Contact topology and holomorphic invariants via elementary combinatorics

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- Much of it is quite involved, requiring:
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  - moduli spaces of pseudo-holomorphic curves
  - delicate differential geometry and topology
  - intricate algebraic structures keeping track of analytic data
- However, *in the simplest cases* some of this structure reduces to some *elementary combinatorics and algebra* which is interesting in its own right.

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- Discuss some of our algebraic and combinatorial results in their own right.
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- Briefly explain how this elementary algebra/combinatorics describes contact topology and arises from holomorphic invariants.

Symplectic manifolds

Definition

A symplectic manifold is a pair

$$(M, \omega)$$

where

- *M is a smooth manifold*
- *ω is a closed 2-form ($d\omega = 0$) which is non-degenerate.*

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Structure of Hamiltonian mechanics:

- Given a smooth function $H : M \longrightarrow \mathbb{R}$ (Hamiltonian) we obtain a 1-form dH and a dual vector field X_H via

$$\omega(X_H, \cdot) = dH$$

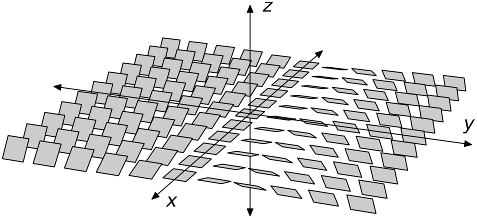
E.g. $M = \mathbb{R}^{2n}$, $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$.

Contact geometry

"The odd-dimensional sibling of symplectic geometry"

Definition

A contact structure ξ on a $(2n+1)$-dimensional manifold M is a totally non-integrable comdimension-1 hyperplane field on M.



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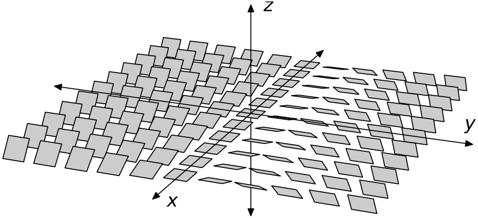

Equivalently, a contact structure is the kernel of a *contact form* α, i.e. satisfying $\alpha \wedge (d\alpha)^n \neq 0$ everywhere.
E.g. $\mathbb{R}^3$ with $\alpha = dz - y\,dx$.

Symplectic vs complex geometry

- Complex geometry also only exists in *even* number of dimensions.
- Gromov (1985): Consider *almost complex structures* on symplectic manifolds and *holomorphic curves*.

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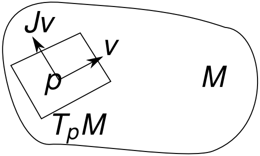
Definition

An almost complex structure on a smooth manifold is a map

$$J : TM \longrightarrow TM$$

preserving each fibre T_pM *and satisfying* $J^2 = -1$.



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- Every symplectic manifold has a *compatible* almost complex structure J, and all choices of compatible J are homotopic.
  (Compatible: J and ω behave in linear algebra like i and $dx \wedge dy$. $\omega(v, w) = \omega(Jv, Jw)$ and $\omega(v, Jv) > 0$)

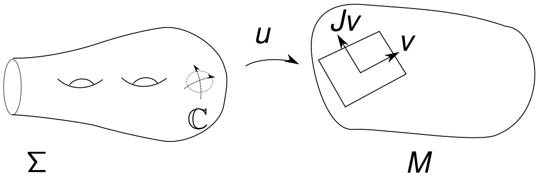
Holomorphic curves

Given symplectic (M, ω) and compatible almost complex J...

Definition

A holomorphic curve is a map $u : \Sigma \longrightarrow M$, where Σ is a Riemann surface, satisfying the Cauchy-Riemann equations

$$Du \circ i = J \circ Du.$$



An *almost* complex structure is sufficient for the equations: "pseudo-holomorphic", "J-holomorphic".

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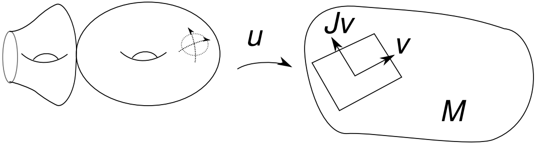

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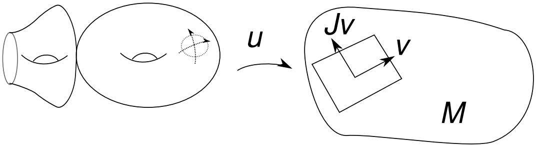

- $\mathcal{M}$ and $\overline{\mathcal{M}}$ encode a great deal of information about M.
- Some powerful invariants use only the *codimension-1 boundary* of $\overline{\mathcal{M}}$.

Homology theories

Floer Homology theories (e.g. contact homology, Heegaard Floer homology), roughly...

- Define a chain complex generated by boundary conditions for holomorphic curves
- A differential counting 0-dimensional families of holomorphic curves between boundary conditions.
- Boundary structure of moduli space gives $\partial^2 = 0$.

