Hyperbolic cone-manifold structures with prescribed holonomy

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Outline

Holonomy

Recall:

- A hyperbolic structure on a manifold M^n is equivalent to an developing map $\mathcal{D} : \widetilde{M}^n \longrightarrow \mathbb{H}^n$.
- A loop $C \in \pi_1(M, x_0)$ lifts to a path in $\mathbb{H}^n$, giving an isometry $\rho(C)$ relating first and last charts around x_0.
- This gives holonomy homomorhism $\rho : \pi_1(M, x_0) \longrightarrow \text{Isom}^+\mathbb{H}^n$.

Notation:

- Capitals denote curves in $\pi_1(M)$, lower case denotes image under ρ, i.e. $\rho(G) = g$.
- All surfaces orientable connected.

2 papers on arxiv

- 1006.5223: Hyperbolic cone-manifold structures with prescribed holonomy I: punctured tori
- 1006.5384: Hyperbolic cone-manifold structures with prescribed holonomy II: higher genus

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Questions

$$\left\{ \begin{array}{c} \text{Hyperbolic structure} \\ \text{on } M \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Algebraic representation} \\ \pi_1(M) \longrightarrow PSL_2\mathbb{R} \end{array} \right\}$$

- Which representations $\pi_1(M) \longrightarrow PSL_2\mathbb{R}$ are holonomy maps of hyperbolic structures?
- Do other representations have a geometric interpretation?
- In general, how does algebra determine geometry?

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Known results

- 3-dimensional hyperbolic/euclidean/spherical geometry, M with boundary (but no boundary control): Leleu 2000
- 2-dimensional complex projective geometry, M closed: Gallo–Kapovich–Marden 2000
- 2-dimensional hyperbolic geometry, M closed/punctured/geodesic boundary: Goldman 1980

Here:

2-dimensional hyperbolic geometry.
Extend (and reprove) Goldman's results.

$\widetilde{PSL_2\mathbb{R}}$

All the previous results involve lifting representations $\pi_1(M) \longrightarrow \mathrm{Isom}^+X$ to the universal cover $\widetilde{\mathrm{Isom}^+X}$.

As unit tangent bundle:

$$PSL_2\mathbb{R} \cong UT\mathbb{H}^2 \cong \mathbb{H}^2 \times S^1$$

$$\widetilde{PSL_2\mathbb{R}} \cong \left\{ \begin{array}{c} \text{“unit tangent bundle but with} \\ \text{angles measured in } \mathbb{R} \text{ not } \mathbb{R}/2\pi\mathbb{Z}\text{"} \end{array} \right\} \cong \mathbb{H}^2 \times \mathbb{R}$$

As classes of paths:

$$\widetilde{PSL_2\mathbb{R}} = \left\{ \begin{array}{c} \text{“Homotopy classes of paths in } PSL_2\mathbb{R} \\ \text{starting at 1, rel endpoints”} \end{array} \right\}$$

Projection to $PSL_2\mathbb{R}$: take a path to its endpoint.

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Lifts to $\widetilde{PSL_2\mathbb{R}}$

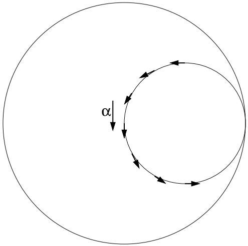
Lifts of:

- $1 \in PSL_2\mathbb{R}$ are $\{\mathbf{z}^n \ : \ n \in \mathbb{Z}\}$ = centre = Z. $\mathbf{z}$ = 2π rotation.
- $g \in PSL_2\mathbb{R}$ are $\tilde{g}Z$ where $\tilde{g}$ is one particular lift.

Lemma

If $g, h \in PSL_2\mathbb{R}$ then $[g, h]$ is well-defined in $\widetilde{PSL_2\mathbb{R}}$.

A *parabolic* or *hyperbolic* $\alpha \in PSL_2\mathbb{R}$ has a "simplest" lift to $\widetilde{PSL_2\mathbb{R}}$: 'minimal twist to tangent vector".



