Sutures, quantum groups and topological quantum field theory

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Outline

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1. Introduction
    - Overview
    - Three key ideas
    - Categorification
    - Topological quantum field theory (TQFT)
    - Quantum groups
    - Motivation
2. Sutured and occupied surfaces
3. Sutured TQFT
4. Quantum actions

Overview

Important ideas in recent developments in topology:

- Categorification
- Quantum group representations
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Some recent work in the areas of

- Contact geometry
- Floer homology
- Homological algebra

leads to a simple model demonstrating all these important ideas.

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and has important features:

- TQFT: Is (almost!) a 2-dimensional *TQFT*.
- Categorification: Is a categorified version of (a generalisation of) the Alexander polynomial
- Quantum groups: Carries representations of $U_q(\mathfrak{sl}(1|1))$. (These representations include, as special cases, a (quantized) Temperley–Lieb algebra.)

Key Idea 1: Categorification

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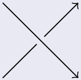
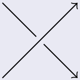
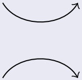
Jones polynomial (Jones, 1984)

$\frac{1}{t} J\Bigg($  $\Bigg) - t\, J\Bigg($  $\Bigg) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) J\Bigg($  $\Bigg)$

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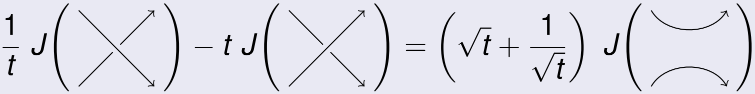
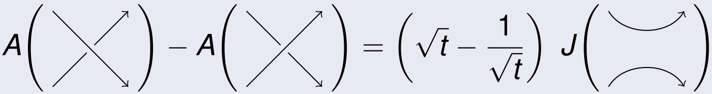


$$\frac{1}{t} J\left(\quad\right) - t\, J\left(\quad\right) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) J\left(\quad\right)$$

Alexander polynomial (Alexander, 1923)



$$A\left(\quad\right) - A\left(\quad\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) J\left(\quad\right)$$

Both are Laurent polynomials in a single variable with integer coefficients.

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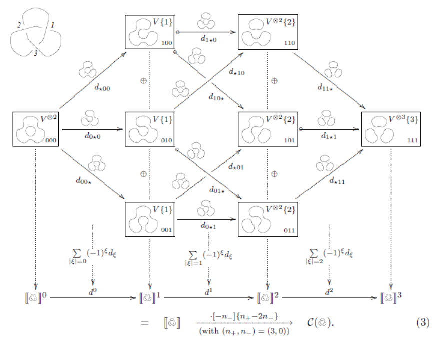
Khovanov homology (Khovanov, late 1990s)

Knot → resolve crossings → arrange resolutions into cube → vertices = groups, edges = homomorphisms based on $U_q(sl(2))$ (1+1)-dimensional TQFT → find differential → Take homology



(Source: Bar–Natan, "On Khovanov's categorification of the Jones polynomial")

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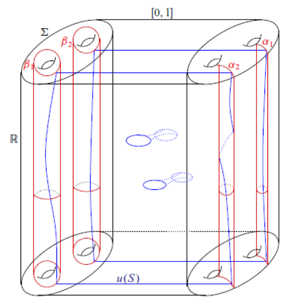
Heegaard Floer homology (Ozsváth–Szabó, Rasmussen, 2003)

Take Heegaard decomposition $(\Sigma, \alpha, \beta) \to$ Form $\Sigma \times I \times \mathbb{R} \to$ Take almost complex structure $\to$ Consider holomorphic curves in $\Sigma \times I \times \mathbb{R} \to$ Prescribe boundary conditions at $\pm\infty$ by $(\alpha \cap \beta) \to$ Form chain complex, groups = boundary conditions, differential = holomorphic curve counts $\to$ Take homology



Source: Lipshitz, “A cylindrical reformulation of Heegaard Floer homology”

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$$\left\{ \begin{array}{c} \text{Khovanov} \\ \text{Knot Floer} \end{array} \right\} \text{homology } \textit{categorifies} \left\{ \begin{array}{c} \text{Jones} \\ \text{Alexander} \end{array} \right\} \text{polynomials.}$$

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A *functor* from a cobordism/topological category to an algebraic category.

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$$\mathfrak{sl}(2,\mathbb{R}) = \{A,\ \mathrm{Tr}\,A = 0\} = \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

$$= \langle E, F, K \mid [E,F] = K, [E,K] = [F,K] = 0 \rangle$$

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Universal enveloping algebra: E.g. $U(\mathfrak{g}) = U(\mathfrak{sl}_2\mathbb{R})$.

- Has multiplication, "Lie brackets become commutators"
  $[X,Y] \rightsquigarrow XY - YX$

$$U(\mathfrak{sl}(2,\mathbb{R})) = \mathbb{R}\langle E, F, K \mid EF - FE = K,\ EK = KE,\ FK = KF \rangle$$

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The *quantum group* $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$ over a "quantum" variable q.

$$U_q(\mathfrak{sl}(2)) = \mathbb{Q}(q) \left\langle E, F, K^{\pm 1} \middle| \begin{array}{c} KE = q^2 EK, \; KF = q^{-2} FK, \\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \end{array} \right\rangle$$

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"Quantum algebra" tends to do things like

- Replace integers n with expressions like

$$\frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$$

- Taking $q \to 1$ gives a "semiclassical limit".