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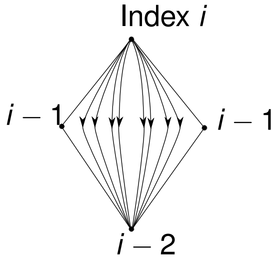
(Analogous Morse construction of singular homology: complex generated by critical points of Morse function, differential counts 0-dimensional families of gradient trajectories.)



The power of holomorphic invariants

- Floer homology theories give very powerful invariants of 3-manifolds, knots, etc...
- Related to Seiberg–Witten theory, Donaldson–Thomas theory, etc...
- E.g., *knot Floer homology* can compute the genus of a knot.

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- E.g., *knot Floer homology* can compute the genus of a knot.
- For a *less complicated* variant called *sutured Floer homology*, and a *simple class* of manifolds $M = \Sigma \times S^1$, we obtain all the combinatorial structure we are about to see, and more...

Outline

Quantum Pawn Dynamics

- Pawns on a finite 1-dimensional chessboard.
- A state of the QPD universe:

- Pawns move from left to right, one square at a time. (No capturing, no en passant, no double first moves.)

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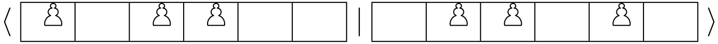
Quantum pawns: "Inner product" $\langle \cdot | \cdot \rangle$ describes the possibility of pawn moves from one state to another.
Valued in $\mathbb{Z}_2$.

Definition (Pawn "inner product")

$$\langle w_0 | w_1 \rangle = \begin{cases} 1 & \textit{if it is possible for pawns to move from } w_0 \textit{ to } w_1 \\ & \quad \textit{(this includes the case } w_0 = w_1\textit{);} \\ 0 & \textit{if not.} \end{cases}$$

Quantum Pawn Dynamics

E.g.



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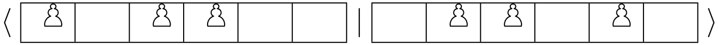
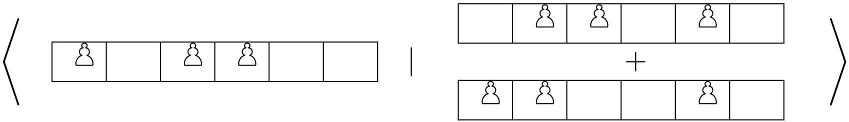


$= 1$

Also, entangled chessboards.



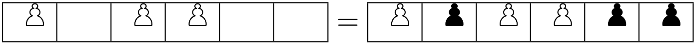
$= 1 + 0 = 1.$

Note asymmetry of $\langle \cdot | \cdot \rangle$.

A “booleanized” partial order. (Complete lattice.)

Dirac Pawn Sea

- Think of an “empty” chessboard as a thriving sea of anti-pawns.
  “Anti-pawn” = “absence of pawn”.



Creation and annihilation operators

The *initial pawn creation operator* $a^*_{p,0}$ adjoins a new *initial* (leftmost) square to the chessboard, containing a pawn.

$a^*_{p,0}$ [♙ | ♟ | ♙ | ♙ | ♟ | ♟] = [♙ | ♙ | ♟ | ♙ | ♙ | ♟ | ♟]

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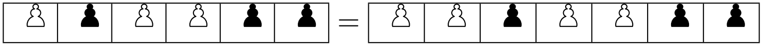
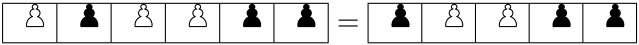
$a^*_{p,0}$ 

The *initial pawn annihilation operator* $a_{p,0}$ deletes the leftmost square from the chessboard, and a pawn on it.

$a_{p,0}$ 

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If no pawn (anti-pawn) in the leftmost square, try to delete...
"error 404 universe not found" mod 2 = 0.

$a_{p,0}$

♟	♙	♙	♟	♟

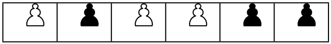
$= 0$

Similar initial anti-pawn annihilation $a_{q,0}$ and creation $a^\dagger_{q,0}$.

Creation of chessboards

- The *vacuum* state of the QPD universe is the null chessboard $\emptyset$.
  (Note $\emptyset \neq 0$.)
- Applying initial creation operators to the vacuum can create any chessboard.

$$a^*_{p,0} a^\dagger_{q,0} a^*_{p,0} a^*_{p,0} a^\dagger_{q,0} a^\dagger_{q,0} \quad \emptyset \quad =$$



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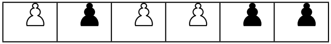

- The $*$ and $\dagger$ refer to *adjoints*.
  (*Galois connections* on partial orders.)

