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An explicit formula for the A-polynomial of twist knots

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ABSTRACT

We extend Hoste–Shanahan's calculations for the A-polynomial of twist knots, to give an explicit formula.

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1. Introduction

Since Cooper–Culler–Gillet–Long–Shalen introduced the A-polynomial in 1994 [1], A-polynomials have found important applications to hyperbolic geometry, the topology of knot complements, and K-theory. More recently, they appear in relation to physics, in particular in the AJ conjecture.

However, calculations of A-polynomials remain relatively difficult. In particular, there are very few infinite families of knots for which A-polynomials are known. In his 1996 thesis, Shanahan [5] gave a formula for A-polynomials of torus knots. In 2004, Hoste–Shanahan [3] gave recursive formulas for the A-polynomials of twist knots and the knots $J(3, 2n)$ described below, and Tamura–Yokota [6] gave a recursive formula for the A-polynomials of $(-2, 3, 1 + 2n)$-pretzel knots. In 2011, Garoufalidis–Mattman [2] showed that the A-polynomials of $(-2, 3, 3 + 2n)$-pretzel knots satisfy a linear recursion relation, effectively demonstrating a recursive formula. Most recently, Petersen [4] gave a description of the A-polynomials of a family of two-bridge knots $J(k, l)$ including the twist knots (illustrated below) as the resultant of two recursively-defined polynomials, and in the cases of twist knots and the family $J(3, 2n)$ recovered the recursive formulas of Hoste–Shanahan. To our knowledge this exhausts the current state of knowledge on formulas for A-polynomials of infinite families of knots.

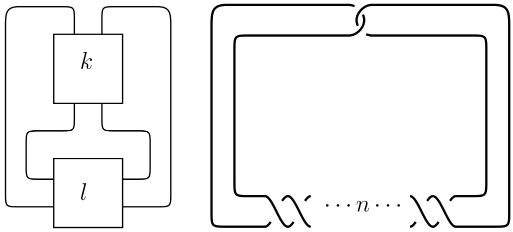

Fig. 1. The knot $J(k,l)$, at left, is given by drawing k and l right-handed half twists in the boxes as shown. The twist knot K_n, shown right, with n full twists, is equal to $J(2,2n)$.

In this short note we give an explicit, non-recursive formula for the twist knots. Let $J(k,l)$ be the family of knots illustrated in Fig. 1; the twist knots are obtained when $k = \pm 2$. Note that $J(-k,-l)$ is the mirror image of $J(k,l)$, so its A-polynomial is obtained by replacing M with M^{-1} (and normalizing appropriately). Further, $J(2,2n+1) = J(-2,2n)$. So it is sufficient to consider the knots $J(2,2n)$; we write K_n for $J(2,2n)$.

Let $A_n(L,M)$ be the A-polynomial of K_n.

Theorem 1.1. *When $n > 0$, we have*

$$A_n(L,M) = M^{2n}(L+M^2)^{2n-1} \sum_{i=0}^{2n-1} \binom{n + \lfloor \frac{i-1}{2} \rfloor}{i} \left(\frac{M^2-1}{L+M^2} \right)^i \times (1-L)^{\lfloor \frac{i}{2} \rfloor} (M^2 - LM^{-2})^{\lfloor \frac{i+1}{2} \rfloor}.$$

When $n \leq 0$, we have

$$A_n(L,M) = M^{-2n}(L+M^2)^{-2n} \sum_{i=0}^{-2n} \binom{-n + \lfloor \frac{i}{2} \rfloor}{i} \left(\frac{M^2-1}{L+M^2} \right)^i \times (1-L)^{\lfloor \frac{i}{2} \rfloor} (M^2 - LM^{-2})^{\lfloor \frac{i+1}{2} \rfloor}.$$

The proof is very direct and based on the methods of Hoste–Shanahan [3].

