Chord diagrams, contact-topological quantum field theory, and contact categories

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Thesis defence
29 May 2009

Outline

1. Background
   - Contact geometry
   - Sutured Floer homology, contact elements, TQFT
2. Contact elements in $SFH(T)$
   - Solid tori, contact structures, Catalan, Narayana
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
3. Main theorems
   - Statements
   - Properties of contact elements
4. Contact geometry applications
   - Stackability
   - Contact categories
5. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Outline

The idea of contact geometry

Contact geometry is the study of

non-integrable plane distributions ξ.

(I.e. no surfaces tangent to ξ).

Plane field can be described as the kernel of a 1-form $\xi = \ker \alpha$.
In 3 dimensions, non-integrability condition: $\alpha \wedge d\alpha \neq 0$.

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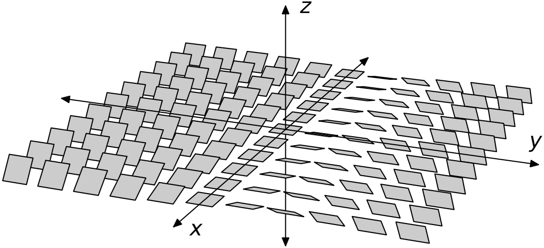

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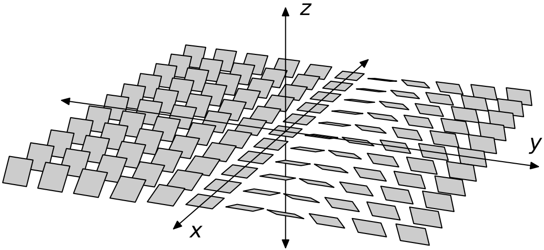

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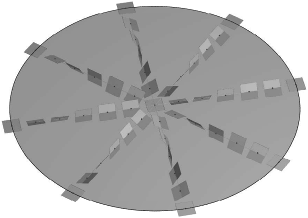
Tight vs. Overtwisted

Eliashberg 1989, *Classification of overtwisted contact structures on 3-manifolds*

An *overtwisted disc* is: Image by P. Massot



Theorem (Eliashberg)

Overtwisted contact geometry reduces to homotopy theory.

Tight contact structure

Contains no overtwisted disc.

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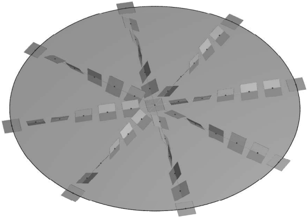

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Convex surfaces

Giroux 1991, *Convexité en topologie de contact*

Generic property for embedded surface in contact 3-manifold.

Convex surface S

There exists a contact vector field X transverse to S.

"Invariant vertical direction".

Dividing set

$$\Gamma = \{x \in S \ : \ X(x) \in \xi\}.$$

"Where ξ is perpendicular".

Dividing set divides S into positive/negative regions $S_\pm$.

Euler class evaluation:

$$e(\xi)[S] = \chi(S_+) - \chi(S_-).$$

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Contact structure near a convex surface

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Honda 2000, *On the classification of tight contact structures. I*

Theorem (Giroux)

The dividing set essentially determines the contact structure near a convex surface.

Given S, Γ, is the nearby contact structure tight?

For $S \neq S^2$:

Contact structure is tight iff Γ has no contractible components.

For $S = S^2$:

Contact structure is tight iff Γ has one component.

If $S^2 = \partial B^3$, tight contact structure near boundary extends uniquely over ball (Eliashberg).

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Corners & rounding

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- When convex surfaces meet transversely along a legendrian curve, dividing sets interleave.
- Corners can be rounded in a standard way.

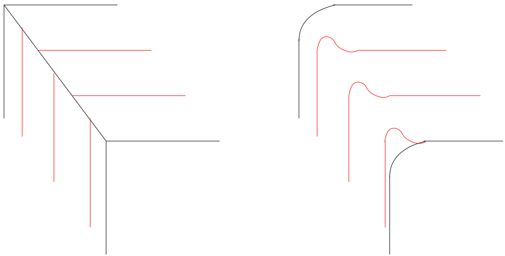

Figure: Rounding corners.

Bypasses

Honda 2000, *On the classification of tight contact structures. I*

- Fundamental building block in contact topology.

"All topologically trivial contact topology is constructed from bypasses."

- Half an overtwisted disc.

"Every step in contact geometry is half way to oblivion."

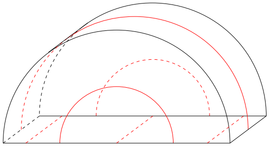

Figure: Bypass with convex boundary.

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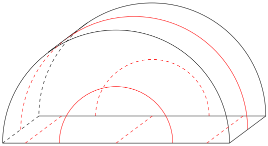
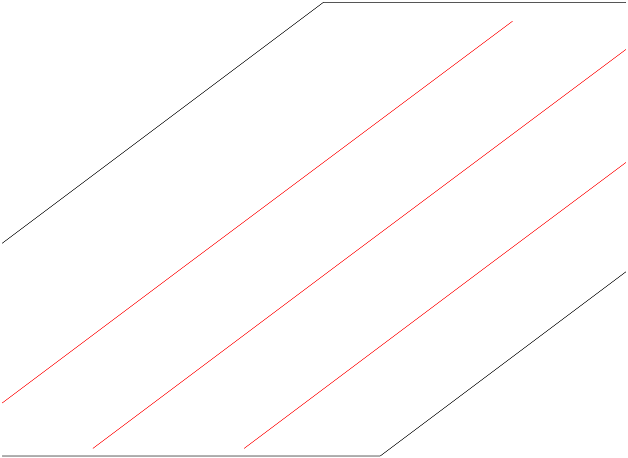


Figure: Bypass with convex boundary.

