Discrete Contact Geometry

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Outline

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This talk is about some interesting recent applications that are *discrete* and *combinatorial*:

- Arrangements & combinatorics of curves on surfaces
- “Topological computation"
- Finite state automata

What is contact geometry?

Definition

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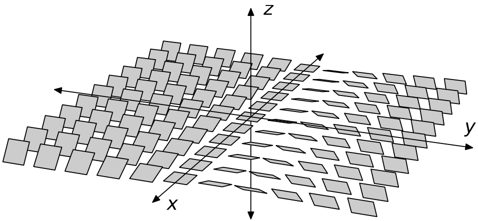

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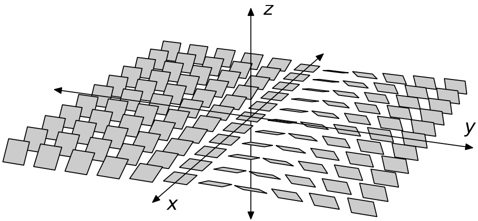

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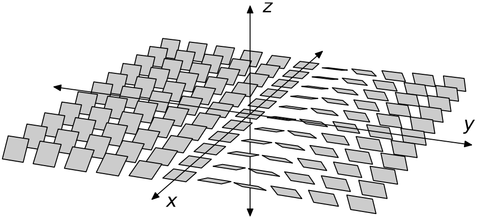

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Such ξ can be given as $\ker \alpha$ where α is a differential 1-form satisfying $\alpha \wedge d\alpha \neq 0$ everywhere.
E.g. $\mathbb{R}^3$ with $\alpha = dz - y\,dx$.

Flexible vs discrete

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But it's also a surprisingly *rigid* type of geometry.

- Any other "nontrivial" contact structure ξ on $\mathbb{R}^3$ is *isotopic* to the standard one ξ_{std}.
  (I.e. ξ can be continuously deformed through contact structures to ξ_{std}.)

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- Giroux (1991): *Convex surfaces* and *dividing sets*.
- Gromov (1986), Eliashberg (1990s), ...: Development of *pseudoholomorphic curve* methods.
- Ozsváth-Szabó (2004), many others... : Development of *Floer homology* methods.

Why we contactify

Some motivations for the study of contact geometry:

- *Topology:* One way to understand the topology of a manifold is to study the contact structures on it.
- *Dynamics:* There are natural *vector fields* on contact manifolds and their dynamics have important applications to classical mechanics.

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- *Physics:* Many recent developments run parallel with physics — Gromov-Witten theory, string theory, etc.
- *Pure mathematical / Structural:* Mathematical structures found in contact geometry connect to other fields...
  - Combinatorics
  - Information theory
  - Discrete mathematics

Outline

Fact #1: Dividing sets

Consider a generic *surface* S in a contact 3-manifold M, possibly with boundary ∂S. (In this talk, $S =$ disc or annulus.)

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A contact structure ξ near S is described exactly by a finite set Γ of non-intersecting smooth curves on S, called its *dividing set*.

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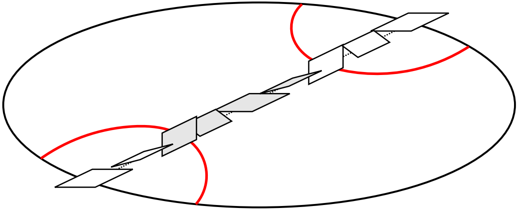

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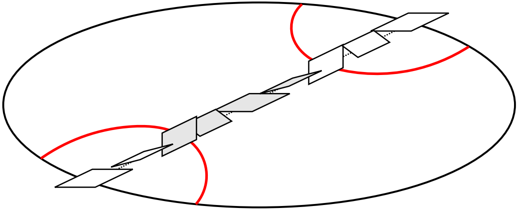

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- Tangent to ∂S
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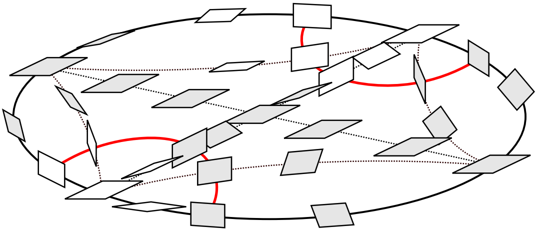

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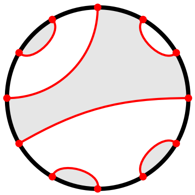

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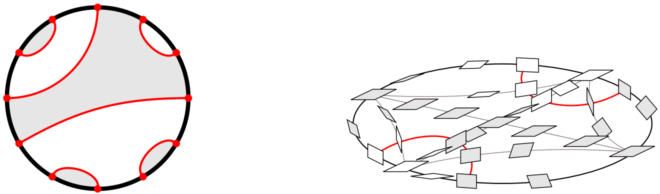

Note: We can shade alternating regions of a chord diagram.

