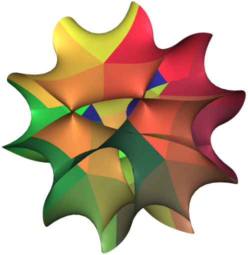
Topology
Shapes of Space, and Space of Shapes

Daniel V Mathews

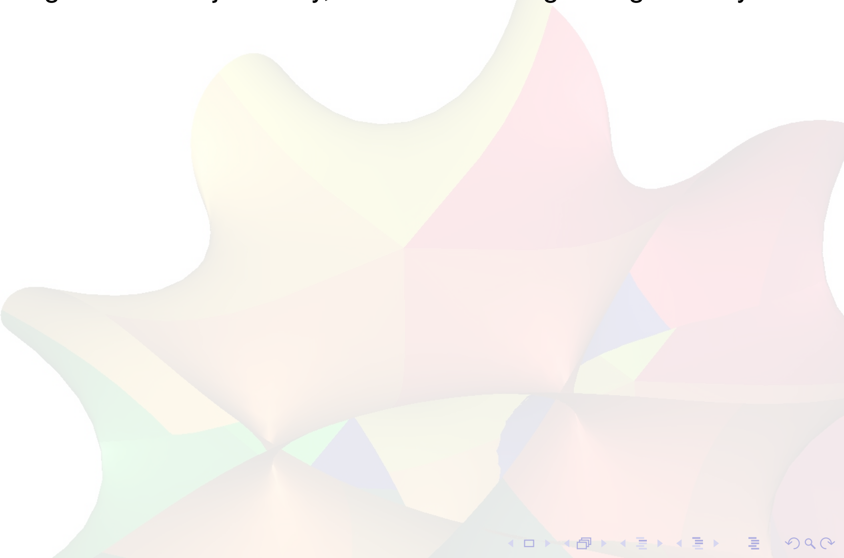
Daniel.Mathews@monash.edu



What is topology?

“The study of the shape of things".

Take a geometric object. Say, a cube. Now forget its geometry.



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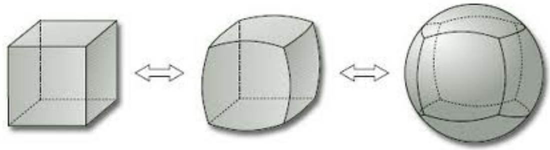

Source: Ágnes Szilárd

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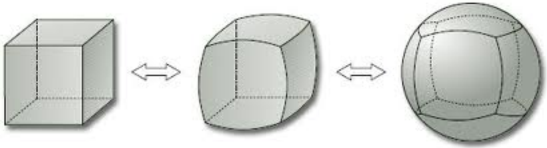

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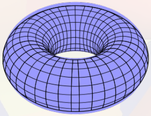

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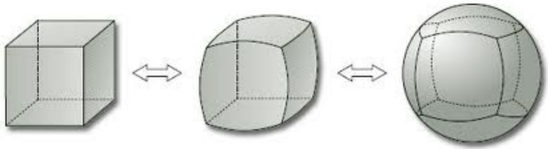

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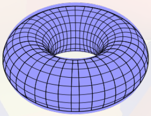

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“Rubber sheet geometry." Animate!

What is topology?

Take it back to basics: 1 dimension! Let's consider *curves*.

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Horizontal line segment
1 m long

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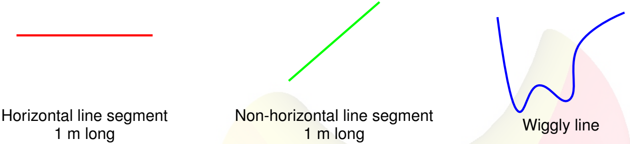

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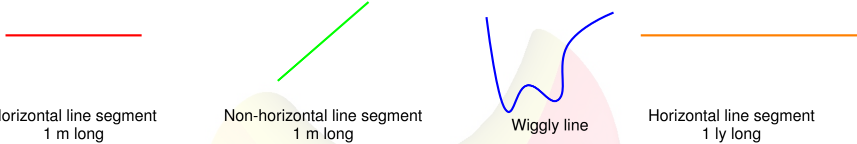

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“essentially line-ish."

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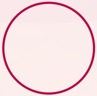
Here is another curve:



Circle

Somehow, the circle is “different" from the others... not line-ish.
“More different" from “lines" than lines are among themselves.

What is topology?

Horizontal line segment
1 m long

Non-horizontal line segment
1 m long

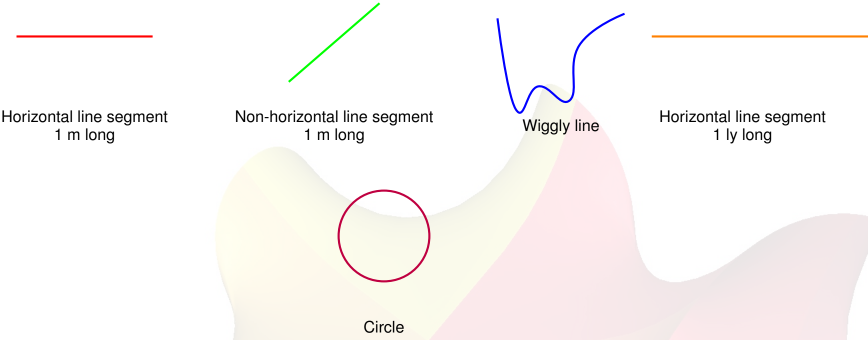
Wiggly line

Horizontal line segment
1 ly long

Circle

All are 1-dimensional curves in the plane.

What is topology?



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- What makes the first four curves “different" from the last?

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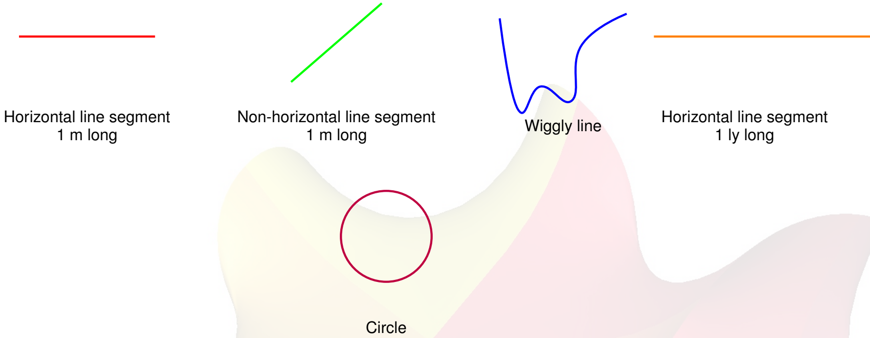

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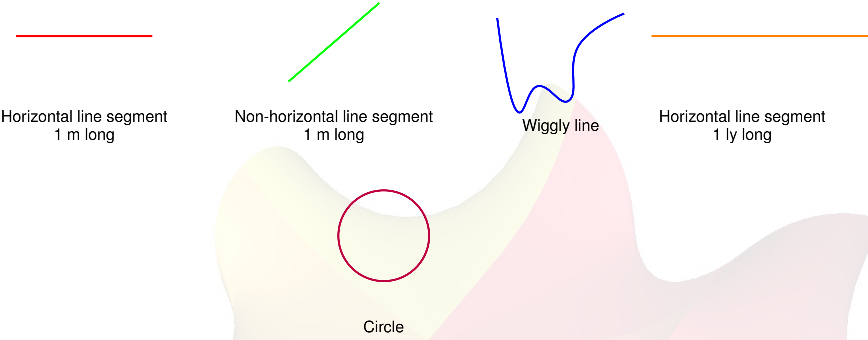

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Think like a mathematician!

- What kind of *functions* are there between these curves?