Adding a bypass



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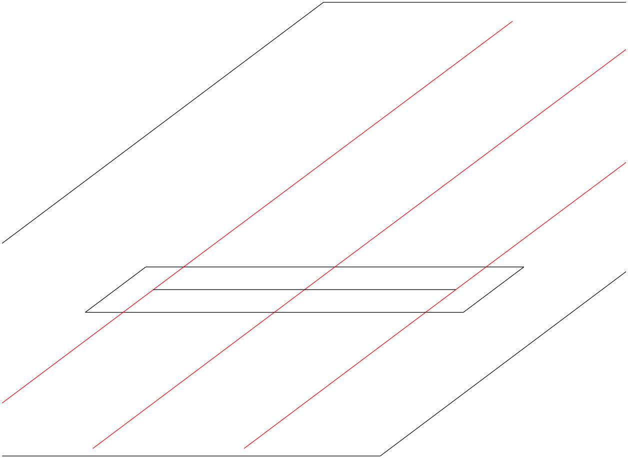

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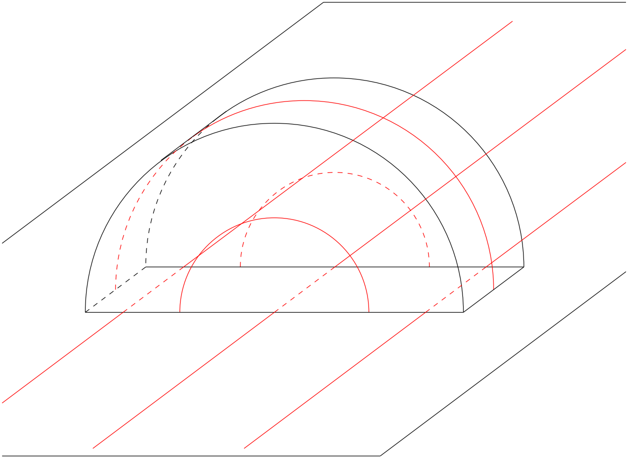

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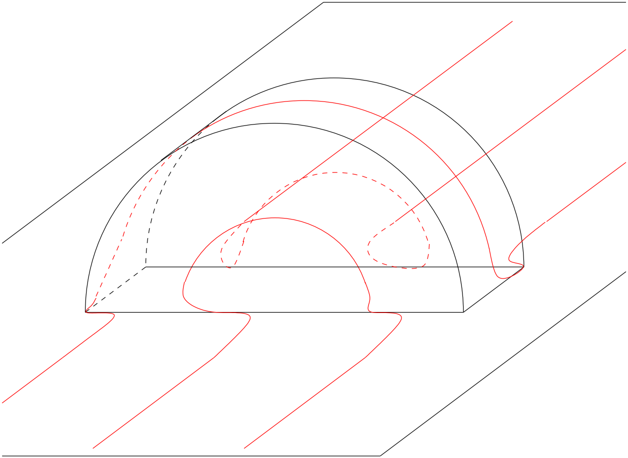

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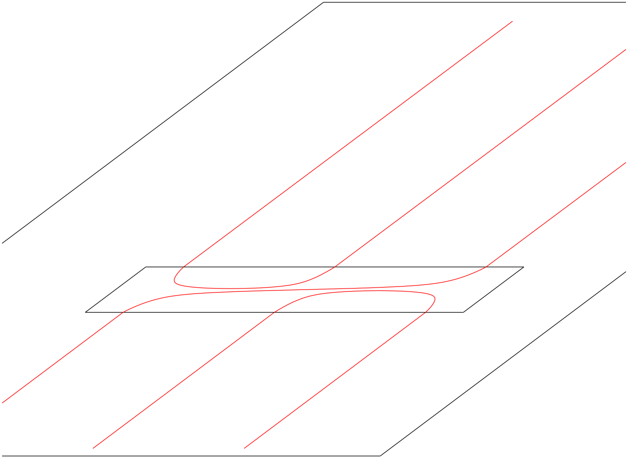

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Sutured manifolds & *SFH*

Juhász 2006, *Holomorphic discs and sutured manifolds*

Sutured manifold (M, Γ)

M 3-manifold with boundary.
Γ collection of disjoint simple closed curves on boundary, dividing ∂M into positive/negative regions.

(Balanced.)

$(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)$

- Take sutured Heegaard decomposition, symmetric product of Heegaard surface.
- Chain complex generated by intersection points of α, β tori.
- Differential counts certain holomorphic curves in symmetric product with certain boundary conditions.
- Invariant of (balanced) sutured manifold.

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Closed case Ozsváth–Szabó 2005, Honda–Kazez–Matić 2007; sutured case Honda–Kazez–Matić 2007, *The contact invariant in sutured Floer homology*

Contact structutre on sutured manifold

ξ contact structure on (M, Γ):

- ∂M convex
- Γ dividing set
- Positive/negative regions.

Theorem (Honda–Kazez–Matić)

A contact structure ξ on (M, Γ) gives a well-defined contact element $c(\xi) \in SFH(-M, -\Gamma)$.

We take $\mathbb{Z}_2$ coefficients. (Otherwise a sign ambiguity.)
ξ overtwisted $\Rightarrow c(\xi) = 0$

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TQFT property

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Theorem

Given

- $(M', \Gamma') \hookrightarrow (M, \Gamma)$ *inclusion of sutured manifolds.*
- ξ'' *contact structure on* $(M - M', \Gamma \cup \Gamma')$

there is a natural map

$$SFH(M', \Gamma') \longrightarrow SFH(M, \Gamma).$$

Further

$$c(\xi') \mapsto c(\xi' \cup \xi'').$$

"TQFT-inclusion".
(Actually $\longrightarrow SFH(M, \Gamma) \otimes V^m$ where m is number of "isolated" components).

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The question

Motivating question:

How do contact elements lie in SFH?

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Solid tori

Only sutured 3-manifolds we consider are *solid tori*.

Sutured manifold (T, n)

- Solid torus $D^2 \times S^1$
- Longitudinal sutures $F \times S^1$, $F \subset \partial D^2$ finite, $|F| = 2n$.

(Notational cover-up: $(T, n) = (-(D^2 \times S^1), -(F \times S^1))$.)

- Part of the $(1+1)$-dimensional TQFT discussed in HKM 2008.

