Strings, fermions and the topology of curves on surfaces

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Outline

Curves on surfaces

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Related to various other important fields:

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- Lie bialgebras and quantization
- String topology — topology of loop spaces
- Symplectic/contact geometry — symplectic field theory, Floer homology

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But the construction itself is very elementary.

String diagrams on marked surfaces

Definition

A marked surface is a pair (Σ, F) where

1. *Σ is a compact oriented surface with nonempty boundary*
2. *F is a set of $2n \geq 0$ distinct points on $\partial\Sigma$, with n points labelled "in" and n points labelled "out".*

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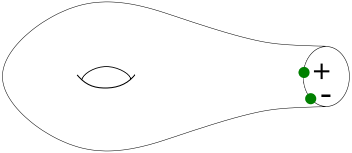

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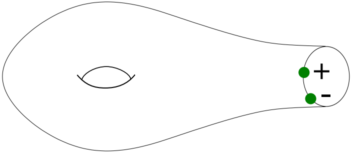

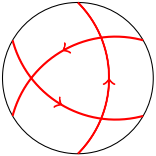

Definition

A string diagram s on (Σ, F) is an immersed oriented compact 1-manifold in Σ such that $\partial s = F$, with all self-intersection in the interior of Σ.

A chain complex

Let $\mathcal{S}(\Sigma, F) = \{$homotopy classes of string diagrams on $(\Sigma, F)\}$
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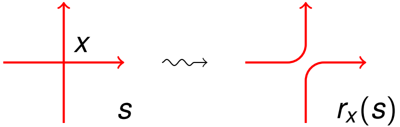

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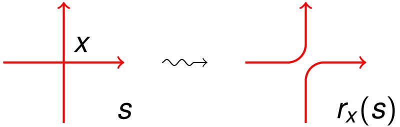

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But first...

- Why this chain complex?

(... apart from being a natural elementary construction...)

Outline

1. Introduction
2. Motivations and connections
   - Goldman bracket, Teichmüller space, and surface group representations
   - The Turaev cobracket
   - Symplectic field theory
3. The chain complex and its homology

Teichmüller space and group representations

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- Turns out

$$\mathcal{T}_g \subset \text{Hom}(\pi, G)/G \quad \text{(algebraic variety of flat connections).}$$

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Poisson bracket: Given two functions $F, G : M \longrightarrow \mathbb{R}$ we obtain another function $\{F, G\} = \omega(X_F, X_G)$.
Dual to Lie bracket of vector fields: $[X_F, X_G] = -X_{\{F,G\}}$.

Symplectic geometry of surface group representations

Goldman in 1984 showed that:

- $T_{[\rho]}\mathrm{Hom}(\pi, G)/G \cong H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho})$
- Group cohomology of π with coefficients in the Lie algebra $\mathfrak{g}$ of G, which is a π-module via $\pi \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathfrak{g}$.

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Theorem (Wolpert, 1982)

Let α be a simple loop on S_g and let $l_\alpha : \mathcal{T}_g \longrightarrow \mathbb{R}$ be the length of the geodesic homotopic to α. The corresponding Hamiltonian vector field X_{l_α} on $\mathcal{T}_g$ is the Fenchel-Nielsen twist flow about α.

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Goldman extended this to:

- general $\mathrm{Hom}\,(\pi, G)/G$ and $\alpha \in \pi$ (not just simple)
- general conjugation-invariant $G \longrightarrow \mathbb{R}$ (not just length).

The Goldman Lie bracket

The results of Wolpert & Goldman thus give maps

$$\zeta : \pi \longrightarrow C^{\infty}\left(\mathcal{T}_g, \mathbb{R}\right), \quad \text{more generally} \longrightarrow C^{\infty}\left(\mathrm{Hom}\left(\pi, G\right)/G, \mathbb{R}\right).$$

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There is a Lie bracket on $\mathbb{Z}\widehat{\pi}$ such that the map $\zeta : \mathbb{Z}\widehat{\pi} \longrightarrow C^{\infty}(Hom(\pi, G)/G, \mathbb{R})$ is a Lie algebra homomorphism.

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- Our $\widehat{CS}(\Sigma, F)$ is a generalisation: multiple curves, endpoints, resolving self-intersections.