Adjoints

- Recall an adjoint f^* of an operator f usually means that

$$\langle fx|y\rangle = \langle x|f^* y\rangle, \quad \langle x|fy\rangle = \langle f^* x|y\rangle.$$

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- As our "inner product" is asymmetric, we have *two distinct adjoints* $f^*, f^\dagger$ of an operator f.

$$\langle fx|y\rangle = \langle x|f^* y\rangle, \quad \langle x|fy\rangle = \langle f^\dagger x|y\rangle.$$

So $f^{*\dagger} = f^{\dagger *} = f$.

Initial creation and annihilation are adjoint

Proposition

$$\langle a_{p,0} x | y \rangle = \langle x | a^*_{p,0} y \rangle$$

Proof.

$a^*_{p,0} y$ begins with a pawn.
If x begins with an anti-pawn, both sides are 0.
If x begins with a pawn, $\langle x | a^*_{p,0} y \rangle \neq 0$ compares two chessboards with initial pawns.
$a_{p,0}$ removes an initial pawn so $\langle a_{p,0} x | y \rangle$ gives the same result. □

Similarly, initial anti-pawn creation/annihilation †-adjoint.

Adjoining adjoints

What is $a_{p,0}^{**}$? What operator f satisfies

$$\langle a_{p,0}^{*} x | y \rangle = \langle x | f y \rangle ?$$

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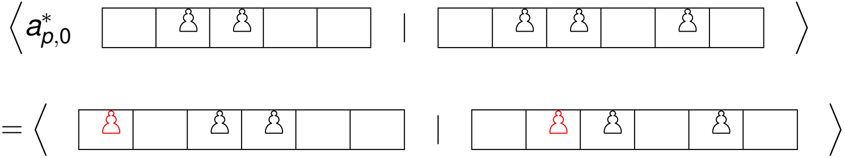

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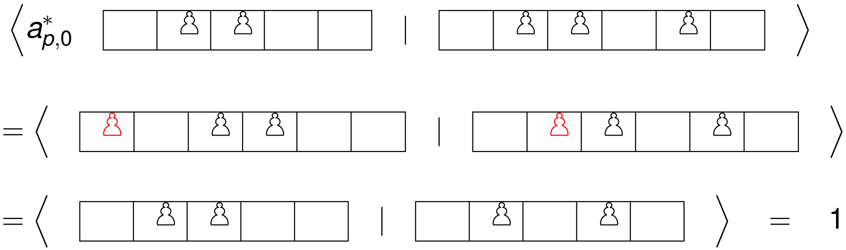

Iterated adjoints

Proposition

The iterated adjoints of $a_{p,0}$ are

$$a_{p,0} \to a^*_{p,0} \to a_{p,1} \to a^*_{p,1} \to a_{p,2} \to \cdots \to a_{p,\Omega} \to a^*_{p,\Omega}$$

where:
$a_{p,i}$ deletes the i'th pawn
$a^*_{p,i}$ doubles the i'th pawn
$a_{p,\Omega}, a^*_{p,\Omega}$ are final pawn creation and annihilation.

Similarly for anti-pawns in the opposite direction.

$$a^\dagger_{q,\Omega} \to a_{q,\Omega} \to \cdots \; a_{q,2} \to a^\dagger_{q,1} \to a_{q,1} \to a^\dagger_{q,0} \to a_{q,0}$$

(A *simplicial structure*.)

Adjoint periodicity

Hence

$$a_{p,0}^{*^{2n_p+2}} = a_{p,\Omega}$$

where n_p = number of pawns.

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Theorem (M.)

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One can also show that the *duality* operator defined by

$$\langle u|v\rangle = \langle v|Hu\rangle$$

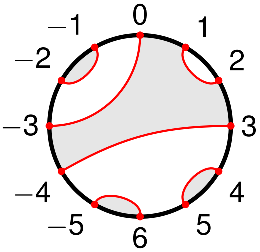
satisfies

Theorem (M.)

$$H^{2n+2} = 1.$$

Chord diagrams

Consider a disc D with some points F marked on ∂D.
A *chord diagram* is a collection of non-intersecting curves on D joining points of F.
E.g.



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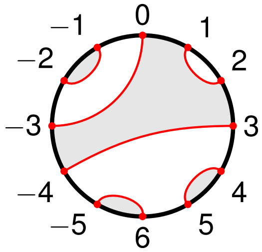
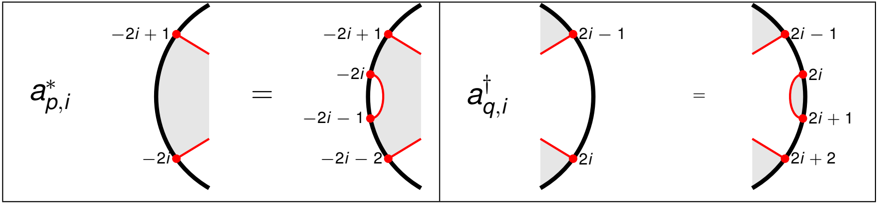


- Curves join points of opposite parity, so shade as shown.
- 0 is a basepoint.

