c exits R either through a positive region on the eastside of Γ_w, or through a negative region on the westside of Γ_w.

In case (i), therefore, c exits R at one end through a positive region p on the eastside; and at the other end through a negative region n on the westside. Now, we again slide the endpoints of c along the dividing set on the sphere S^2, until they lie in R; the result is depicted in figure 6.31(a). If either p or n is enclosed by an outermost chord in Γ_w, then it is clear that performing an upwards bypass move along c will create a dividing set with multiple components. Thus we may assume neither p nor n is enclosed by an outermost chord. Hence the middle intersection point of c lies on a non-outermost chord γ of Γ_w. Since Γ_w is a basis diagram, γ runs from the eastside to the westside and its endpoints are adjacent to where c exits R. There are

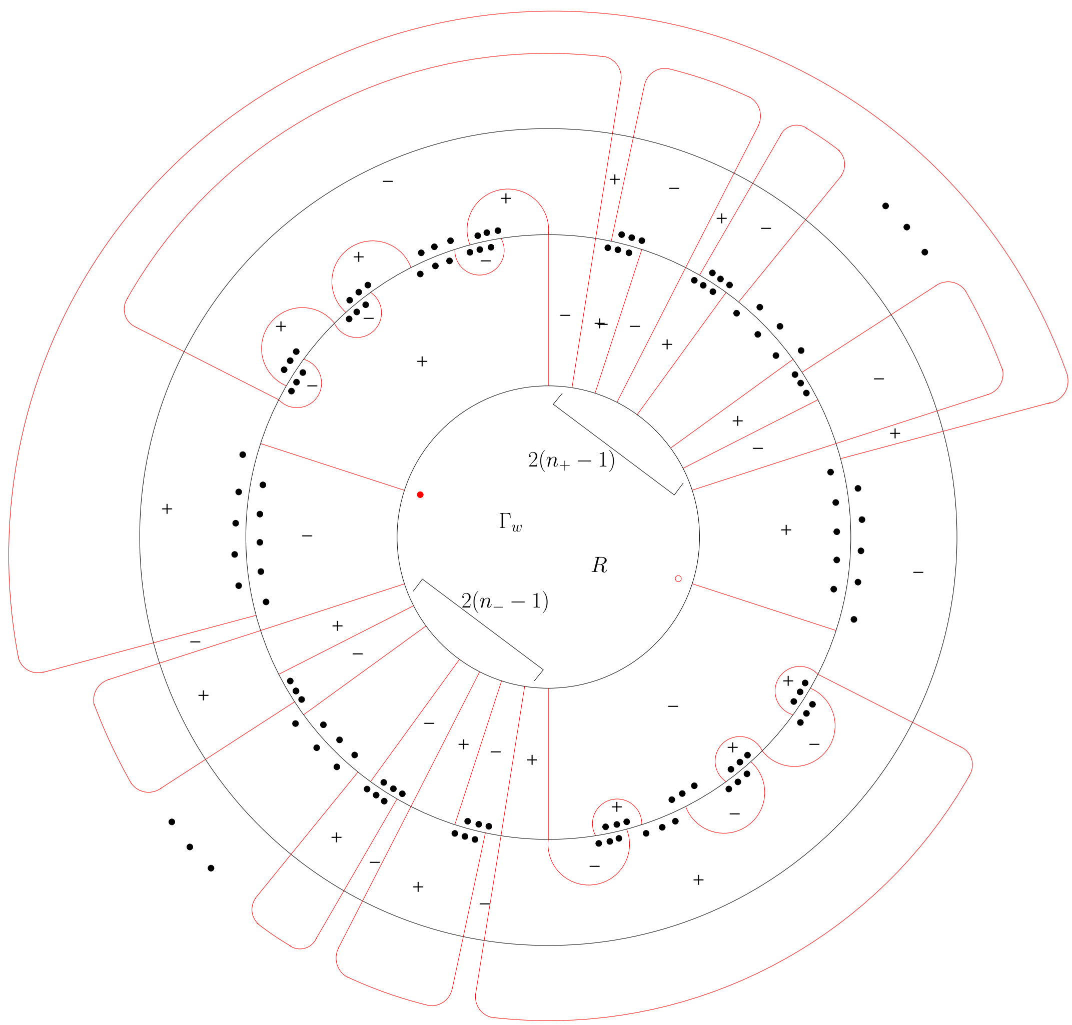


Figure 6.28: General dividing set on $\mathcal{M}(G_w, \Gamma_1)$ within a general bypass cobordism. Each set of black dots (...) represents extra chords; corresponding sets of black dots contain copies of the same arrangements of chords.

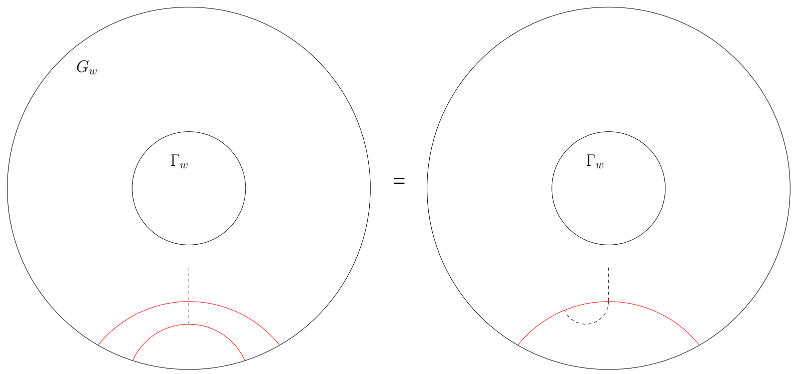


Figure 6.29: Middle intersection point of c lies outside R.

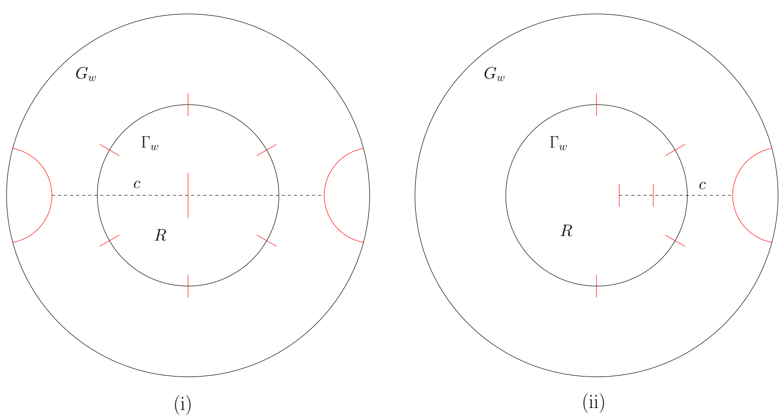


Figure 6.30: Middle intersection point of c lies inside R: cases (i) and (ii).

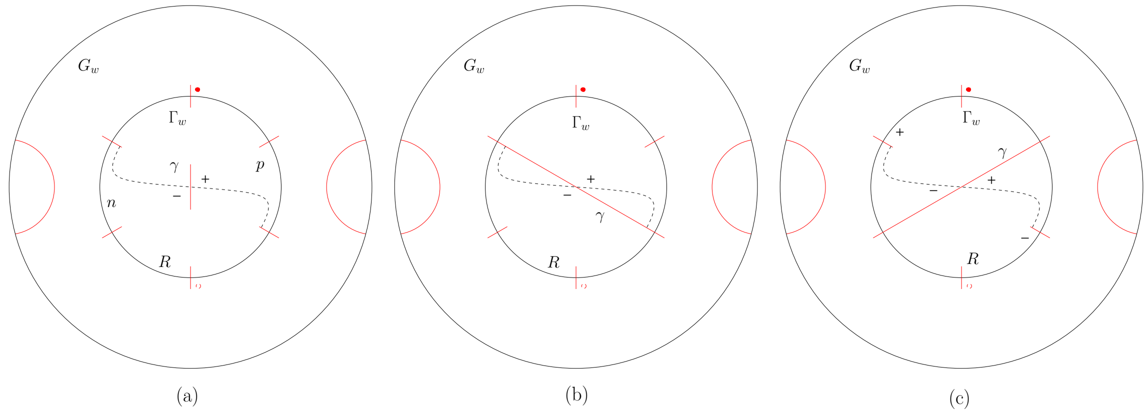


Figure 6.31: Arrangements in case (i).

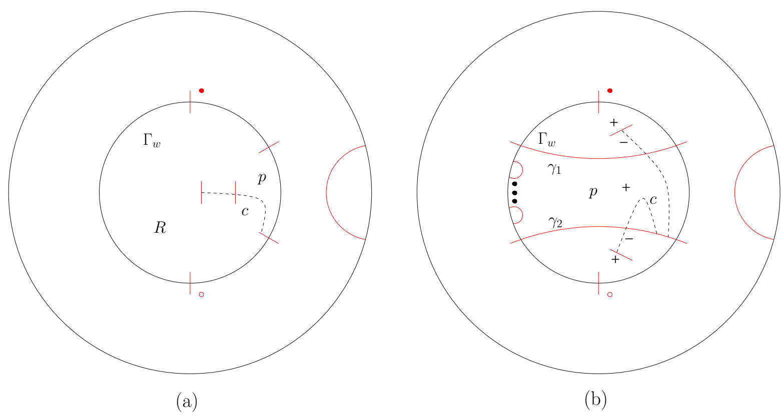


Figure 6.32: Arrangements in case (ii).

two possibilities, depicted in 6.31(b) and (c). In the situation of figure 6.31(b), again clearly an upwards bypass move along c disconnects the dividing set. In the situation of figure 6.31(c), we see that c has become a backwards attaching arc on Γ_w; and hence no bypass exists above it in $\mathcal{M}(\Gamma_w, \Gamma_{(+)^{n_+}(-)^{n_-}})$; and hence not in $\mathcal{M}(G_w, \Gamma_1)$ either.

In case (ii), without loss of generality we may assume c exits R through a positive region p on the eastside; the case of exiting through a negative westside region is similar. Again we slide the endpoint of c outside R along the dividing set on the sphere S^2 until it lies in R; the result is depicted in figure 6.32(a). If p is enclosed by an outermost chord in Γ_w, then performing an upwards bypass move along c

disconnects the dividing set. So the two chords γ_1, γ_2 of Γ_w adjacent to the exit point of c are non-outermost chords, hence proceed to the westside. The region p may have several components of Γ_w on its boundary; since Γ_w is a basis chord diagram, in addition to γ_1, γ_2 on the boundary of p, there may also be chords of Γ_w enclosing negative outermost regions on the westside. However if c intersects any of these then its final intersection point must also lie on the same chord, contradicting nontriviality of c. Thus the middle intersection point of c lies on γ_1 or γ_2, and lies in one of the two situations depicted in figure 6.32(b). Thus c has become either trivial or backwards; in neither case can a bypass exist above it. ∎

Chapter 7

Main results and consequences

We now prove our main theorems 1.2.16 and 1.2.15, and then consider some consequences and further properties of contact elements.

7.1 Proof of main results

We now prove theorem 1.2.16, that there is a bijection between chord diagrams and pairs of comparable words $w_1 \preceq w_2$. This map takes a chord diagram to the lexicographically first and last elements occurring in its basis decomposition.

Corollary 6.1.43 above shows that there is a map

$$\left\{ \begin{array}{c} \text{Comparable pairs of} \\ \text{words } w_1 \preceq w_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Chord} \\ \text{Diagrams} \end{array} \right\}$$

taking (w_1, w_2) to a chord diagram in which Γ_{w_1} is lexicographically the first in its basis decomposition, and Γ_{w_2} the last. Since basis decompositions are unique, this map is clearly injective. Moreover, by proposition 1.2.14, proved in section 3.2, these two sets have the same cardinality. Thus we have the desired bijection.