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A *Lie coalgebra* *is an abelian group* $\mathfrak{g}$ *with a map* $\nu : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ *such that*

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where $\tau : \mathfrak{g}^{\otimes 3} \longrightarrow \mathfrak{g}^{\otimes 3}$ *and* $\sigma : \mathfrak{g}^{\otimes 2} \longrightarrow \mathfrak{g}^{\otimes 2}$ *are order 3 and 2 permutations.*

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A Lie bialgebra is a Lie algebra and coalgebra such that

$$\nu([a, b]) = a\nu(b) - b\nu(a).$$

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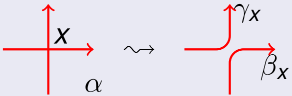

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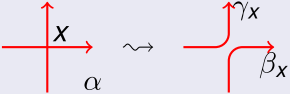

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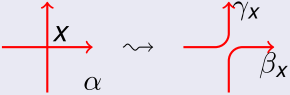

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Theorem (Turaev 1991)

With the Goldman bracket and Turaev cobracket, $\mathbb{Z}\hat{\pi}$ is a Lie bialgebra.

Multiple curves and symmetric algebra

From any abelian group $\mathfrak{g}$ we can form the symmetric algebra

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Theorem (Turaev 1991)

There is a Poisson algebra homomorphism $S(\mathbb{Z}\widehat{\pi}) \longrightarrow C^\infty(Hom(\pi, G)/G, \mathbb{R})$.

Symplectic field theory

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The essential idea:

Understand contact and symplectic manifolds by studying holomorphic curves in them, in the spirit of topological quantum field theory.

- Study moduli spaces of holomorphic curves in symplectic/contact manifolds.
- Develop algebraic machinery encoding structure of moduli spaces.

Includes Gromov-Witten theory, many previous symplectic invariants as special cases.

Contact/symplectic notions

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- Let q_i be coordinates on Q, corresponding fibre coordinates p_i.
- Tautological 1-form $\alpha = \sum_i p_i dq_i$.

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- Let q_i be coordinates on Q, corresponding fibre coordinates p_i.
- Tautological 1-form $\alpha = \sum_i p_i dq_i$.
- Symplectic form $\omega = d\alpha$.
- One end $\cong UT^*Q \times [0, \infty)$: unit cotangent bundle is contact.

Symplectic cobordisms

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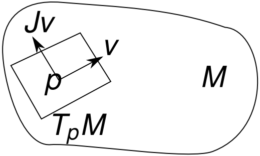
(For any contact (Y, λ), $Y \times \mathbb{R}$ has a natural symplectic structure $d(e^t \lambda)$.)

Holomorphic curves

Symplectic vs complex structures



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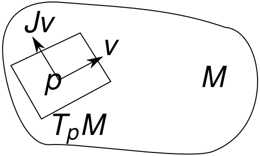
Almost complex structure: $J : TM \longrightarrow TM$ such that $J^2 = -1$.

(Generally non-integrable: $(M, J) \neq (\mathbb{C}^n, i)$ locally.)



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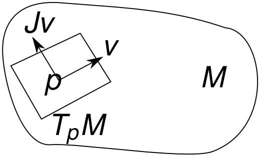
For a Riemann surface (S, i), $u : S \longrightarrow M$ is (pseudo-)holomorphic if $Du \circ i = J \circ Du$.



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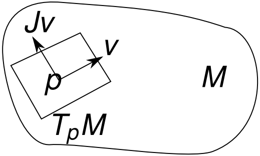
Thus

Moduli spaces exist : Prescribing sufficient data on holomorphic curves gives finite-dimensional moduli spaces, Riemann-Roch holds



Holomorphic curves in symplectic cobordisms

Consider a symplectic manifold (X, ω) of dimension $2n$ with:

- positive end $(\mathbb{R}_+ \times Y^+, \lambda^+)$, negative end $(\mathbb{R}_- \times Y^-, \lambda^-)$
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Consider collections of closed Reeb orbits:

- s^+ orbits in Y^+, $\Gamma^+ = (\gamma_1^+, \dots, \gamma_{s^+}^+)$
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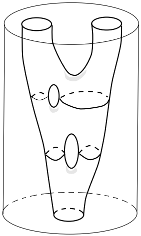
Definition

$$\mathcal{M}_g^A(X; \Gamma^-, \Gamma^+)$$

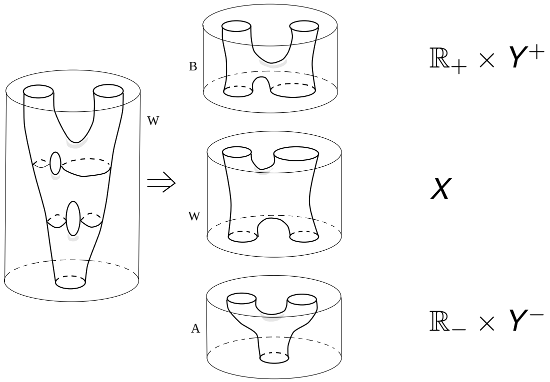
is the moduli space of connected genus g J-holomorphic curves in homology class $A \in H_2(M)$ with $s^+ + s^-$ punctures asymptotic to the $\gamma_i^\pm$.