To classify contact structures:

- consider dividing sets on convex meridian disc and boundary torus

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Contact structures on (T, n) and chord diagrams

Dividing set Γ on meridional disc (convex, leg. b'dy)

- Interleaves with sutures $F \times S^1$ on boundary; $2n$ endpoints.
- For tight contact structure, Γ has no closed components.

Chord diagram

Collection of disjoint properly embedded arcs on disc.
Up to homotopy rel endpoints.

E.g.

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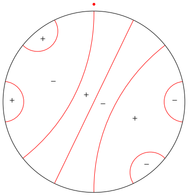

Euler class of chord diagram

Chord diagram has relative euler class e:

$$e = \text{signed sum of regions of } D - \Gamma.$$

So $|e| \leq n - 1$, and e opposite parity to n.

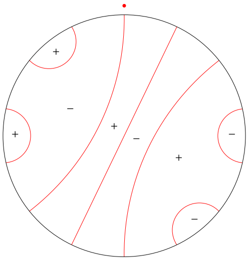

Figure: $n = 7$, $e = 0$

Contact structures on (T, n) *are* chord diagrams

Honda 2000, *On the classification of tight contact structures. II*;
Honda 2002, *Gluing tight contact structures*;
Giroux 2001, *Structures de contact sur les variétés fibrées en cercles...*

A chord diagram determines a contact structure on $D^2 \times S^1$:

- Place it on top and bottom of cylinder, unique tight contact structure on ball, glue ends together.

For solid tori in general:

- A chord diagram may give an overtwisted contact structure on $D^2 \times S^1$
- Distinct chord diagrams may give isotopic contact structures.

However with *longitudinal sutures* of (T, n), neither occurs.

Theorem (Honda, Giroux)

$$\left\{ \begin{array}{c} \textit{Tight contact structures} \\ \textit{on } (T, n) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Chord diagrams} \\ \textit{with } n \textit{ chords} \end{array} \right\}$$

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Catalan and Naryana numbers

$$\#\left\{\begin{array}{c}\text{Tight contact}\\ \text{structures on } (T,n)\end{array}\right\} = \#\left\{\begin{array}{c}\text{Chord diagrams}\\ n \text{ chords}\end{array}\right\}$$

Catalan numbers $C_n = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

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Narayana numbers C_n^e:

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The Catalan disease
and Narayana symptoms

Catalan:

- tight ct. str's on (T, n)
- chord diagrams, n chords
- pairings of n brackets
- Dyck paths length $2n$
- rooted planar bin. trees
- Recursion

$$C_{n+1} = \sum_{n_1+n_2=n} C_{n_1} C_{n_2}.$$

Narayana:

- # with euler class e
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- # with k occurrences of "()"
- # with k peaks
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Computation of $SFH(T, n)$

Juhász 2008, "Floer homology and surface decompositions"

Honda–Kazez–Matić 2008, *Contact structures, sutured Floer homology and TQFT*

Theorem

$$SFH(T, n+1) = \mathbb{Z}_2^{2^n}.$$

Split over spin-c structures:

$$SFH(T, n+1) = \bigoplus_k \mathbb{Z}_2^{\binom{n}{k}}.$$

For ξ with euler class e,

$$c(\xi) \in \mathbb{Z}_2^{\binom{n}{k}} \quad \text{where} \quad k = \frac{e+n}{2}$$

so let

$$SFH(T, n+1, e) = \mathbb{Z}_2^{\binom{n}{k}}.$$

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“Categorified Pascal triangle”

$$
\begin{array}{lcccccccc}
SFH(T,1) & = & & & & SFH(T,1,0) & & & \\
SFH(T,2) & = & & & SFH(T,2,-1) & \oplus & SFH(T,2,1) & & \\
SFH(T,3) & = & & SFH(T,3,-2) & \oplus & SFH(T,3,0) & \oplus & SFH(T,3,2) & \\
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\cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

$$
\begin{array}{lcccccccc}
SFH(T,1) & = & & & & \mathbb{Z}_2^{\binom{0}{0}} & & & \\
SFH(T,2) & = & & & \mathbb{Z}_2^{\binom{1}{0}} & \oplus & \mathbb{Z}_2^{\binom{1}{1}} & & \\
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SFH(T,4) & = & \mathbb{Z}_2^{\binom{3}{0}} & \oplus & \mathbb{Z}_2^{\binom{3}{1}} & \oplus & \mathbb{Z}_2^{\binom{3}{2}} & \oplus & \mathbb{Z}_2^{\binom{3}{3}}
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Catalan and Pascal triangle

Contact elements in each $SFH(T, n, e)$ form a distinguished subset of order C_n^e.

$$\left\{ \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 6 & & 6 & & 1 \end{array} \right\} \subset \left\{ \begin{array}{ccccccc} & & & \mathbb{Z}_2^1 & & & \\ & & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^1 & & \\ & \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^2 & \oplus & \mathbb{Z}_2^1 & \\ \mathbb{Z}_2^1 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^3 & \oplus & \mathbb{Z}_2^1 \end{array} \right\}$$

Question:

How do the C_n^e contact elements lie in $\mathbb{Z}_2^{\binom{n}{k}}$?

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Addition and the bypass relation

Are the contact elements a subgroup?

- No.
- Closure under addition described by bypasses.

Bypass-related chord diagrams naturally come in triples.

Proposition

Suppose $a, b \in SFH(T, n, e)$ are contact elements. Then $a + b$ is a contact element if and only if a, b are related by a bypass surgery. In this case, $a + b$ is the third chord diagram in the triple.