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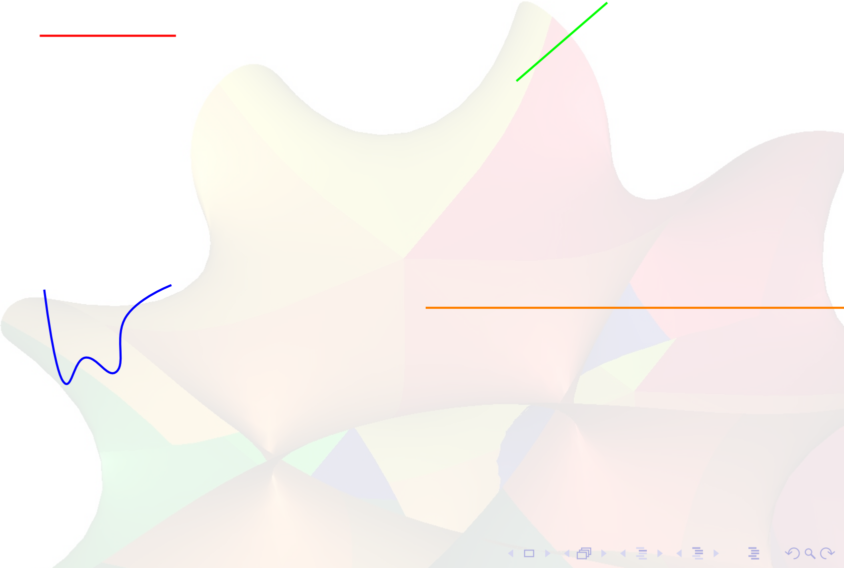

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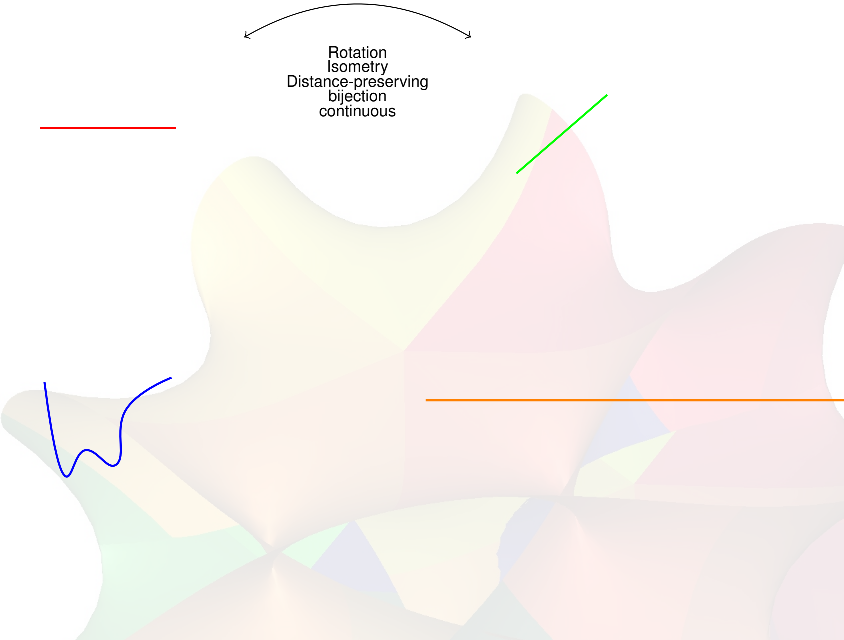

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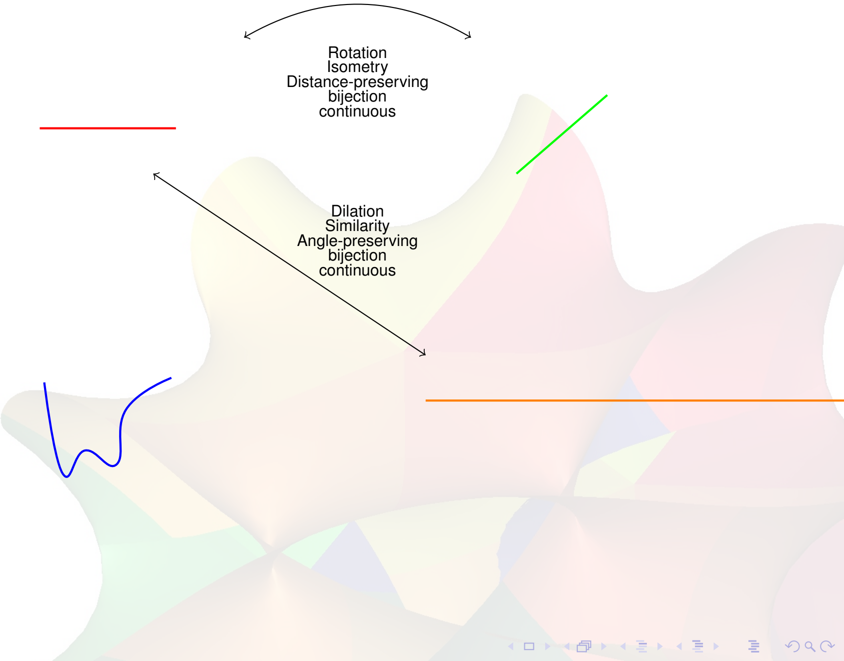

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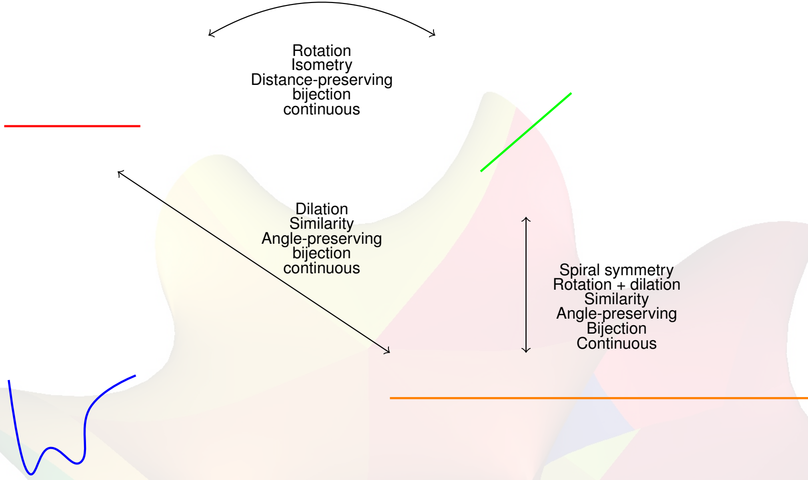

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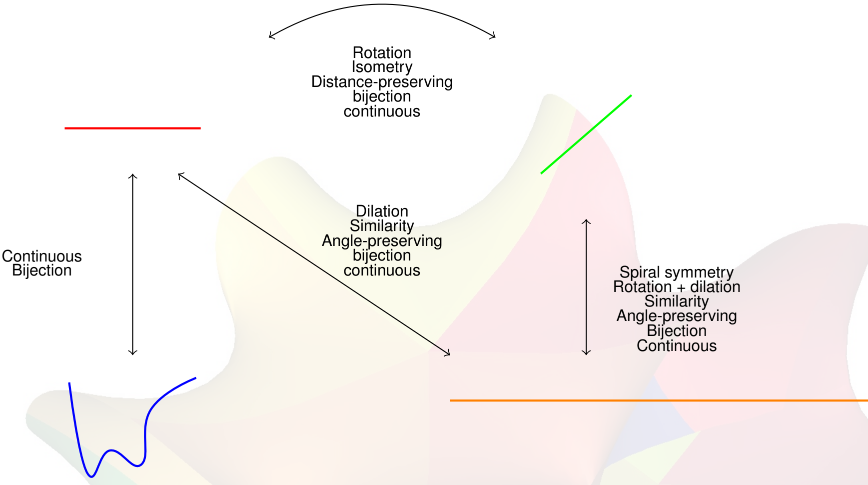

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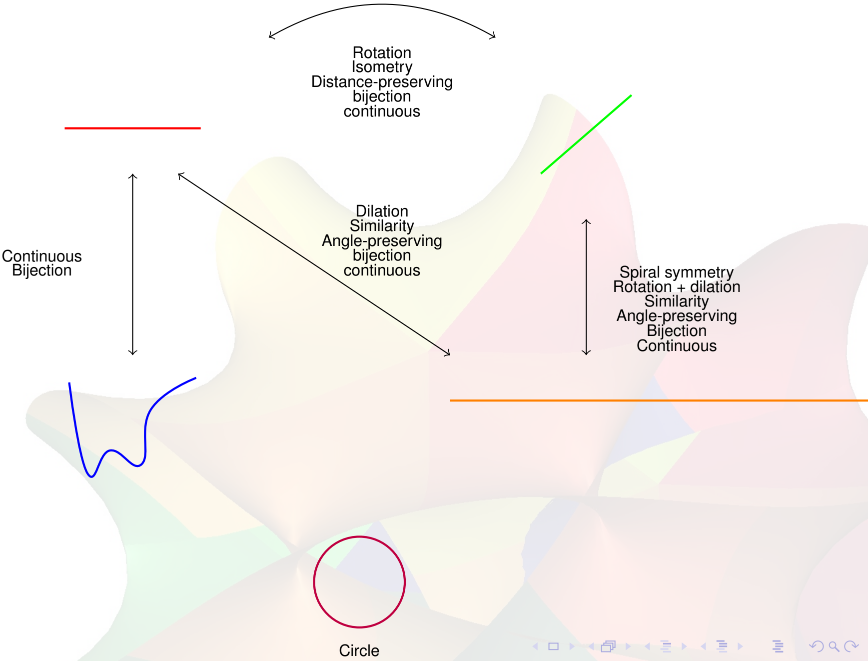