- Colour = visible side of contact plane.

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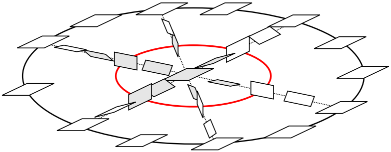

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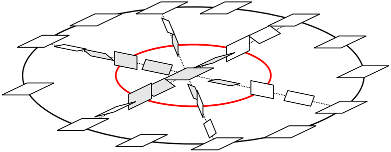

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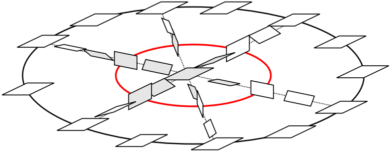
An overtwisted disc is: 

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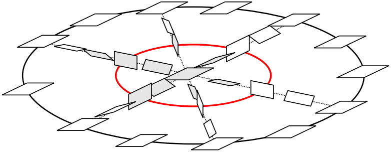

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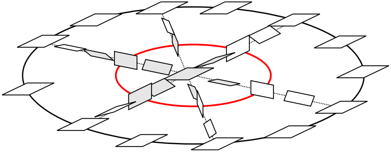

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- On a *sphere*, when there is *more than one* dividing curve.

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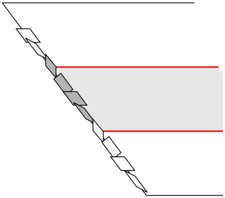

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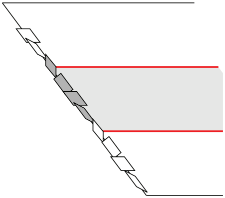

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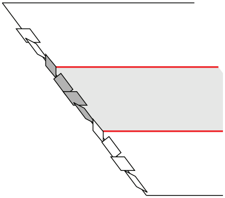

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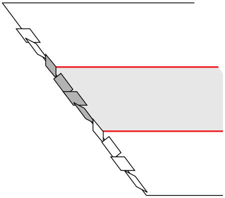

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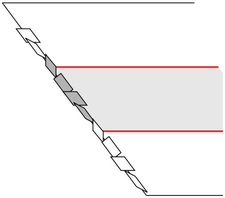

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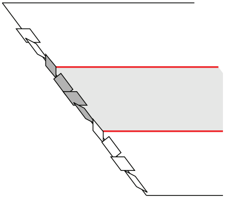

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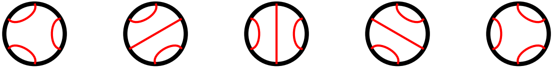
E.g. $n = 3$: 

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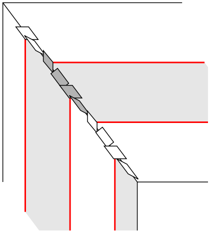

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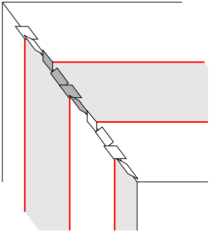

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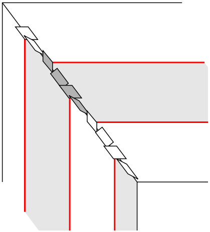

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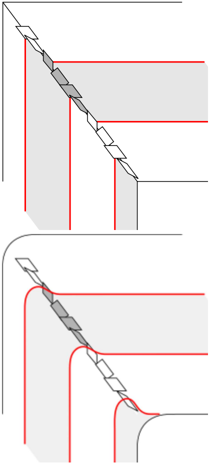

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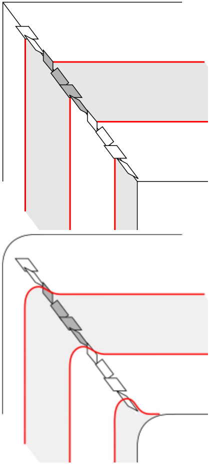

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Fact # 3 (Honda 2000)

When surfaces intersect transversely, dividing sets interleave.
Rounding corners, “turn right to dive" and “turn left to climb".

This leads to interesting combinatorics of curves...

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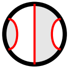

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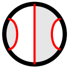

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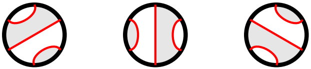

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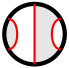

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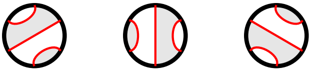

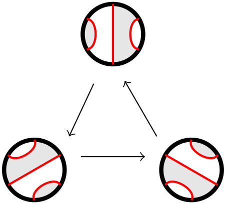

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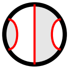

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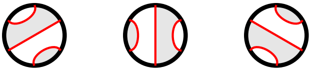

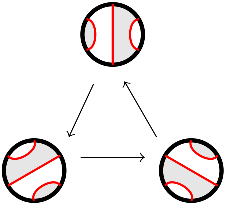

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Bypass surgery is a natural order-3 operation on dividing sets.

