Itsy Bitsy Topological Field Theory

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Outline

1. Background
   - Origins
   - Sutured Floer homology
2. Decompositions & quadrangulations
   - Decomposing sutured surfaces
   - Occupied surfaces
   - Quadrangulation
   - Sutured quadrangulated surfaces
   - Sutured quadrangulated field theory
3. Itsy bitsy topology
   - Sutures store information
   - Bypass relation
   - Digital creation and annihilation
   - Structure theorem
   - Physical connections

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Motivation

- Work in SFH & contact geometry gives results which
  - can be described purely topologically/combinatorially; and
  - have striking physical analogies.
- We describe an object very like a $(2+1)$-dimensional TQFT (based on work of Honda–Kazez–Matić) which is:
  - Simple as a “toy model” ; and
  - Algebraic structure can be interpreted as processing information (*bits*),
  - or as analogous to creation/annihilation operators in QFT (*its*).
- John Archibald Wheeler: “it from bit”.

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Topological Quantum Field Theory

"Classically" (Witten, Segal, Atiyah 1980s) an $(n+1)$-dimensional TQFT assigns

$$\begin{aligned} n\text{-manifold } M &\rightsquigarrow \text{Vector space } Z(M) \\ (n+1)\text{-manifold } W \text{ "filling" } M &\rightsquigarrow c(W) \in Z(M) \end{aligned}$$

$$\left\{ \begin{array}{c} (n+1)\text{-dim cobordism} \\ \partial W = M_{in} \cup M_{out} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{Linear map} \\ \mathcal{D}_W : Z(M_{in}) \to Z(M_{out}) \end{array} \right\}$$

$$Z\left(\sqcup_i M_i\right) = \bigotimes_i Z(M_i)$$

$$Z\left(\bar{M}\right) = Z(M)^*$$

(Note $\text{Hom}(Z(M_{in}), Z(M_{out})) \cong Z(M_{in})^* \otimes Z(M_{out})$.)
A *functor* from a cobordism/topological category to an algebraic category.

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A TQFT with sutures

Honda–Kazez–Matić defined a TQFT-like object based on 2-dimensional surfaces (*always with nonempty boundary*) and *sutures*.

Definition

A set of sutures Γ on an oriented surface Σ is a set of disjoint oriented curves on Σ, cutting Σ into coherently oriented pieces

$$\Sigma \backslash \Gamma = R_+ \cup R_-, \quad \partial R_\pm \backslash \partial \Sigma = \Gamma.$$

Every component of $\partial\Sigma$ is required to intersect Γ.

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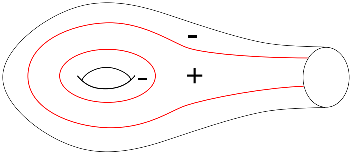
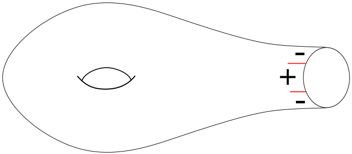


Boundaries of sutures

The restriction of Γ to $\partial\Sigma$ forms a set of *signed points* $F \subset \partial\Sigma$. Note $\partial\Sigma \backslash F = C_+ \cup C_-$ where $C_\pm$ forms part of $\partial R_\pm$.



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Sutures Γ "fill in" (Σ, F), $\partial\Gamma = F$.

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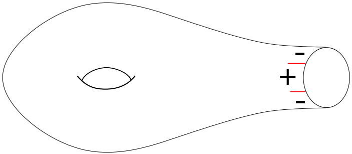

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The itsy bitsy TQFT

Our "TQFT" will assign (always over $\mathbb{Z}_2$):

$$\begin{array}{rcl} \text{Sutured background } (\Sigma, F) & \rightsquigarrow & \text{(Graded) vector space } Z(\Sigma, F) \\ \text{Sutures } \Gamma \text{ "filling" } (\Sigma, F) & \rightsquigarrow & \text{Suture element } c(\Gamma) \in Z(\Sigma, F) \end{array}$$

$$\left\{ \begin{array}{c} \text{Decorated morphism} \\ (\phi, \Gamma_c) : (\Sigma, F) \to (\Sigma', F') \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{(Graded) linear map} \\ \mathcal{D}_{\phi,\Gamma} : Z(\Sigma, F) \to Z(\Sigma', F') \end{array} \right\}$$

Decorated morphism is painful to define... roughly consists of:

- an *inclusion* $\phi : \Sigma \to \Sigma'$, together with
- sutures Γ_c on the complement $(\Sigma' \backslash \Sigma, F \cup F')$

(So sutures Γ on (Σ, F) give rise to sutures $\Gamma \cup \Gamma_c$ on (Σ', F').)