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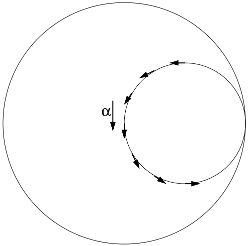

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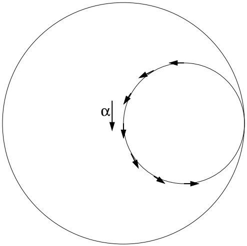
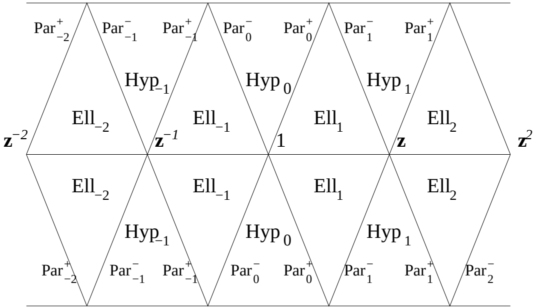


Regions in $\widetilde{PSL_2\mathbb{R}}$

$$\begin{aligned}
\mathrm{Hyp}_0 &= \{\text{“simplest lifts of hyperbolics''}\} \\
\mathrm{Par}_0 &= \{\text{“simplest lifts of parabolics''}\} = \mathrm{Par}_0^+ \cup \mathrm{Par}_0^- \\
\mathrm{Ell}_1 &= \{\text{“rotations by } \theta \in (0, 2\pi)\text{''}\} \\
\mathrm{Ell}_{-1} &= \{\text{“rotations by } \theta \in (-2\pi, 0)\text{''}\}
\end{aligned}$$



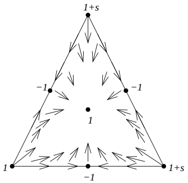
Euler class of a representation

Algebraic definition:

$$\pi_1(S) = \langle G_1, H_1, \ldots, G_k, H_k \mid [G_1, H_1] \cdots [G_k, H_k] = 1 \rangle$$

Consider $\rho([G_1, H_1] \cdots [G_k, H_k]) = \begin{cases} 1 \in PSL_2\mathbb{R} \\ \mathbf{z}^m \in \widetilde{PSL_2\mathbb{R}} \end{cases}$

m = Euler class of $\rho = e(\rho)$
Also obstruction-theoretic: "$e(\rho)$ is the obstruction to an equivariant developing map with vector field $\mathcal{D} : \tilde{S} \longrightarrow UT\mathbb{H}^2$"



Proposition

S closed, ρ holonomy representation. Then $e(\rho) = \pm\chi(S)$.

For surfaces with boundary, need to trivialize.

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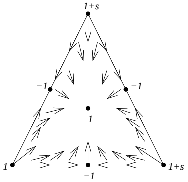

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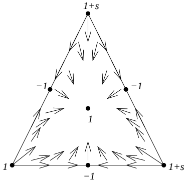

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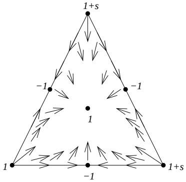

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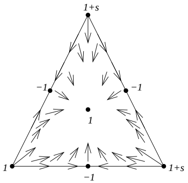

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Theorem (Milnor–Wood inequality 1958)

When $\chi(S) < 0$, for $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$

$$\chi(S) \le e(\rho) \le -\chi(S).$$

e is a continuous map from the *representation variety* to $\mathbb{Z}$.

$$R(S) = \{\text{representations } \pi_1(S) \longrightarrow PSL_2\mathbb{R}\}$$

Theorem (Goldman 1988)

Suppose S closed, $\chi(S) < 0$. Then $R(S)$ has $2|\chi(S)| + 1$ components, parametrized by Euler class.

$$e = \chi(S), \chi(S) + 1, \ldots, -\chi(S) - 1, -\chi(S).$$

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Geometric interpretation of representations

Above: for S closed, ρ holonomy representation $\Rightarrow$ $e(\rho) = \pm\chi(S)$ extremal. The converse is also true.

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1. *ρ holonomy of a complete hyperbolic structure on S with geodesic / cusped boundary components (resp. as ρ is hyperbolic or parabolic)*
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Geometric interpretation for other components? Holonomy of *cone-manifold structures*.

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Hyperbolic cone surfaces

Definition

A surface locally isometric to $\mathbb{H}^2$ except at finitely many singular of cone points. Singular points have neighbourhoods which are:

- *a cone on a circle of length θ; interior cone point.*
- *a cone on an arc of angle θ; boundary cone point or corner point.*

Order of cone point: excess angle in multiples of 2π.