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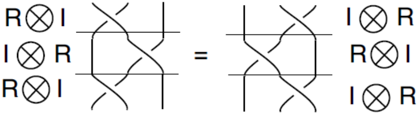

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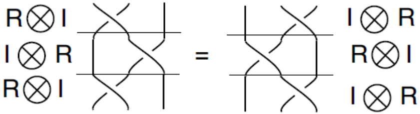

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- A braid on m strands gives a map of quantum group representations

$$\underbrace{V \otimes \cdots \otimes V}_{m} \longrightarrow \underbrace{V \otimes \cdots \otimes V}_{m}.$$

Closing the braid we obtain a *knot*; taking a trace we obtain a *quantum knot invariant*.

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Quantum group	Rep'n	Invariant	
$U_q(\mathfrak{sl}(2))$	V_2	Jones	(Witten 1989, Reshetikhin-Turaev 1990)
$U_q(\mathfrak{sl}(2))$	V_n	Coloured Jones	(Turaev 1994, Melvin-Morton 1995)
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- The definition of Khovanov homology “contains” $U_q(\mathfrak{sl}(2))$.
- The definition of Heegaard Floer homology *does not* obviously contain $U_q(\mathfrak{sl}(1|1))$.

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- Recent work of Yin Tian gives a "categorification of (some variants of) $U_q(\mathfrak{sl}(1|1))$" through *contact topology*.
- We can apply this to our own work... which is just about curves on surfaces...

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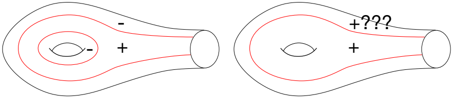
Definition

A set of sutures Γ on Σ is a set of disjoint oriented curves on Σ, cutting Σ into coherently oriented pieces

$$\Sigma \backslash \Gamma = R_+ \cup R_-, \quad \partial R_\pm \backslash \partial \Sigma = \Gamma.$$

Every component of ∂Σ is required to intersect Γ.

This notion goes back to Gabai, 1983.



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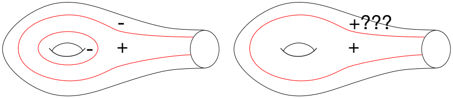

Definition

The Euler class $e(\Gamma) = \chi(R_+) - \chi(R_-)$.

Decomposing sutures

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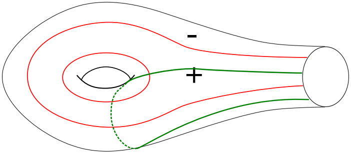
- Cut along a properly embedded arc *a* transverse to sutures (*decomposing* arc).



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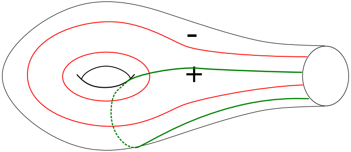

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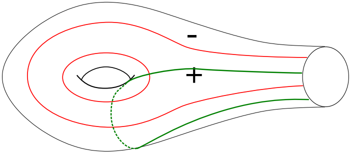

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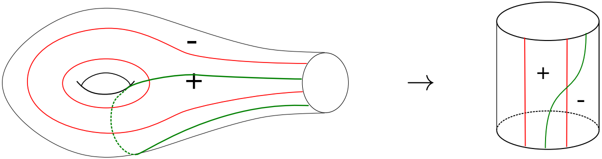

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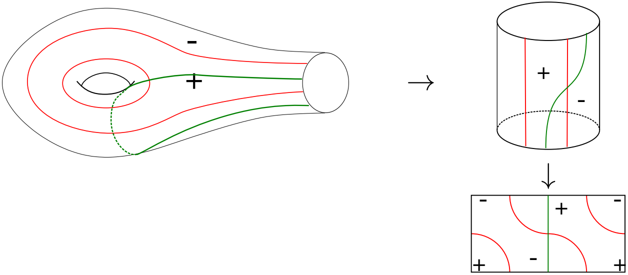

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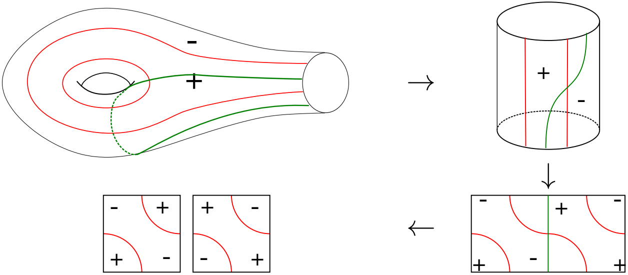