Summary

Fact #1: Dividing sets (Giroux, 1991)

A contact structure ξ near S is described exactly by a finite set Γ of non-intersecting smooth curves on S, called its *dividing set*.

Fact #2: Giroux's criterion

Dividing sets detect trivial contact structures (OT discs).

- On a *disc D*, via a *closed dividing curve*.
- On a *sphere*, when there is *more than one* dividing curve.

Fact #3: Edge rounding (Honda 2000)

When surfaces intersect transversely, dividing sets interleave. Rounding edges, “turn right to dive" and “turn left to climb".

Fact #4: Bypass surgery (Honda 2000)

Bypass surgery is a natural order-3 operation on dividing sets.

Outline

Cylinders

A combinatorial construction using dividing sets (fact #1), edge rounding (#3) and Giroux's criterion (#2):

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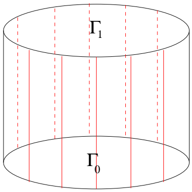

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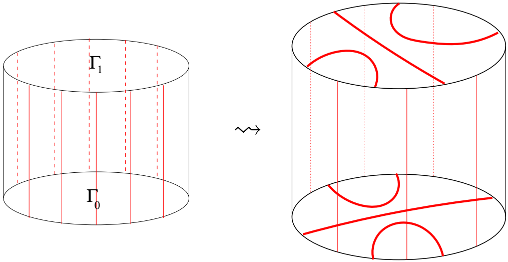

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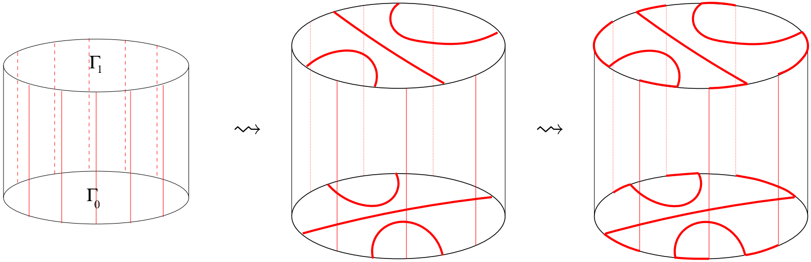

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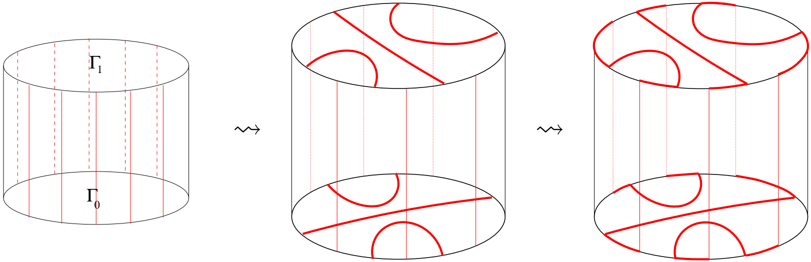

By Giroux's criterion, the contact structure obtained on S^2 is:

- *Trivial* (OT) if it is disconnected, i.e. contains $>$ 1 curve.
- *Nontrivial* (tight) if it is connected, i.e. contains 1 curve.

An “inner product" on chord diagrams

Define an “inner product" function based on this construction.

Definition

$$\langle \cdot | \cdot \rangle \;:\; \{\textit{Div sets on } D^2\} \times \{\textit{Div sets on } D^2\} \longrightarrow \mathbb{Z}_2$$

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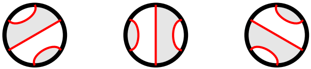

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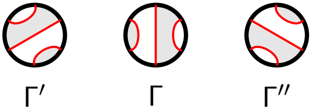


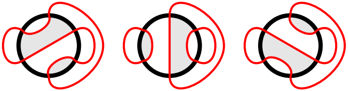
Proposition (M.)

For any $\Gamma, \Gamma', \Gamma''$ *as above and any* Γ_1,

$$\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma'' | \Gamma_1 \rangle = 0.$$

A vector space of chord diagrams

Idea of proof:

 $= 1 + 0 + 1 = 0$

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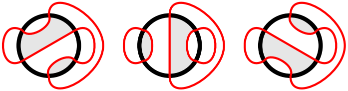 $= 1 + 0 + 1 = 0$

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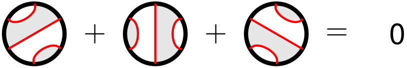

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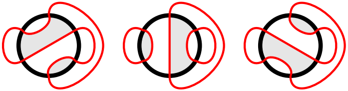
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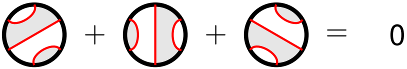

Leads to the definition of a *vector space* (over $\mathbb{Z}_2$).