Creation and annihilation of chords

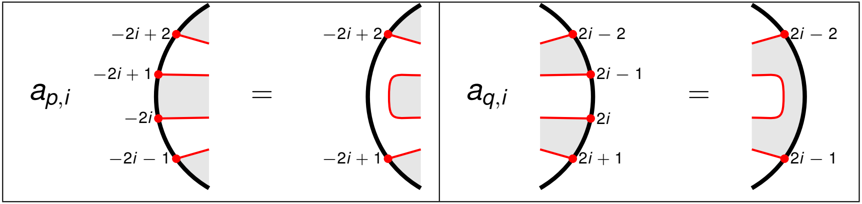
Define *creation operators* $a^*_{p,i}$, $a^\dagger_{q,i}$ to insert a new chord in a specific place in a chord diagram as shown.



$a^*_{p,i}$ creates a *white* region i spots down on the left.
$a^\dagger_{q,i}$ creates a *black* region i spots down on the right.

Creation and annihilation of chords

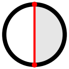
Define *annihilation operators* $a_{p,i}$, $a_{q,i}$ to close off chords in a chord diagram as shown.



$a_{p,i}$ closes off a black region i spots down on the left.
$a_{q,i}$ closes off a white region i spots down on the right.

Diagrams of chessboards

The simplest chord diagram is called the *vacuum* $\Gamma_\emptyset$.



Build up more complicated diagrams with creation operators.

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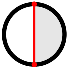
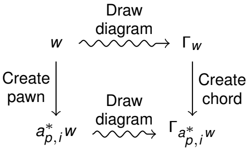


Build up more complicated diagrams with creation operators.

Proposition (M.)

For any chessboard w, there is a chord diagram Γ_w such that creation and annihilation operators agree (are equivariant):

$$\Gamma_{a^*_{p,i} w} = a^*_{p,i} \Gamma_w.$$



Ski slopes

Construction of the *slalom skiing* chord diagram of a chessboard.

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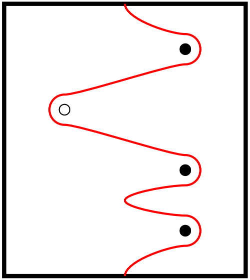
$\leftrightarrow$ 

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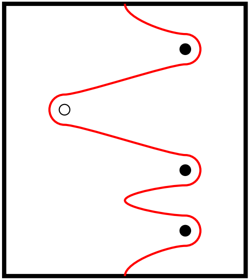
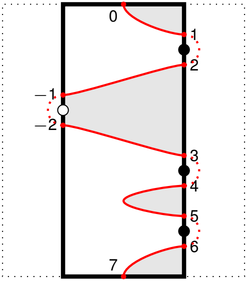
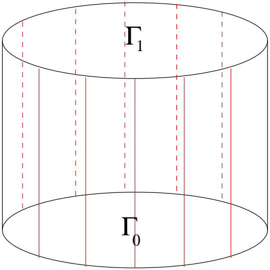
$\leftrightarrow$  $\leftrightarrow$ 

An “Inner product” on chord diagrams

There’s a bilinear form on chord diagrams defind by *entering into a cylinder*.



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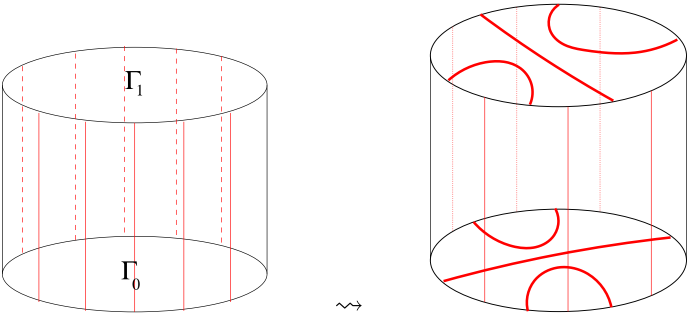

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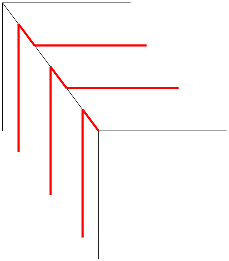
Note curves don't meet at corners! We treat corners as shown.



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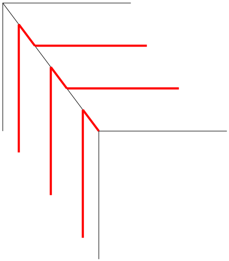

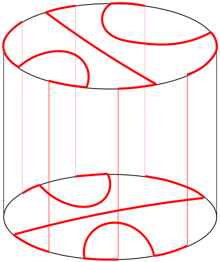

Definition

$$\langle \Gamma_0 | \Gamma_1 \rangle = \begin{cases} 1 & \textit{if the resulting curves on the cylinder form a single connected curve;} \\ 0 & \textit{if the result is disconnected.} \end{cases}$$

Theorem (M.)