(Source: Eliashberg-Givental-Hofer)

Moduli spaces

Eliashberg-Givental-Hofer considered the compactification of these moduli spaces: “multi-level" holomorphic curves including curves in the ends $\mathbb{R}_\pm \times Y^\pm$.



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Theorem (Eliashberg–Givental–Hofer)

The Hamiltonian $\mathbf{H} \in \frac{1}{\hbar}\mathcal{W}$ *satisfies*

$$ \mathbf{H} * \mathbf{H} = 0. $$

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Weyl algebras $\mathcal{W}^\pm$ act as differential operators on $\mathcal{D}$ via

$$q^+_\gamma \mapsto \kappa_\gamma \hbar \overleftarrow{\frac{\partial}{\partial p^+_\gamma}}, \quad p^-_\gamma \mapsto \kappa_\gamma \hbar \overrightarrow{\frac{\partial}{\partial q^-_\gamma}}.$$

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Theorem (Eliashberg-Givental-Hofer)

The potential $\mathbf{F} \in \frac{1}{\hbar}\mathcal{D}$ satisfies the master equation

$$e^{\mathbf{F}} \overleftarrow{\mathbf{H}^+} - \overrightarrow{\mathbf{H}^-} e^{\mathbf{F}} = 0.$$

Curves with boundary

Cieliebak–Latschev in 2007 generalised to consider

$$\mathcal{M}_{g,k}(X, Q; \Gamma^-, \Gamma^+)$$

curves in (X, ω) between $(Y^\pm, \lambda^\pm)$ with k boundary components on a Lagrangian submanifold Q.

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SFT formalism meets string topology

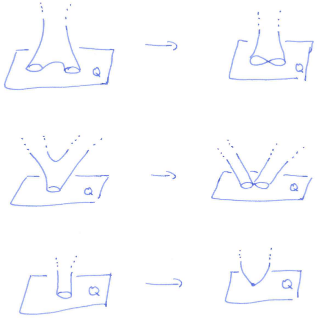
Theorem (Cieliebak–Latschev)

The potential $\mathbf{L}$ *satisfies the generalised master equation*

$$(\partial + \Delta + \hbar\nabla)\, e^{\mathbf{L}} = e^{\mathbf{L}}\overleftarrow{\mathbf{H}^{+}} - \overrightarrow{\mathbf{H}^{-}} e^{\mathbf{L}},$$

Here Δ, ∇ are string operations.

- Δ resolves a string at a self-intersection.
- ∇ glues two strings at an intersection.



(Source: Cieliebak-Latschev)

Codimension-1 phenomena at the boundary of holomorphic curves are Goldman bracket and Turaev cobracket.

Outline

Back to curves on surfaces

Recall our marked surfaces (Σ, F), string diagrams, and the chain complex:

Definition

$$\widehat{CS}(\Sigma, F) = \frac{\mathbb{Z}_2\langle \mathcal{S}(\Sigma, F)\rangle}{\mathbb{Z}_2\langle \mathcal{S}_C(\Sigma, F)\rangle}, \quad \partial s = \sum_{x \text{ crossing of } s} r_x(s).$$

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We see that ∂:

- reduces to the Goldman bracket $[\alpha, \beta]$ for two simple curves
- reduces to and Turaev cobracket $\nu(\alpha)$ for a single curve
- allows multiple curves: incorporates symmetric algebra of loops $S(\mathbb{Z}\hat{\pi})$ in Turaev's Poisson algebra homomorphism
- describes holomorphic curve boundary phenomena, as in Cieliebak–Latschev generalised master equation.