Sutured backgrounds and decorated morphisms form a *category*.

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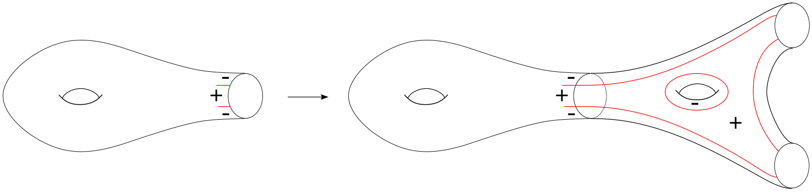

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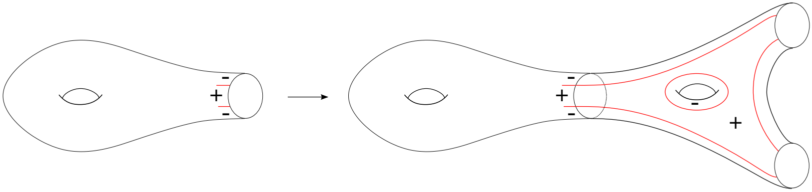

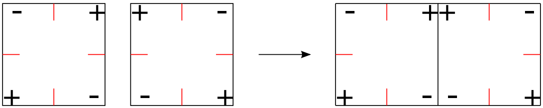

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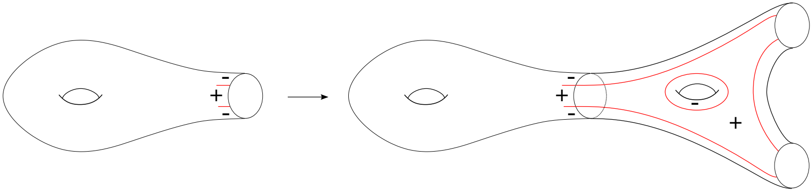

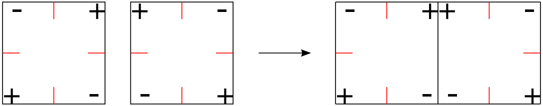

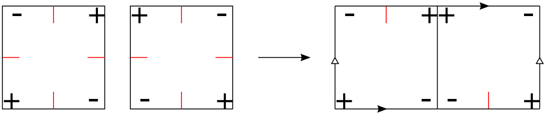

Sutured Floer homology

Definition (Gabai)

A sutured 3-manifold (M, Γ) is a 3-manifold M with ∂ such that $(\partial M, \Gamma)$ is a sutured surface.

SFH assigns:

Balanced sutured (M, Γ)	$\rightsquigarrow$	(Graded) abelian gp. $SFH(M, \Gamma)$
Contact structure ξ on (M, Γ)	$\rightsquigarrow$	$c(\xi) \in SFH(M, \Gamma)$

(Balanced: $\chi(R_+) = \chi(R_-)$; every component of ∂M has sutures.)
(Counting holomorphic curves in symmetric product of Heegaard surface with boundary conditions prescribed by Heegaard circles.)

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Dimensionally reduced *SFH*

Our TQFT originates from *SFH* of product manifolds

$$Z(\Sigma, F) = SFH(\Sigma \times S^1, F \times S^1)$$

Contact structures on these manifolds are described by

Theorem (Giroux, Honda)

$$\left\{ \begin{array}{c} \textit{Tight contact str's} \\ \textit{on } (\Sigma \times S^1, F \times S^1) \\ \textit{up to isotopy rel } \partial \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Sutures } \Gamma \textit{ on } (\Sigma, F) \\ \textit{without contractible loops} \\ \textit{up to isotopy rel } \partial \end{array} \right\}$$

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$$\xi_\Gamma \leftrightarrow \Gamma$$

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Our $c(\Gamma)$ is the contact element of the contact structure ξ_Γ corresponding to Γ.

TQFT property of SFH

Theorem (Honda–Kazez–Matić)

Given (M, Γ) and (M', Γ'), an inclusion $M \hookrightarrow \text{Int } M'$, and a contact structure ξ_c on $(M' \backslash M, \Gamma \cup \Gamma')$, there is a natural map

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which sends each

$$c(\xi) \mapsto c(\xi \cup \xi_c).$$

Our "decorated morphisms" derive from inclusions and complementary contact structures, but are more general...