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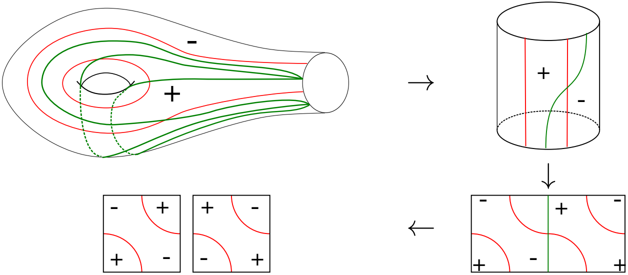

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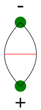

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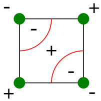

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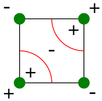

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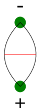

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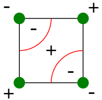

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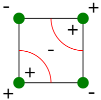

Basic square Γ_+

Some facts about these decompositions...
Let Γ intersect $\partial\Sigma$ in $2N$ points.

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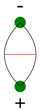

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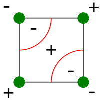

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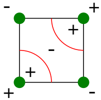

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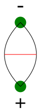

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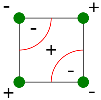

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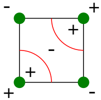

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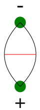

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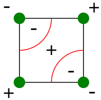

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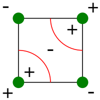

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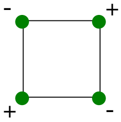
Lemma

If Γ is non-isolating, Σ decomposes into $N - \chi$ basic squares.

Occupied surfaces

Take endpoints of decomposing (green) arcs — vertices — to alternate with suture endpoints around $\partial\Sigma$. Vertices are *signed*.



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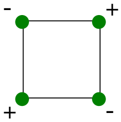

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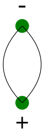

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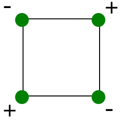

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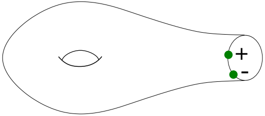

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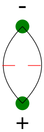

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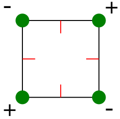

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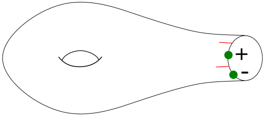

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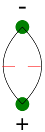

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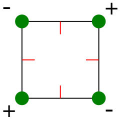

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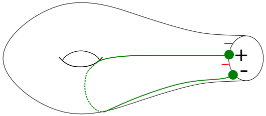

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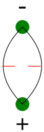

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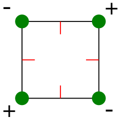

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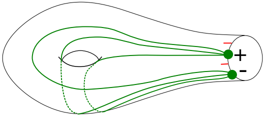

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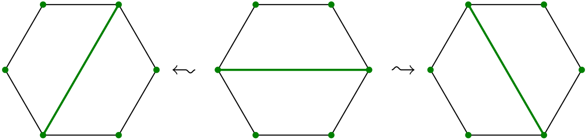
Any two quadrangulations of (Σ, V) are related by diagonal slides.



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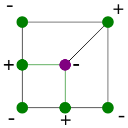

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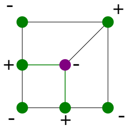

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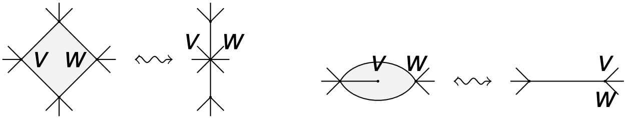

Outline

The idea of sutured TQFT

Witten, Segal, Atiyah 1980s:

An $(n+1)$-dimensional TQFT assigns

$$\begin{array}{rcl} n\text{-manifold } M & \rightsquigarrow & \text{Vector space } Z(M) \\ (n+1)\text{-manifold } W \text{ ``filling'' } M & \rightsquigarrow & c(W) \in Z(M) \end{array}$$

satisfying properties such as...

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$$\begin{array}{rcl} Z\left(\sqcup_i M_i\right) & = & \bigotimes_i Z(M_i) \\ Z\left(\bar{M}\right) & = & Z(M)^* \end{array}$$

A *functor* from a cobordism/topological category to an algebraic category.

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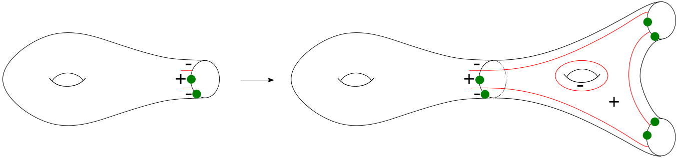
A *functor* from “occupied surface category” to graded $\mathbb{Z}_2$ vector spaces.

Some occupied surface morphisms

Occupied surface morphisms are “almost embeddings, with sutures on the complement of the image”.