This proves the main theorem 1.2.16. Moreover, the proof of the main theorem shows that every chord diagram can be constructed by the methods of section 6.1. In particular, if we take any contact element and write it as a sum of basis vectors v_w, and take the lexicographically first and last basis elements v_{w_-}, v_{w_+} among them, we have $w_- \preceq w \preceq w_+$. This proves theorem 1.2.15.

We may therefore formalise the notation described in section 1.2.5.

Definition 7.1.1 (Notation for contact elements) *Given two words $w_- \preceq w_+$, we write $[w_-, w_+]$ or $[v_{w_-}, v_{w_+}]$ to denote the unique contact element which has v_{w_-} and v_{w_+} respectively as first and last basis element. We write $[\Gamma_{w_-}, \Gamma_{w_+}]$ to denote the corresponding chord diagram.*

In practice, as in the foregoing, we will abuse this notation, identifying contact elements with chord diagrams and basis contact elements with words.

Note that in this definition, "first and last" could be according to the lexicographic order or the partial order $\preceq$; it makes no difference.

7.2 Consequences of main results

7.2.1 Up and Down

The main theorem, along with the idea of proposition 1.2.19, gives the following corollary, which will be needed subsequently.

Corollary 7.2.1 (Upwards vs. downwards bypass moves)

(i) Suppose there is a minimal bypass system B on Γ_{w_1} such that attaching bypasses above Γ_{w_1} along B gives a tight $\mathcal{M}(\Gamma_{w_1}, \Gamma_{w_2})$. (Minimality here means: B has no trivial attaching arcs; no proper subset $B' \subset B$ satisfies $\text{Up}_{B'}(\Gamma_{w_1}) = \Gamma_{w_2}$).

Then $\text{Down}_B(\Gamma_{w_1}) = [\Gamma_{w_1}, \Gamma_{w_2}]$.

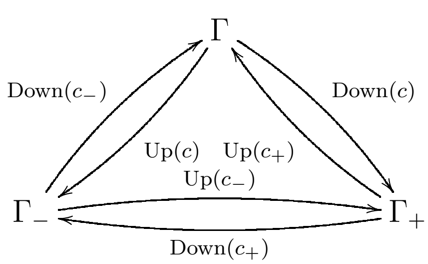
(ii) Suppose there is a minimal bypass system B on Γ_{w_2} such that attaching bypasses below Γ_{w_2} along B gives a tight $\mathcal{M}(\Gamma_{w_1}, \Gamma_{w_2})$. (Minimality here means: B has no trivial attaching arcs; no proper subset $B' \subset B$ satisfies $\text{Down}_{B'}(\Gamma_{w_2}) = \Gamma_{w_1}$).

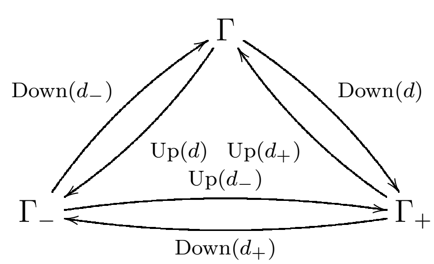
Then $\text{Up}_B(\Gamma_{w_2}) = [\Gamma_{w_1}, \Gamma_{w_2}]$.

Proof We use minimality of B as in the proof of proposition 1.2.19. From tightness we have immediately $w_1 \preceq w_2$. Moreover by the bounded contact category computation (proposition 1.3.7), we have that all chord diagrams occurring inside are basis diagrams; so every attaching arc of B is forwards. Expanding the downwards bypass system as a sum over all subsets of upwards bypasses (lemma 4.2.13) and using proposition 6.1.16, we have a sum of basis chord diagrams, where w_1 is the minimal element occurring in this sum and w_2 the maximum; and w_1 and w_2 occur only once by minimality, hence do not cancel. ■

7.2.2 Generalised bypass triple Γ_-, Γ_+, $[\Gamma_-, \Gamma_+]$

We now consider in more detail the relationship between the chord diagrams Γ_-, Γ_+ and $\Gamma = [\Gamma_-, \Gamma_+]$. By the construction of the main theorem, we see that these three chord diagrams form a generalised bypass triple, which we may regard as a type of "exact triangle" in the contact category, as described in sections 4.2.5 and 4.2.10. We have a bypass system $c_- = FBS(w_-, w_+)$ on Γ_- such that $\text{Up}_{c_-}(\Gamma_-) = \Gamma_+$ and $\text{Down}_{c_-}(\Gamma_-) = \Gamma$ (and which is minimal); moreover there are corresponding bypass systems c_+ on Γ_+ and c on $[\Gamma_-, \Gamma_+]$ obtained by regarding the bypass moves as local 60° rotations. In addition, we have a minimal bypass system $d_+ = BBS(w_-, w_+)$ on Γ_+ such that $\text{Down}_{d_+}(\Gamma_+) = \Gamma_-$ and $\text{Up}_{d_+}(\Gamma_+) = \Gamma$; along with a corresponding d_- on Γ_- and d on Γ. These various bypass systems take the three chord diagrams to each other:





(Note: although c_- is minimal, c and c_+ need not be. Similarly, d_+ is minimal, but d and d_- need not be.)

The morphisms in these triples correspond to cobordisms (sutured cylinders)

$$\mathcal{M}(\Gamma_-, \Gamma_+), \quad \mathcal{M}(\Gamma_+, \Gamma) \quad \text{and} \quad \mathcal{M}(\Gamma, \Gamma_-).$$

Moreover, performing bypass attachments along c, c_-, c_+ or along d, d_-, d_+, up or down, give contact structures on these sutured solid cylinders.

Proposition 7.2.2 (Contact generalised bypass triple)

(i) *The contact structure on $\mathcal{M}(\Gamma, \Gamma_-)$ obtained from performing upwards bypass attachments on c or downwards bypass attachments on c_- is tight.*

(ii) *The contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$ obtained from performing upwards bypass attachments on c_- or d_- or downwards bypass attachments on c_+ or d_+ is tight.*

(iii) *The contact structure on $\mathcal{M}(\Gamma_+, \Gamma)$ obtained from performing upwards bypass attachments on d_+ or downwards bypass attachments on d is tight.*

Note that our computation of the bounded contact category of a basis cobordism implies that the contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$ by attaching bypasses above c_- or below d_+ is tight; but our proof will be independent of this result, and instead use pinwheels.

Proof We show that $c_- = FBS(w_-, w_+)$ has no pinwheels (see section 4.1.5 above or [29]), upwards or downwards. Suppose there were an upwards pinwheel P; the downwards case is similar. Recall (definition 4.1.2) the boundary of P consists of arcs α_i and γ_i, where the γ_i run along the dividing set Γ_-, and the α_i run along the attaching arcs of the bypass system c_-. Since Γ_- is a basis chord diagram, its chords are ordered by the stage of the base point construction algorithm at which they are constructed; let s_i denote the stage at which γ_i is constructed.

Now, $FBS(w_-, w_+)$ consists of nontrivial forwards attaching arcs, and hence each attaching arc has a negative prior outer region. Moreover, proceeding along each attaching arc from prior endpoint to latter endpoint, the numberings of the chords of Γ_- which it intersects strictly increase. If P is a negative region, this implies that $s_k < s_{k-1} < \cdots < s_1 < s_k$, a contradiction. Similarly, if P is a positive region, we have $s_1 < s_2 < \cdots < s_k < s_1$, also a contradiction.

So there are no pinwheels in c_-, upwards or downwards. This implies that performing upwards (resp. downwards) bypass attachments along c_- gives a tight contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$ (resp. $\mathcal{M}(\Gamma, \Gamma_-)$). Since c_+ is the corresponding bypass system on Γ_+, c_- and c_+ give two sets of bypasses lying in the same location, which "undo each other"; so performing downwards bypass attachments along c_+ gives a tight contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$. Similarly, performing upwards bypass attachments along c gives a tight contact structure on $\mathcal{M}(\Gamma, \Gamma_-)$.

Similar arguments for $d_+ = BBS(w_-, w_+)$ give the remaining desired tight contact structures. ∎

The weaker result that

$$m(\Gamma, \Gamma_-) = m(\Gamma_+, \Gamma) = m(\Gamma_-, \Gamma_+) = 1$$

can be proved by other means. For instance, it can be proved simply by expanding out over basis elements: the only Γ_w in the decomposition of Γ satisfying $\Gamma_w \preceq \Gamma_-$ is $\Gamma_w = \Gamma_-$ itself, and similarly for Γ_+.

This weaker result can also be seen directly, by considering the decomposition algorithms of section 5.2, which decompose Γ into its basis elements by performing bypass moves. In the algorithms, these bypass moves were considered purely combinatorially; but of course we can consider the contact manifolds obtained by attaching bypasses. Every basis element is obtained by attaching some bypasses to Γ. Most basis elements are obtained by attaching some bypasses above and some below. But Γ_- is obtained by *attaching only upwards bypasses*, and is the only such basis element in Γ. Similarly, Γ_+ is the one and only basis element in Γ obtained by *attaching only downwards bypasses*. Thus the basis decomposition algorithm (either one) naturally constructs contact structures on $\mathcal{M}(\Gamma, \Gamma_-)$ and $\mathcal{M}(\Gamma_+, \Gamma)$. However it does this by attaching bypasses along arcs that may in general intersect; it will not always give a nice bypass system.

Nonetheless, consider $\mathcal{M}(\Gamma, \Gamma_-)$. We may perform rounding and un-rounding of corners and isotopies, and each chord created in the base point construction algorithm for Γ_- can be successively isotoped off the top of the cylinder, and pushed down the cylinder into the bottom disc, where it simplifies Γ to the chord diagram on the unused disc of an appropriate $\Gamma_{w'}$. In this way we see $m(\Gamma, \Gamma_-) = 1$ directly; and similarly we may see $m(\Gamma_+, \Gamma) = 1$.

7.2.3 Categorical meaning of main theorems

Interpreting the main theorems and the above remarks in the language of the contact category, we now easily obtain the categorical results announced in the introductory section 1.3.4.

First, we have proposition 1.3.10.

Proposition (Tight basis cobordisms elementary) *Let Γ_0 and Γ_1 be basis chord diagrams, and suppose $\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight. Then $\mathcal{M}(\Gamma_0, \Gamma_1)$ is elementary.* ■

Indeed, this follows immediately from proposition 7.2.2(ii).

We can now regard

$$\longrightarrow \Gamma_1 \longrightarrow [\Gamma_0, \Gamma_1] \longrightarrow \Gamma_0 \longrightarrow$$

as an exact triangle in $\mathcal{C}(D^2, n)$. Taking the (somewhat unsatisfactory) notion of "cone" described in section 4.2.10, we can regard $[\Gamma_0, \Gamma_1]$ as the "cone" of the morphism $\Gamma_0 \longrightarrow \Gamma_1$ arising from the bypass system $FBS(\Gamma_0, \Gamma_1)$.

Our main theorems show that the bypass system $FBS(\Gamma_0, \Gamma_1)$ is in some sense canonical, or rather, has canonical effect: although $FBS(\Gamma_0, \Gamma_1)$ can be any minimal subsystem of $CFBS(\Gamma_0, \Gamma_1)$, any choice gives $\text{Down}_{FBS(\Gamma_0,\Gamma_1)} \Gamma_0 = [\Gamma_0, \Gamma_1]$. So we will describe $[\Gamma_0, \Gamma_1]$ as *the* cone of the tight morphism $\Gamma_0 \longrightarrow \Gamma_1$. We then obtain a well-defined cone of any tight morphism between basis chord diagrams.