For any two chessboards w_0, w_1,

$$\langle w_0 | w_1 \rangle = \langle \Gamma_{w_0} | \Gamma_{w_1} \rangle.$$

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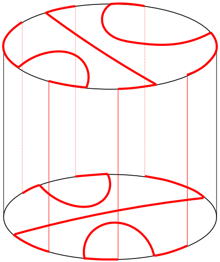
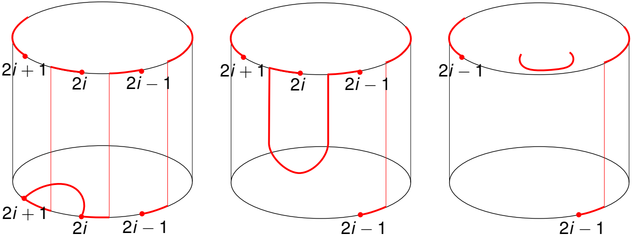
E.g.

$= 1 =$



Adjoints

Adjoint relations can be seen *topologically* as “finger moves”.

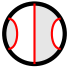


$$\langle a^{\dagger}_{q,i}\,\Gamma_0 \mid \Gamma_1\rangle \quad = \quad \langle \Gamma_0 \mid a_{q,i}\,\Gamma_1\rangle$$

Now perhaps believable that adjoint is periodic.

Bypass surgery

In a chord diagram on disc D, consider a sub-disc B as shown:



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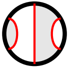
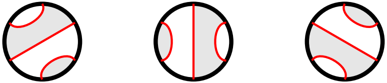


Two natural ways to adjust this chord diagram, consistent with the colours: *bypass surgeries*.



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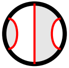

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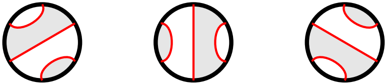

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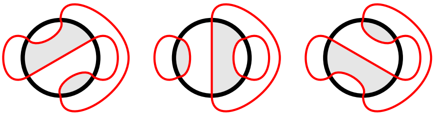
Proposition

With $\Gamma, \Gamma', \Gamma''$ *as above, for any* Γ_1*,*

$$\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle = 0.$$

Bypass surgery

Idea of proof:



$$1 \quad + \quad 0 \quad + \quad 1 \quad = \quad 0$$

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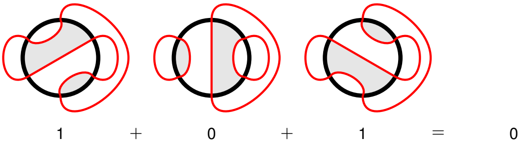
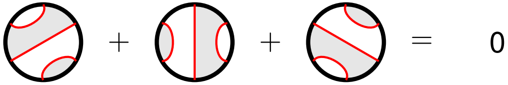


If $\langle \cdot | \cdot \rangle$ is to be nondegenerate, any three chord diagrams related by bypass surgery should sum to 0: *bypass relation*.



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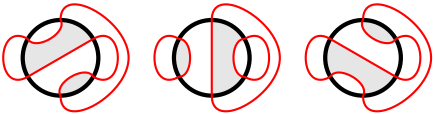

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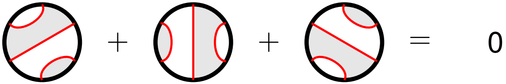

So we define a vector space

$$V_n = \frac{\mathbb{Z}_2 \langle \text{Chord diagrams with } n \text{ chords} \rangle}{\text{Bypass relation}}$$

A vector space of chord diagrams

Theorem (M.)

V_n has dimension 2^{n-1} and the diagrams from chessboards of $n-1$ squares form a basis.

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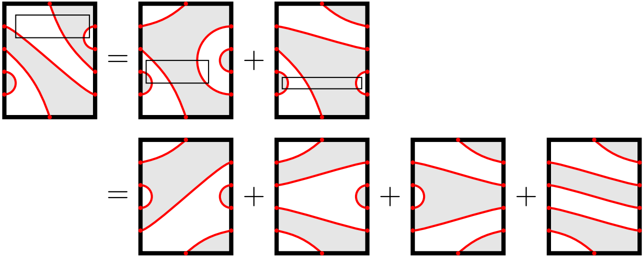
E.g.



$$= \Gamma_{ppqq} + \Gamma_{pqqp} + \Gamma_{qppq} + \Gamma_{qpqp}$$

Outline

Chord diagrams and contact structures

Giroux (1991): theory of *convex surfaces*.
A chord diagram Γ / *dividing set* on a disc D describes a contact structure ξ_Γ on a neighbourhood $D \times I$ of D.