2. Proof of Theorem

We follow Hoste–Shanahan's notation for convenience and refer there for further details. The relevant fundamental group is

$$\pi_1(S^3 \backslash K_n) = \langle a, b \,|\, a(ab^{-1}a^{-1}b)^n = (ab^{-1}a^{-1}b)^n b \rangle = \langle a, b \,|\, aw^n = w^n b \rangle,$$

where $w = ab^{-1}a^{-1}b$. Both a, b are meridians. A general irreducible representation $\rho : \pi_1(S^3 \backslash J(2,2n)) \to SL(2,\mathbb{C})$ may be conjugated to be of the form

$$\rho(a) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} M & 0 \\ Z & M^{-1} \end{pmatrix},$$

where M, Z are both nonzero. (Our Z is $-z$ in [3].) The equation $\rho(aw^n) = \rho(w^n b)$ gives four polynomial relations in M and Z, which as discussed in [3] reduces to a single one $r_n = 0$. Writing

$$\rho(w^n) = \begin{pmatrix} w_{11}^n & w_{12}^n \\ w_{21}^n & w_{22}^n \end{pmatrix} \quad \text{we have } r_n = (M - M^{-1})w_{12}^n + w_{22}^n.$$

We compute

$$\rho(w) = \rho(ab^{-1}a^{-1}b) = \begin{pmatrix} w_{11}^1 & w_{12}^1 \\ w_{21}^1 & w_{22}^1 \end{pmatrix}$$
$$= \begin{pmatrix} M^2 Z + (1-Z)^2 & M - M^{-1} + ZM^{-1} \\ -ZM^{-1} + ZM + Z^2 M^{-1} & 1 + \imath Z M^{-2} \end{pmatrix}$$

so that, by the Cayley–Hamilton identity (noting the above matrix has determinant 1)

$$\rho(w^n) = \chi \rho(w^{n-1}) - \rho(w^{n-2}) \quad \text{where } \chi = \operatorname{Tr} \rho(w) = Z^2 + (M - M^{-1})^2 Z + 2.$$

Hence each entry w_{ij}^n satisfies $w_{ij}^n = \chi w_{ij}^{n-1} - w_{ij}^{n-2}$. As $r_n = (M - M^{-1})w_{12}^n + w_{22}^n$, we also have a recurrence relation

$$r_n = \chi r_{n-1} - r_{n-2}. \tag{2.1}$$

On the other hand, a longitude is given by $\lambda = w^n \overline{w}^n$, where $\overline{w} = ba^{-1}b^{-1}a$. We have

$$\rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix},$$

where $L = w_{11}^n \overline{w}_{22}^n + Z w_{12}^n \overline{w}_{12}^n$. Here $\overline{w}_{ij}^n$ is obtained from w_{ij}^n by replacing M with M^{-1}. We then have the relation $s_n = 0$, where

$$s_n = w_{12}^n L + \overline{w}_{12}^n.$$

Note r_n is a polynomial satisfied by M and Z, while s_n is a polynomial satisfied by L, M and Z. Eliminating Z from $r_n = 0$ and $s_n = 0$ gives the A-polynomial of $J(2, 2n)$.

We may simplify $s_n = 0$ to the relation $s'_n = 0$ where

$$s'_n = w_{12}^1 L + \overline{w}_{12}^1 = (M - M^{-1} + ZM^{-1})L + M^{-1} - M + ZM.$$

Thus $s'_n = 0$ is equivalent to

$$Z = \frac{(M - M^{-1})(1 - L)}{M + LM^{-1}}. \tag{2.2}$$

All of the above is in [3]. Our strategy is simply to find an explicit formula for r_n in terms of M, Z, and then substitute Z for the expression above in terms of L and M.

Inspection of the r_n reveals that these expressions simplify when written in terms of Z and $(M-M^{-1})^2$, rather than Z and M; and the resulting coefficients are products of binomial coefficients. This leads to the formulas for r_n in the following lemma.