Thus, chord diagrams, or objects of $\mathcal{C}(D^2, n+1)$, correspond precisely, via this cone construction, to morphisms $\Gamma_0 \longrightarrow \Gamma_1$ of basis elements with $\Gamma_0 \preceq \Gamma_1$, which are precisely the morphisms of the bounded contact category $\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ of the universal cobordism. Once we phrase this categorically, we immediately obtain the following proposition.

Proposition 7.2.3 (Chord diagrams as cones) *Chord diagrams with $n+1$ chords and euler class e are in bijective correspondence with the morphisms of the bounded* *contact category $\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ of the universal cylinder:*

$$Mor\left(\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right)\right) \cong Ob\left(\mathcal{C}(D^2, n+1, e)\right).$$

Moreover, under the inclusion

$$\iota \ : \ \mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right) \hookrightarrow \mathcal{C}(D^2, n+1, e),$$

every morphism of $\mathcal{C}^b(\mathcal{U}(n_-, n_+))$ has a well-defined cone in $\mathcal{C}(D^2, n+1, e)$; the mapping

$$\begin{aligned} Cone \circ \iota \ : \ Mor\left(\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right)\right) &\stackrel{\cong}{\longrightarrow} Ob\left(\mathcal{C}(D^2, n+1, e)\right) \\ (\Gamma_{w_0} \to \Gamma_{w_1}) &\mapsto Cone\left(\iota(\Gamma_{w_0} \to \Gamma_{w_1})\right) = [\Gamma_{w_0}, \Gamma_{w_1}] \end{aligned}$$

gives the bijection explicitly. ■

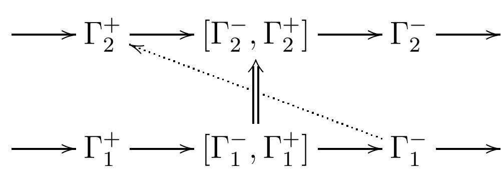
We also have the following "snake lemma", which is not particularly profound in terms of contact geometry, but might nonetheless be of interest from the categorical perspective.

Lemma 7.2.4 ("Snake lemma") *Consider a tight cobordism $\mathcal{M}(\Gamma_1, \Gamma_2)$, where*

$$\Gamma_1 = [\Gamma_1^-, \Gamma_1^+] \quad and \quad \Gamma_2 = [\Gamma_2^-, \Gamma_2^+].$$

Then there is a tight morphism between basis chord diagrams $\Gamma_1^- \longrightarrow \Gamma_2^+$.

The reason for the name "snake lemma" is from the following diagram, regarding cobordism as a map of "exact triangles". All the arrows represent tight cobordisms.



Proof Since $m([\Gamma_1^-, \Gamma_1^+], [\Gamma_2^-, \Gamma_2^+]) = 1$, by proposition 1.3.4, the number of pairs (w_1, w_2) with $\Gamma_{w_1} \in [\Gamma_1^-, \Gamma_1^+]$, $\Gamma_{w_2} \in [\Gamma_2^-, \Gamma_2^+]$ and $w_1 \preceq w_2$ is odd. Hence there is at least one such pair of comparable words in these decompositions. But by theorem 1.2.15, Γ_1^- is an absolute minimum for $[\Gamma_1^-, \Gamma_1^+]$ with respect to $\preceq$, and Γ_2^+ is an absolute maximum for $[\Gamma_2^-, \Gamma_2^+]$ with respect to $\preceq$. Hence $\Gamma_1^- \preceq \Gamma_2^+$, and so there is a morphism $\Gamma_1^- \longrightarrow \Gamma_2^+$ which represents a tight cobordism. ■

7.3 Properties of contact elements

We can now give some further properties of contact elements; in particular, about which basis elements occur in the decomposition of a contact element.

7.3.1 How many basis elements in a decomposition?

A natural first question to ask about contact elements is how many basis elements they contain in their decomposition. The answer is given by proposition 1.2.20:

Proposition (Size of basis decomposition) *Every chord diagram which is not a basis element has an even number of basis elements in its decomposition.*

We found a proof of this result in section 6.2.1, by "skiing"; we can now give a more direct proof.

Proof Consider a non-basis chord diagram Γ. Then as we perform a decomposition algorithm (either one from section 5.2) on Γ, we obtain chord diagrams $\Gamma_{w\cdot}$ (or $\Gamma_{\cdot w}$) in sets Υ_k associated to words of length k. For each basis chord diagram in the decomposition of Γ, it first appears at some stage of this algorithm (possibly not the last). But when it does appear, it comes from a non-basis chord diagram which is related to it by a bypass move. However, by lemma 6.1.14, the only non-basis chord diagrams related by a bypass move to a given basis chord diagram are sums of two basis chord diagrams. It must be this pair of basis chord diagrams which appears; and so the basis elements come in pairs. ■

Actually we essentially already gave this proof in our remark after lemma 5.2.3, that the base point decomposition algorithm gives a binary tree of chord diagrams.

In fact, we have proved a little more: if we write out the basis elements of Γ in lexicographic order, then the $(2j-1)$'th and $2j$'th are bypass-related (and as we group and sum according to the binary tree, we obtain more bypass-related diagrams). A similar result also holds if we use a right-to-left lexicographic order.

This proposition also immediately gives a criterion for whether an element is a basis element or not; unsurprisingly, it is identical to a criterion for existing in a universal cobordism (see section 6.2.1), and can alternatively be proved by skiing as in first proof of lemma 6.2.1.

Lemma 7.3.1 (Test for basis element) *For any chord diagram Γ with n chords and euler class $e = n_+ - n_-$,*

$$m\left(\Gamma_{(-)^{n_-}(+)^{n_+}}, \Gamma\right) = m\left(\Gamma, \Gamma_{(+)^{n_+}(-)^{n_-}}\right) = \begin{cases} 1 & \text{if } \Gamma \text{ is a basis chord diagram} \\ 0 & \text{otherwise} \end{cases}$$

■

7.3.2 Symbolic interpretation of outermost regions

As it turns out, the appearance of certain symbols in both words w_-, w_+ implies the appearance of certain symbols in all the basis elements of $\Gamma = [w_-, w_+]$, and means that Γ has an outermost chord in a specific place.

Lemma 7.3.2 (Outermost regions at base point) *Let $\Gamma = [\Gamma_-, \Gamma_+] = [w_-, w_+]$. The following are equivalent.*

(i) Γ has an outermost chord enclosing a negative region (resp. positive region) at the base point.

(ii) For every Γ_w in the basis decomposition of Γ, w begins with a $-$ (resp. $+$).

(iii) Γ_- and Γ_+ both have outermost chords enclosing a negative region (resp. positive region) at the base point, i.e. w_-, w_+ both begin with a $-$ (resp. begin with a $+$).

PROOF That (i) implies (ii) follows immediately from considering the decomposition algorithm. That (ii) implies (iii) is obvious. That (iii) implies (i) follows immediately from

$$B_-[\Gamma_{w_1}, \Gamma_{w_2}] = [B_-\Gamma_{w_1}, B_-\Gamma_{w_2}] = [\Gamma_{-w_1}, \Gamma_{-w_2}].$$ ■

There is a similar result at the root point.

Lemma 7.3.3 (Outermost regions at root point) *Let* $\Gamma = [\Gamma_-, \Gamma_+] = [w_-, w_+]$*. The following are equivalent.*

(i) Γ *has an outermost chord enclosing a negative region (resp. positive region) at the root point.*

(ii) For every Γ_w *in the basis decomposition of* Γ*,* w *ends with a* $-$ *(resp.* $+$*).*

(iii) Γ_- *and* Γ_+ *both have outermost chords enclosing a negative region (resp. positive region) at the root point, i.e.* w_-, w_+ *both end with a* $-$ *(resp. end with a* $+$*).* ■

We can detect other outermost chords in a similar way; in particular, outermost chords enclosing negative regions on the westside, and outermost chords enclosing positive regions on the eastside. First, we note from lemma 5.1.9 that a basis chord diagram Γ_w has a negative outermost chord on the westside, at $(-2j-1, -2j)$, if and only if the $(j+1)$'th $-$ sign in w is following. Similarly, Γ_w has a positive outermost chord on the eastside, at $(2j, 2j+1)$, if and only if the $(j+1)$'th $+$ sign in w is following.

As mentioned in section 3.1.2, creation and annihilation operators can be defined, not just at the base point, but elsewhere. One may define linear operators on SFH which have the effect of adding or removing an outermost chord at any given site.

Definition 7.3.4 (Eastside/westside creation operators)

(i) For each $i = 0, \ldots, n_-$*, the operator*

$$B_-^{west,i} : SFH(T, n+1, e) \rightarrow SFH(T, n+2, e-1)$$

takes a chord diagram with $n+1$ *chords and relative euler class* e*, and produces a chord diagram with* $n+2$ *chords and relative euler class* $e-1$*, adding an outermost chord enclosing a negative region on the westside, between points* $(-2i-3, -2i-2)$ *(as labelled on the chord diagram with* $n+2$ *chords).*

(ii) For each $j = 0, \ldots, n_+$*, the operator*

$$B_+^{east,j} : SFH(T, n+1, e) \to SFH(n+2, e+1)$$

takes a chord diagram with $n+1$ *chords and relative euler class* e*, and produces a chord diagram with* $n+2$ *chords and relative euler class* $e+1$*, adding an outermost chord enclosing a positive region on the eastside, between marked points* $(2j+2, 2j+3)$*.*

Note that $B_-^{west,0}$ adds an outermost negative region not at the base point, but just west of it ("$B_- = B_-^{west,-1}$"); and then the various $B_-^{west,i}$ add outermost regions further anticlockwise, until B_-^{west,n_-} adds an outermost region "east" of the original root point, to create a new root point further anticlockwise of the original one. Similarly for the $B_+^{east,j}$.

Definition 7.3.5 (Eastside/westside annihilation operators)

(i) For each $i = 0, \ldots, n_-$*, the operator,*

$$A_+^{west,i} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e+1)$$

takes a chord diagram with $n+1$ *chords and relative euler class* e*, and produces a chord diagram with* n *chords and relative euler class* $e+1$*, by joining the chords at positions* $(-2i-2, -2i-1)$*.*

(ii) For each $j = 0, \ldots, n_+$*, the operator*

$$A_-^{east,j} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e-1)$$

takes a chord diagram with $n+1$ *chords and relative euler class* e*, and produces* *a chord diagram with n chords and relative euler class $e-1$, by joining the chords at positions $(2j+1, 2j+2)$.*

So $A_+^{west,0}$ joins chords not at the base point, but just west of it ("$A_- = A_-^{west,-1}$"); and the various $A_+^{west,i}$ join chords further anticlockwise, until A_+^{west,n_-} actually joins chords on the "east" of the original chord diagram, namely those at the root point and immediately east of it.

We see that, as with our original annihilation and creation operators at the base point, we have

$$A_+^{west,j} \circ B_-^{west,j} = 1, \quad A_-^{east,j} \circ B_+^{east,j} = 1,$$

and we will investigate further relations in section 8.2. The numbering of these operators may seem a little strange, but the reasons for it will become apparent in section 8.2.