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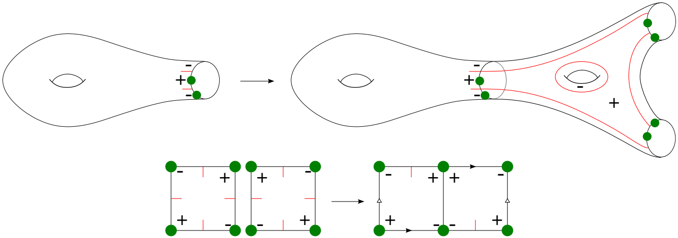

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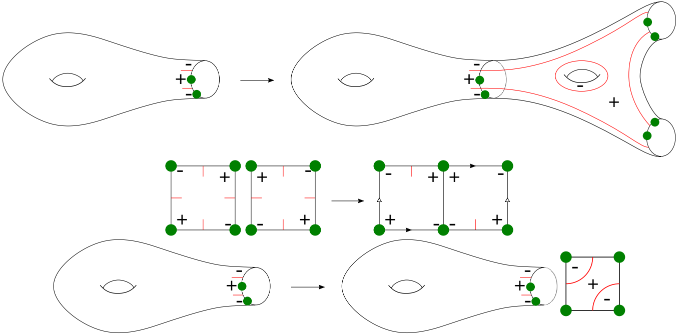

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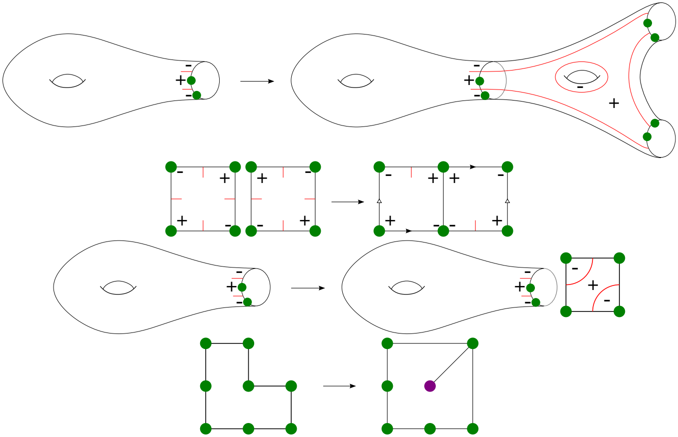

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A decorated occupied surface morphism $(\phi, \Gamma_c) : (\Sigma, V) \to (\Sigma', V')$ *satisfies*

- $\phi : \Sigma \to \Sigma'$ *is an embedding on the interior of* Σ
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Note:

- Morphisms turn sutures on (Σ, V) into sutures on (Σ', V'): $\Gamma \rightsquigarrow \Gamma \cup \Gamma_c$.
- Naturality of sutured TQFT requires linear maps $Z(\Sigma, V) \longrightarrow Z(\Sigma', V')$ to send $c(\Gamma) \mapsto c(\Gamma \cup \Gamma_c)$.

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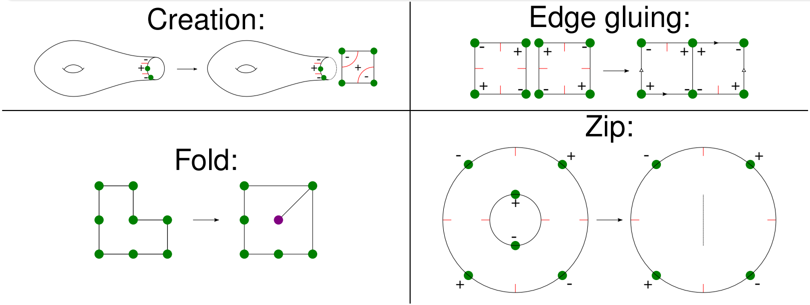

Creation:

Edge gluing:

Fold:

Zip:

Occupied surface morphisms are very combinatorial: “gluing up squares”.

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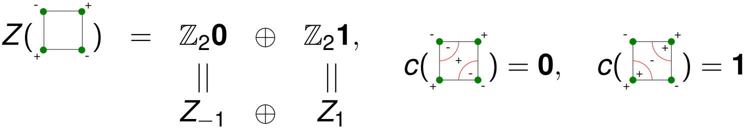

$$\begin{array}{ccccc} Z(\quad) & = & \mathbb{Z}_2\mathbf{0} & \oplus & \mathbb{Z}_2\mathbf{1}, \\ & & || & & || \\ & & Z_{-1} & \oplus & Z_1 \end{array} \qquad c(\quad) = \mathbf{0}, \qquad c(\quad) = \mathbf{1}$$

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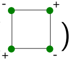

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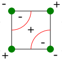

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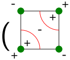


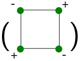
$$c(\quad) = \mathbf{1}$$

- “Basic means basic” on all occupied surfaces (not just squares): for a quadrangulation $(\Sigma, V) = \cup_i \square_i$ with basic sutures $\Gamma = \cup_i \Gamma_i$, $c(\Gamma) = \bigotimes_i c(\Gamma_i)$.

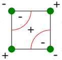
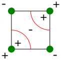
Normalization

We normalize sutured TQFT by requiring:

- Basic sutures on a square form a basis ("qubits").