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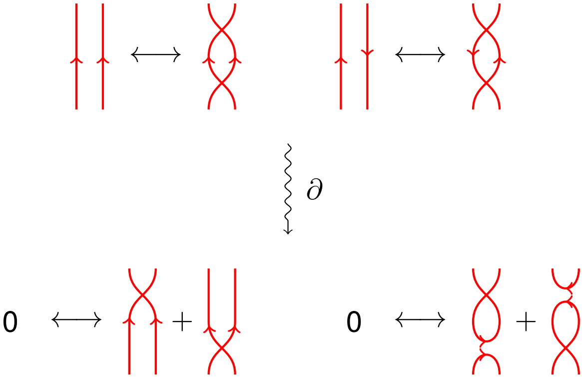

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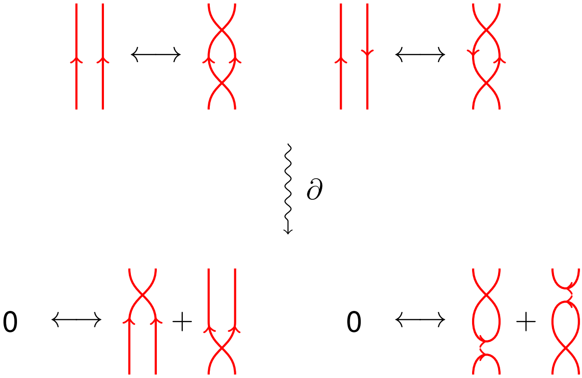
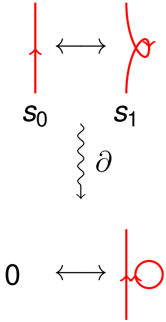


This shows why mod 2 is so useful...

Well-definition

And also...



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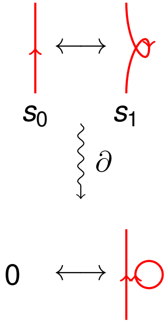

This shows why contractible strings must be set to zero.

Calculations of homology

Some results are known for discs and annuli:

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- Turns out that whether points F are *alternating* is important.

Theorem (M.)

If F is not alternating and

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Question

For any non-alternating F, is $\widehat{HS}(\Sigma, F) = 0$?

Non-alternating marked points

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If F has two consecutive points of the same sign, then $\widehat{HS}(\Sigma, F) = 0$.

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Proof

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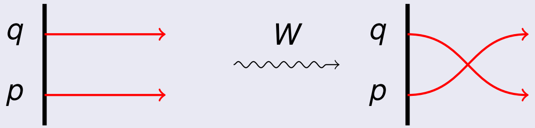

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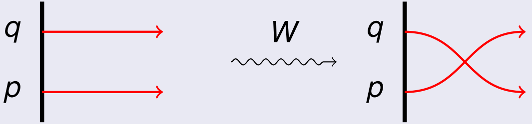

Now consider ∂Ws:

$\square$

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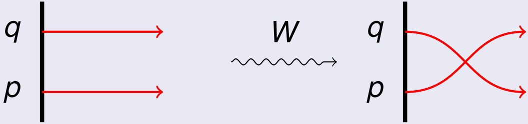

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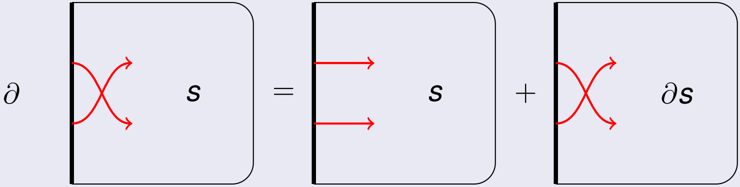


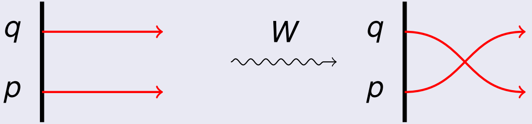
$$\partial W s = s + W \partial s$$

□

Non-alternating marked points

Proof.