It's easy from lemma 5.1.9 to see that $B_-^{west,j}$ has the effect on Γ_w of producing $\Gamma_{w'}$, where w' is obtained from w as follows. If $0 \leq j \leq n_- - 1$, then we insert a $-$ sign in w immediately before or after the $(j+1)$'th $-$ sign; we can also regard this as "splitting the $(j+1)$'th $-$ sign into two $-$ signs". If $j = n_-$, then we add a $-$ sign at the end of w. Similarly, $B_+^{east,j}$ adds a $+$ sign immediately before or after the $(j+1)$'th $+$ sign, "splitting the $(j+1)$'th $+$ sign in two", if $0 \leq j \leq n_+ - 1$; and adds a $+$ sign at the end, if $j = n_+$.

We also observe the effect of annihilation operators. The operator $A_+^{west,j}$ has the effect of deleting the $(j+1)$'th $-$ sign, for $0 \leq j \leq n_- - 1$; and for $j = n_-$, it deletes the $-$ sign at the end of the word, if there is one, or returns 0 if the word ends in a $+$. The operator $A_-^{east,j}$ has the effect of deleting the $(j+1)$'th $+$ sign, for $0 \leq j \leq n_+ - 1$; and for $j = n_+$, it deletes the $+$ sign at the end of the word, if there is one, else returns 0.

Lemma 7.3.6 (Outermost negative regions on westside) *Let $\Gamma = [\Gamma_-, \Gamma_+]$ be a chord diagram with $n+1$ chords. Let j be an integer from 1 to $n_- - 1$. The following are equivalent.*

(i) Γ has an outermost chord enclosing a negative region on the westside, between the points $(-2j-1, -2j)$.

(ii) *For every Γ_w in the basis decomposition of Γ, Γ_w has an outermost chord enclosing a negative region between $(-2j-1, -2j)$. Equivalently, every such w has the $(j+1)$'th $-$ sign following (i.e. not the first in its block).*

(iii) *Both Γ_- and Γ_+ have an outermost chord enclosing a negative region between $(-2j-1, -2j)$, i.e. w_-, w_+ both have $(j+1)$'th $-$ sign following.*

Proof The proof is similar to the previous two lemmas, after noting

$$B_-^{west,j-1}\,\Gamma = B_-^{west,j-1}[\Gamma_-, \Gamma_+] = [B_-^{west,j-1}\,\Gamma_-, B_-^{west,j-1}\,\Gamma_+],$$

adding a $-$ sign immediately after the j'th $-$ sign in every word occurring in the decomposition of Γ, so that the $(j+1)$'th $-$ sign is following. ■

There is a similar lemma on the eastside.

Lemma 7.3.7 (Outermost positive regions on eastside) *Let $\Gamma = [\Gamma_-, \Gamma_+]$ be a chord diagram with $n+1$ chords. Let j be an integer from 1 to $n_+ - 1$. The following are equivalent.*

(i) *Γ has an outermost chord enclosing a positive region on the eastside, between the points $(2j, 2j+1)$.*

(ii) *For every Γ_w in the basis decomposition of Γ, Γ_w has an outermost chord enclosing a positive region between $(2j, 2j+1)$. Equivalently, every such w has the $(j+1)$'th $+$ sign following (i.e. not the first in its block).*

(iii) *Both Γ_- and Γ_+ have an outermost chord enclosing a positive region between $(2j, 2j+1)$, i.e. w_-, w_+ both have $(j+1)$'th $+$ sign following.* ■

All the lemmas in this section say that, if a chord diagram has an outermost region in a specific place, then so do all the basis chord diagrams in its decomposition. In particular, as we proceed through the decomposition algorithm, there is no decomposition at that chord. It is not difficult to see this explicitly from the decomposition algorithm.

7.3.3 Relations from bypass systems

We now examine the various basis elements within the decomposition of a chord diagram $\Gamma = [\Gamma_-, \Gamma_+] = [w_-, w_+]$, using the bypass systems constructed in chapter 6.

Definition 7.3.8 (Basis chord diagram in Γ) *If Γ_w appears in the decomposition of Γ, we write $\Gamma_w \in \Gamma$.*

After all, $\mathbb{Z}_2$ addition can be regarded as boolean addition of sets.

First, we observe that in addition to the tight contact cylinders of proposition 7.2.2, we have many more tight cylinders within them.

Lemma 7.3.9 (More tight cylinders) *Let $\Gamma = [\Gamma_-, \Gamma_+] = [w_-, w_+]$, and take bypass systems $c_- = FBS(w_-, w_+)$ on Γ_- and $d_+ = FBS(w_-, w_+)$ on Γ_+.*

(i) For every Γ_w obtained by performing upwards bypass moves on Γ_- along some subset a_- of c_-,

$$m(\Gamma_-, \Gamma_w) = m(\Gamma_w, \Gamma_+) = m(\Gamma, [\Gamma_-, \Gamma_w]) = m([\Gamma_-, \Gamma_w], \Gamma_-) = 1,$$

and tight contact structures on these cylinders can be obtained by bypass attachments related to c_-.

(ii) For every Γ_w obtained by performing downwards bypass moves on Γ_+ along some subset b_+ of d_+,

$$m(\Gamma_-, \Gamma_w) = m(\Gamma_w, \Gamma_+) = m(\Gamma_+, [\Gamma_w, \Gamma_+]) = m([\Gamma_w, \Gamma_+], \Gamma) = 1,$$

and tight contact structures on these cylinders can be obtained by bypass attachments related to d_+.

Note in particular that any $\Gamma_w \in \Gamma$ satisfies the hypotheses of both halves of this lemma; the lemma is more general, because the $\Gamma_w = \mathrm{Up}_{a_-} \Gamma_-$ may come in an even number of copies and hence cancel.

The meaning of "bypass attachments related to" c_- and d_+ will be clear from the proof.

Proof We prove part (i); part (ii) is similar. We use proposition 7.2.2, and construct contact manifolds which can be embedded inside the tight contact manifolds constructed in that proposition. For instance, we saw in proposition 7.2.2 that the contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$ obtained by attaching bypasses above Γ_- along c_- is tight. Attaching only bypasses along the subset a_- then gives the contact manifold $\mathcal{M}(\Gamma_-, \Gamma_w)$ as a contact submanifold of this tight $\mathcal{M}(\Gamma_-, \Gamma_+)$; hence it is also tight.

Now take the corresponding bypass systems $a_+ \subseteq c_+$ on Γ_+ (which need not be minimal) corresponding to $a_- \subseteq c_-$. We know from proposition 7.2.2 that performing downwards bypass attachments along c_+ gives a tight contact structure on $\mathcal{M}(\Gamma_-, \Gamma_+)$, and moreover that the downwards bypasses of c_+ "undo" the upwards bypasses of c_-; hence performing downwards bypass attachments along the complement of a_+ gives a tight contact structure on $\mathcal{M}(\Gamma_w, \Gamma_+)$.

To construct a tight contact structure on $\mathcal{M}([\Gamma_-, \Gamma_w], \Gamma_-)$, we first take a *minimal* sub-system a_-^0 of a_- (as in section 6.1.15). That is, a_-^0 still satisfies $\text{Up}(a_-^0)\Gamma_- = \Gamma_w$, and still contains no trivial attaching arcs, but upwards bypass moves on any proper subset of a_-^0 does not give Γ_w. As noted in section 6.1.15, not every bypass system has a minimal sub-system; but a bypass system with no trivial attaching arcs, such as a_-, does.

Then we may apply corollary 7.2.1: performing downwards bypass moves on Γ_- along a_-^0 gives $[\Gamma_-, \Gamma_w]$. Moreover, we know from proposition 7.2.2 that performing downwards bypass moves on Γ_- along all of c_- gives a tight contact structure on $\mathcal{M}(\Gamma, \Gamma_-)$. Thus attaching bypasses below Γ_- along a_-^0 gives a tight contact structure on $\mathcal{M}([\Gamma_-, \Gamma_w], \Gamma_-)$.

Finally, we construct a contact $\mathcal{M}(\Gamma, [\Gamma_-, \Gamma_w])$. Corresponding to $a_-^0 \subset c_-$ on Γ_- there is $a \subset c$ on Γ; upwards bypasses along c "undo" downwards bypasses along c_-. Since performing downwards bypass moves on Γ_- along a_-^0 gives $[\Gamma_-, \Gamma_w]$, performing upwards bypass moves on Γ along the complement $a^c = c - a$ also gives $[\Gamma_-, \Gamma_w]$. And these bypass attachments all lie inside the tight $\mathcal{M}(\Gamma, \Gamma_-)$; hence attaching bypasses above Γ along a^c gives a tight contact structure on $\mathcal{M}(\Gamma, [\Gamma_-, \Gamma_w])$. ■

Now we can take these bypass system shenanigans a little further.

Proposition 7.3.10 *Let* $\Gamma = [\Gamma_-, \Gamma_+]$ *and consider the bypass systems*

$$c_- = FBS(w_-, w_+) \text{ on } \Gamma_-, \quad d_+ = BBS(w_-, w_+) \text{ on } \Gamma_+.$$

Let Γ_w *be a basis chord diagram obtained by either performing upwards bypass moves on* Γ_- *along a subset* a_- *of* c_-*, or by performing downwards bypass moves on* Γ_+ *along a subset* b_+ *of* d_+*. Then*

$$m(\Gamma, \Gamma_w) = \begin{cases} 1 & \Gamma_w = \Gamma_-, \\ 0 & \text{otherwise}; \end{cases}$$
$$m(\Gamma_w, \Gamma) = \begin{cases} 1 & \Gamma_w = \Gamma_+, \\ 0 & \text{otherwise}. \end{cases}$$

In particular, the above proposition applies to any $\Gamma_w \in \Gamma = [\Gamma_-, \Gamma_+]$.

Proof We consider $m(\Gamma, \Gamma_w)$ and $a_- \subset c_-$; the other cases are similar. Without loss of generality assume a_- is minimal; we may take a minimal sub-system since $a_- \subset c_-$ contains no trivial attaching arcs. Thus a_- is empty in the case $\Gamma_w = \Gamma_-$ and nonempty otherwise. Expanding "up as a sum over down" (lemma 4.2.13), we have

$$\Gamma_w = \text{Up}_{a_-} \Gamma_- = \sum_{b_- \subseteq a_-} \text{Down}_{b_-} \Gamma_-.$$

For each $b_- \subseteq a_-$, we have

$$\text{Down}_{b_-} \Gamma_- = \sum_{e_- \subseteq b_-} \text{Up}_{e_-} \Gamma_-.$$

By proposition 6.1.16, this last sum is a sum of basis elements; the least term is Γ_-, which appears precisely once, when e_- is the empty set. (This is because a_- is minimal, hence contains no trivial arcs, hence neither does b_- or e_-). Moreover, if b_- is not the empty set, then $\text{Down}_{b_-} \Gamma_- \neq \Gamma_-$; hence

$$\text{Down}_{b_-} \Gamma_- = [\Gamma_-, \Gamma_x] \quad \text{for some word } x, \quad \Gamma_- \prec \Gamma_x \preceq \Gamma_+.$$

Moreover, Γ_x is obtained from upwards bypass moves from Γ_- along some subset of c_-. (Note that the maximum term need not be $\text{Up}_{b_-} \Gamma_-$, which may appear several times and cancel; minimality of a_- does not imply minimality of each b_-.)