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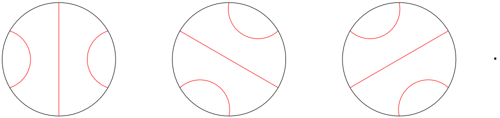

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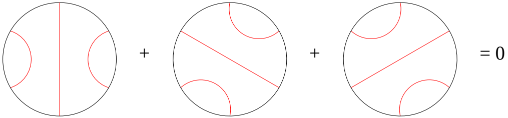

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Reduction to kindergarten

In fact one can show:

Proposition (SFH is combinatorial)

$$SFH(T, n, e) = \frac{\mathbb{Z}_2 \langle \textit{Chord diag's, n chords, euler class e} \rangle}{\textit{Bypass relation}}$$

Outline

1. Background
   - Contact geometry
   - Sutured Floer homology, contact elements, TQFT
2. Contact elements in $SFH(T)$
   - Solid tori, contact structures, Catalan, Narayana
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
3. Main theorems
   - Statements
   - Properties of contact elements
4. Contact geometry applications
   - Stackability
   - Contact categories
5. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Creation operators

A well-defined way to create chords, enclosing positive/negative regions at the basepoint.

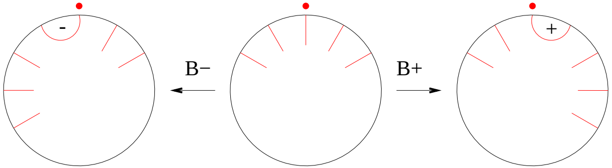

Figure: Creation operators.

We obtain maps

$$B_{\pm} : SFH(T, n, e) \longrightarrow SFH(T, n+1, e \pm 1).$$

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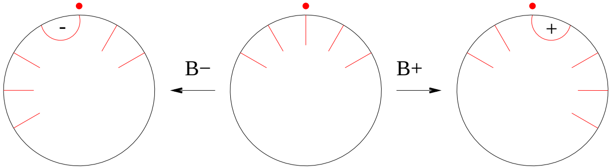

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Origin of creation operators

$B_{\pm}$ arise from TQFT-inclusion

$$(T, n) \hookrightarrow (T, n+1)$$

with intermediate contact structure

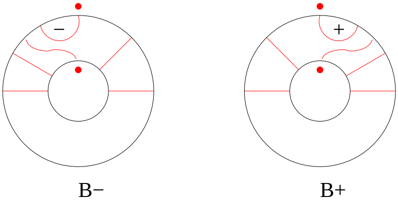
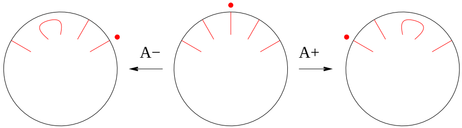


Figure: Creation operator inclusion.

Annihilation operators

Similarly

$$A_{\pm} : SFH(T, n+1, e) \longrightarrow SFH(T, n, e \pm 1)$$



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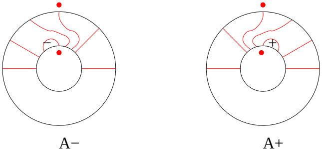
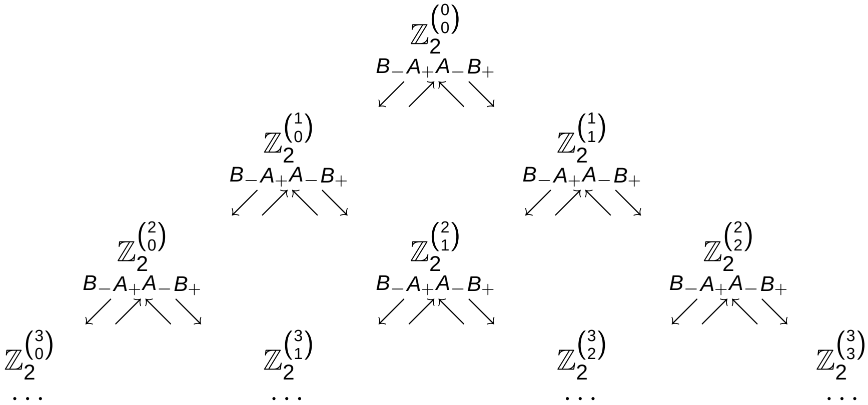


Figure: Annihilation operator inclusion.

Morphisms in Pascal's triangle

"Full categorification"



Operator relations, Pascal recursion

Proposition (Creation/annihilation relations)

$$A_+ \circ B_- = A_- \circ B_+ = 1$$
$$A_+ \circ B_+ = A_- \circ B_- = 0$$

Proposition (Categorification of Pascal recursion)

$$SFH(T, n+1, e) = B_+ SFH(T, n, e-1) \oplus B_- SFH(T, n, e+1)$$

I.e.:

$$\mathbb{Z}_2^{\binom{n+1}{k}} = B_+ \mathbb{Z}_2^{\binom{n}{k-1}} \oplus B_- \mathbb{Z}_2^{\binom{n}{k}}$$

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QFT analogy

$SFH(T, n+1, e) =$ "n-particle states of charge e"

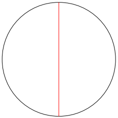

Figure: "The vacuum" $v_\emptyset \in SFH(T, 1, 0) = \mathbb{Z}_2$.

"Basis: apply creation operators to the vacuum"

$W(n_-, n_+) = \{\text{Words on } \{-,+\}, n_- \; - \text{ signs}, n_+ \; + \text{ signs}\}$

For $w \in W(n_-, n_+)$, form $v_w \in SFH(T, n+1, e)$.