$Z($  $)$

$$= \begin{array}{ccc}\mathbb{Z}_2\mathbf{0} & \oplus & \mathbb{Z}_2\mathbf{1}, \\ \| & & \| \\ Z_{-1} & \oplus & Z_1\end{array}$$

$c($  $) = \mathbf{0},$ $c($  $) = \mathbf{1}$

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- Vector spaces are *Graded*: $c(\Gamma)$ grading $e(\Gamma)$.

$$Z(\Sigma, V) = \bigotimes_i (\mathbb{Z}_2\mathbf{0} \oplus \mathbb{Z}_2\mathbf{1})^{\otimes n} = \mathbb{Z}_2^{\binom{n}{0}} \oplus \mathbb{Z}_2^{\binom{n}{1}} \oplus \cdots \oplus \mathbb{Z}_2^{\binom{n}{n}}$$

$$= Z_{-n} \oplus Z_{-n+2} \oplus \cdots \oplus Z_n$$

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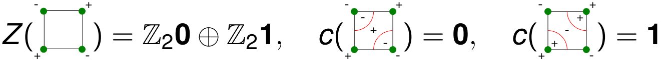

$$Z(\square) = \mathbb{Z}_2\mathbf{0} \oplus \mathbb{Z}_2\mathbf{1}, \quad c(\square) = \mathbf{0}, \quad c(\square) = \mathbf{1}$$

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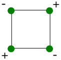
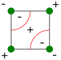
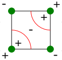
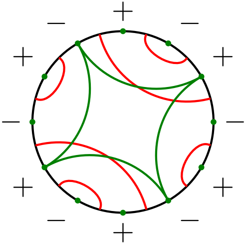
$Z($  $) = \mathbb{Z}_2 \mathbf{0} \oplus \mathbb{Z}_2 \mathbf{1}$, $c($  $) = \mathbf{0}$, $c($  $) = \mathbf{1}$

4. (*Euler grading*) $Z(\Sigma, V) = \bigoplus_e Z_e$ and $c(\Gamma) \in Z_{e(\Gamma)}$

Suture elements: An Example

Take $(\Sigma, V) = (D^2, 12 \text{ pts})$ and Γ basic as shown.



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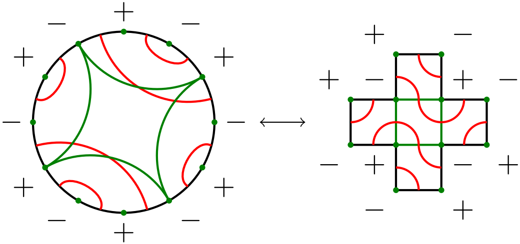

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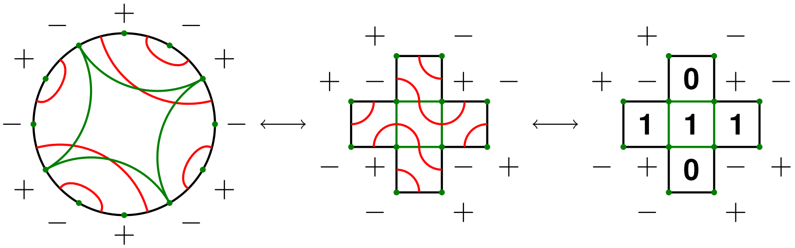

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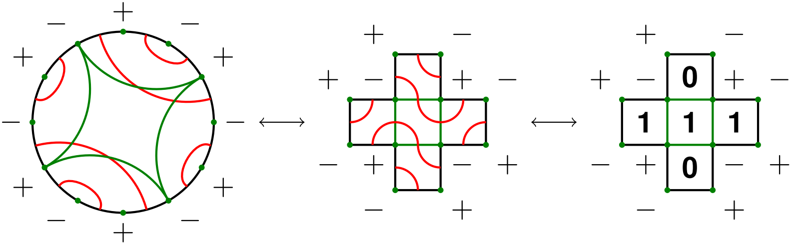


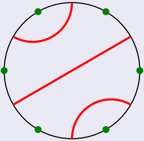
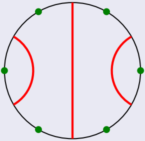
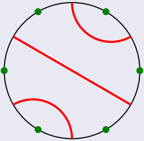
$$c(\Gamma) = \mathbf{1} \otimes \begin{array}{c} \mathbf{0} \\ \otimes \\ \mathbf{1} \\ \otimes \\ \mathbf{0} \end{array} \otimes \mathbf{1} \in \mathbf{V} \otimes \begin{array}{c} \mathbf{V} \\ \otimes \\ \mathbf{V} \\ \otimes \\ \mathbf{V} \end{array} \otimes \mathbf{V} = Z(D^2, 12 \text{ pts})$$

Here $\mathbf{V} = Z(\square) = \mathbb{Z}_2 \mathbf{0} \oplus \mathbb{Z}_2 \mathbf{1}$.

SQFT reads a basic quadrangulation “in binary format”.