Definition

$$V_n = \frac{\mathbb{Z}_2 \langle \textit{Chord diagrams with n chords} \rangle}{\textit{Bypass relation}}$$

(One can show V_n is a rudimentary form of *Floer homology*...)

A vector space of chord diagrams

Theorem (M.)

1. *V_n has dimension 2^{n-1}, with natural bases indexed by binary strings of length $n-1$.*

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1. The *Slalom basis* $\{S_b\}_{b \in B_{n-1}}$
2. The *Turing tape basis* $\{T_b\}_{b \in B_{n-1}}$

The slalom basis

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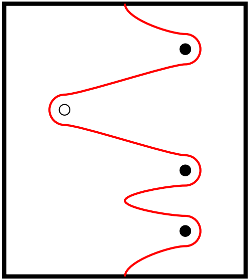
1011 $\leftrightarrow$ 

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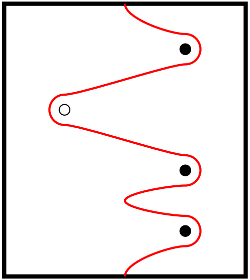
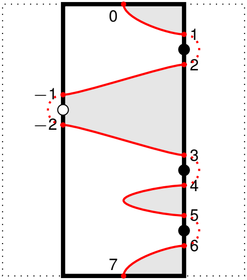
$1011 \leftrightarrow$  $\leftrightarrow$ 

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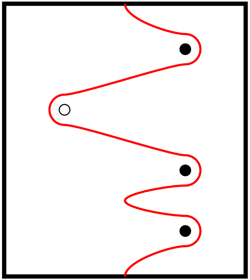
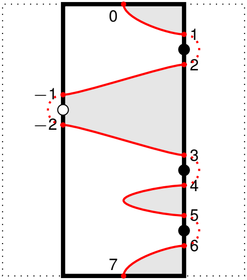
1011 $\leftrightarrow$  $\leftrightarrow$ 

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In this basis, the bilinear form $\langle \cdot | \cdot \rangle$ has a simple description:

Theorem (M.)

$$\langle S_a | S_b \rangle = \begin{cases} 1 & \textit{if } a \preceq b \\ 0 & \textit{otherwise,} \end{cases}$$

where $\preceq$ is a certain partial order on binary strings.

A partial order on binary strings

Definition

For two binary strings a, b, the relation $a \preceq b$ holds if

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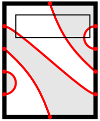
Inserting chord diagrams into a cylinder is a "topological machine" for comparing binary strings with respect to $\preceq$.

Properties of $\preceq$

Recall we said the slalom chord diagrams form a *basis* for V_n.

E.g. 

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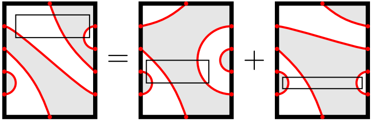

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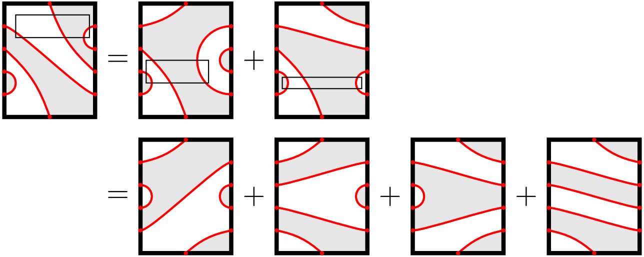

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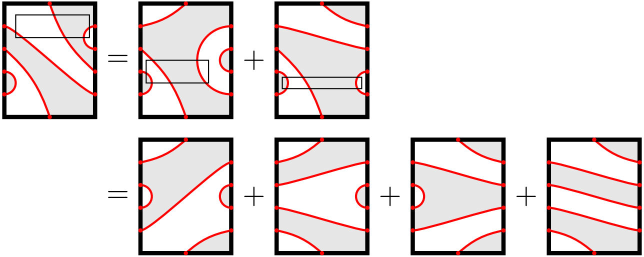

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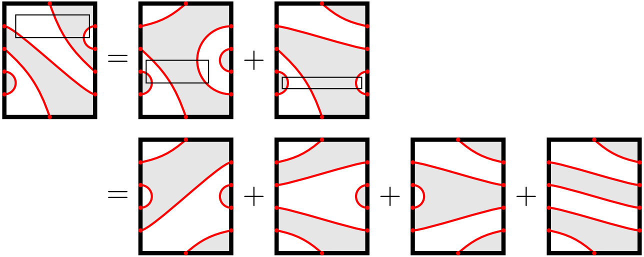

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- We say the *component strings* of Γ are 0011, 0110, 1001, 1010.
- Given a chord diagram Γ, let $b_-(\Gamma)$ denote the numerically least, and $b_+(\Gamma)$ the numerically greatest, component string.