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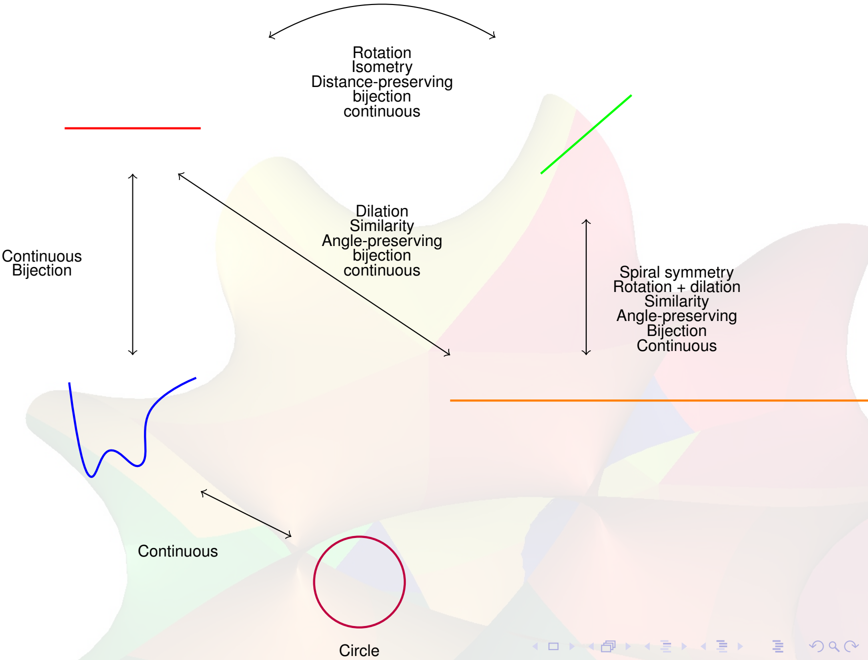

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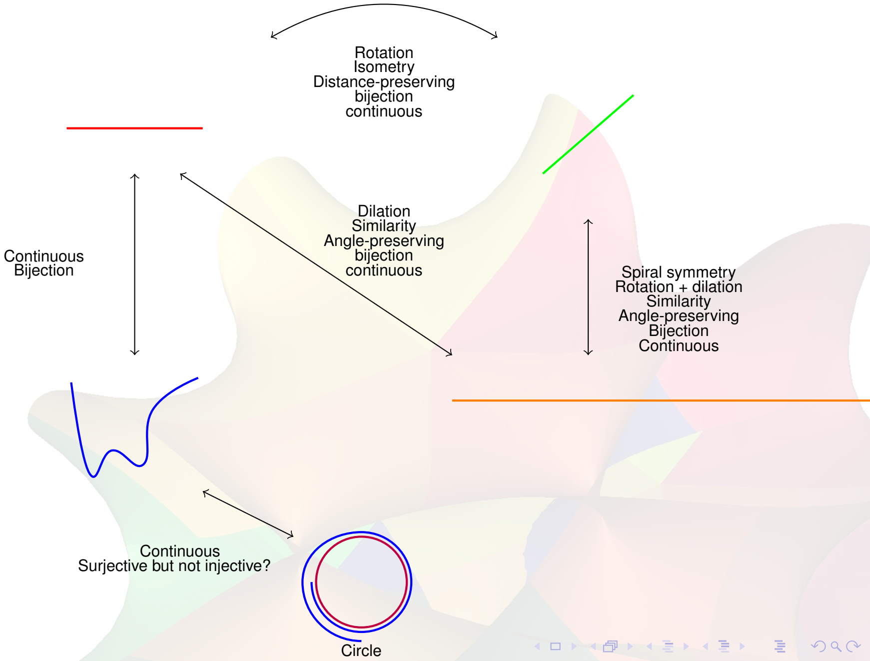

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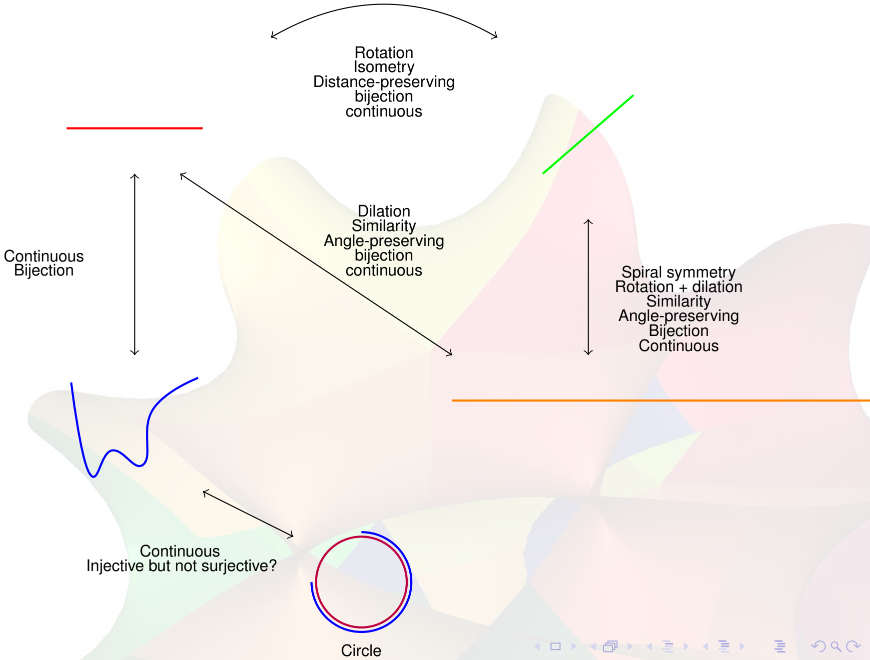

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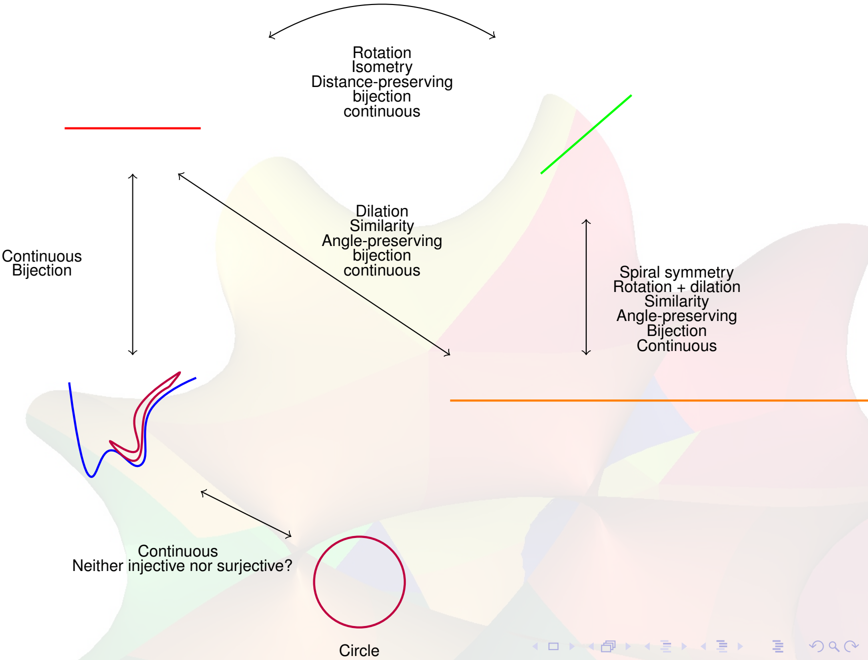

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What about our cube, sphere, and donut?