Properties of suture elements

Proposition (Bypass relation)

$c\Big($  $\Big) + c\Big($  $\Big) + c\Big($  $\Big) = 0.$

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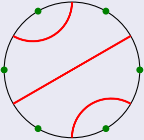
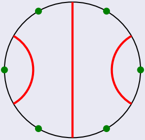
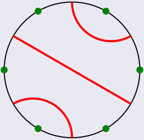
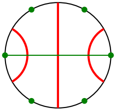
$c\Bigg($  $\Bigg) + c\Bigg($  $\Bigg) + c\Bigg($  $\Bigg) = 0.$

Allows us to decompose suture elements into basis elements.



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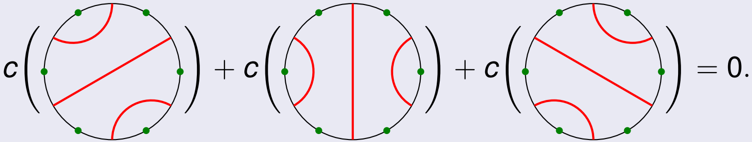

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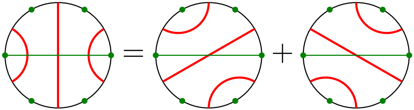

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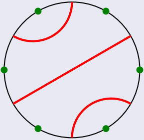
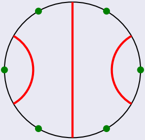
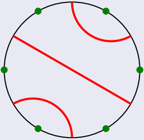
$c\Big($  $\Big) + c\Big($  $\Big) + c\Big($  $\Big) = 0.$

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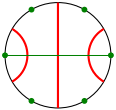
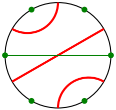
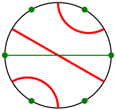
 $=$  $+$  $= \mathbf{0} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{0}$

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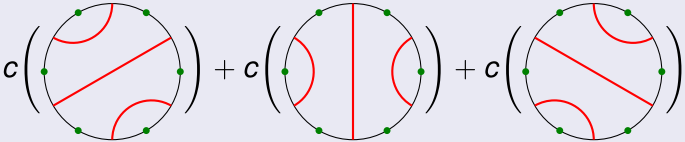

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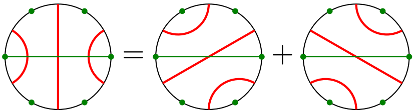

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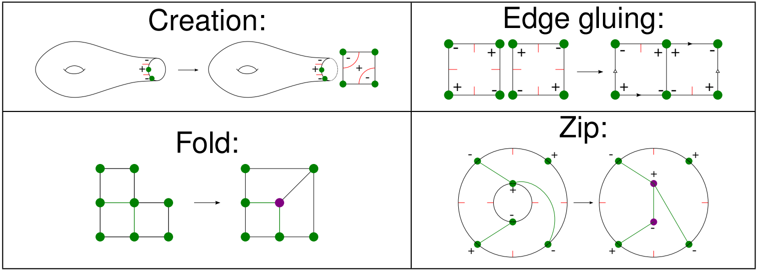
Proposition

If Γ is isolating, i.e. some component of $\Sigma \backslash \Gamma$ does not intersect $\partial\Sigma$, then $c(\Gamma) = 0$.

Elementary linear maps in SQFT

An occupied surface morphism adjoins and glues up squares:



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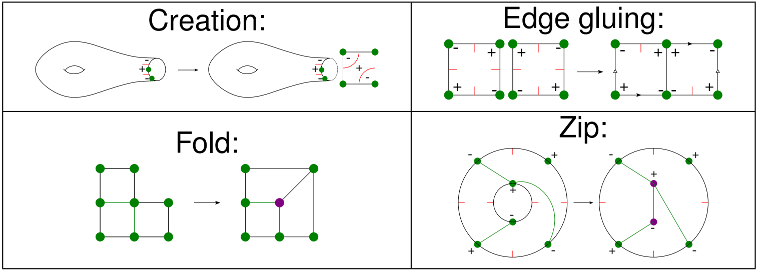


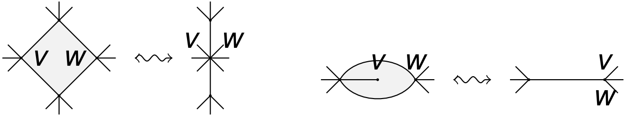
Creation:

Edge gluing:

Fold:

Zip:

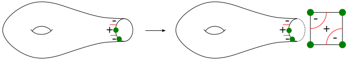
After gluing, we still have a quadrangulation, possibly *slack*.
Obtain a true quadrangulation by collapsing slack squares.



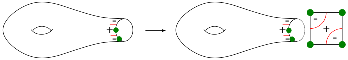
We find the algebraic effect of *creation* and *slack square collapse*.