Now taking $m(\Gamma, \cdot)$ we obtain

$$m(\Gamma, \Gamma_w) = \sum_{b_- \subseteq a_-} m\left(\Gamma, \text{Down}_{b_-} \Gamma_-\right).$$

For b_- empty, the term is $m(\Gamma, \Gamma_-) = 1$ as part of a generalised bypass triple. For b_- nonempty, each term is of the form $m(\Gamma, [\Gamma_-, \Gamma_x])$, where Γ_x is obtained from upwards bypass moves from Γ_- along some subset of c_-. Hence by lemma 7.3.9 above, this term is 1. So the sum is

$$m(\Gamma, \Gamma_w) = \sum_{b_- \subseteq a_-} 1 = 2^{|a_-|}$$

which (mod 2) is 0 when a_- is nonempty, and 1 when a_- is empty. ∎

Proposition 7.3.11 *For every Γ_w occurring in the basis decomposition of Γ, other than $\Gamma_\pm$, the number of basis elements of Γ which precede Γ_w (with respect to $\preceq$) is even, and the number of basis elements which follow it (with respect to $\preceq$) is also even.*

Proof Expand out $m(\Gamma, \Gamma_w) = 0$ and $m(\Gamma_w, \Gamma) = 0$ over the basis elements of Γ. ∎

We can now prove theorem 1.2.21:

Theorem *Suppose v_w occurs in the basis decomposition of $v = [v_{w_-}, v_{w_+}]$ and is comparable, with respect to $\preceq$, with every other basis element occurring in the decomposition. Then $w = w_-$ or w_+.*

Proof Let Γ and Γ_w denote the chord diagrams corresponding to v and v_w. If v is a basis element, it is clear. Otherwise, the number of elements comparable to v_w is $m(\Gamma, \Gamma_w) + m(\Gamma_w, \Gamma) + 1$. (We overcount Γ_w in the sum, so correct by adding 1.) If v_w is comparable to every basis element in v then this number must be even, since Γ contains an even number of basis elements (proposition 1.2.20). But by proposition 7.3.10 it is odd. ∎

7.3.4 Complete(ly unwieldy) descriptions

Perhaps the most complete question we could ask about contact elements, given theorem 1.2.16, is:

Given basis chord diagrams $\Gamma_- \preceq \Gamma_+$ or words $w_- \preceq w_+$, what is the basis decomposition of $[\Gamma_-, \Gamma_+]$?

It is worth noting that, although we have given partial answers in the previous sections, we do have an answer to this question; it is just an unwieldy answer. We know that if we construct a bypass system $FBS(\Gamma_-, \Gamma_+)$ on Γ_-, then performing downwards bypass moves along it gives $[\Gamma_-, \Gamma_+]$. Expanding "down as a sum of ups", we obtain that

$$[\Gamma_-, \Gamma_+] = \sum_{c \subseteq FBS(\Gamma_-, \Gamma_+)} \mathrm{Up}_c \, \Gamma_-.$$

We can therefore compute $[\Gamma_-, \Gamma_+]$ algorithmically by analysing all the bypass subsystems of $FBS(\Gamma_-, \Gamma_+)$. The computation need not be done on chord diagrams; it can be made into an algorithm concerning only words in $W(n_-, n_+)$. However, we have found that writing the algorithm solely in terms of words amounts to little more than codifying the algorithm on chord diagrams, and so we do not include it here.

Another important question we could ask about contact elements is:

Given two contact elements $v_1, v_2 \in SFH(T, n, e)$, are they related by a bypass move?

Obviously, if we have the corresponding chord diagrams Γ_1, Γ_2, it is easy to tell if they are bypass-related, by inspecting the positions of chords. Our question is whether we can tell bypass-relatedness from other data. For example, if we are given the contact elements in the form $v_1 = [\Gamma_1^-, \Gamma_1^+]$, $v_2 = [\Gamma_2^-, \Gamma_2^+]$, are they bypass related?

Again, there is a complete answer, which is also completely unwieldy. We saw earlier (proposition 1.2.4) that contact elements are bypass-related if and only if their sum is also a contact element. So the question reduces to being able to compute contact elements. Namely, we compute the decompositions of $[\Gamma_1^-, \Gamma_1^+]$ and $[\Gamma_2^-, \Gamma_2^+]$; then we sum them; and we determine whether the sum is a contact element.

We can also give much weaker, simpler conditions, such as the following. We note that if we have three contact elements which sum to zero, then every basis element which occurs in them, must occur in precisely two of them. Thus, in any nontrivial bypass triple $[\Gamma_1^-, \Gamma_1^+]$, $[\Gamma_2^-, \Gamma_2^+]$, $[\Gamma_3^-, \Gamma_3^+]$, the set $\{\Gamma_1^-, \Gamma_2^-, \Gamma_3^-\}$ contains precisely two distinct elements; and similarly for $\{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+\}$.

In a categorical direction, we could ask the question:

Which chord diagrams / contact elements occur in $\mathcal{C}^b(\Gamma_0, \Gamma_1)$?

We have given an answer to this question in lemma 4.2.9; for the existence of a chord diagram Γ, we require the existence of successive bypasses attached above Γ_0, obtaining successive chord diagrams G_i such that $m(G_i, \Gamma_1) = 1$. We know how to compute m, so the question reduces to which chord diagrams are obtained from a given one by bypass attachments; hence to the previous question. In any case, using this method, it is possible, in principle, algorithmically, to give a complete description of any $\mathcal{C}^b(\Gamma_0, \Gamma_1)$.

Chapter 8

Further considerations

8.1 The rotation operator

We now consider the operation of rotating chord diagrams, or equivalently, moving the base point.

This will give rise to a linear operator R on SFH. Since it simply corresponds to shifting the base point, it is clear that $m(\Gamma_0, \Gamma_1) = m(R\Gamma_0, R\Gamma_1)$. While we have yet to find interesting applications of this fact, the operator R itself seems to contain interesting structure.

If we are to keep a negative region anticlockwise from the base point, and a positive region clockwise, then we must move the base point by two marked points.

Such a rotation corresponds to the inclusion of sutured manifolds $(T, n) \hookrightarrow (T, n)$ given by thickening the solid torus along its boundary. On the intermediate manifold, which is an annulus $\times S^1$, we specify an S^1-invariant contact structure by giving a dividing set on the annulus, as shown in figure 8.1.

By TQFT-inclusion, we obtain a linear operator

$$\begin{array}{rccc} R: & SFH(T, n+1, e) & \longrightarrow & SFH(T, n+1, e) \\ & \mathbb{Z}_2^{\binom{n}{k}} & \longrightarrow & \mathbb{Z}_2^{\binom{n}{k}} \end{array}$$

Obviously R^{n+1} is the identity, R takes contact elements to contact elements, and R

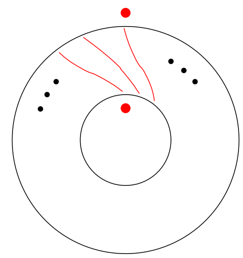


Figure 8.1: The rotation operator.

rotates chord diagrams anticlockwise (or, equivalently, moves the base point 2 marked points clockwise). As with other operators, there is actually a separate R for each n and e; when we wish to refer to a particular n, k we write $R_{n,k}$ for the above map on $\mathbb{Z}_2^{\binom{n}{k}}$

8.1.1 Small cases

For $SFH(T, 1, 0) = \mathbb{Z}_2$, there is only one nonzero element, corresponding to the vacuum $v_\emptyset$, and it is fixed by rotation. Thus R is the identity in this case.

Similarly, for an extremal euler class,

$$SFH(T, n+1, e = \pm n) = \left\{ \begin{array}{c} \mathbb{Z}_2^{\binom{n}{0}} \\ \mathbb{Z}_2^{\binom{n}{n}} \end{array} \right\} = \mathbb{Z}_2.$$

Hence there is only one nonzero element, corresponding to $\Gamma_{(\pm)^n}$, consisting only of outermost chords. Again R is the identity.

The smallest non-identity case is $SFH(T, 3, 0) = \mathbb{Z}_2^{\binom{2}{1}} = \mathbb{Z}_2^2$. And $C_3^0 = 3$, with the three chord diagrams being a bypass triple. We easily obtain

$$v_{-+} \mapsto v_{+-} \mapsto v_{-+} + v_{+-} \mapsto v_{-+}$$

under R, and hence R is given as follows.

$$R_{2,1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{array}{r cc} & \underbrace{-+} & \underbrace{+-} \\ -+\} & 0 & 1 \\ +-\} & 1 & 1 \end{array}$$

Here we write matrices using the lexicographically ordered basis, as shown in the final expression above.

In a similar way we obtain

$$R_{3,1} = R_{3,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$R_{4,1} = R_{4,3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note it appears from these examples that perhaps $R_{n,k} = R_{n,n-k}$; but that is not the case in general, as the following examples show.

$$R_{5,2} \quad = \quad \begin{array}{r c|c|c|c} & \underbrace{---+\cdot} & \underbrace{--+\cdot} & \underbrace{-+\cdot} & \underbrace{+\cdot} \\ ---+\cdot\} & 0 & 1 \,\vdots\, 0 & 0\;0\;0 & 0\;0\;0\;0 \\ \hline --+\cdot\ \Big\} & 0 & 0\;0 & 0\;1 \,\vdots\, 0 & 0\;0\;0\;0 \\ & 0 & 0\;0 & 1\;1 \,\vdots\, 0 & 0\;0\;0\;0 \\ \hline & 0 & 0\;0 & 0\;0\;0 & 0\;1\;0 \,\vdots\, 0 \\ -+\cdot\ \Big\} & 0 & 0\;0 & 0\;0\;0 & 0\;0\;1 \,\vdots\, 0 \\ & 0 & 0\;0 & 0\;0\;0 & 1\;1\;1 \,\vdots\, 0 \\ \hline & 0 & 0\;1 & 0\;1\;0 & 0\;1\;0\;0 \\ +\cdot\ \Big\} & 0 & 0\;0 & 0\;0\;1 & 0\;0\;1\;0 \\ & 0 & 0\;0 & 0\;0\;0 & 0\;0\;0\;1 \\ & 1 & 1\;1 & 1\;1\;1 & 1\;1\;1\;1 \end{array}$$

$$
R_{5,3} \quad = \quad
\begin{array}{r}
--+\cdot\} \\
\\
-+\cdot\;\Big\} \\
\\
\\
\\
+\cdot\;\Big\} \\
\\
\\
\\
\end{array}
\overset{\overbrace{--+\cdot}\quad\overbrace{-+\cdot}\qquad\quad\overbrace{+\cdot}\qquad}{
\begin{array}{c|ccc|cccccc}
0 & 1 & \vdots\, 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & \vdots\, 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \vdots\, 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & \vdots\, 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}}
$$

We have blocked these matrices in a way that will shortly become meaningful.

8.1.2 Computation of R

We now compute R. The computation is recursive: we define $R_{n,k}$ in terms of smaller R matrices. We will need to choose basis elements with certain properties: if x is a word in $+$ and $-$, we will say *x-basis elements* to mean those v_w where the string w begins with the string x (and possibly $w = x$). We will also say *x-rows* or *x-columns* to mean the columns correspond to all x-basis elements. For two strings x and y, the *$x \times y$ minor* of R is the submatrix consisting of the intersection of the x-rows with the y-columns.