So for the example Γ above, $b_-(\Gamma) = 0011$ and $b_+(\Gamma) = 1010$.

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... and produces the Catalan numbers again.

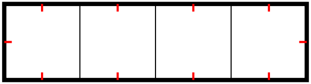
Corollary

The number of pairs of strings s_-, s_+ of length n such that $s_- \preceq s_+$ is C_{n+1}.

Outline

The Turing tape basis

Divide the disc with $|F| = 2n$ into $n-1$ *squares*:



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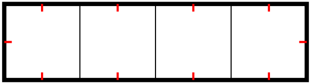
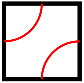
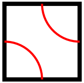


On each square there are two “basic" possible sets of sutures

0: , **1:** 

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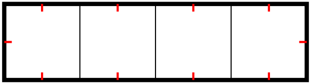

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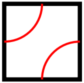
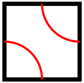
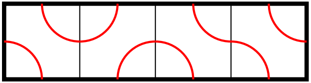
0: , **1**: 

Draw them according to a string b to obtain Turing tape basis diagrams T_b — another basis for V_n.
E.g.

$T_{1011} =$ 

Cubulated inner product

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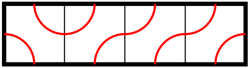

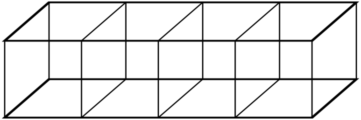

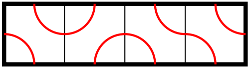

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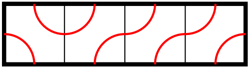

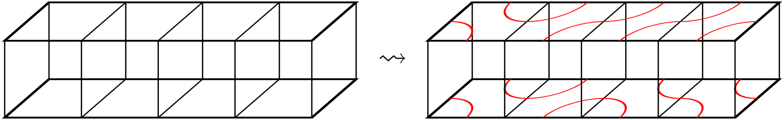

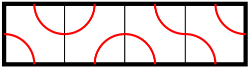

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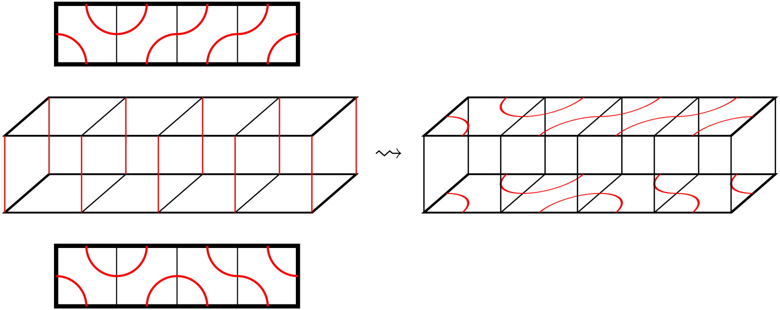

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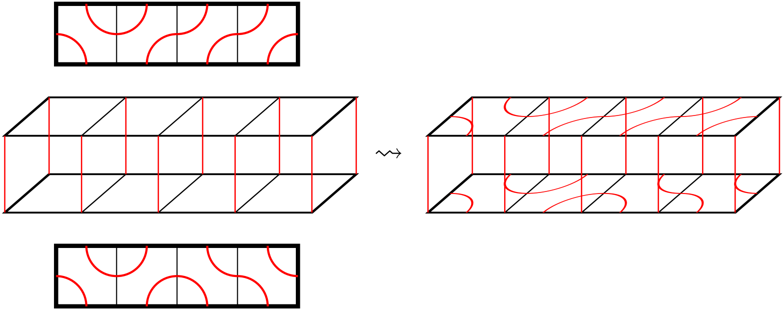

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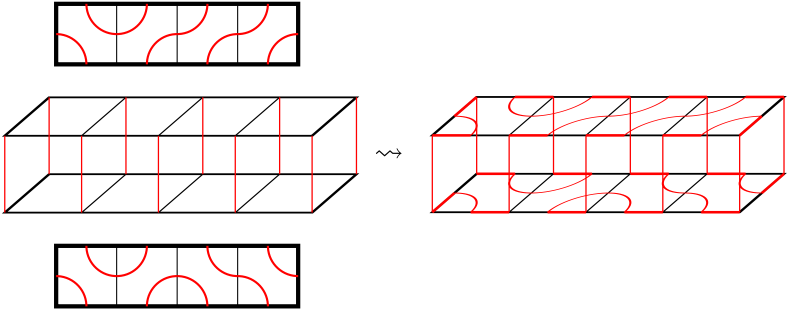