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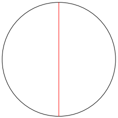

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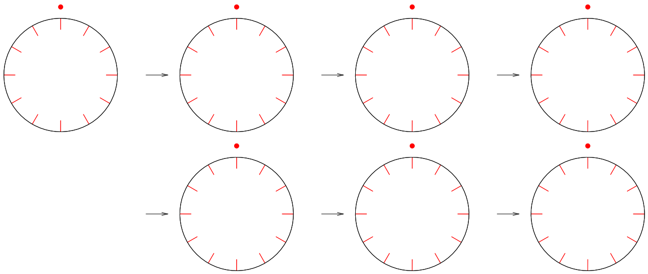
"QFT basis"

E.g.

$$v_{-+-++} = B_- B_+ B_- B_+ B_+ v_\emptyset \in SFH(T, 6, 1).$$



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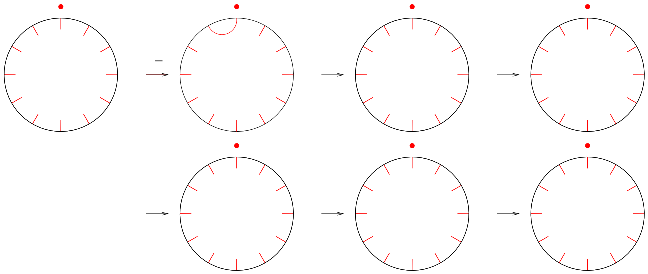

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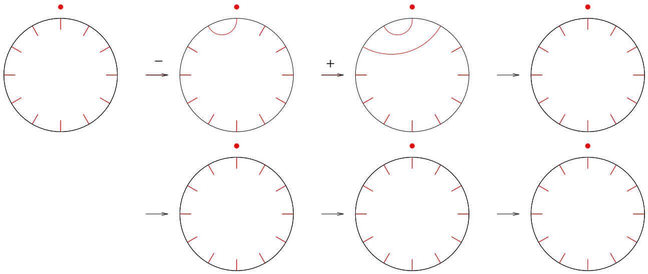

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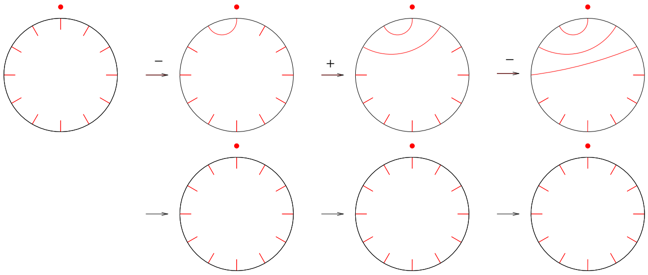

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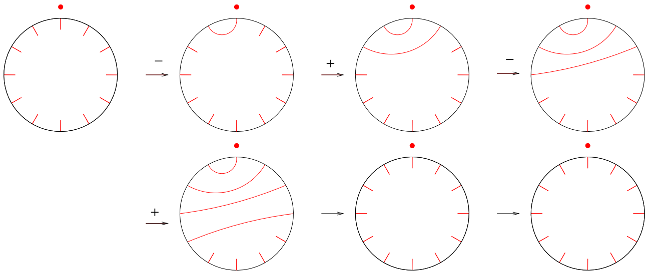

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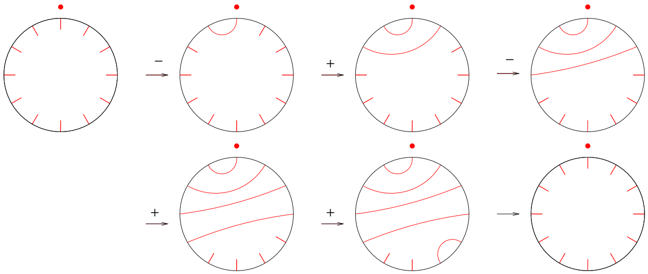

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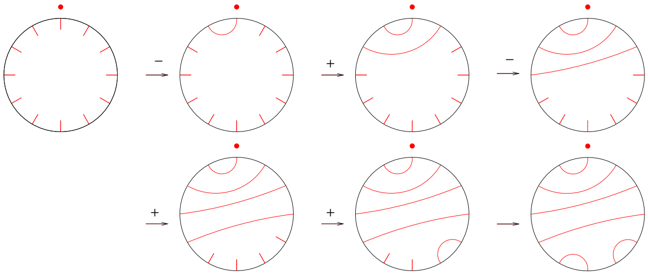

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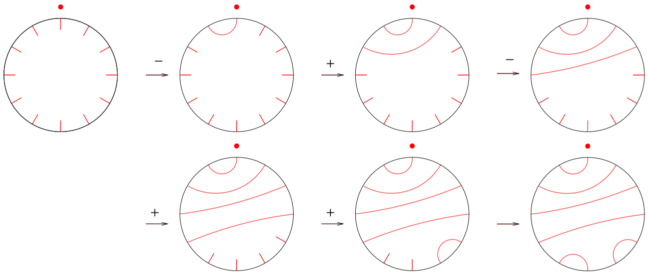

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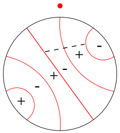
Basis decomposition



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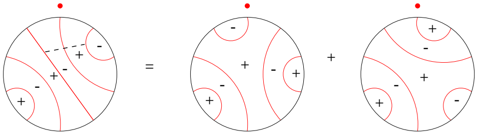

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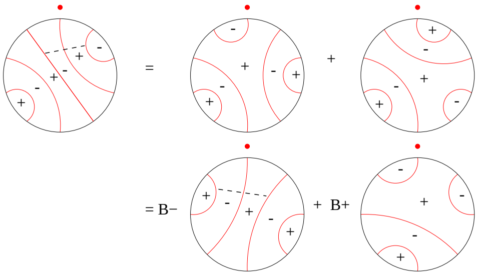

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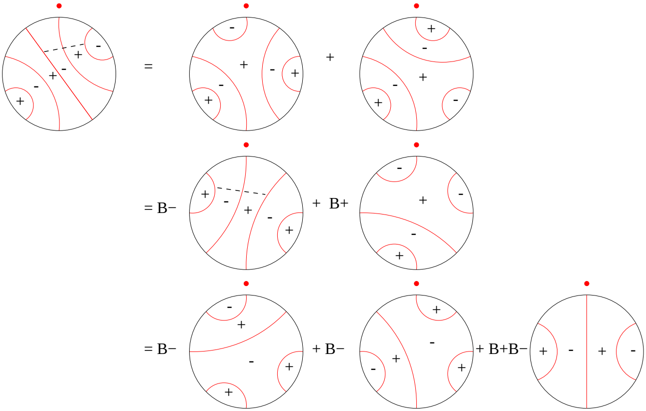