For example, we have written $R_{5,2}$ and $R_{5,3}$ above to suggest the following decompositions in terms of such minors.

$$
R_{5,2} \quad = \quad
\begin{array}{r|c|c|c|c}
 & \overbrace{---+\cdot} & \overbrace{--+\cdot} & \overbrace{-+\cdot} & \overbrace{+\cdot} \\
---+\cdot\} & 0 & R_{1,1}\,\vdots\, 0 & 0 & 0 \\
\hline
--+\cdot\} & 0 & 0 & R_{2,1}\,\vdots\, 0 & 0 \\
\hline
-+\cdot\} & 0 & 0 & 0 & R_{3,1}\,\vdots\, 0 \\
\hline
+\cdot\;\Big\} & \begin{array}{c}(---\cdot)\text{-cols}\\ \text{of } R_{4,1}\end{array} & \begin{array}{c}(--\cdot)\text{-cols}\\ \text{of } R_{4,1}\end{array} & \begin{array}{c}(-\cdot)\text{-cols}\\ \text{of } R_{4,1}\end{array} & R_{4,1}
\end{array}
$$

$R_{5,3} =$

	$\overbrace{--+\cdot}$	$\overbrace{-+\cdot}$	$\overbrace{+\cdot}$
$--+\cdot\}$	0	$R_{2,2} \mid 0$	0
$-+\cdot\}$	0	0	$R_{3,2} \mid 0$
$+\cdot\}$	$(--\cdot)$-cols of $R_{4,2}$	$(-\cdot)$-cols of $R_{4,2}$	$R_{4,2}$

We will now prove that something similar occurs for all $R_{n,k}$.

Proposition 8.1.1 (Recursive computation of *R*) *$R_{n,k}$ is described as follows.*

(i) The $(+) \times (+)$ minor of $R_{n,k}$ consists of $R_{n-1,k-1}$.

(ii) The $(+)\times(-+)$ minor of $R_{n,k}$ contains the $(-)$-columns of $R_{n-1,k-1}$. More generally, the $(+)\times((-)^j+)$ minor of $R_{n,k}$ contains the $((-)^j)$-columns of $R_{n-1,k-1}$, for any $j = 1, \dots, n-k$.

(iii) The $(-+) \times (+-)$ minor of $R_{n,k}$ consists of $R_{n-2,k-1}$. More generally, for any $j = 0, \dots, n-k-1$, the $((-)^j - +) \times ((-)^j + -)$ minor of $R_{n,k}$ consists of $R_{n-j-2,k-1}$.

(iv) All other entries are zero. To write these remaining entries out exhaustively (with some overlap):

(a) ("Below and on the diagonal, in the $(-)$ rows.") The $(-+)\times(-)$ minor of $R_{n,k}$ is zero. More generally, the $((-)^j+) \times ((-)^j)$ minor of $R_{n,k}$ is zero, for any $j = 1, \dots, n-k$.

(b) ("Above the diagonal and the submatrices $R_{n-j-2,k-1}$.") The $(--) \times (+)$ minor of $R_{n,k}$ is zero. More generally, the $((-)^j --) \times ((-)^j+)$ minor of $R_{n,k}$ is zero, for any $j = 0, \dots, n-k-2$.

(c) (The pieces in the $(-)$ rows just to the right of the submatrices $R_{n-j-2,k-1}$.) The $(-)\times(++)$ minor of $R_{n,k}$ is zero. More generally, the $(-)\times((-)^j++)$ minor of $R_{n,k}$ is zero, for any $j = 0, \dots, n-k$.

PROOF We simply verify all these conditions. The conditions given are equivalent to the following equations on operators:

(i) $A_- R B_+ = R$.

(ii) $A_- R (B_-)^j B_+ = R(B_-)^j$, for $j = 1, \ldots, n-k$.

(iii) $A_- A_+ (A_+)^j R (B_-)^j B_+ B_- = R$, for $j = 0, \ldots, n-k-1$.

(iv) (a) $A_-(A_+)^j R (B_-)^j = 0$, for $j = 1, \ldots, n-k$.

(b) $A_+ A_+ (A_+)^j R (B_-)^j B_+ = 0$, for $j = 0, \ldots, n-k-2$.

(c) $A_+ R (B_-)^j B_+ B_+ = 0$, for $j = 0, \ldots, n-k$.

These are now easily proved by examining the corresponding chord diagrams. ■

These matrices have interesting combinatorial properties: for instance, for every row, there is precisely one column which has its highest nonzero element in that row.

8.1.3 An explicit description

From the above recursive form of the matrix for R, we can write down a recursive formula.

$$\begin{aligned} R &= \sum_{n=0}^{\infty} B_+ R B_-^n A_- A_+^n + B_-^{n+1} B_+ R A_+ A_- A_+^n \\ &= \sum_{n=0}^{\infty} \left(B_+ R B_-^n + B_-^{n+1} B_+ R A_+ \right) A_- A_+^n \\ &= \sum_{n=0}^{\infty} \left[B_+ R A_+, B_-^{n+1} \right] A_- A_+^n \end{aligned}$$

We can also describe explicitly $R(v_w)$ for each basis vector v_w. Write w in the form

$$w = (-)^{a_1} (+)^{b_1} \cdots (-)^{a_k} (+)^{b_k}$$

where possibly $a_1 = 0$ or $b_k = 0$, but all other a_i, b_i are nonzero. Interpreting the formula for R above as a set of instructions for operating on w, removing or adding $+$ and $-$ signs, and proceeding by induction, we obtain the following.

Proposition 8.1.2 (Explicit computation of R) *If $k \geq 2$ then $R(v_w)$ is given by taking*

$$(+)^{b_1-1}(-)^{a_1+1}(+)^{b_2}(-)^{a_2}\cdots(+)^{b_{k-1}}(-)^{a_{k-1}}(+)^{b_k+1}(-)^{a_k-1}$$
$$= (+)^{b_1-1}(-)^{a_1+1}\left(\prod_{j=2}^{k-1}(+)^{b_j}(-)^{a_j}\right)(+)^{b_k+1}(-)^{a_k-1}$$

and then, for each possible way of grouping $(1,2,\ldots,k)$ into the form

$$((1,2,\ldots,l_1),(l_1+1,l_1+2,\ldots,l_2),\ldots,(l_{T-1}+1,l_{T-1}+1,\ldots,l_T=k)),$$

(including the trivial grouping $((1),(2),\ldots,(k))$, taking the term

$$(+)^{b_1+\cdots+b_{l_1}-1}(-)^{a_1+\cdots+a_{l_1}+1}(+)^{b_{l_1+1}+\cdots+b_{l_2}}(-)^{a_{l_1+1}+\cdots+a_{l_2}}\cdots$$
$$\cdots(+)^{b_{l_{T-2}+1}+\cdots+b_{l_{T-1}}}(-)^{b_{l_{T-2}+1}+\cdots+b_{l_{T-1}}}(+)^{b_{l_{T-1}+1}+\cdots+b_{l_T}+1}(-)^{a_{l_{T-1}+1}+\cdots+a_{l_T}-1}$$

$$= (+)^{b_1+\cdots+b_{l_1}-1}(-)^{a_1+\cdots+a_{l_1}+1}\left(\prod_{m=2}^{T-1}(+)^{b_{l_{m-1}+1}+\cdots+b_{l_m}}(-)^{a_{l_{m-1}+1}+\cdots+a_{l_m}}\right)$$
$$(+)^{b_{l_{T-1}+1}+\cdots+b_{l_T}+1}(-)^{a_{l_{T-1}+1}+\cdots+a_{l_T}-1}$$

obtained by grouping factors of the first expression accordingly, and summing all the corresponding basis elements.

If $k=1$, so that w is of the form $(-)^a$ or $(+)^b$ or $(-)^a(+)^b$, then $R(v_w)$ is given by a single term $v_{w'}$ where w' is given by:

(i) for $w=(-)^a$, $w'=(-)^a$ also;

(ii) for $w=(+)^a$, $w'=(+)^a$ also;

(iii) for $w=(-)^a(+)^b$, $w'=(+)^b(-)^a$; ■

Note that every chord diagram has an outermost region: after some rotation, every chord diagram has an outermost region at the base point. And a chord diagram with an outermost region at the base point is of the form $B_\pm\Gamma$, for some smaller Γ. Thus, these rotation matrices give a quick way to compute all the contact elements in $SFH(T,n+1,e)$ recursively. If we know all the contact elements of $SFH(T,n,e\pm1)$, then we apply B_- to all contact elements in $SFH(T,n,e+1)$ and B_+ to all contact elements in $SFH(T,n,e-1)$. Applying B_- (resp. B_+) to a contact element (or any element) simply prepends a $-$ (resp. $+$) to all of the words in its basis decomposition. And then, applying R will generate all contact elements in $SFH(T, n+1, e)$. In fact, when the euler class is not extremal, there are outermost regions of both signs; so it is sufficient to look at only one of $B_\pm$.

8.2 Simplicial structures

Recall that in section 7.3.2, we defined eastside and westside creation and annihilation operators

$$B_-^{west,i},\ A_+^{west,i},\ B_+^{east,j},\ A_-^{east,j}$$

for $0 \leq i \leq n_-$ and $0 \leq j \leq n_+$, where:

(i) $B_-^{west,i}$ inserts a chord $(-2i-3, -2i-2)$.

(ii) $A_+^{west,i}$ joins the chords at positions $(-2i-2, -2i-1)$

(iii) $B_+^{east,j}$ inserts a chord $(2i+2, 2i+3)$.

(iv) $A_-^{east,j}$ joins the chords at positions $(2j+1, 2j+2)$

Note that it is perfectly compatible with these conditions to take i or $j = -1$ and obtain our original operators,

$$B_- = B_-^{west,-1},\ A_+ = A_+^{west,-1},\ B_+ = B_+^{east,-1},\ A_- = A_-^{east,-1}.$$

We have seen that

$$B_-^{west,j} \circ A_+^{west,j} = 1, \quad B_+^{east,j} \circ A_-^{east,j} = 1,$$

but there are other relations as well.

Lemma 8.2.1 (Westside simplicial structure) *For all $0 \leq i, j \leq n_-$, we have*

$$\begin{aligned} A_+^{west,i} \circ A_+^{west,j} &= A_+^{west,j-1} \circ A_+^{west,i} \quad i < j \\ A_+^{west,i} \circ B_-^{west,j} &= \begin{cases} B_-^{west,j-1} \circ A_+^{west,i} & i < j \\ 1 & i = j, j+1 \\ B_-^{west,j} \circ A_+^{west,i-1} & i > j+1 \end{cases} \\ B_-^{west,i} \circ B_-^{west,j} &= B_-^{west,j+1} \circ B_-^{west,i} \quad i \leq j \end{aligned}$$

Proof Clear, either from considering the effect on words, or the effect on chord diagrams. Perhaps only the cases involving the extremal A_+^{west,n_-} require some explanation, since the operator A_+^{west,n_-} joins the points $(-2n_- - 2, -2n_- - 1)$, where $-2n_- - 1$ is the root point (and becomes part of the east side), and $-2n_- - 2$ is on the eastside (and remains so), so that this is not a very western operator. If $n_+ > 0$, then the relations clearly follow, but if $n_+ = 0$, then A_+^{west,n_-} actually connects the root point to the base point. However in the case $n_+ = 0$ we have $e = -n$, and there is only one possible chord diagram, i.e. the one with $n+1$ outermost negative regions. With only one chord diagram to check, the relations are easily verified. ∎

It follows that there is a *simplicial structure* on our vector spaces $SFH(T, n+1, e)$, with face maps $d_i^+ = A_+^{west,i}$ and degeneracy maps $s_j^+ = B_-^{west,j}$ for $0 \leq i, j \leq n_-$. Hence the map

$$d^+ = \sum_{i=0}^{n_-} d_i^+ = \sum_{i=0}^{n_-} A_+^{west,i}$$

satisfies $(d^+)^2 = 0$ (recall we have $\mathbb{Z}_2$ coefficients), and we obtain chain complexes

$$SFH(T, n+1, e) \xrightarrow{d^+} SFH(T, n, e+1) \xrightarrow{d^+} \cdots \xrightarrow{d^+} SFH\left(T, \frac{n+e}{2} + 1, \frac{n+e}{2}\right)$$

along which the pairs (n_-, n_+) proceed

$$(n_-, n_+) \mapsto (n_- - 1, n_+) \mapsto (n_- - 2, n_+) \mapsto \cdots \mapsto (0, n_+),$$

and hence n_- can be regarded as the "dimension". This is a "northeast–southwest" diagonal of Pascal's triangle. We can call the chain complex C_*^{+,n_+}, so the i-dimensional part (dimension is $i = n_-$) is

$$C_i^{+,n_+} = C_{n_-}^{+,n_+} = SFH\left(T,\ \frac{n+e}{2} + 1 + i,\ \frac{n+e}{2} - i\right)$$
$$= SFH\left(T,\ n_+ + 1 + i,\ n_+ - i\right).$$

Recall from section 7.3.2 the effect of $d_i^+ = A_+^{west,i}$ on words (more precisely, on the corresponding basis vectors, but to simplify notation we simply write the words themselves):

(i) For $0 \leq i \leq n_- - 1$, the effect of $A_+^{west,i}$ is to delete the $(i+1)$'th $-$ sign in a word.