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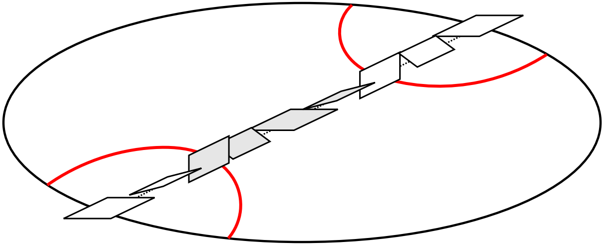
- Tangent to ∂D
- "Perpendicular" to D precisely along Γ



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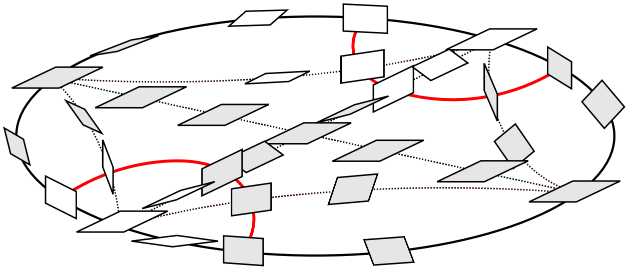

Colours in chord diagram = visible side of contact plane.

Overtwisted contact structures

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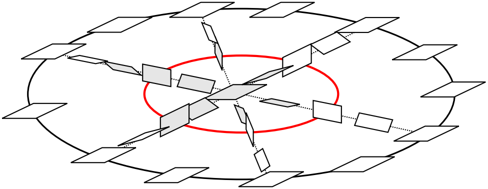

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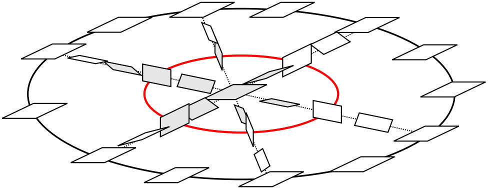

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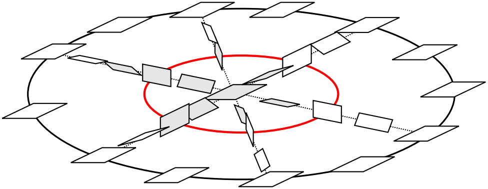
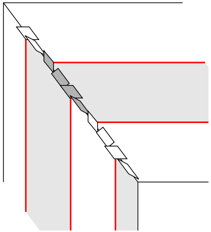


- Overtwisted contact geometry reduces to (well-understood) homotopy theory. Tight contact structures offer important topological information.
- Eliashberg (1992): contact structure near an S^2 is tight iff dividing set is *connected*. If so, contact structure extends uniquely (up to isotopy) to a tight contact structure on B^3.

Contact corners

When two convex surfaces meet along a boundary, contact planes are arranged as shown.



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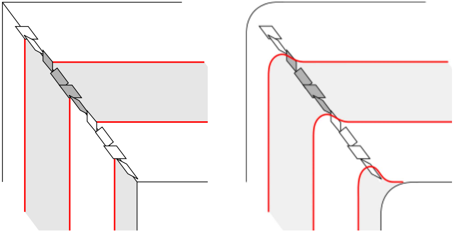


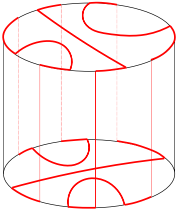
Proposition

Let Γ_0, Γ_1 be chord diagrams. The following are equivalent:

- $\langle \Gamma_0 | \Gamma_1 \rangle = 1$.
- *The solid cylinder with dividing set Γ_0 on the bottom and Γ_1 on the top has a tight contact structure.*

Bypasses

Honda (2000's): any 3-manifold can be built up from a surface and dividing set by adding *bypasses*.



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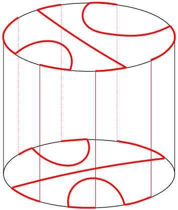

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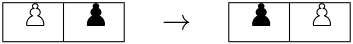
Corresponds to

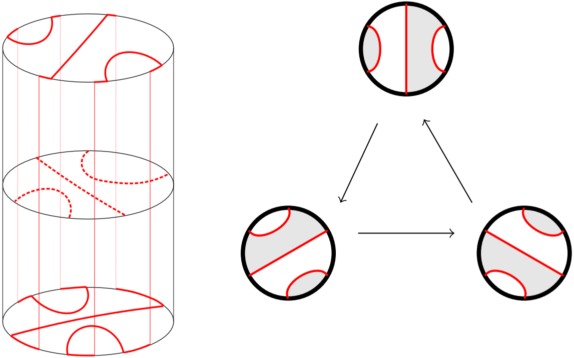
$$\langle \Gamma_{pq} | \Gamma_{qp} \rangle = 1$$

or



Bypasses

Stacking two bypasses on top of each other produces an overtwisted contact structure!



Can build a *triangulated category* out of dividing sets and contact structures (Honda, M.). V_n is the *Grothiendick group*.

Contact TQFT = Quantum pawn dynamics

These definitions give many of the properties of a (2+1)-dimensional *topological quantum field theory*.