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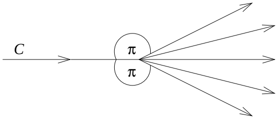

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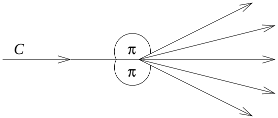

Holonomy of hyperbolic cone surfaces

Lemma (from Gauss–Bonnet)

If S is a hyperbolic cone surface, orders of cone points s_i, then $\sum s_i < -\chi(S)$.

A loop C around an interior cone point is contractible! So if ρ holonomy, $\rho(C) = 1 \in \mathrm{Isom}^+\mathbb{H}^2$. But ρ is also rotation by θ. So $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ can be the holonomy of a hyperbolic cone-manifold structure on S, but all interior cone angles must be $\in 2\pi\mathbb{N}$.

From obstruction-theoretic definition of Euler class:

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Statements

When S is a punctured torus...

Theorem (M.)

S punctured torus, $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ homomorphism. TFAE:

1. *ρ holonomy for a hyperbolic cone-manifold structure on S with geodesic boundary except at most one corner point, and no interior cone points;*
2. *ρ is not virtually abelian.*

Two punctured tori make a closed surface!

Theorem (M.)

S closed genus 2, $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$, $e(\rho) = \pm 1$. Suppose ρ takes a separating curve to a non-hyperbolic. Then ρ is the holonomy of a hyperbolic cone surface with one 4π cone point.

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Two punctured tori make a closed surface!

Theorem (M.)

S closed genus 2, $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$, $e(\rho) = \pm 1$. Suppose ρ takes a separating curve to a non-hyperbolic. Then ρ is the holonomy of a hyperbolic cone surface with one 4π cone point.

Results

Theorem (M.)

S closed, genus ≥ 2. Consider representations $\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{R}$ with $e(\rho) = \pm(\chi(S) + 1)$, sending some non-separating simple closed curve to an elliptic. Almost every such representation is the holonomy of a hyperbolic cone-manifold structure on S with a single cone point, angle 4π.

Almost? Thre's a *measure* on the character variety of representations. Arising from its *symplectic structure* (Goldman 1984).

It's *not* true that every component of $R(S)$ contains only cone-manifold holonomy representations.

Counterexample (Ser Peow Tan 1994): S closed genus 3, $e(\rho) = \pm 2$.

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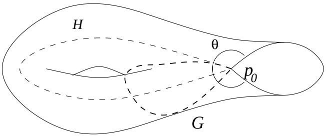
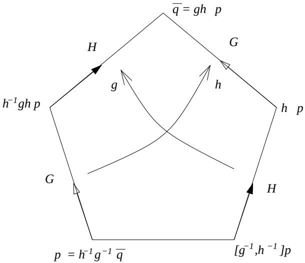
Punctured tori and pentagons

Let S be a hyperbolic punctured torus, with no interior cone points, one corner point q, corner angle $\theta \in (0, 3\pi)$.

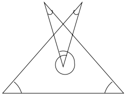




Can find two geodesic loops G, H, intersecting only at q, cutting S into a pentagon; interior angle sum θ.

Punctured tori and pentagons

Pentagon need not be embedded in $\mathbb{H}^2$...



Definition

Given $g, h \in PSL_2\mathbb{R}$, $p \in \mathbb{H}^2$, the pentagon $\mathcal{P}(g, h; p)$ is

$$p \to h^{-1}ghp \to ghp \to hp \to g^{-1}h^{-1}ghp \to p.$$

Lemma (Construction lemma)

ρ is the holonomy of a punctured torus with a corner if and only if $\exists$ a free basis G, H of $\pi_1(S, q)$ and $p \in \mathbb{H}^2$ such that $\mathcal{P}(g, h; p)$ is nondegenerate bounding an immersed disc.

To construct punctured tori: just find a good pentagon.

Punctured tori and pentagons

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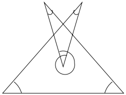

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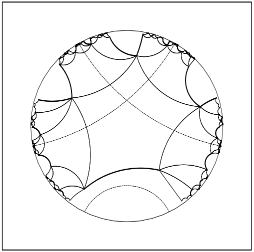
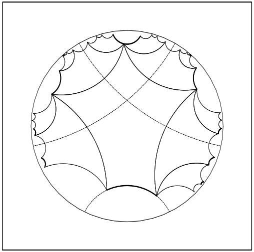
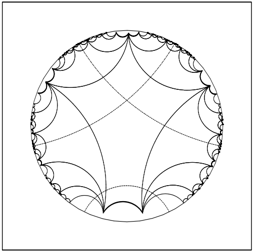
Non-rigidity

If $\mathcal{P}(g, h; p)$ works, can vary p and it still works!
Obtain many punctured tori with different hyperbolic cone-manifold structures, but same holonomy ρ.
Cone angle is determined by g, h, p as "twist of commutator".