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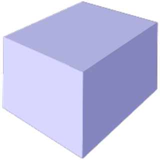

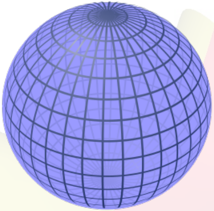

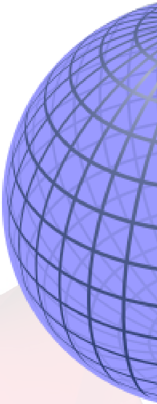

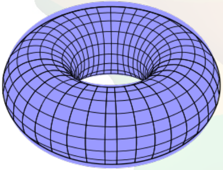

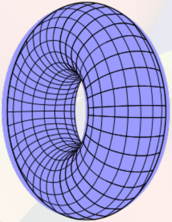

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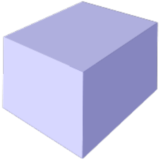

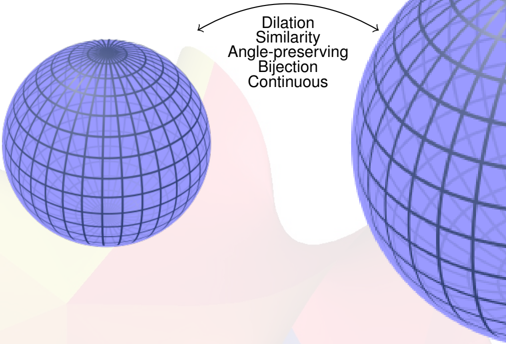

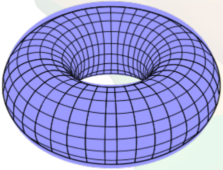

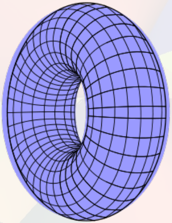

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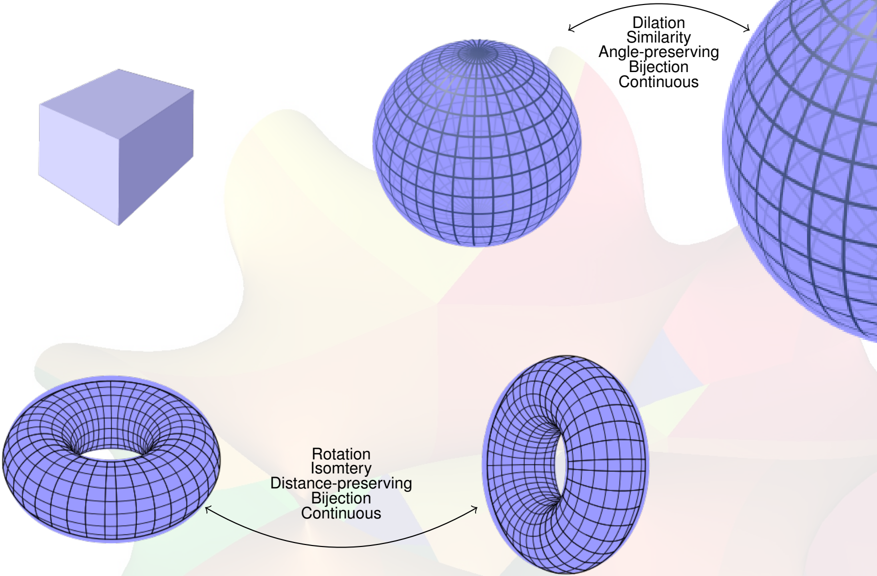

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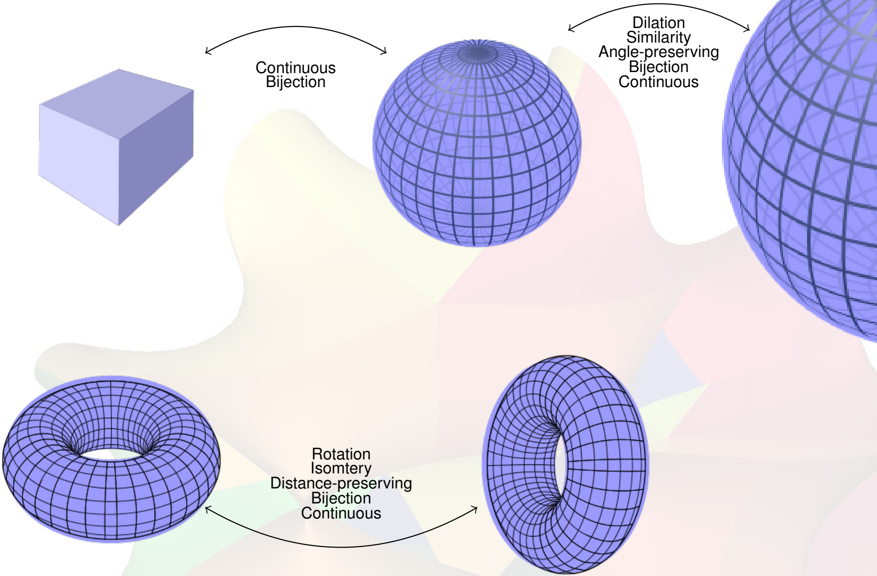

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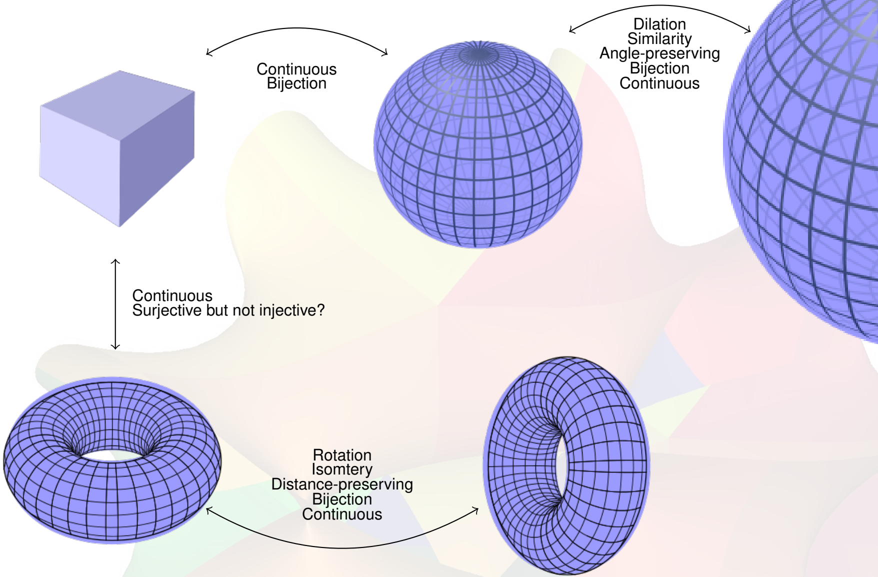

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Conformal geometry

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Conformal geometry

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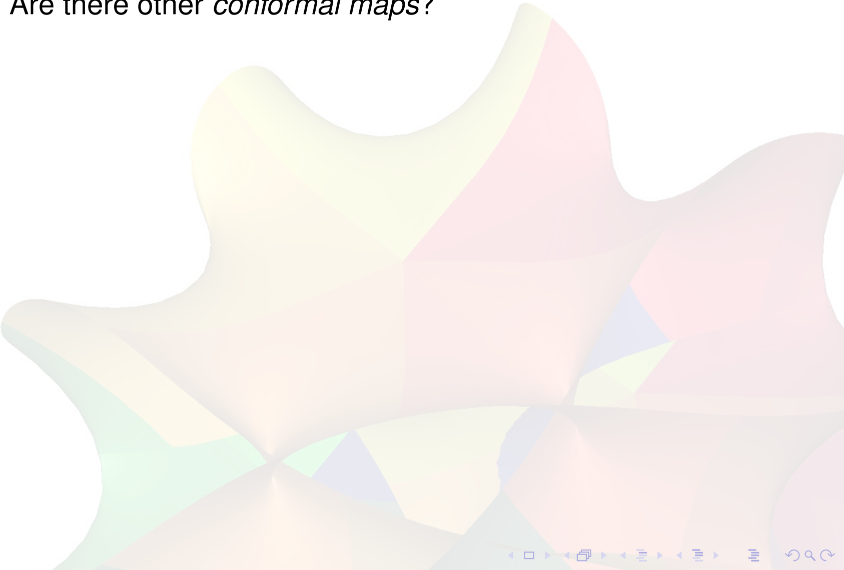

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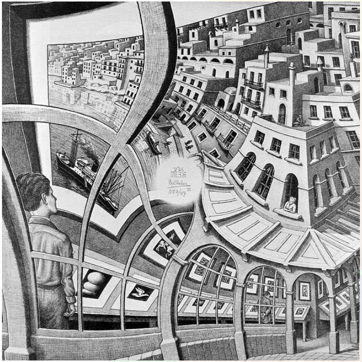