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Our "decorated morphisms" derive from inclusions and complementary contact structures, but are more general...

Outline

1. Background
   - Origins
   - Sutured Floer homology
2. Decompositions & quadrangulations
   - Decomposing sutured surfaces
   - Occupied surfaces
   - Quadrangulation
   - Sutured quadrangulated surfaces
   - Sutured quadrangulated field theory
3. Itsy bitsy topology
   - Sutures store information
   - Bypass relation
   - Digital creation and annihilation
   - Structure theorem
   - Physical connections

Decomposing sutured surfaces

A natural way to decompose a sutured surface (Σ, Γ):

- Cut along a properly embedded arc a from C_- to C_+, transverse to Γ.

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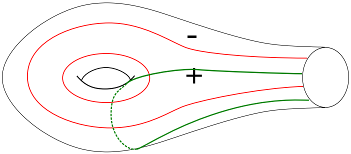

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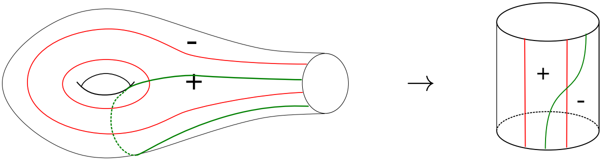

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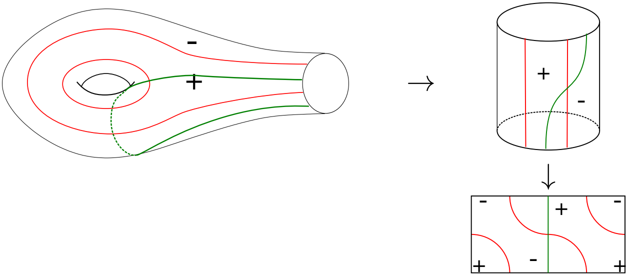

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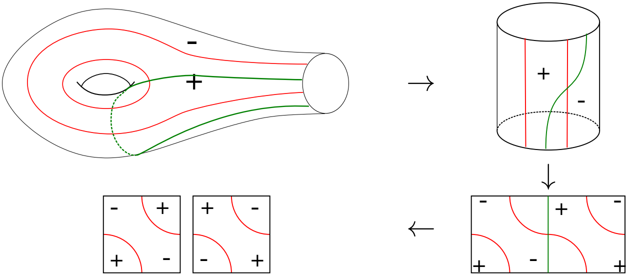

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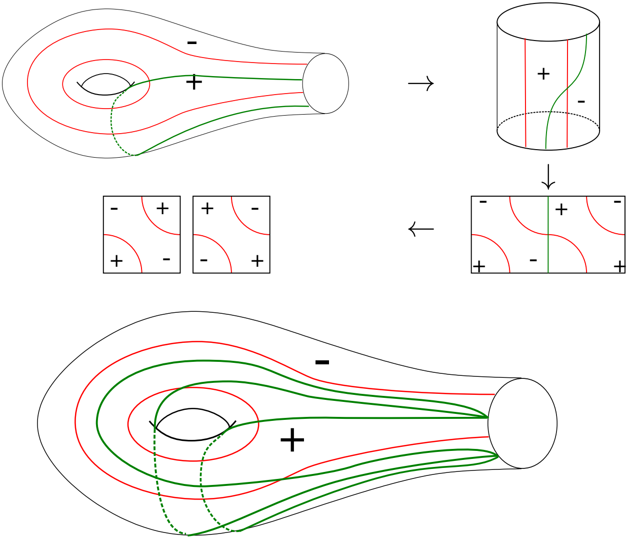

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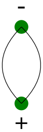

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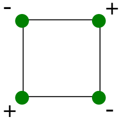

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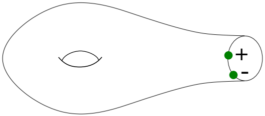

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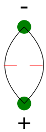

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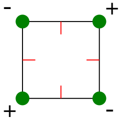

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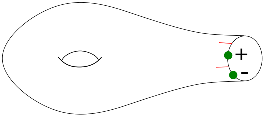

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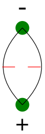

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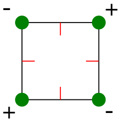

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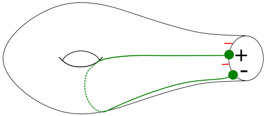

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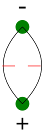

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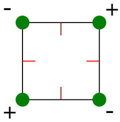

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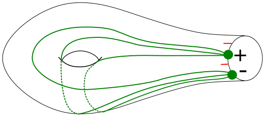

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- Any (Σ, V) without vacuum components decomposes into occupied squares — i.e. has a *quadrangulation*.
- Any quadrangulation of (Σ, V) has precisely $N - \chi(\Sigma)$ occupied squares, where $|V| = 2N$.
- A simple way to adjust quadrangulations: *diagonal slide*.