- Contact structure near disc (2-dim) $\rightsquigarrow$ "states" in V_n
- Contact structure over cylinder (2+1-dim) $\rightsquigarrow$ element of $\mathbb{Z}_2$.
- "Possibility of a tight contact structure from one state to another" $\rightsquigarrow$ inner product $\langle \cdot | \cdot \rangle : V_n \otimes V_n \longrightarrow \mathbb{Z}_2$.

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Theorem (M.)

"Contact TQFT is isomorphic to quantum pawn dynamics."

Outline

1. Introduction
2. Combinatorial and algebraic structure
3. Contact topology
4. Holomorphic invariants
    - Sutured Floer homology
    - A “computation”

Sutured Floer homology

Actually all the above comes from *sutured Floer homology*, a holomorphic invariant of sutured manifolds.
Very roughly... (Ozsváth–Szabó 2004, Juhasz 2006)

- A *sutured manifold* is a 3-manifold M with boundary, and some curves Γ on ∂M dividing ∂M into alternating positive and negative regions.

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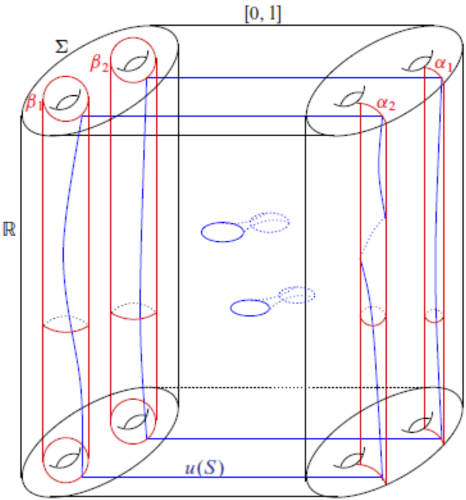
- Consider $\Sigma \times I \times \mathbb{R}$ as a symplectic manifold with an almost complex structure and consider holomorphic curves

$$u : S \longrightarrow \Sigma \times I \times \mathbb{R}$$

  where S is a Riemann surface.
- Boundary conditions based on Heegaard curves α_i and β_i.

Sutured Floer homology

Cylindrical picture of Lipshitz (2006):



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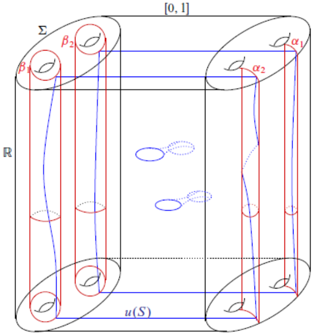


$$\mathrm{ind}\,(D\bar{\partial}) = k - \chi(S) + \sum_{i=1}^{k} \mu(a_i) - \sum_{i=1}^{k} \mu(b_i).$$

Sutured Floer homology

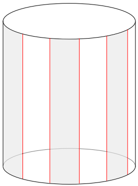
- Chain complex generated by boundary conditions, which are *intersections* of boundary curves.

$$z_1 \in \alpha_1 \cap \beta_{\sigma(1)}, \; z_2 \in \alpha_2 \cap \beta_{\sigma(2)}, \; \ldots, \; z_k \in \alpha_k \cap \beta_{\sigma(k)}.$$

- Differential counting index-1 holomorphic curves between boundary conditions.
- Resulting homology is $SFH(M, \Gamma)$.
- Etnyre–Honda (2009): Any *contact structure* ξ on (M, Γ) defines a natural *element* $c(\xi) \in SFH(M, \Gamma)$.

Solid tori

We consider the *sutured solid torus* $D^2 \times S^1$ with $2n$ longitudinal curves $F_n \times S^1$. ($|F_n| = 2n$)



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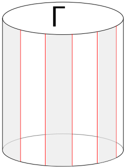
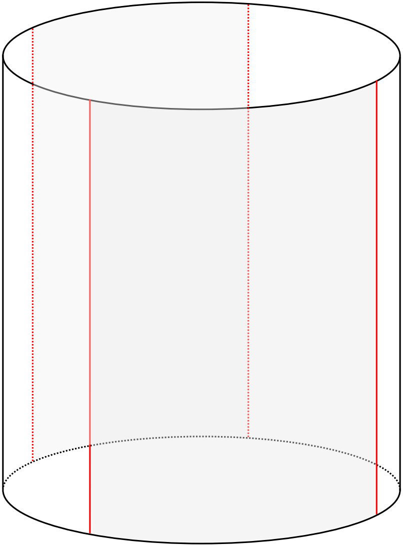


Theorem (M.)

$$SFH(D^2 \times S^1, F_n \times S^1) \cong V_n = \frac{\mathbb{Z}_2 \langle \textit{Chord diagrams w/ n chords} \rangle}{\textit{Bypass relation}}$$

Any chord diagram Γ *in* V_n *corresponds to a a contact structure* ξ_Γ *on* $D^2 \times S^1$ *and maps to* $c(\xi_\Gamma)$.