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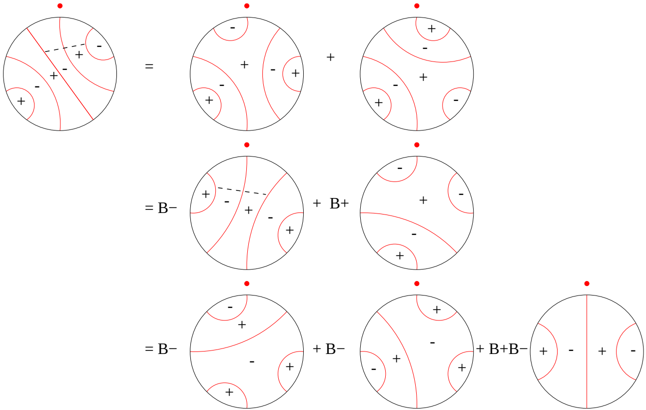
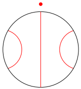


$$= B_-B_-B_+B_+v_\emptyset + B_-B_+B_+B_-v_\emptyset + B_+B_-(B_-B_+v_\emptyset + B_+B_-v_\emptyset)$$

$$= v_{--++} + v_{-++-} + v_{+--+} + v_{+-+-}$$

Examples of basis decomposition



$$\begin{matrix} - & + \\ + & - \end{matrix}$$

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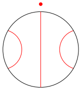
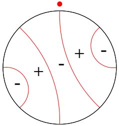


−	+
+	−



−	−	+
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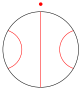

−	+
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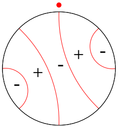
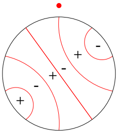


−	−	+
+	−	−



−	−	+	+
−	+	+	−
+	−	−	+
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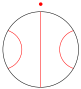
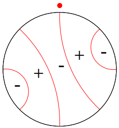
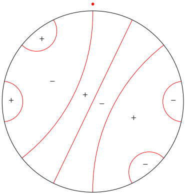
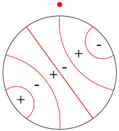


−	+
+	−



−	−	+
+	−	−





−	−	+	+
−	+	+	−
+	−	−	+
+	−	+	−

−	+	−	+	−	+
−	+	−	+	+	−
−	+	+	−	−	+
−	+	+	−	+	−
+	−	−	+	−	+
+	−	−	+	+	−
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+	+	−	−	+	−
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Outline

Orderings on $W(n_-, n_+)$

- Lexicographic ordering: Total order.
- Partial order $\preceq$: "All minus signs move right (or stay where they are)."

E.g.

$$- - ++ \preceq + - +-$$

but

$$- + +-, \quad + - - + \text{ not comparable.}$$

Theorem

Write a contact element v as a sum of basis vectors

$$v = \sum_w v_w, \quad w \in W(n_-, n_+).$$

Let w_-, w_+ be (lex.) first and last words occurring. Then for all words w in decomposition, $w_- \preceq w \preceq w_+$.

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Chord diagram = comparable pair

Now have

$$\Phi : \{\text{Contact elements}\} \longrightarrow \{\text{Comparable pairs of words}\}$$

$$v = \sum_w v_w \mapsto (w_-, w_+)$$

Proposition

These sets have the same cardinality.
I.e. # comparable pairs of words = C_n^e.

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Outline

Properties of contact elements

Notation $v = [w_-, w_+]$.

Proposition

The number of terms in the basis decomposition of a contact element v is

$$\begin{cases} 1 & \text{if } v \text{ is a basis element.} \\ \text{even} & \text{otherwise.} \end{cases}$$

Theorem (Not much comparability)

Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.
Suppose w is comparable with every other element in the decomposition.
Then $w = w_-$ or w_+.

Properties of contact elements

Notation $v = [w_-, w_+]$.

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The number of terms in the basis decomposition of a contact element v is

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Suppose v_w occurs in the basis decomposition of the contact element $v = [w_-, w_+]$.

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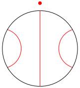
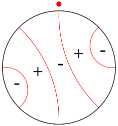
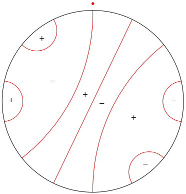
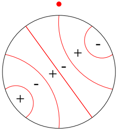


−	+
+	−



−	−	+
+	−	−





−	−	+	+
−	+	+	−
+	−	−	+
+	−	+	−

−	+	−	+	−	+
−	+	−	+	+	−
−	+	+	−	−	+
−	+	+	−	+	−
+	−	−	+	−	+
+	−	−	+	+	−
+	−	+	−	−	+
+	−	+	+	−	−
+	+	−	−	+	−
+	+	−	+	−	−

Summary of results

- Distinct chord diagrams/contact structures give distinct contact elements.
- Contact elements not a subgroup, but “addition means bypasses”.
- Can give a basis for each $SFH(T, n, e)$ consisting of chord diagrams / contact elements.
- There is a partial order $\preceq$ on each basis.
- Chord diagrams / contact structures correspond precisely to comparable pairs of basis elements.

Outline

1. Background
   - Contact geometry
   - Sutured Floer homology, contact elements, TQFT
2. Contact elements in $SFH(T)$
   - Solid tori, contact structures, Catalan, Narayana
   - Computation, addition of contact elements
   - Creation operators, basis of contact elements
3. Main theorems
   - Statements
   - Properties of contact elements
4. Contact geometry applications
   - Stackability
   - Contact categories
5. Idea of proof of main theorems
   - Comparable pairs and bypass systems

Stacking construction

Given Γ_0, Γ_1 chord diagrams, consider $\mathcal{M}(\Gamma_0, \Gamma_1)$:

- sutured solid cylinder $D \times I$
- Γ_i sutures along $D \times \{i\}$
- Vertical interleaving sutures along $\partial D \times I$.

Figure: $\mathcal{M}(\Gamma_0, \Gamma_1)$.