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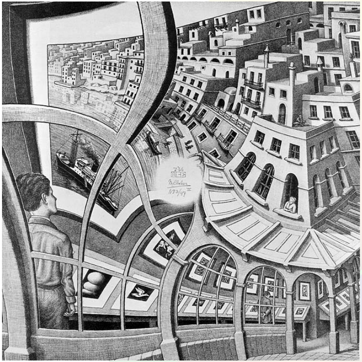

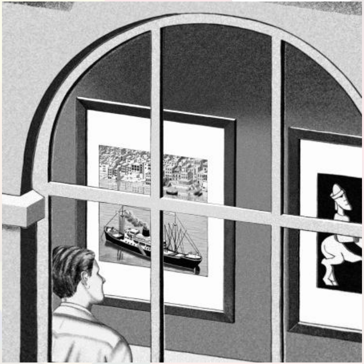

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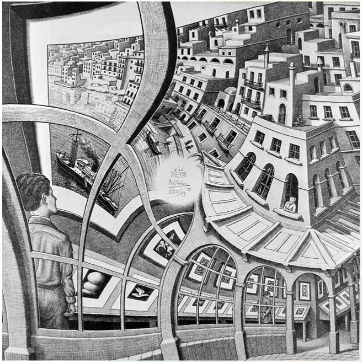

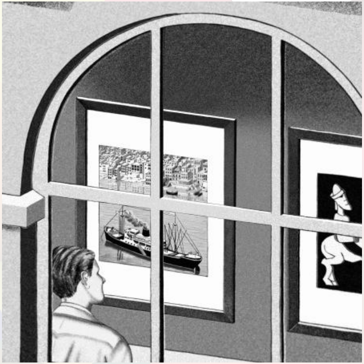

Zoom in! Animate!

1-dimensional Topology

Can we write down *all* possible 1-dimensional spaces?

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Line Circle

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That’s it.

1-dimensional Topology

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What about 2-dimensional objects?

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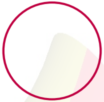

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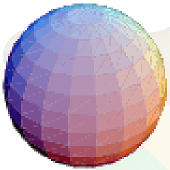

Sphere
Genus 0

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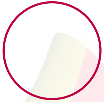

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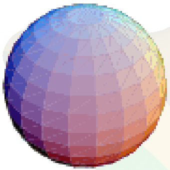

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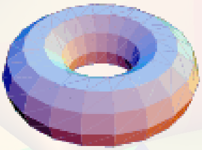

Torus
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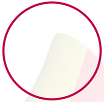

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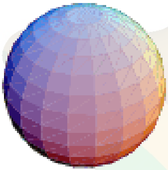

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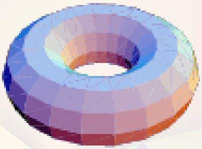

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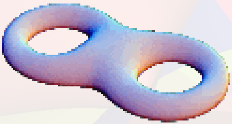

Double torus
Genus 2

. . .

1-dimensional Topology

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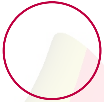

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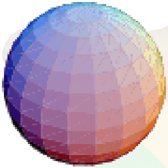
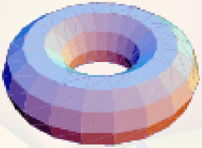
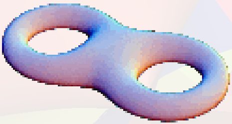
What about 2-dimensional objects?



Sphere
Genus 0



Torus
Genus 1



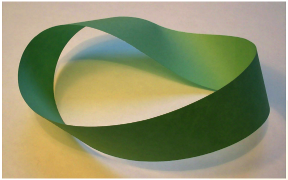
Double torus
Genus 2

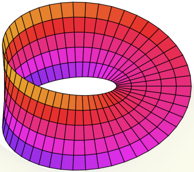
. . .

But wait!

2-dimensional topology

There's more! For instance, the *Möbius strip*.





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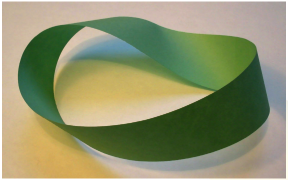

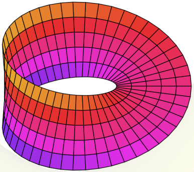

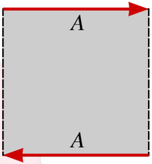

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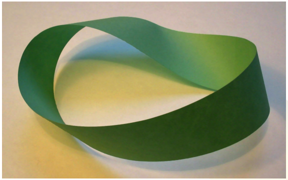
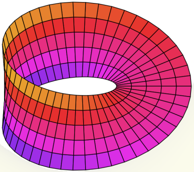
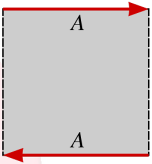
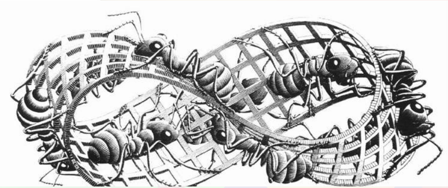






Non-orientable.

Escher:



Now animated!

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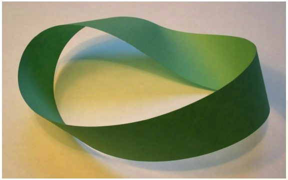

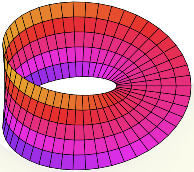

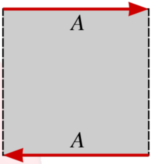

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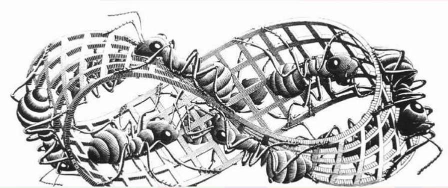

Now animated!

Why did the chicken cross the Mobius strip?

2-dimensional topology

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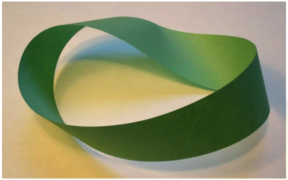

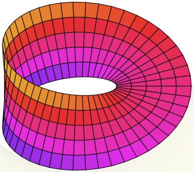

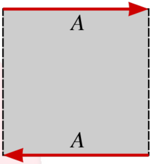

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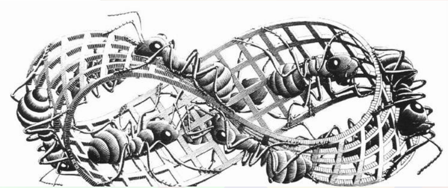

Now animated!

Why did the chicken cross the Mobius strip?
From now on, only consider surfaces *without boundary*...

2-dimensional topology

Are there non-orientable surfaces *without boundary*? YES!

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Felix Klein, 1882

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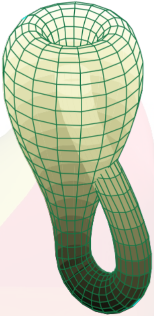

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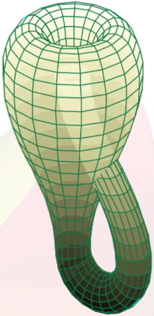

Klein bottle for rent – inquire within!

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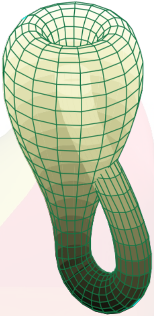

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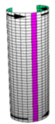

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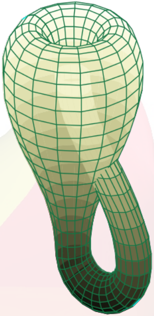

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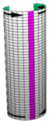

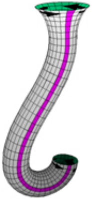

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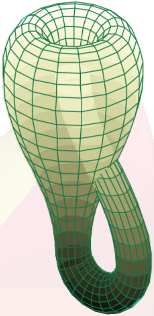

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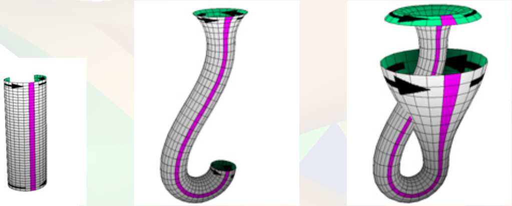

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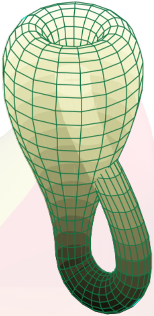

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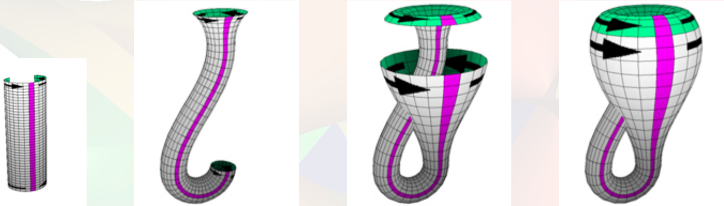

Now animated!