Creation operators

Effect of creation is a digital creation operator.
We create the digit/“particle”/qubit **0** or **1** according to the sutures on the created square.



$$\begin{array}{rccc} a_{\mathbf{0}}^* : & \mathbf{V}^{\otimes n} & \rightarrow & \mathbf{V} \otimes \mathbf{V}^{\otimes n} \\ & x & \mapsto & \mathbf{0} \otimes x \end{array}$$



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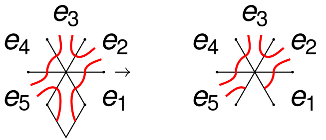
Annihilation operators

Slack square collapse performs digital annihilation. May be

Simple...	
 $\mathbf{V} \otimes \mathbf{V}^{\otimes n} \longrightarrow \mathbf{V}^{\otimes n}$ $\mathbf{0} \otimes \mathbf{x} \mapsto \mathbf{x}$	

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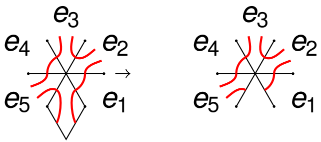
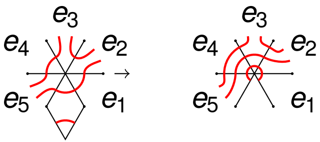
 $\mathbf{V} \otimes \mathbf{V}^{\otimes n} \longrightarrow \mathbf{V}^{\otimes n}$ $\mathbf{0} \otimes \mathbf{x} \mapsto \mathbf{x}$	

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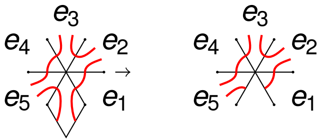
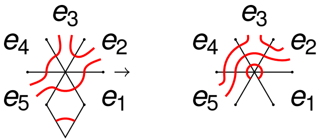
 $\mathbf{V} \otimes \mathbf{V}^{\otimes n} \longrightarrow \mathbf{V}^{\otimes n}$ $\mathbf{0} \otimes \mathbf{x} \mapsto \mathbf{x}$	 $\mathbf{V} \otimes \mathbf{V}^{\otimes n} \longrightarrow \mathbf{V}^{\otimes n}$ $\mathbf{1} \otimes \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_n \mapsto$ $\sum_{\mathbf{e}_i=\mathbf{0}} \mathbf{e}_1 \otimes \cdots \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{e}_n$

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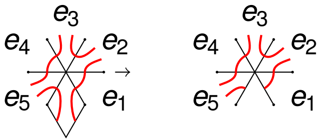	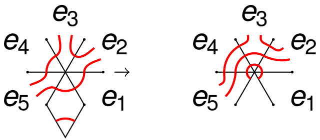
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The **0**-annihilation operator is the map

$$\begin{array}{rlcc} a_{\mathbf{0}} : & \mathbf{V}^{\otimes (n+1)} = \mathbf{V} \otimes \mathbf{V}^{\otimes n} & \rightarrow & \mathbf{V}^{\otimes n} \\ & \mathbf{0} \otimes e_1 \otimes \cdots \otimes e_n & \mapsto & e_1 \otimes \cdots \otimes e_n \\ & \mathbf{1} \otimes e_1 \otimes \cdots \otimes e_n & \mapsto & \sum_{e_i=\mathbf{0}} e_1 \otimes \cdots \otimes \mathbf{1} \otimes \cdots \otimes e_n \end{array}$$

(Similarly, **1**-annihilation $a_{\mathbf{1}}$.)

Structure theorem of SQFT

A slack square collapse only affects squares adjacent to the collapse.

A generalised digital annihilation operator is $\text{Id} \otimes a_0$ or $\text{Id} \otimes a_1$.

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Any map of vector spaces in SQFT (over $\mathbb{Z}_2$) is a composition of digital creation and generalised digital annihilation operators.

SQFT maps can be interpreted as creating/annihilating

- "particles of topology" in occupied squares
- manipulating binary information "qubits" on each square

John Archibald Wheeler: "It from bit"

Comments

- Quantum information: It’s possible to perform some (conservative) *topological quantum computation* in SQFT.
  - E.g. “quantum conservative logic gates”.

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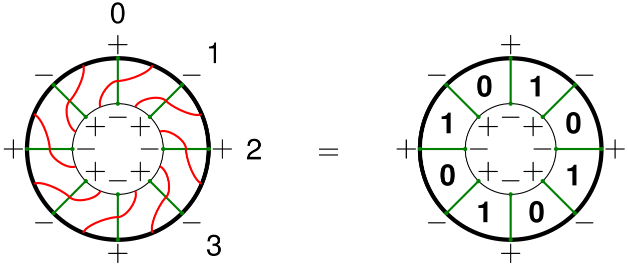
- Representation theory: Tensor powers of 2-dimensional **V** recall the representation theory of $\mathfrak{sl}(2)$... or $\mathfrak{sl}(1|1)$
- Curves Γ joining **V**'s recalls spin networks, diagrammatic representation theory, categorification...