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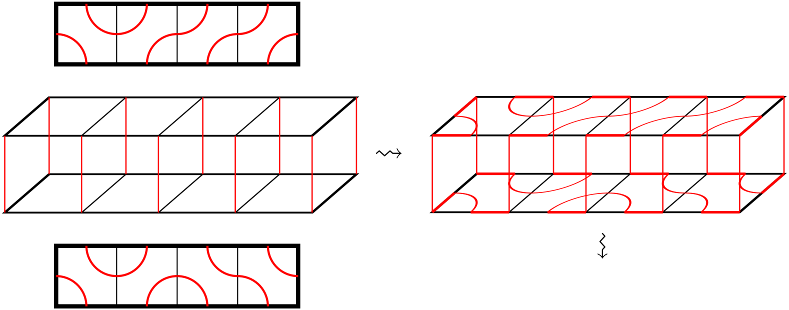

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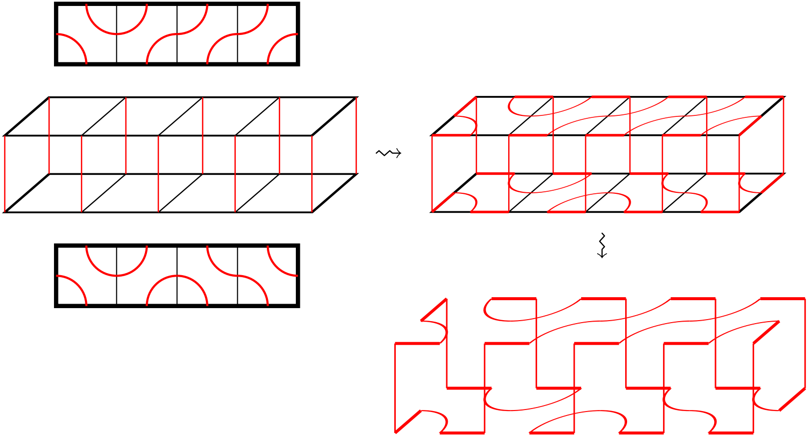

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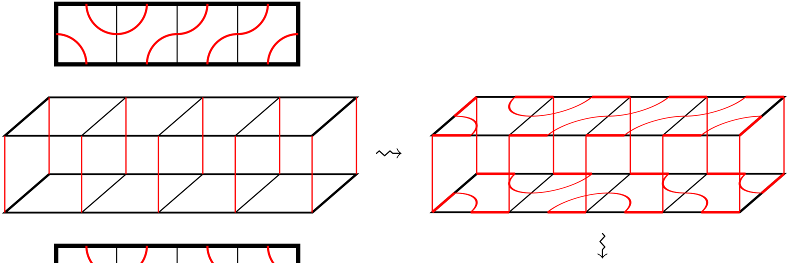

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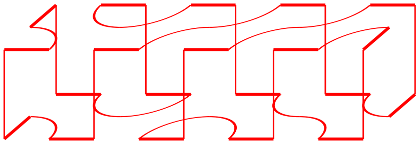

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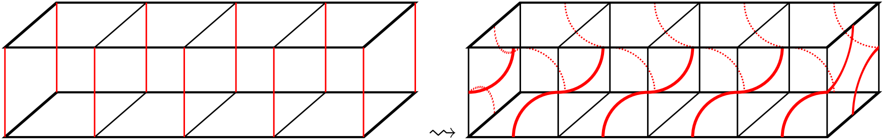

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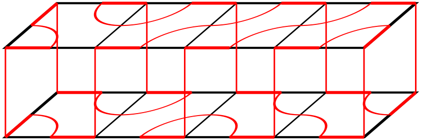
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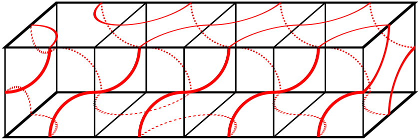

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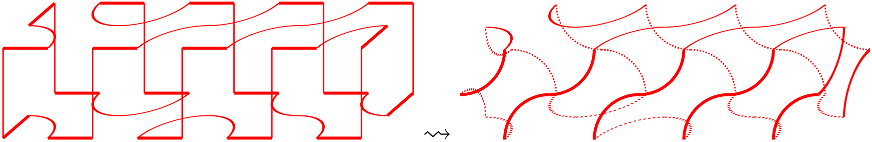

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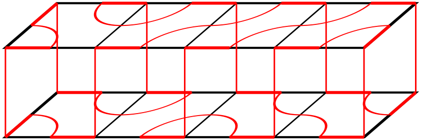

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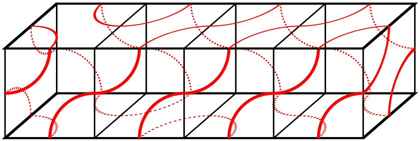