Topological video games...

Wraparound: a gameplay variation on video games...

Topological video games...

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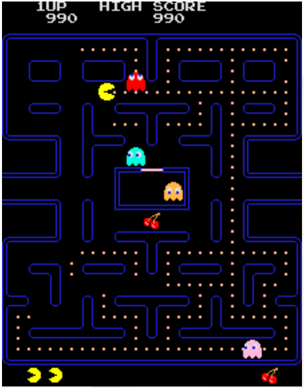

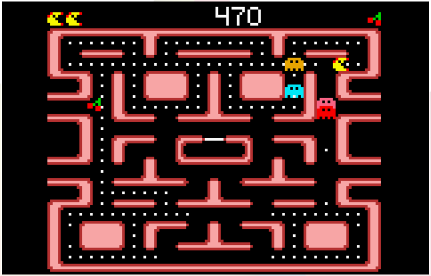

Pac Man & Ms. Pac Man

Topological video games...

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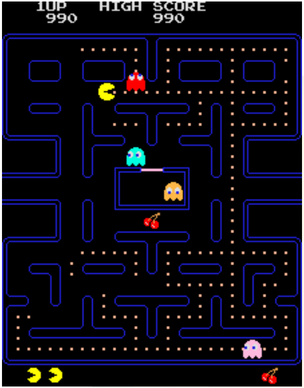

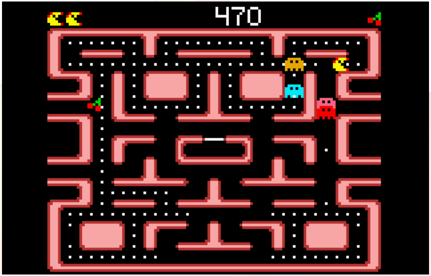

Pac Man & Ms. Pac Man

Edges are *glued* together... What is the topology of Pac Man games?

Topological video games

Idealised Pac Man: exit the screen at any point & re-enter opposite...

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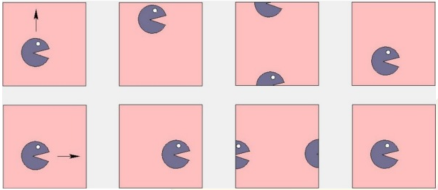

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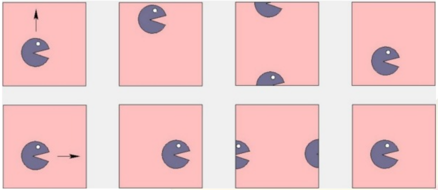

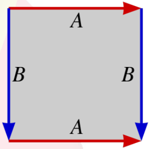

Opposite edges are identified, so should be glued together.

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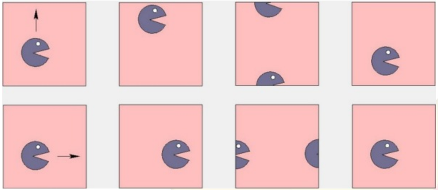

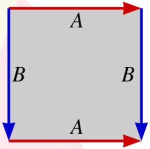

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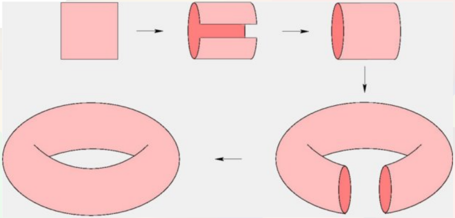
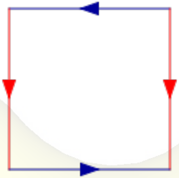


Pac man & Ms. Pac man live on a torus.

Topological video games

Pac man variant: after exiting top, re-enter bottom differently...



What surface is this?

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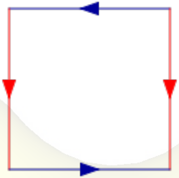

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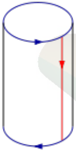

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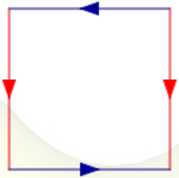

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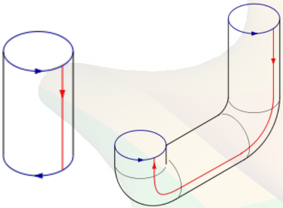

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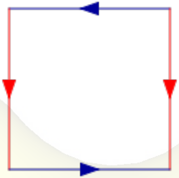

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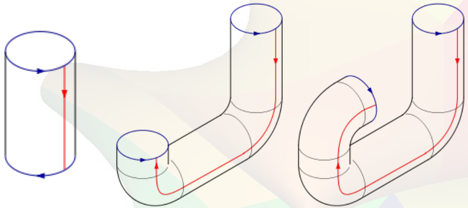

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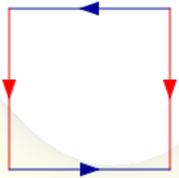

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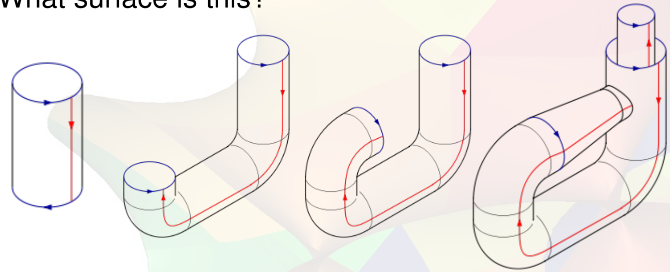

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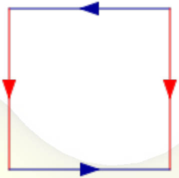

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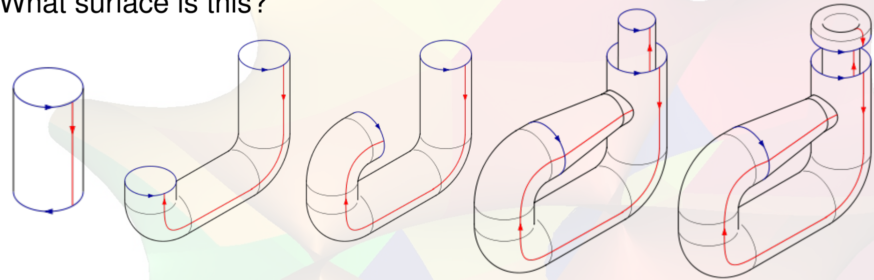

Pac Man von Klein!

Video games classify surfaces

In fact, *any surface* can be expressed as a video game with boundary entries & exits specified.

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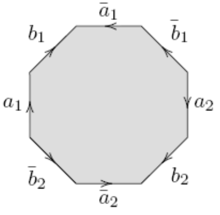

$=$

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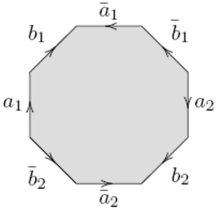

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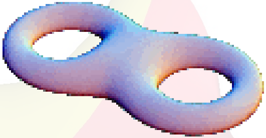

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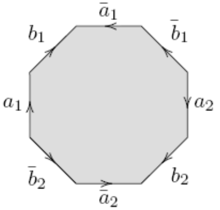

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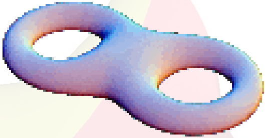

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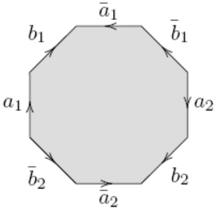

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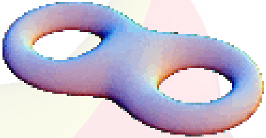

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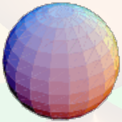
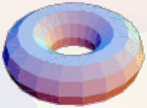
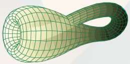
Orientable:    ...

Non-orientable: ...  ...

Aside: Dimension?

What is dimension anyway? Does it even make sense?

- “n coordinates specify a point in n-dimensional space."