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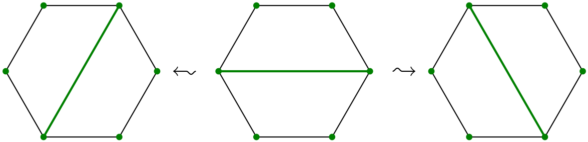

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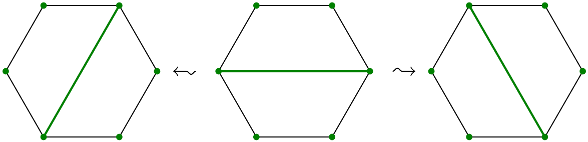

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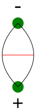

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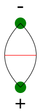

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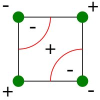

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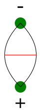

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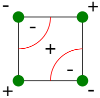

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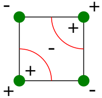

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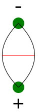

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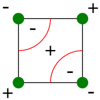

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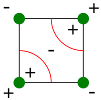

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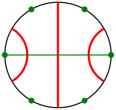

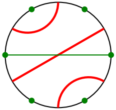

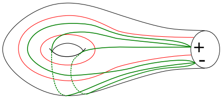

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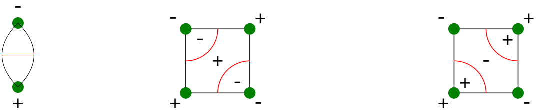

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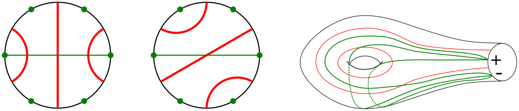

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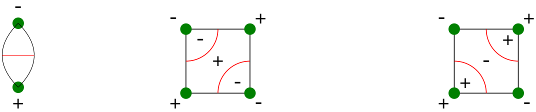

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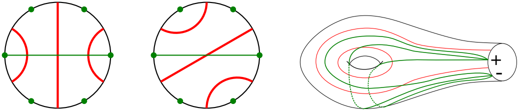

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Whenever Γ is *nonconfining* (each component of $\Sigma \backslash \Gamma$ intersects $\partial\Sigma$), we can find a basic quadrangulation.

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A *sutured quadrangulated field theory* (SQFT) is a pair $(\mathcal{D}, c)$ where

- $\mathcal{D}$ is a functor

$$\left\{ \begin{array}{c} \text{Occupied surfaces \&} \\ \text{decorated morphisms} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Graded } \mathbb{Z}_2 \\ \text{vector spaces} \end{array} \right\}$$

  i.e.

$$(\Sigma, V) \rightsquigarrow Z(\Sigma, V) \text{ and}$$
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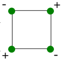
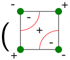
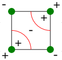
$Z($  $) = \mathbb{Z}_2 \mathbf{0} \oplus \mathbb{Z}_2 \mathbf{1}$, $c($  $) = \mathbf{0}$, $c($  $) = \mathbf{1}$

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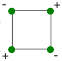
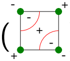
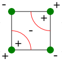
$Z($  $) = \mathbb{Z}_2 \mathbf{0} \oplus \mathbb{Z}_2 \mathbf{1}$, $c($  $) = \mathbf{0}$, $c($  $) = \mathbf{1}$

4. (*Grading*) $c(\Gamma)$ has grading $e(\Gamma)$, where e is the Euler class

Morphisms are gluing together squares

Occupied surface morphisms allow us to glue sides of squares in combinatorial fashion.