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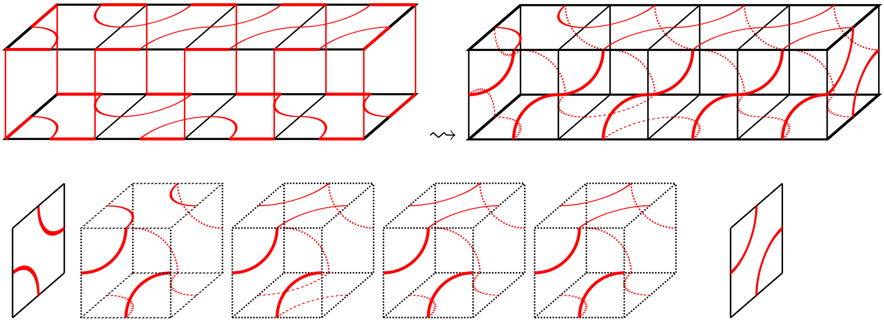

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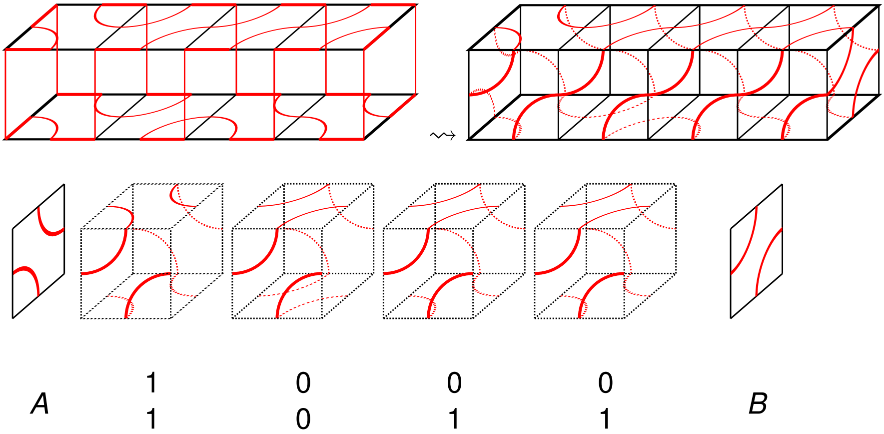

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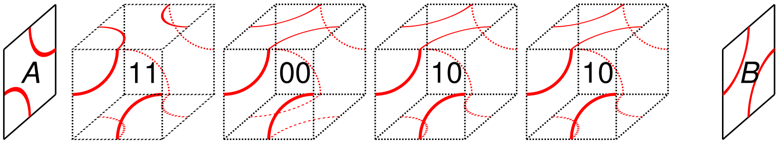

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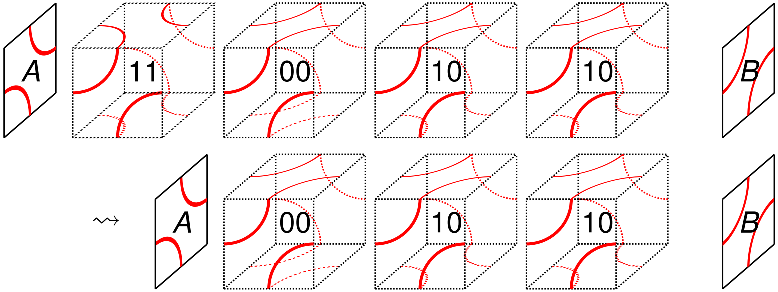

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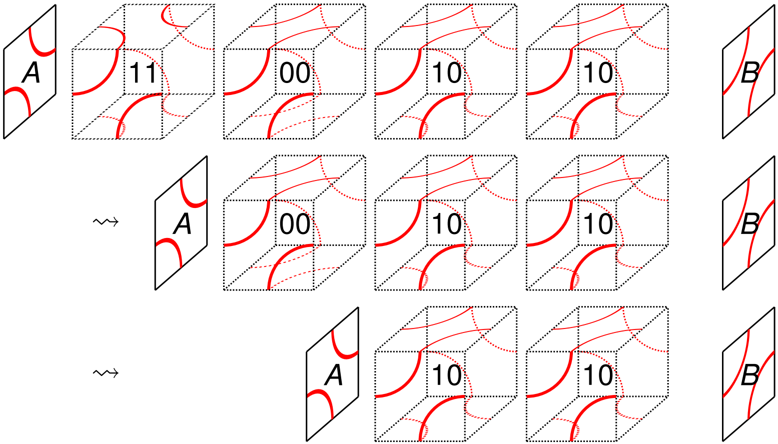