$W : \widehat{CS}(\Sigma, F) \longrightarrow \widehat{CS}(\Sigma, F)$



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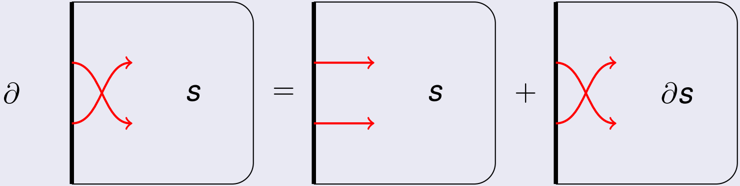

$$\partial Ws = s + W\partial s$$

Thus W is a chain homotopy from 1 to 0, and $\widehat{HS}(\Sigma, F) = 0$. □

String homology of discs

Theorem (M.–Schoenfeld)

For alternating F,

$$\widehat{HS}(D^2, F) \cong \frac{\mathbb{Z}_2\langle \text{Sutures on } (\Sigma, F)\rangle}{\text{Bypass relation}}$$

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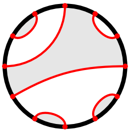
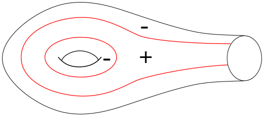
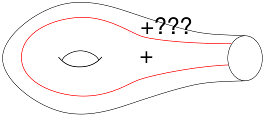
A set of sutures Γ on (Σ, F) is an embedded string diagram that splits Σ into alternating positive and negative regions.







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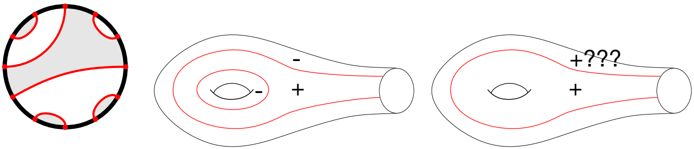
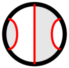


Sets of sutures only exist for *alternating* (Σ, F).

Bypass relation

If a set of sutures contains a disc which looks like



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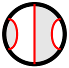

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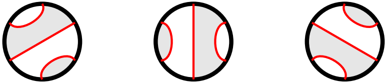

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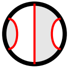

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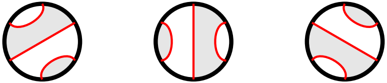

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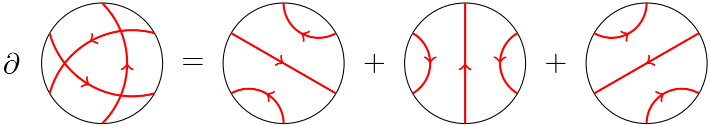
The bypass relation says bypass triples sum to zero.

$$\Gamma' + \Gamma + \Gamma'' = 0.$$

So $\widehat{HS}(D^2, F)$ is generated by string diagrams of sutures, modulo this relation.

Bypass relation as a boundary

A “reason" why the theorem is plausible:



Hence relations in $\widehat{HS}(\Sigma, F)$ are like the bypass relation.

Another relationship to holomorphic curves

Theorem (M.–Schoenfeld)

$$\widehat{HS}(D^2, F) \cong SFH(D^2 \times S^1, F \times S^1) \quad \left(\cong \frac{\mathbb{Z}_2 \langle \textit{Sutures on } (\Sigma, F) \rangle}{\textit{Bypass relation}} \right)$$

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($SFH(D^2 \times S^1, F \times S^1) \cong \frac{\text{sutures}}{\text{bypasses}}$ was known earlier.)

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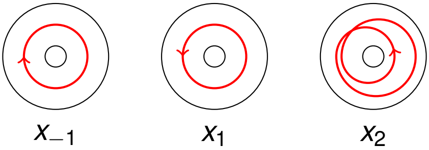
I.e. a polynomial ring in infinitely many variables squaring to 0. For each $n \in \mathbb{Z}$, x_n is the string which traverses the annulus n times; $\bar{x}_n$ its homology class.



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The result $H(\mathcal{X}) = \frac{\mathbb{Z}_2[\ldots,\bar{x}_{-3},\bar{x}_{-1},\bar{x}_1,\bar{x}_3,\ldots]}{(\ldots,\bar{x}_{-3}^2,\bar{x}_{-1}^2,\bar{x}_1^2,\bar{x}_3^2,\ldots)}$ is very “fermionic":

- only “odd spin" strings survive in homology
- in homology, two odd strings annihilate each other, $\bar{x}_{2j+1}^2 = 0$ (“Pauli exclusion principle")

Further computations

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Question

Can $\mathbb{M}$ be given an explicit presentation?

Adding more marked points is easy

While the $(2, 2)$ case is complicated, it never gets more complicated!

Theorem

For any $m, n \geq 0$,

$$\widehat{HS}(\mathbb{A}, F_{2m+2,2n+2}) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\otimes(m+n)} \otimes_{\mathbb{Z}_2} \mathbb{M}.$$

More work to be done... and connections to be drawn.

Thanks for listening!

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