Definition

A decorated occupied surface morphism $(\phi, \Gamma_c) : (\Sigma, V) \to (\Sigma', V')$ satisfies

- $\phi : \Sigma \to \Sigma'$ *is an embedding on the interior of* Σ
- ϕ *is a homeomorphism on boundary edges*
- *Distinguished arcs in* Σ' *(i.e. boundary edges of* Σ' *or* ϕ*(boundary edges of* Σ*) which intersect other than at endpoints, coincide*
- $\phi(V_+) \cup V'_+$ *and* $\phi(V_-) \cup V'_-$ *disjoint*
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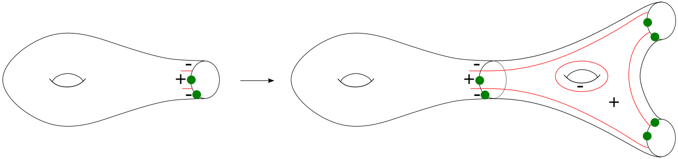

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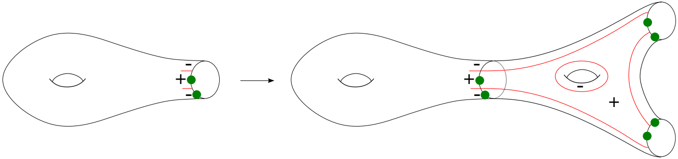

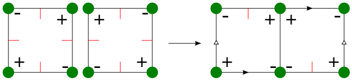

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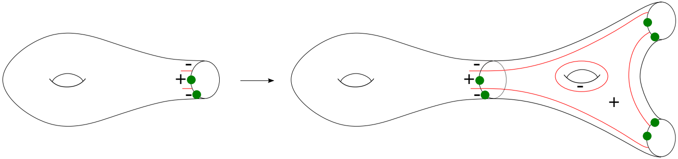

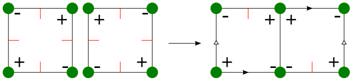

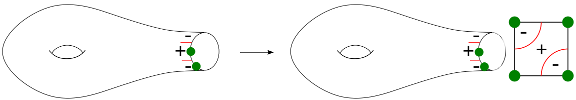

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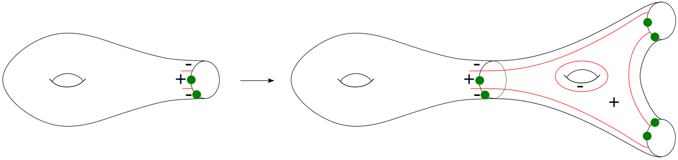

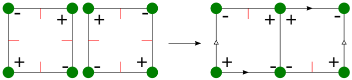

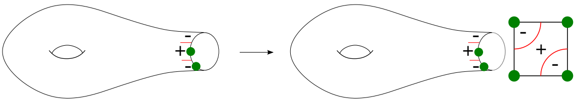

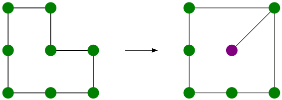

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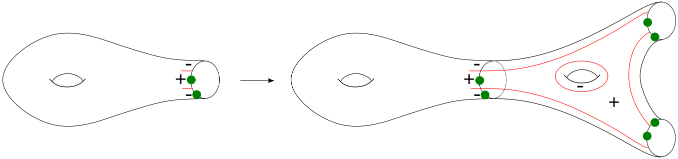

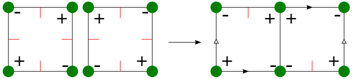

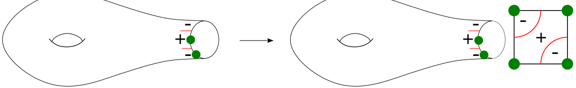

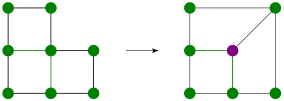

Dimensionally reduced SFH is an SQFT

Theorem

$SFH(\Sigma \times S^1, F \times S^1)$, together with its TQFT properties, form an SQFT.

(All known properties of SQFT:

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- (Honda–Kazez–Matić) TQFT property
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Outline

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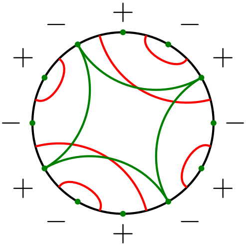

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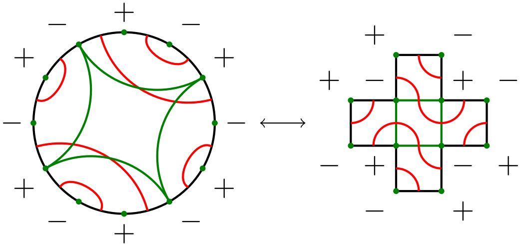