(ii) For $i = n_-$, the effect of A_+^{west,n_-} is to delete the final $-$ sign, if the word ends in a $-$ sign; else return 0.

Hence the effect of d^+ on a word w is to give a sum over all of the above, and we easily obtain the following. Let a word w be written

$$w = (-)^{a_1}(+)^{b_1} \cdots (-)^{a_k}(+)^{b_k}$$

where possibly $a_1 = 0$ or $b_k = 0$, but all other a_i, b_i are nonzero.

Lemma 8.2.2 (Effect of d^+) *If $b_k > 0$, i.e. w ends in a $+$, then*

$$d^+ w = a_1 (-)^{a_1 - 1}(+)^{b_1} \cdots (-)^{a_k}(+)^{b_k} + \cdots + a_k (-)^{a_1}(+)^{b_1} \cdots (-)^{a_k - 1}(+)^{b_k}.$$

If $b_k = 0$, so that w ends in a $-$, then

$$\begin{aligned} d^+ w \;=\; & a_1 (-)^{a_1-1}(+)^{b_1} \cdots (-)^{a_k} + \cdots + a_{k-1} (-)^{a_1} \cdots (-)^{a_{k-1}-1}(+)^{b_{k-1}}(-)^{a_k} \\ & + (a_k + 1)(-)^{a_1}(+)^{b_1} \cdots (-)^{a_{k-1}}(+)^{b_{k-1}}(-)^{a_k - 1}. \end{aligned}$$

■

That is, when w ends in a $+$, d^+ behaves just like a (non-commutative) "partial differentiation by $-$", "$d^+ = \frac{\partial}{\partial -}$". When w ends in a $-$, there is an extra term. From this it is easy to see that $(d^+)^2 = 0$ directly.

All this applies analogously on the eastside. We have a simplicial structure with face maps $d_i^- = A_-^{east,i}$, degeneracy maps $s_j^- = B_+^{east,j}$, and boundary operator

$$d^- = \sum_{i=0}^{n_+} d_i^- = \sum_{i=0}^{n_+} A_-^{east,i}.$$

We then have $(d^-)^2 = 0$, giving a chain complex

$$SFH(T, n+1, e) \xrightarrow{d^-} SFH(T, n, e-1) \xrightarrow{d^-} \cdots \xrightarrow{d^-} SFH\left(T, \frac{n-e}{2} + 1, \frac{-n+e}{2}\right).$$

In this complex the pairs (n_-, n_+) proceed

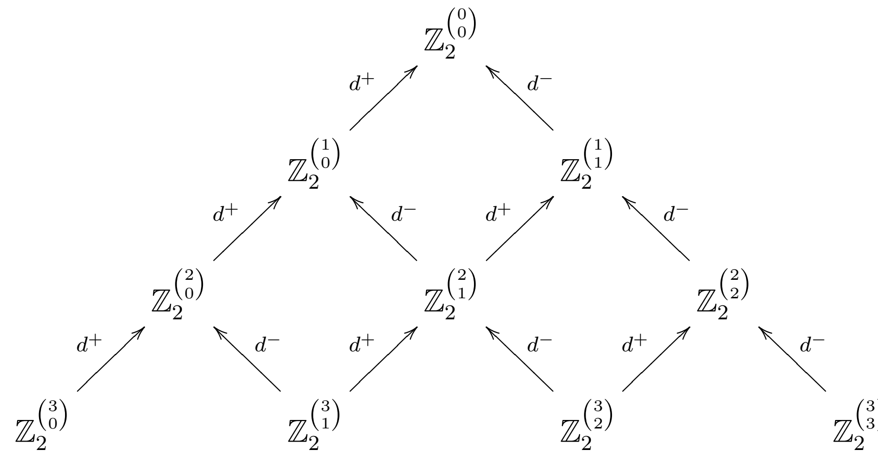
$$(n_-, n_+) \mapsto (n_-, n_+ - 1) \mapsto \cdots \mapsto (n_-, 0)$$

and hence n_+ can be regarded as the "dimension". This is a "northwest–southeast" diagonal of Pascal's triangle. The chain complex can be denoted C_*^{-,n_-}, so that the i-dimensional part ($i = n_+$) is

$$\begin{aligned} C_i^{-,n_-} = C_{n_+}^{-,n_-} &= SFH\left(T,\ \frac{n-e}{2} + 1 + i,\ \frac{-n+e}{2} + i\right) \\ &= SFH\left(T,\ n_- + 1 + i,\ -n_- + i\right). \end{aligned}$$

Thus the chain complex groups $C_{n_-}^{+,n_+}$ and $C_{n_+}^{-,n_-}$ are both equal to the SFH vector space corresponding to (n_-, n_+). The effect of $A_-^{east,i}$ is similar to $A_+^{west,i}$, except that $+$ signs are deleted. The effect of d^- is therefore identical to that of d^+, except that the role of $+$ and $-$ signs is reversed. In particular, "$d^- = \frac{\partial}{\partial +}$" on words w which end in a $-$; if w ends in a $+$ then we have an analogous extra term.

It is not difficult to see that the two boundary operators d^-, d^+ commute; they are essentially partial differentiation by different variables, though some consideration must be paid to the final term. Thus we obtain a double complex structure on the categorified Pascal's triangle:



It is also not too difficult to see that the homology of the chain complexes is rather uninteresting.

Proposition 8.2.3 (Westside/eastside homology)

(i) For all i, the homology of the complex $\left(C_*^{+,n_+}, d^+\right)$ is zero:

$$H_i\left(C_*^{+,n_+}, d^+\right) = 0.$$

(ii) For all i, the homology of the complex $\left(C_*^{-,n_-}, d^-\right)$ is zero:

$$H_i\left(C_*^{-,n_-}, d^-\right) = 0.$$

Here we use a similar argument to Frabetti in [12], in the context of planar binary trees. (Planar binary trees have a nice bijective correspondence with chord diagrams.)

Proof We prove (i); (ii) is similar. We note that our original creation operator $B_- = B_-^{west,-1} : C_*^{+,n_+} \longrightarrow C_{*+1}^{+,n_+}$ satisfies

$$A_+^{west,0} B_- = 1 \quad \text{and} \quad A_+^{west,i} B_- = B_- A_+^{west,i-1} \text{ for } i > 0.$$

Hence for a word $w \in W(n_-, n_+)$,

$$\begin{aligned}\left(B_- d^+ + d^+ B_-\right) w &= B_- \sum_{i=0}^{n_-} A_+^{west,i} w + \sum_{i=0}^{n_-+1} A_+^{west,i} B_- w \\ &= A_+^{west,0} B_- w + \sum_{i=1}^{n_-+1} \left(A_+^{west,i} B_- + B_- A_+^{west,i-1}\right) w \\ &= w\end{aligned}$$

so $B_- d^+ + d^+ B_- = 1$. So B_- is a chain homotopy from the chain maps 1 to 0 on C_*^{-,n_-}, and the homology is zero. Even more directly, if we have a cycle, $d^+ w = 0$, then we have $B_- d^+ w + d^+ B_- w = d^+ B_- w = w$, so $w = d^+(B_- w)$ is a boundary. ■

We have now proved proposition 1.2.22.

8.3 QFT and higher categorical considerations

8.3.1 Dimensionally-reduced TQFT

We have seen that sutured Floer homology obeys some of the properties of a topological quantum field theory. Moreover, in the case of sutured manifolds of the type

$$(\Sigma \times S^1, F \times S^1),$$

where Σ is a surface with boundary, and F is a finite collection of points on $\partial\Sigma$, sutured Floer homology can be regarded as a $(1+1)$-dimensional TQFT via dimensional reduction (see [33]). Clearly, in the case that $\Sigma = D^2$, these sutured manifolds are precisely our (T, n).

In [33], it is noted that

$$SFH(\Sigma \times S^1, F \times S^1) = \mathbb{Z}_2^{2^{n-\chi(\Sigma)}},$$

where $|F| = 2n$ is the number of boundary sutures. As in the case $\Sigma = D^2$, contact structures on such sutured manifolds correspond bijectively to dividing sets K drawn on Σ without any contractible components: see [23, 19]. Note, however, that in higher genus surfaces, K may have closed components.

The dimensionally-reduced TQFT has the following properties:

(i) To every pair (Σ, F), where F divides $\partial\Sigma$ into positive and negative arcs, we associate the vector space $V(\Sigma, F) = \mathbb{Z}_2^{2^{n-\chi(\Sigma)}}$.

(ii) To every properly embedded 1-manifold $K \subset \Sigma$ with boundary F, dividing Σ into positive and negative regions, consistent with the signs on $\partial\Sigma - F$, we associate an element $c(K)$ of this vector space $V(\Sigma, F)$.

See [33] for further details.

We have given a fairly explicit description of the mechanics of this topological quantum field theory, in the case where Σ is a disc.

In their paper [33], Honda–Kazez–Matić prove some properties of this TQFT; we have seen these properties for discs.

(i) $V(\Sigma, F)$ is generated by contact elements;

(ii) $c(K) = 0$ if and only if K is *separating* in the sense that $\Sigma - K$ has components which do not intersect $\partial\Sigma$.

8.3.2 A contact 2-category

We have seen the notion of contact category, and some generalisations. We have noted that in our case $\Sigma = D^2$, the various K corresponding to nontrivial contact structures are just chord diagrams Γ. And we have shown that such Γ are naturally described by pairs (Γ_-, Γ_+) of basis chord diagrams, corresponding to words $w_- \preceq w_+$; we have seen that the objects can be considered as morphisms in a universal category, or as cones of morphisms (proposition 7.2.3). That is, "objects are morphisms"; so morphisms become "morphisms between morphisms". This leads us towards 2-categories, where the idea of "morphisms between morphisms" is formalised. See [2] for a general introduction to higher category theory.

In this spirit, we define a *contact 2-category*, which is a generalisation of the contact category, in the sense that the objects of Honda's category become our 1-morphisms; and its 1-morphisms become our 2-morphisms. It is a specialisation of Honda's contact category, in the sense that it only applies to $\Sigma = D^2$. (There are isomorphisms between any $V(\Sigma, F)$ and some $V(D^2, F')$, but there is no canonical isomorphism; and such an isomorphism is not bijective on contact elements. See section 8.3.4 below. Hence, for the time being at least, we restrict ourselves to discs.)