A "computation" of Sutured Floer homology



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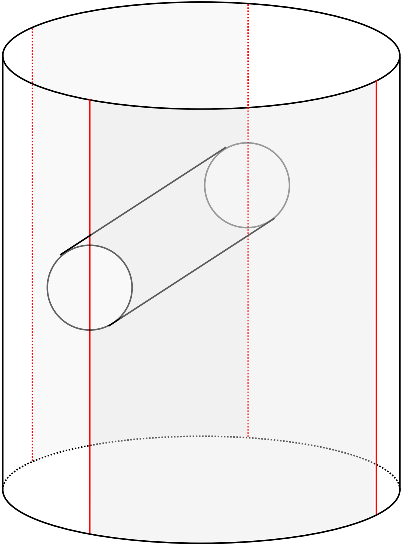

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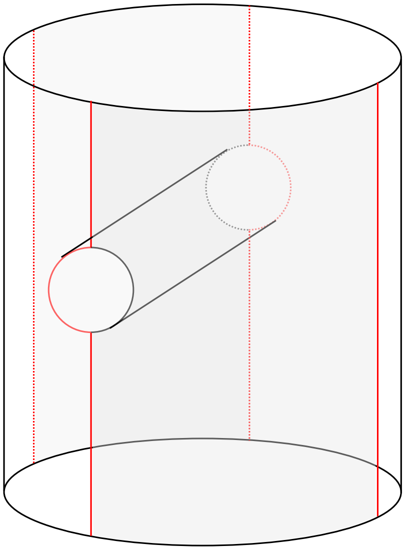

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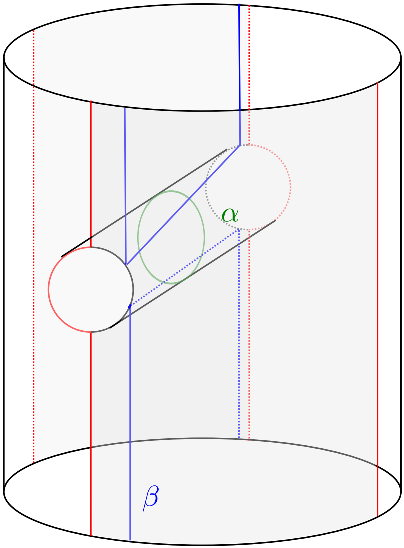

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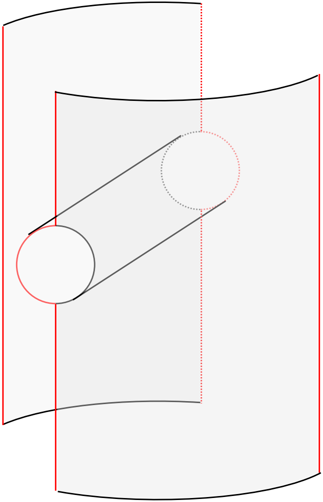

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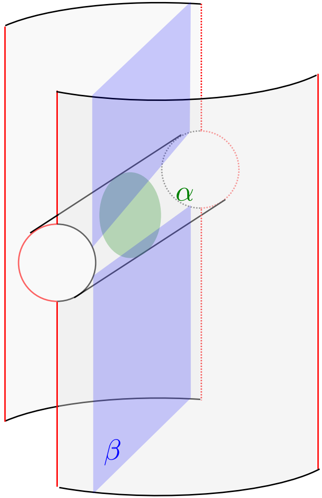

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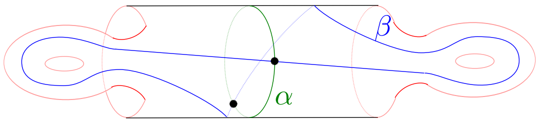

Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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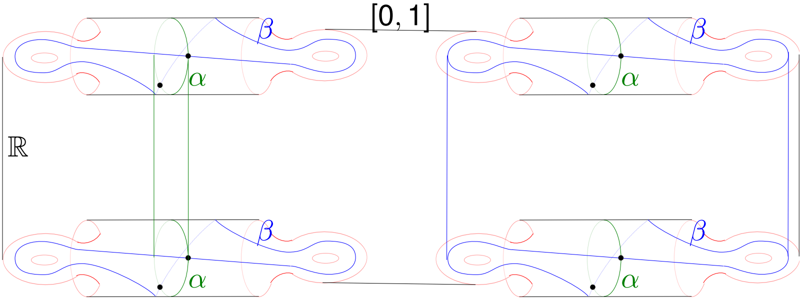

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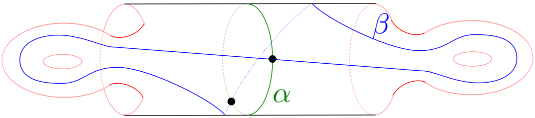

Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
Nowhere for holomorphic curves to go! $\partial = 0$.

$$SFH = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = V_2$$

Thanks for listening!

References:

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