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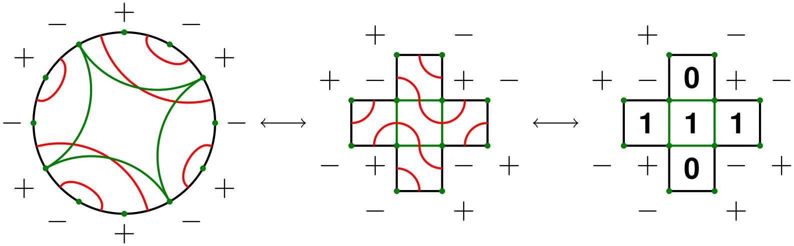

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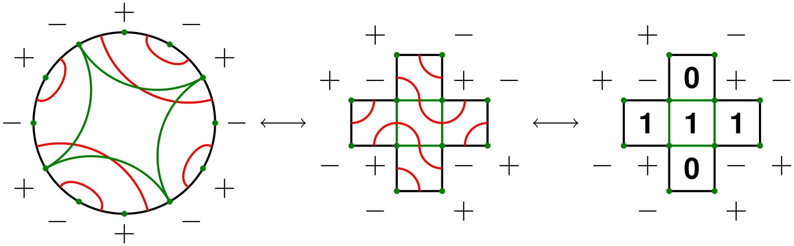

$$\begin{array}{ccccccccccc} & & \mathbf{0} & & & & & & \mathbf{V} & & \\ & & \otimes & & & & & & \otimes & & \\ c(\Gamma) = \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \in & \mathbf{V} & \otimes & \mathbf{V} & \otimes & \mathbf{V} \\ & & \otimes & & & & & & \otimes & & \\ & & \mathbf{0} & & & & & & \mathbf{V} & & \end{array}$$

Bypass relation

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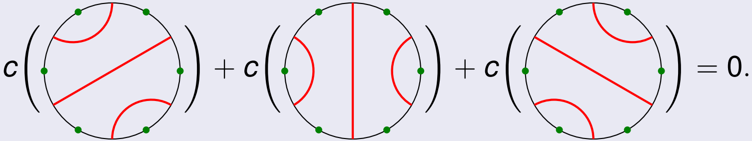

(A *bypass* is a contact-geometric object introduced by Honda, giving rise to such dividing set alterations.)

This allows us to write any suture element as a sum of basis elements.

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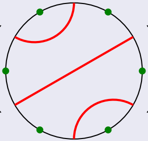
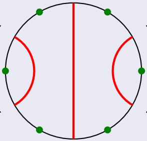
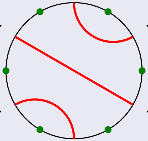
$c\Big($  $\Big) + c\Big($  $\Big) + c\Big($  $\Big) = 0.$

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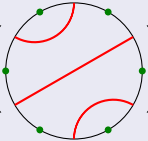
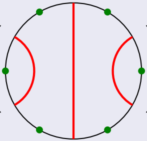
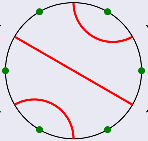
$c\Big($  $\Big) + c\Big($  $\Big) + c\Big($  $\Big) = 0.$

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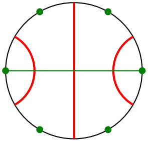

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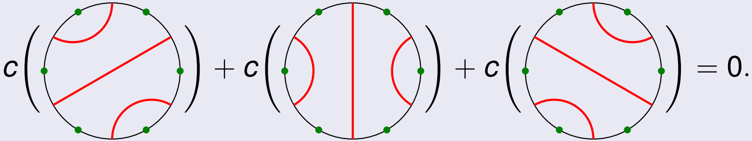

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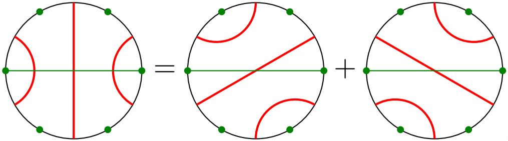

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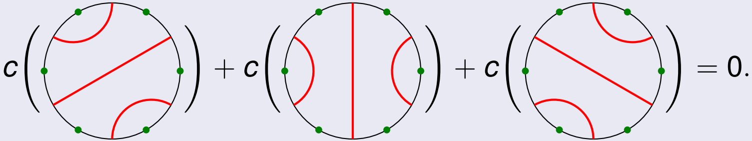