Definition 8.3.1 (Contact 2-category) *The* contact 2-category $\mathcal{C}(n+1, e)$ *is defined as follows.*

(i) The objects are words on $\{-, +\}$ *with* n_- $-$ *signs and* n_+ $+$ *signs.*

(ii) The 1-morphisms $w_0 \longrightarrow w_1$ *are those arising from the partial order* $\preceq$. *There is one 1-morphism* $w_0 \longrightarrow w_1$ *if* $w_0 \preceq w_1$, *and none otherwise.*

- *The composition of two morphisms* $w_0 \longrightarrow w_1 \longrightarrow w_2$ *is the unique morphism* $w_0 \longrightarrow w_2$.
- *Thus 1-morphisms correspond precisely to chord diagrams* $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$ *on the disc with* $n+1$ *chords; the composition of the two chord diagrams* $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$ *and* $\Gamma' = [\Gamma_{w_1}, \Gamma_{w_2}]$ *is*

$$\Gamma' \circ \Gamma = [\Gamma_{w_0}, \Gamma_{w_2}].$$

(iii) The 2-morphisms $\Gamma_0 \longrightarrow \Gamma_1$ *are the tight contact structures on* $\mathcal{M}(\Gamma_0, \Gamma_1)$, *along with one extra 2-morphism* $\{*\}$ *for overtwisted contact structures. There are two types of composition of 2-morphisms.*

- *Given two 2-morphisms*

$$\Gamma_0 \stackrel{\xi_0}{\to} \Gamma_1 \stackrel{\xi_1}{\to} \Gamma_2,$$

their vertical composition $\xi_0 \cdot \xi_1$ *is the 2-morphism* $\Gamma_0 \longrightarrow \Gamma_2$ *which is the contact structure on* $\mathcal{M}(\Gamma_0, \Gamma_2)$ *obtained by stacking* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *and* $\mathcal{M}(\Gamma_1, \Gamma_2)$ *with contact structures* ξ_0, ξ_1 *respectively.*

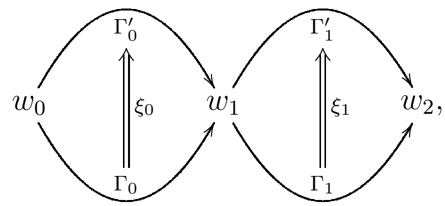
- *Given three objects* w_0, w_1, w_2*, two pairs of 1-morphisms between them*

$$w_0 \overset{\Gamma_0,\Gamma_0'}{\longrightarrow} w_1, \quad w_1 \overset{\Gamma_1,\Gamma_1'}{\longrightarrow} w_2,$$

and two 2*-morphisms*

$$\Gamma_0 \overset{\xi_0}{\longrightarrow} \Gamma_0', \quad \Gamma_1 \overset{\xi_1}{\longrightarrow} \Gamma_1',$$

i.e. the situation



the horizontal composition $\xi_0\xi_1$ *is a morphism* $(\Gamma_1 \circ \Gamma_0) \to (\Gamma_1' \circ \Gamma_0')$ *defined as follows. Since the 1-morphisms arise from a partial order,* $\Gamma_0 = \Gamma_0'$ *and* $\Gamma_1 = \Gamma_1'$*. Thus* ξ_0 *is a contact structure on* $\mathcal{M}(\Gamma_0, \Gamma_0)$ *and* ξ_1 *on* $\mathcal{M}(\Gamma_1, \Gamma_1)$*. If these are both the unique tight contact structures, then we define* $\xi_0\xi_1$ *to be the unique tight contact structure on* $\mathcal{M}(\Gamma_1 \circ \Gamma_0, \Gamma_1 \circ \Gamma_0)$*. Otherwise* $\xi_0\xi_1 = \{*\}$*.*

Note that, considered as a 1-category, $\mathcal{C}(n+1, e) \cong W(n_-, n_+) \cong \mathcal{C}^b(\mathcal{U}(n_-, n_+))$. We can therefore regard this category as a 2-category structure on the bounded contact category of a universal cobordism.

Lemma 8.3.2 (Existence of contact 2-category) $\mathcal{C}(n+1, e)$ *is a 2-category.*

Proof We verify the axioms of a 2-category as stated in [2]. That the objects and 1-morphisms form a category is clear. That vertical composition is associative is clear, since it just corresponds to a union of contact structures. Note that $\{*\}$ acts as a zero for this composition; any composition involving $\{*\}$ is again $\{*\}$.

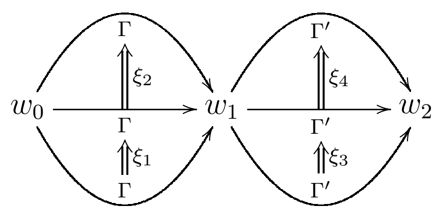
That horizontal composition is associative is also clear: if any of the ξ_i being composed is overtwisted $\{*\}$, then the horizontal composition is $\{*\}$; else associativity follows immediately since 1-morphisms arise from a partial order. Again $\{*\}$ acts as a zero.

There is an identity 2-morphism for each 1-morphism Γ: the identity 2-morphism $\Gamma \xrightarrow{1_\Gamma} \Gamma$ is the tight contact structure on $\mathcal{M}(\Gamma, \Gamma)$. This is just a thickened neighbourhood of a convex surface, so its vertical composition is indeed the identity; and since it is not $\{*\}$, its horizontal composition is also the identity.

The "interchange law"

$$(\xi_1 \cdot \xi_2)(\xi_3 \cdot \xi_4) = (\xi_1 \xi_3) \cdot (\xi_2 \xi_4),$$

perhaps best understood from the diagram,



is only defined when the 2-morphisms ξ_1, ξ_2 are contact structures on some $\mathcal{M}(\Gamma, \Gamma)$, where $\Gamma = [\Gamma_{w_0}, \Gamma_{w_1}]$; and similarly the 2-morphisms ξ_3, ξ_4 are contact structures on some $\mathcal{M}(\Gamma', \Gamma')$, where $\Gamma' = [\Gamma_{w_1}, \Gamma_{w_2}]$. If any of these is $\{*\}$, we have $\{*\}$ on both sides. If not, then $\xi_1 = \xi_2 = 1_\Gamma$ and $\xi_3 = \xi_4 = 1_{\Gamma'}$, being standard neighbourhoods of chord diagrams; thus both sides are equal to $1_{\Gamma' \circ \Gamma}$, the unique tight contact structure on $\mathcal{M}(\Gamma'', \Gamma'')$, where $\Gamma'' = \Gamma' \circ \Gamma = [\Gamma_{w_0}, \Gamma_{w_2}]$. ■

We have now proved proposition 1.3.11.

8.3.3 Improving the 2-category structure

As we have discussed previously in sections 4.2.10, 7.2.2 and 7.2.3, contact categories possess some of the properties of triangulated categories. We now consider these issues for our 2-category, and raise some questions for further investigation.

Since the 1-morphisms in our contact 2-category come from a partial order, there are *no triangles* among 1-morphisms, let alone distinguished triangles. Hence, if our 2-category is to satisfy anything like the axioms of a triangulated category in a non-vacuous way, it is only meaningful to look at the 2-morphisms. But as we have discussed in section 4.2.10, since not every cobordism is elementary (lemma 4.2.14), potential notions of exact triangles like generalised bypass triples are unsatisfactory.

We can ask: Is there a more general definition of distinguished triangle, or improvement of the category structure, for which every 2-morphism (or non-trivial 2-morphism) extends to an exact triangle?

Honda shows that SFH gives a functor from his contact category to the category of vector spaces. Here, of course, our 1- and 2-morphisms are the objects and morphisms of that category. In this functor, our objects map to basis elements of those vector spaces.

Note also that our contact 2-category $\mathcal{C}(n+1,e)$ is specific to an n and e; in fact, all its objects and morphisms relate to $SFH(T,n,e)$. If we consider these $SFH(T,n,e)$ over all n and e, we obtain a family of 2-categories. Moreover, the 0-cells of this 2-category are words on $\{-,+\}$. But these can themselves be regarded as *paths* on Pascal's triangle.

This suggests the construction of a 3-category where:

(0) objects are points of Pascal's triangle, pairs $(n+1,e)$, or perhaps more generally the integer lattice, or perhaps just a point;

(i) 1-morphisms are finite paths on Pascal's triangle, or the lattice, generated by unit southeast and southwest moves on the triangle;

(ii) 2-morphisms are generated by the partial order $\preceq$; equivalently, pairs of paths on Pascal's triangle from the origin to the same endpoint, one always lying left of the other; equivalently, chord diagrams Γ or contact elements;

(iii) 3-morphisms are contact structures on $\mathcal{M}(\Gamma_0,\Gamma_1)$;

This is a question for further investigation.

8.3.4 QFT remarks

The dimensionally-reduced TQFT described above in section 8.3.1 has certain gluing isomorphisms, proved in [33].

Proposition 8.3.3 (Honda–Kazez–Matić [33] lemma 7.9) *Let γ, γ' be disjoint arcs of $\partial\Sigma$, with endpoints not in F, each intersecting F precisely once. Let the gluing τ of γ to γ' produce (Σ', F'). Then the gluing τ gives an isomorphism*

$$\Phi_\tau : V(\Sigma, F) \longrightarrow V(\Sigma', F').$$

Moreover this isomorphism takes contact elements $c(K) \mapsto c(\bar{K})$, where $\bar{K}$ is obtained by performing the gluing τ on K.

Such a gluing decreases $|F| = 2n$ by 2 and decreases χ by 1, so the dimension $2^{n-\chi(\sigma)}$ of the vector spaces is preserved.

By repeated application of such gluing, or the reverse procedure of cutting, we can obtain many isomorphisms between different $V(\Sigma, F)$. In particular, if we have any (Σ, F) and K, and we can cut Σ into a disc along properly embedded arcs or closed curves, each of which intersects K precisely once, then we have an isomorphism $V(D^2, F') \cong V(\Sigma, F)$.

In [33] it is proved that contact elements $c(K)$ generate $V(\Sigma, F)$. In fact, it's easy to see that under a gluing isomorphism $V(D^2, F') \cong V(\Sigma, F)$, a generating set (or basis) of contact elements is obtained by gluing contact elements in $V(D^2, F')$.

However, in general, although there may be an isomorphism between any $V(\Sigma, F)$ and some $V(D^2, F')$, this need not not give a bijection between contact elements; or between nonzero contact elements. Every contact element in $V(D^2, F')$ gives a corresponding contact element in $V(\Sigma, F)$; but not all contact elements in $V(\Sigma, F)$ arise in this way; only those arising from dividing sets which intersect every gluing arc precisely once. That is, the isomorphism $V(D^2, F') \to V(\Sigma, F)$ induces an injective but not surjective map on contact elements.

A simple case of such an isomorphism is when $\partial\Sigma$ has components with two points of F. We may simply glue up such a boundary component of Σ and obtain a surface with one fewer boundary component. In a standard topological quantum field theory picture, this is a good reason why a chord diagram with 1 chord can be regarded as "the vacuum". It may be glued up, or "filled in", or "capped off", without any effect. It is, effectively, not there. A cobordism from vacua is equivalent to a cobordism from the empty set, in this TQFT.

We also remark that our chord diagrams are bijective, in an explicit fashion, with *planar binary trees*, and the vector space generated by such objects has been considered previously; they have also been considered in physical contexts. See, e.g., [5, 12, 13, 20, 39, 40]. The bypass relation translates into a similar linear relation on trees, which appears not to have been considered previously, so far as the author could find.

The upshot is that this story is not finished yet.

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