$\mathcal{M}(\Gamma_0, \Gamma_1)$ is tight if it admits a tight contact structure.
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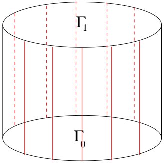

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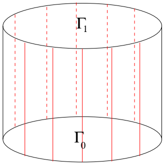

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Properties of stackability

Proposition (Stackability map)

There is a linear map

$$m : SFH(T,n) \otimes SFH(T,n) \longrightarrow \mathbb{Z}_2$$

taking (Γ_0, Γ_1) *to* 1 *if* $\mathcal{M}(\Gamma_0, \Gamma_1)$ *is tight, and* 0 *if overtwisted.*

Proposition (Euler class orthogonality)

If Γ_0, Γ_1 *have distinct Euler class then* $m(\Gamma_0, \Gamma_1) = 0$.

So only interesting part of m on each summand

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Like a metric? *m* is neither symmetric nor antisymmetric...

Lemma (Independence of irrelevant chords)

Suppose Γ_0, Γ_1 have an outermost chord γ in the same position. Then

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$\mathcal{M}(\Gamma_{w_0}, \Gamma_{w_1})$ is tight iff $w_0 \preceq w_1$.

Proposition (General stackability)

Γ_0, Γ_1 chord diagrams, n chords, euler class e.

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Outline

Contact category

Honda (unpublished...)

Σ surface.

Contact category $\mathcal{C}(\Sigma)$

Objects:

- Dividing sets Γ on Σ

Morphisms $\Gamma_0 \longrightarrow \Gamma_1$: "Contact cobordisms"

- Contact structures on $\Sigma \times I$ with $\Gamma_{\Sigma \times \{i\}} = \Gamma_i$.

Composition of morphisms $\Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \Gamma_2$:

- Glue contact structures.

- For surface Σ with boundary: fix marked points on $\partial\Sigma$, vertical sutures on "cobordisms".
- One "zero" morphism for all overtwisted structures.

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- Behaves functorially w.r.t. *SFH*.
- Has something like exact triangles:
  - Bypass triples?
  - "Generalised bypass triples" — multiple bypass attachments?
- Has something like cones.
  - "Cone of up bypass is down bypass".
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Our work computes $\mathcal{C}(D^2, n)$.

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Idea of chord diagrams "existing in" $\mathcal{M}(\Gamma_0, \Gamma_1)$ leads to:

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$\mathcal{C}^b(\Gamma_0, \Gamma_1) =$ "Sub-category of chord diagrams and cobordisms which occur in tight contact $\mathcal{M}(\Gamma_0, \Gamma_1)$."

Proposition

$$\mathcal{C}^b(\Gamma, \Gamma) = \{\Gamma\}$$

I.e. no chord diagrams exist in $\mathcal{M}(\Gamma, \Gamma)$ other than Γ.

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$\mathcal{C}^b(\Gamma_0, \Gamma_1)$ is partially ordered.

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Bounded contact category computations: on basis

We can compute $\mathcal{C}^b(\Gamma_0, \Gamma_1)$ in some cases.

Proposition (Bounded contact category of basis cobordism)

For basis chord diagrams $\Gamma_{w_0}, \Gamma_{w_1}$ *for words* $w_0, w_1 \in W(n_-, n_+)$,

$$\mathcal{C}^b\left(\Gamma_{w_0}, \Gamma_{w_1}\right) \cong \{w \in W(n_-, n_+) \ : \ w_0 \preceq w \preceq w_1\}$$

If we take w_0 minimal and w_1 maximal...

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$$\mathcal{C}^b\left(\mathcal{U}(n_-, n_+)\right) \cong W\left(n_-, n_+\right)$$

This means:

- Chord diagrams in $\mathcal{U}(n_-, n_+)$ are precisely basis diagrams.
- "Universal cobordism" "geometrically realises" $W(n_-, n_+)$.

Note tight $\mathcal{U}(n_-, n_+)$ obtained by a single bypass attachment.

Figure: Bypass move $- - - - + + + +$ to $+ + + + - - - -$.

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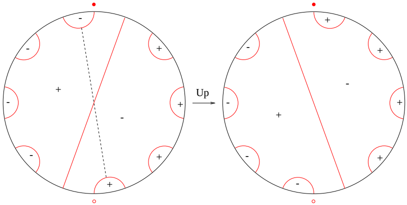

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Bypasses within a bypass

$\mathcal{C}^b(\mathcal{U})$ = "all bypasses within this bypass". In fact:

Theorem (Bypasses within any bypass)

Let Γ_1 be obtained from a single upwards bypass move on Γ_0. Then

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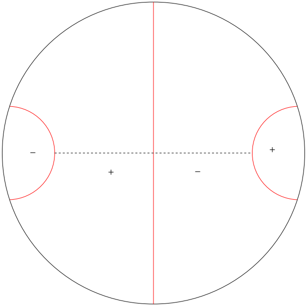

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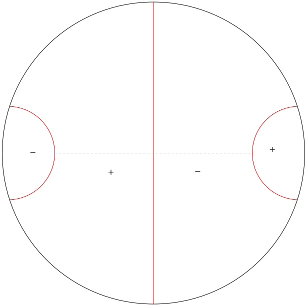

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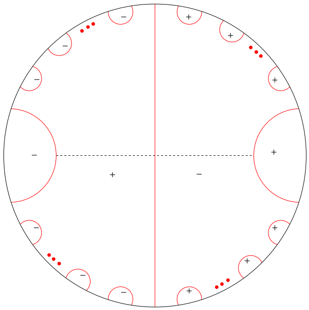

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$\mathcal{C}^b(\mathcal{U})$ = “all bypasses within this bypass”. In fact:

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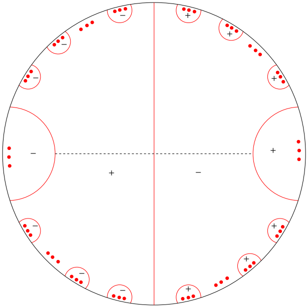

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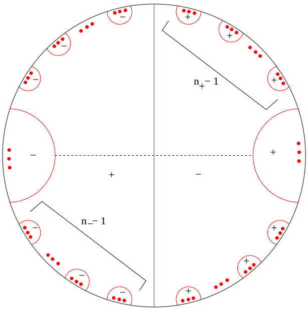

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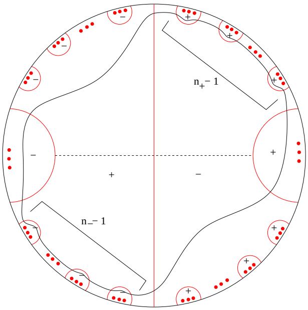

A 2-category

Categorical interpretation of main theorem:

- Inclusion $\iota : \mathcal{C}^b(\mathcal{U}(n_-, n_+)) \longrightarrow \mathcal{C}(D^2, n, e)$.