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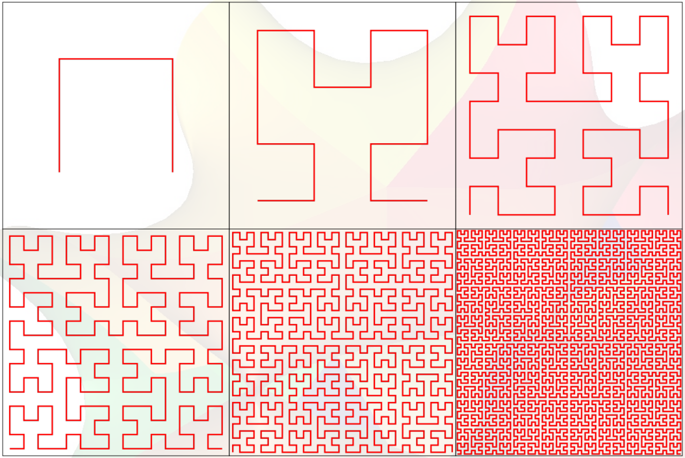

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There are also curves which fill up 3-dimensional space!

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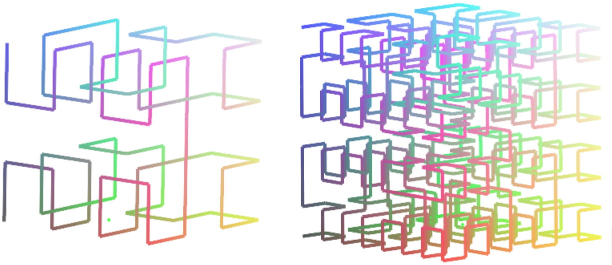

Continuous surjective functions $(0,1) \to (0,1)^2$ and $(0,1)^3$.

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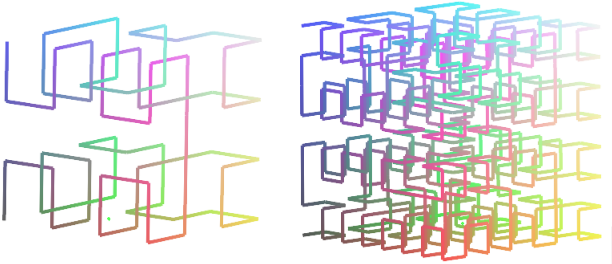

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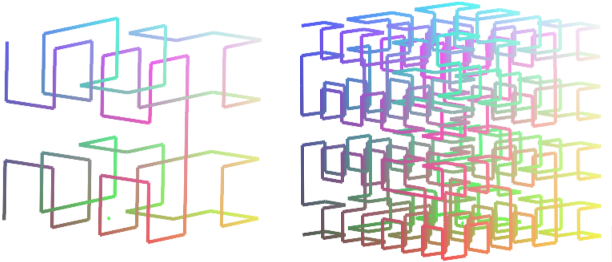

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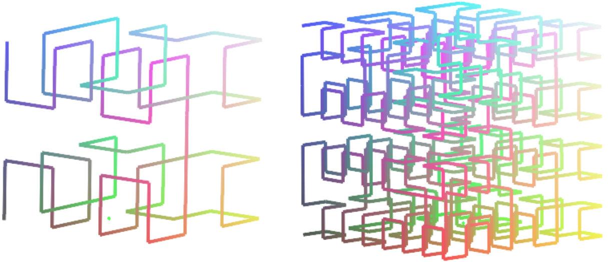

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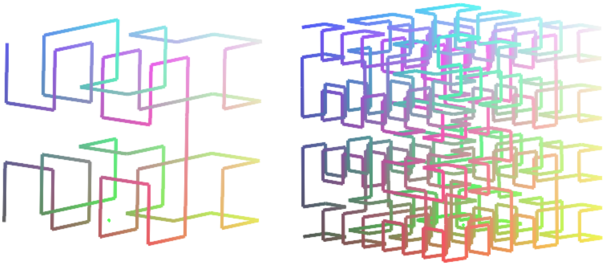

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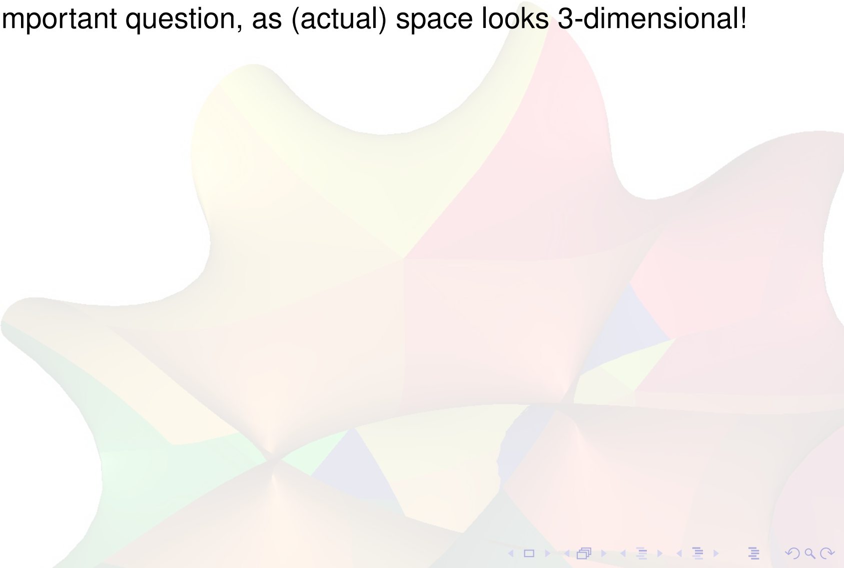
Invariance of dimension theorem (1912):
If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism (continuous bijection w/ continuous inverse) then $m = n$.

Shape of space: Large version

Having classified all 1 and 2-dimensional shapes...

- What are the possible shapes of 3-dimensional spaces?

An important question, as (actual) space looks 3-dimensional!



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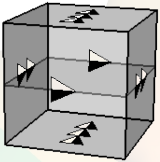

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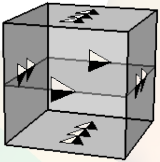
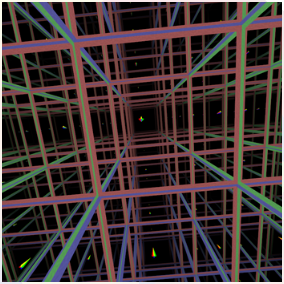
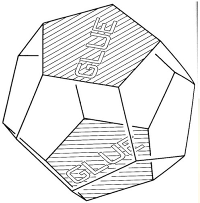




The *3-torus*.

Shape of space: Large version

Why stop at cubes?



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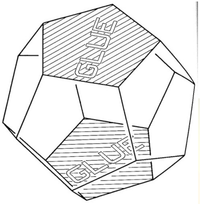

Poincaré dodecahedral space

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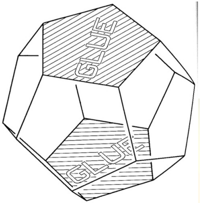

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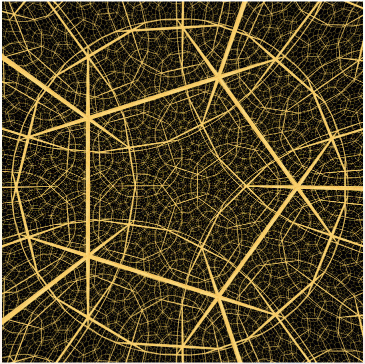

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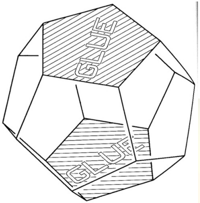

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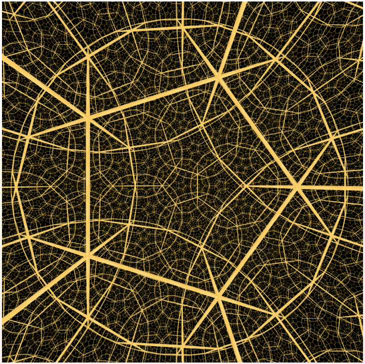

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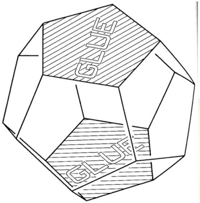

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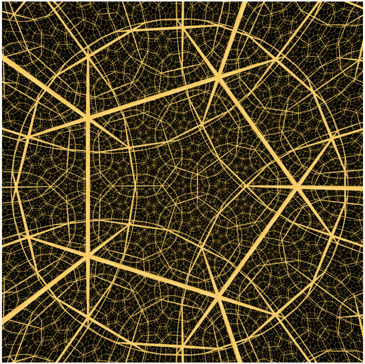

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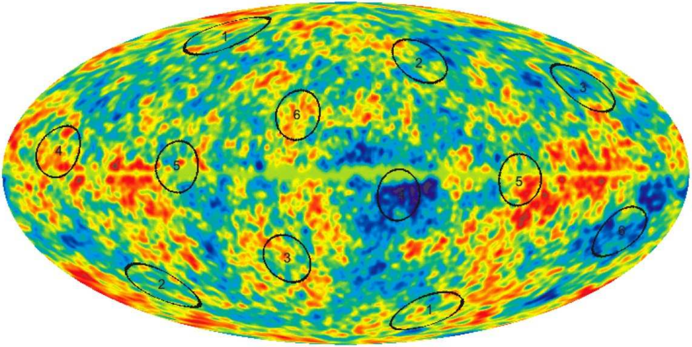
Several studies of *Cosmic Microwave Background Radiation*.