Constructing pentagons

Use 2 results.

Theorem (Nielsen 1918)

Any automorphism of $\langle G, H\rangle$ takes $[G, H]$ to a conjugate of itself or its inverse.

Proposition (Goldman)

$Tr\,[g, h] < 2$ iff g, h are both hyperbolic and their axes cross.

- By Nielsen, $\mathrm{Tr}\,(\rho([G, H])) = t$ is invariant of choice of basis G, H. Go case-by-case on t.
- By Goldman, obtain geometric information from g, h.

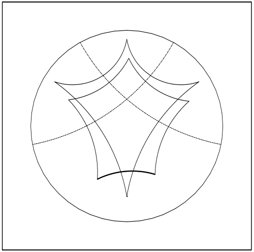
Various cases

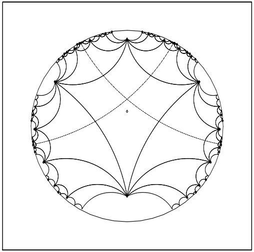
Case $t < -2$:

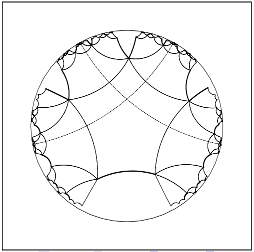
- g, h hyperbolic, axes cross, $[g, h]$ hyperbolic also. In fact ρ discrete, complete hyp structure with geodesic ∂.

Case $t \in (-2, 2)$:

- ρ holonomy of a (non-punctured!) torus with a cone point. Pentagon degenerate — perturb to nondegenerate.



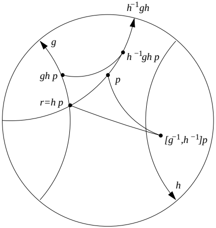


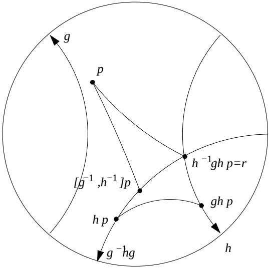


Various cases

Case $t > 2$:
Need to choose a good basis. Consider action of $MCG(S)$ on character variety.

- Use *Markoff triples* to get basis with good character.
- Use good character, Goldman & more for explicit construction.





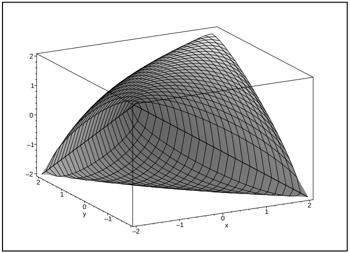
Representation and character varieties

Character of a ($PSL_2\mathbb{R}$)-representation ρ:

$$X : \pi_1(S) \longrightarrow \mathbb{R}, \quad X(G) = \mathrm{Tr}\,(\rho(G)).$$

Trace relations $\Rightarrow X$ determined by values on a finite subset.
Character variety $X(S)$ = {characters of all representations}.
When S is a punctured torus:

- $(x, y, z) = (\mathrm{Tr}\, g, \mathrm{Tr}\, h, \mathrm{Tr}\, gh)$ enough: $X(S) \subset \mathbb{R}^3$.



Punctured torus case

$MCG(S) \cong \operatorname{Out} \pi_1(S) \cong GL_2\mathbb{Z}$
Action of $GL_2\mathbb{Z}$ on $X(S) \Rightarrow$ *Markoff triples*.
$(x, y, z) \sim (x', y', z')$:

- corresponding representations ρ, ρ' are conjugate in $PSL_2\mathbb{R}$ after applying an automorphism of $\pi_1(S)$.

Proposition

For irreducible representations, $(x, y, z) \sim (x', y', z')$ *iff they can be related by the moves*

$$(x, y, z) \mapsto \left\{ \begin{array}{c} (x, y, xy - z), (-x, -y, z), \\ \textit{coordinate permutations} \end{array} \right\}.$$

Dynamics of this $GL_2\mathbb{Z}$-action are *ergodic* in certain regions (Goldman 2003).
This is the key to structures “almost everywhere”.