Theorem (Main theorem, abstract nonsense version)

$$Ob\left(\mathcal{C}(D^2, n, e)\right) \cong Cone \circ \iota\left(Mor\left(\mathcal{C}^b(\mathcal{U}(n_-, n_+))\right)\right)$$

"Chord diagrams are precisely the cones of morphisms in the universal cobordism".
I.e. "objects are morphisms". Hence...

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- Objects = words in $W(n_-, n_+)$ = basis chord diagrams
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Outline

An explicit construction

Prove correspondence

$$\left\{ \text{Chord diagrams} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Comparable pairs} \\ \text{of words} \end{array} \right\}$$

Essential idea:

Given $w_1 \preceq w_2$, construct a chord diagram Γ whose decomposition has w_1 first and w_2 last.

- Along the way, show that every other word w in the decomposition has $w_1 \preceq w \preceq w_2$.
- Elementary combinatorics gives $\# \{\text{pairs } (w_1 \preceq w_2)\} = C_n^e$.
- Done.

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Bypass systems

Take $\Gamma_{w_1}, \Gamma_{w_2}$ basis chord diagrams, $w_1 \preceq w_2$.

Proposition

1. *On Γ_{w_1} there exists a bypass system $FBS(\Gamma_{w_1}, \Gamma_{w_2})$ such that performing upwards bypass moves along it gives Γ_{w_2}.*
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Proposition

Performing either:

1. *downwards bypass moves on Γ_{w_1} along $FBS(\Gamma_{w_1}, \Gamma_{w_2})$, or*
2. *upwards bypass moves on Γ_{w_2} along $BBS(\Gamma_{w_1}, \Gamma_{w_2})$*

gives a chord diagram containing w_1, w_2 in decomposition and:

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Proof by increasingly difficult example

Easy level

“Elementary move” on word = Bypass move on “attaching arc”.

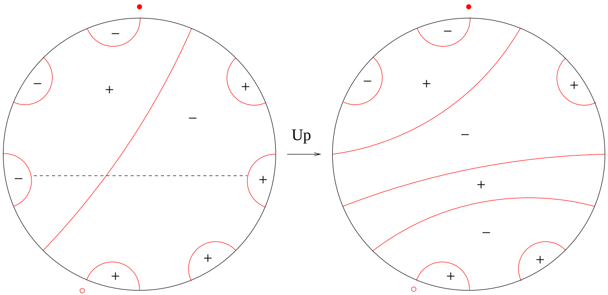

Figure: Upwards move from $\Gamma_{---++++}$ to $\Gamma_{--++-++}$.

Proof by incresingly difficult example

Medium level

$$\left\{ \begin{array}{c} \text{“Generalized elementary} \\ \text{move” on word} \end{array} \right\} = \left\{ \begin{array}{c} \text{Bypass moves on} \\ \text{“generalized attaching arc”} \end{array} \right\}$$

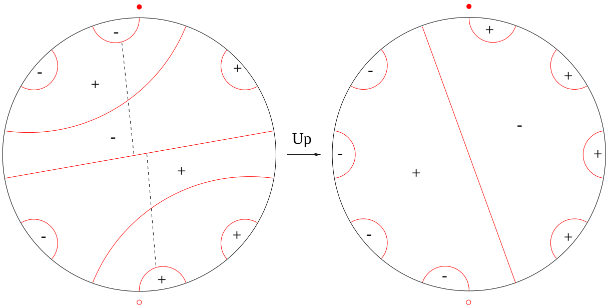

Figure: Upwards moves from $\Gamma_{--++--++}$ to $\Gamma_{++++----}$.

Proof by increasingly difficult example

Hard level

$$\left\{ \begin{array}{c} \text{“Nicely ordered sequence”} \\ \text{of “generalized elementary} \\ \text{moves” on word} \end{array} \right\} = \left\{ \begin{array}{c} \text{Bypass moves on} \\ \text{“well placed sequence” of} \\ \text{“generalized attaching arcs”} \end{array} \right\}$$

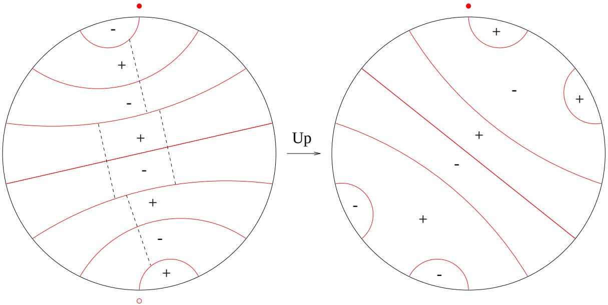

Figure: Upwards moves from Γ_{-+-+-+} to Γ_{++-+--}.

Questions and directions

From perspective of *Floer homology and TQFT*:

- Solid tori with sutures of different slope?
- General surfaces $\Sigma \times S^1$ — dimensionally-reduced TQFT.
- $\mathbb{Z}$ coefficients? Twisted coefficients?
- Relation to bordered Heegaard Floer theory?
- More physics analogies/interpretations?

From the perspective of *category theory*:

- Contact 1-category $\rightsquigarrow$ 2-category $\rightsquigarrow$ 3-category?
- Better triangulated structure? Triangles, cones, kernels?

From the perspective of *contact topology*:

- More computations of bounded contact categories?
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Thanks!

Thanks for coming!