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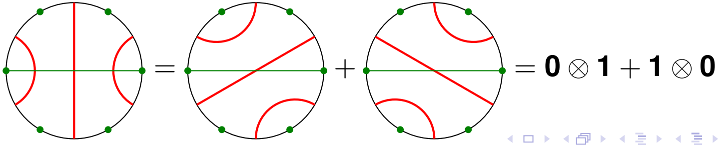

Two curious maps

Adjoining an extra square (with a **0** or a **1**) gives a map

$$\begin{aligned} a_{\mathbf{0}}^* : \quad \mathbf{V}^{\otimes n} &\to \mathbf{V} \otimes \mathbf{V}^{\otimes n} \\ x &\mapsto \mathbf{0} \otimes x \end{aligned}$$

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We call this a *digital creation* operator: "creation of **0**".
Other operations give *digital annihilation*, "deletion of **0**".

$$\begin{aligned} a_{\mathbf{0}} : \quad \mathbf{V}^{\otimes(n+1)} = \mathbf{V} \otimes \mathbf{V}^{\otimes n} &\to \mathbf{V}^{\otimes n} \\ \mathbf{0} \otimes e_1 \otimes \cdots \otimes e_n &\mapsto e_1 \otimes \cdots \otimes e_n \\ \mathbf{1} \otimes e_1 \otimes \cdots \otimes e_n &\mapsto \textstyle\sum_{e_i=\mathbf{0}} e_1 \otimes \cdots \otimes e_{i-1} \otimes \mathbf{1} \otimes \cdots \otimes e_n \end{aligned}$$

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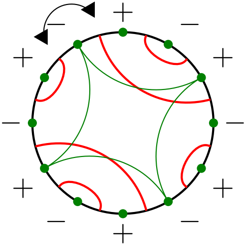

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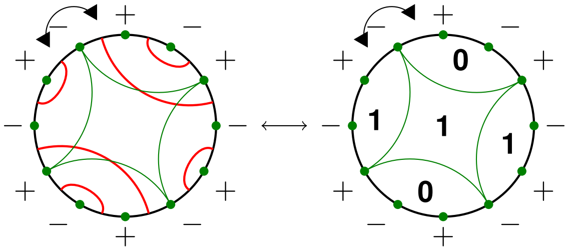

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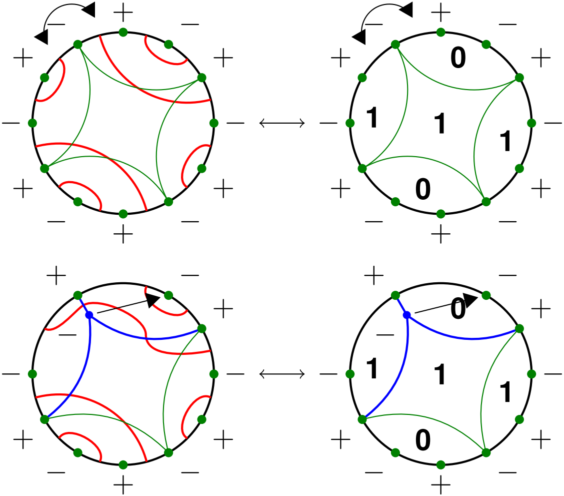

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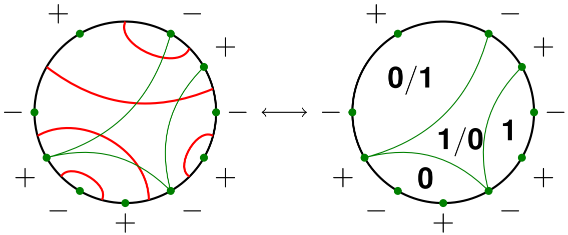

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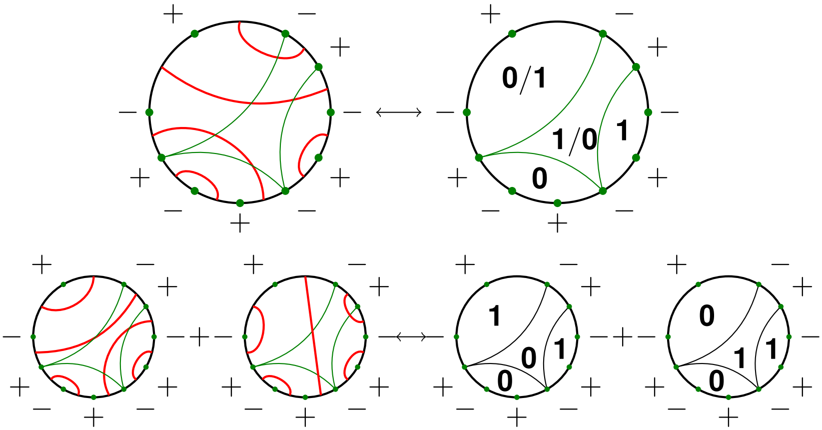