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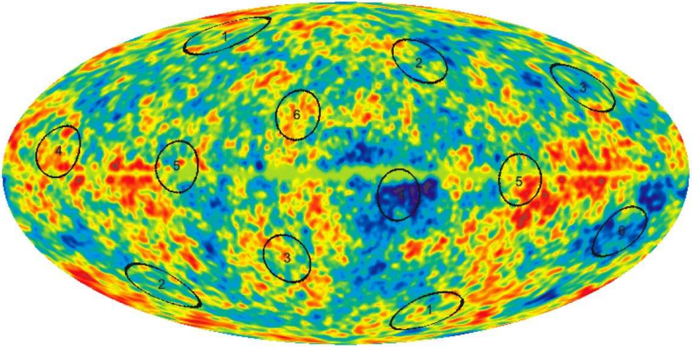

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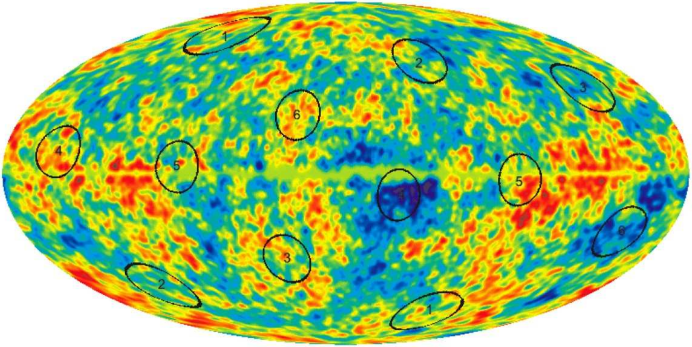

... but evidence is weak.

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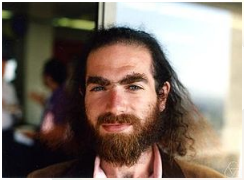

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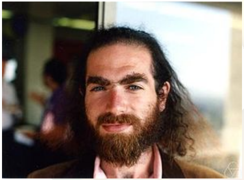

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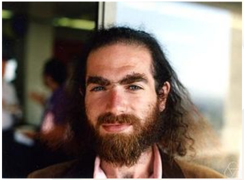

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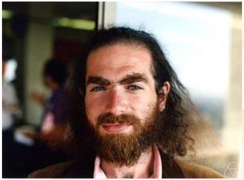

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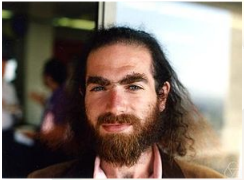

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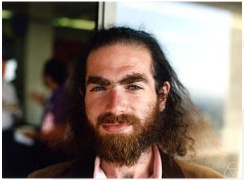

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Shape of space: Small version

At the other end of the length scale... *Superstring theory.*

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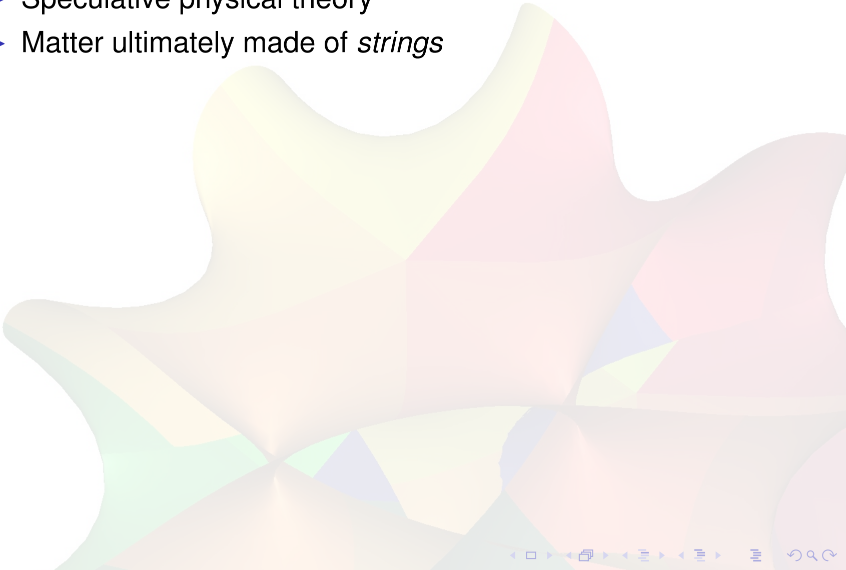

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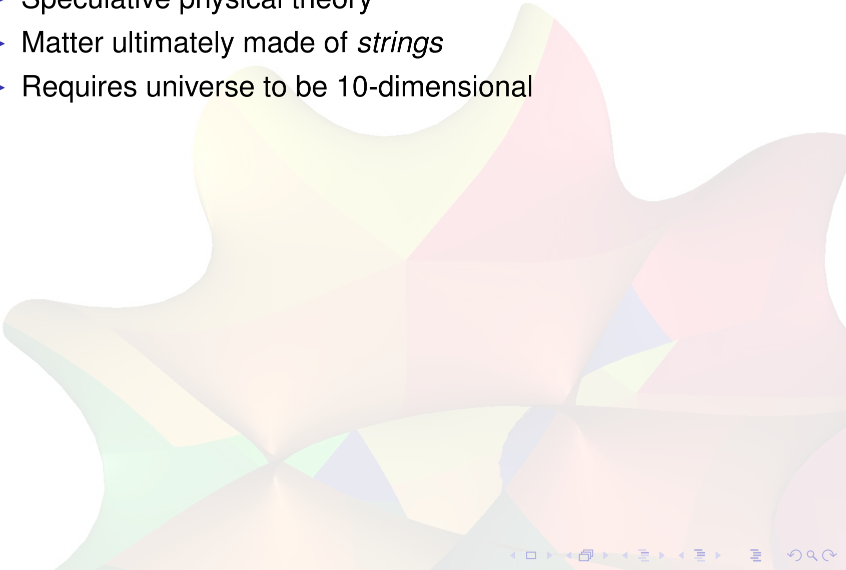

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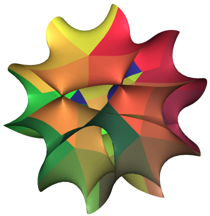

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Useful to understand *all possible shapes* of a space:

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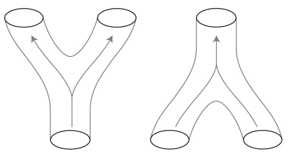

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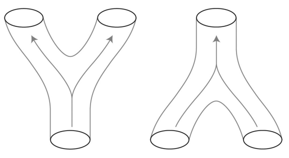

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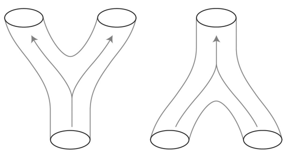

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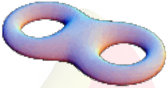

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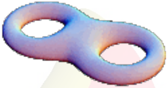

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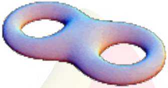

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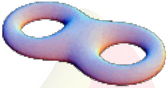

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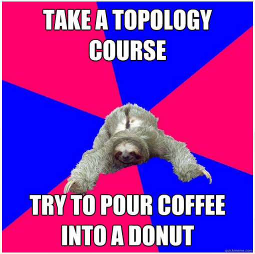
- Computed volumes of all conformal moduli spaces
- Fields medal last week.

Thanks for listening!



Daniel.Mathews@monash.edu

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