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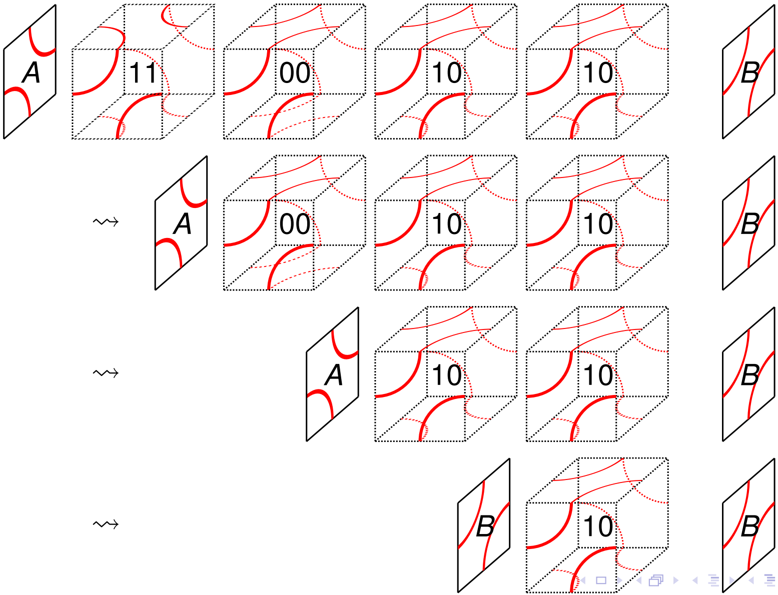

A finite state automaton

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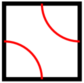
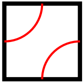
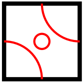
3 states: A:  B:  $\perp$:  (or anything with a closed curve)

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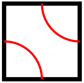
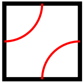
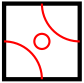
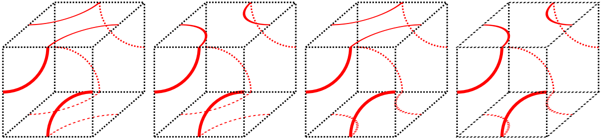
3 states: *A*:  *B*:  $\perp$:  (or anything with a closed curve)

4 inputs: 

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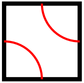
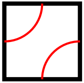
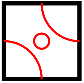
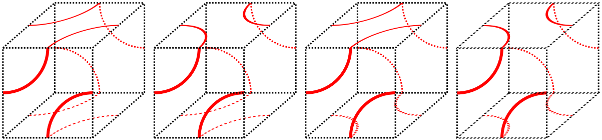
3 states: *A*:  *B*:  $\perp$:  (or anything with a closed curve)

4 inputs: 

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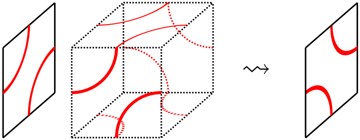

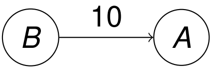

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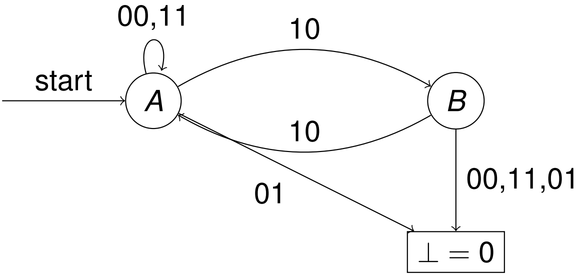

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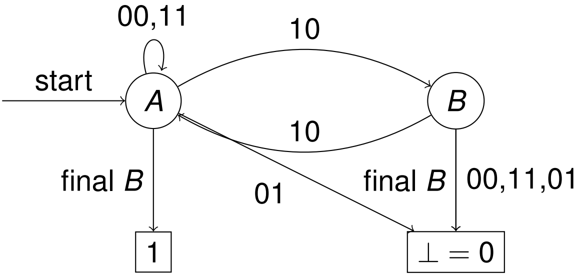

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A finite state automaton is contact-representable if:

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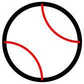
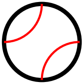
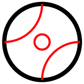
$\Gamma_A =$  $\Gamma_B =$  $\Gamma_\perp$:  (or anything with a closed curve)

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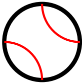
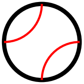
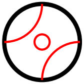
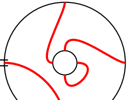
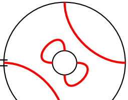
3 states: $\Gamma_A =$  $\Gamma_B =$  $\Gamma_\perp$:  (or anything with a closed curve)

4 inputs: $\Gamma_{00} =$  $\Gamma_{01} =$  $\ldots$

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Thanks for listening!

References:

- D. Mathews, *Chord diagrams, contact-topological quantum field theory, and contact categories*, Alg. & Geom. Top. 10 (2010) 2091–2189
- D. Mathews, *Itsy bitsy topological field theory* Annales Henri Poincaré (2013) DOI 10.1007/s00023-013-0286-0
- D. Mathews, *Contact topology and holomorphic invariants via elementary combinatorics*, Expos. Math. (2013) DOI 10.1016/j.exmath.2013.09.002