Lemma 2.1.

$$r_n = \sum_{i=0}^{2n-1} \binom{n+\lfloor \frac{i-1}{2} \rfloor}{i} Z^i (1+Z^{-1}(M-M^{-1})^2)^{\lfloor \frac{i+1}{2} \rfloor} \quad \textit{when } n > 0 \qquad (2.3)$$

$$= \sum_{i=0}^{-2n} \binom{-n+\lfloor \frac{i}{2} \rfloor}{i} (-Z)^i (1+Z^{-1}(M-M^{-1})^2)^{\lfloor \frac{i+1}{2} \rfloor} \quad \textit{when } n \leq 0. \qquad (2.4)$$

Proof. Write f_n for the claimed formula above; we show $f_n = r_n$. We give the proof for $n > 0$; for $n \leq 0$ the method is similar. Note that the range $0 \leq i \leq 2n-1$ is precisely the range of integers for which $0 \leq i \leq n + \lfloor \frac{i-1}{2} \rfloor$, so we can regard the sum as an infinite one, with all undefined binomial coefficients as zero.

We compute r_0, r_1 directly. As $\rho(w^0)$ is the identity, $r_0 = (M-M^{-1})w_{12}^0 + w_{22}^0 = 1 = f_0$. Noting the computation of w_{ij}^1 above, we have $r_1 = (M-M^{-1})w_{12}^1 + w_{22}^1 = (M-M^{-1})(M-M^{-1}+ZM^{-1}) + 1 + ZM^{-2} = 1 + Z + (M-M^{-1})^2 = f_1$. For convenience write $U = (M-M^{-1})^2$, so $\chi = Z^2 + UZ + 2 = (1+Z^{-1}U)Z^2 + 2$. We now show that f_n satisfies the recurrence (2.1).

$$\begin{aligned}
\chi f_{n-1} - f_{n-2} &= ((1+Z^{-1}U)Z^2+2) \sum_i \binom{n-1+\lfloor \frac{i-1}{2} \rfloor}{i} Z^i (1+Z^{-1}U)^{\lfloor \frac{i+1}{2} \rfloor} \\
&\quad - \sum_i \binom{n-2+\lfloor \frac{i-1}{2} \rfloor}{i} Z^i (1+Z^{-1}U)^{\lfloor \frac{i+1}{2} \rfloor} \\
&= \sum_i \left[2\binom{n-1+\lfloor \frac{i-1}{2} \rfloor}{i} + \binom{n-2+\lfloor \frac{i-1}{2} \rfloor}{i-2} \right. \\
&\quad \left. - \binom{n-2+\lfloor \frac{i-1}{2} \rfloor}{i} \right] Z^i (1+Z^{-1}U)^{\lfloor \frac{i+1}{2} \rfloor} \\
&= \sum_i \binom{n+\lfloor \frac{i-1}{2} \rfloor}{i} Z^i (1+Z^{-1}U)^{\lfloor \frac{i+1}{2} \rfloor} = f_n
\end{aligned}$$

In the second line we collect the sums together, shifting i to make a sum over $Z^i(1+Z^{-1}U)^{\lfloor \frac{i+1}{2} \rfloor}$. In the last line we apply the binomial relation $\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}$ three times. ∎

Now substituting (2.2) for Z into r_n, for $n > 0$, gives

$$\sum_{i=0}^{2n-1} \binom{n + \lfloor \frac{i-1}{2} \rfloor}{i} \left(\frac{(M^2-1)(1-L)}{L+M^2} \right)^i \left(1 + \frac{(M+LM^{-1})(M-M^{-1})}{1-L} \right)^{\lfloor \frac{i+1}{2} \rfloor}.$$

We observe that

$$1 + \frac{(M+LM^{-1})(M-M^{-1})}{1-L} = \frac{M^2 - LM^{-2}}{1-L},$$

and $\lfloor \frac{i}{2} \rfloor + \lfloor \frac{i+1}{2} \rfloor = i$ for all integers i. The resulting expression,

$$\sum_{i=1}^{2n-1} \binom{n + \lfloor \frac{i-1}{2} \rfloor}{i} \left(\frac{M^2-1}{M^2+L} \right)^i (1-L)^{\lfloor \frac{i}{2} \rfloor} (M^2 - LM^{-2})^{\lfloor \frac{i+1}{2} \rfloor},$$

once denominators are cleared to give a polynomial, gives the A-polynomial $A_n(L, M)$. As explained in [3], we multiply by $M^{2n}(L + M^2)^{2n-1}$. Similarly, for $n \leq 0$, we multiply by $M^{-2n}(L + M^2)^{-2n}$. This gives the desired formula, proving the theorem.

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