Outline

1. Introduction
2. Sutured and occupied surfaces
3. Sutured TQFT
4. Quantum actions
    - The idea
    - An annular Temperley–Lieb action
    - Quantizing the action
    - The quantum group action

The idea of the action

Consider a boundary component C of (Σ, V) with $2n$ vertices. Take a nice (possibly slack) quadrangulation of (Σ, V) near C:



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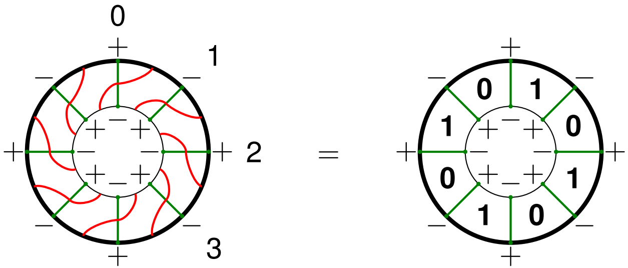


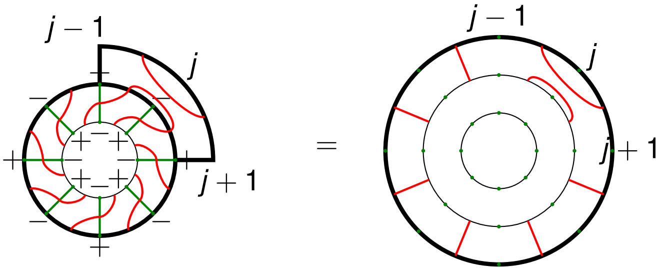
Definition

The integral form $\mathbf{U}_n$ *of* $U_q(\mathfrak{sl}(1|1))$ *is*

$$\mathbb{Z}[q^{\pm1}] \left\langle E, F \;\middle|\; \begin{array}{c} E^2 = F^2 = 0 \\ EF + FE = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n} \end{array} \right\rangle$$

An annular Temperley–Lieb action

Consider gluing a square at position j; let the SQFT map be a_j.



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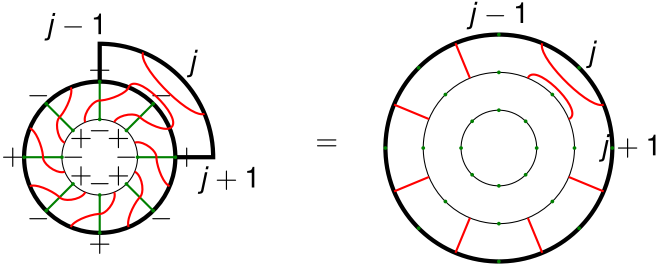

These operations form the *annular Temperley–Lieb algebra*.
Usual Temperley–Lieb relations:

$$\begin{aligned} U_j^2 &= \delta U_j \\ U_j U_{j+1} U_j &= U_j \\ U_i U_j &= U_j U_i \text{ for } |i-j| > 1. \end{aligned}$$

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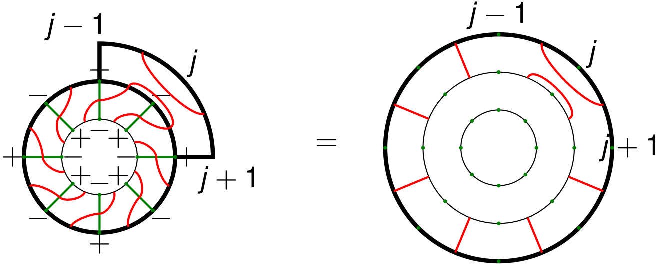

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Quantizing the action

Taking basis elements $\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_{2n}$ we find

$$a_{2j-1}(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_{2n}) = \sum_{k=2j-1,2j}^{\mathbf{e}_k=\mathbf{1}} \mathbf{e}_1 \otimes \cdots \otimes \underbrace{\mathbf{0}}_{k} \otimes \cdots \otimes \mathbf{e}_{2n}$$

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Following ideas of Tian categorifying $U_q(\mathfrak{sl}(1|1))$ we lift from $\mathbb{Z}_2$ to $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ — quantization. Now $\mathbf{V} = \mathbb{Z}[q^{1/2}, q^{-1/2}]\,\langle \mathbf{0}, \mathbf{1} \rangle$.

Quantizing the action

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These then form a quantized Temperley–Lieb algebra

$$\begin{aligned} a_j^2 &= 0 \\ a_j a_{j+1} a_j &= q^{n-2j+1} a_j \\ a_i a_j &= -a_j a_i \text{ for } |i-j| > 1. \end{aligned}$$

Also... $a_j a_{j+1} + a_{j+1} a_j = q^{n-2j+1}$

The quantum group action

Again following ideas of Tian we define

$$E = \sum_{j \text{ odd}} a_j = \sum_{1 \leq k \leq 2n}^{\mathbf{e}_k = \mathbf{1}} (-1)^{\beta_k} q^{n-k+\frac{1}{2}} \mathbf{e}_1 \otimes \cdots \otimes \underbrace{\mathbf{0}}_{k} \otimes \cdots \otimes \mathbf{e}_{2n}$$

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We find $E^2 = F^2 = 0$ and
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Theorem (M.)

For each boundary component of (Σ, V) with $2n$ vertices, there is an action of $\mathbf{U}_{2n}$ on $\mathbf{V}^{\otimes 2n}$ (where $\mathbf{V} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle \mathbf{0}, \mathbf{1} \rangle$), which projects to the SQFT maps induced by annular Temperley–Lieb algebra upon setting $q = 1$ and reducing mod 2.

Thanks for listening.