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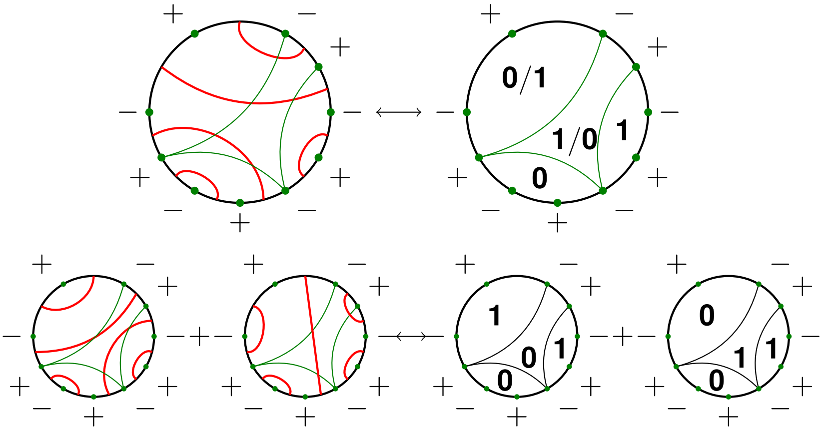

Result $\mathbf{0} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{0} \mapsto (\mathbf{1} \otimes \mathbf{0} + \mathbf{0} \otimes \mathbf{1}) \otimes \mathbf{1} \otimes \mathbf{0}$.

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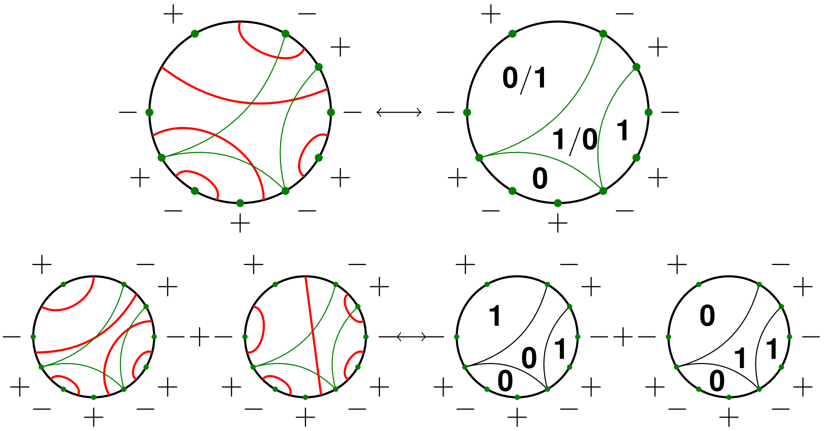

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I.e. $a_1\otimes 1^{\otimes 2}$.

A structure theorem for SQFT

Theorem (M.)

Any map of vector spaces in SQFT (over $\mathbb{Z}_2$) is a composition of digital creation and generalised digital annihilation operators.

Corollary

Any map $SFH(\Sigma \times S^1, F \times S^1) \to SFH(\Sigma' \times S^1, F' \times S^1)$ induced by a surface inclusion is a composition of digital creation and generalised digital annihilation operators.

A structure theorem for SQFT

Theorem (M.)

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Physical connections

Maps in SQFT can be interpreted as

- Creating/annihilating "particles" in occupied squares
- Manipulating binary information/"qubits" on each square.

Itsy and bitsy...

Also...

- It's possible to construct some analogous objects to "quantum logic gates" (over $\mathbb{Z}_2$...)
- Similar to *topological quantum computation* via *anyons*.
- A quadrangulation of (Σ, V) gives Σ the structure of a *ribbon graph*:
  - squares of quadrangulation $\rightsquigarrow$ vertices
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Further, a sutured quadrangulated surface gives a ribbon graph with:

- a *number of points* where sutures intersecting each edge
- a *diagram connecting points* on each vertex

Very reminiscent of *spin networks*, *diagrammatic representation theory*, *categorification*, etc...

Thanks for listening.