Strings, fermions and the topology of curves on surfaces

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Curves on surfaces

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- Relations to several other fields.
- The construction itself is very elementary.

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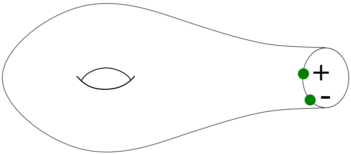
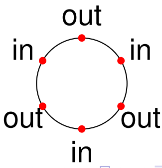
Definition

A marked surface is a pair (Σ, F) where

1. *Σ is a compact oriented surface with nonempty boundary*
2. *F is a set of $2n \geq 0$ distinct points on $\partial\Sigma$, with n points labelled “in" and n points labelled “out".*

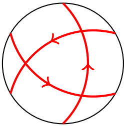




Curves on surfaces

Definition

A string diagram s on (Σ, F) is an immersed oriented compact 1-manifold in Σ such that $\partial s = F$, with all self-intersection in the interior of Σ.



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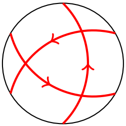

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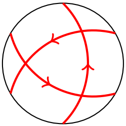

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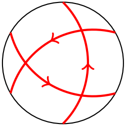

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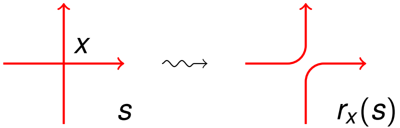
$$\widehat{CS}(\Sigma, F) = \frac{\mathbb{Z}_2\langle \mathcal{S}(\Sigma, F)\rangle}{\mathbb{Z}_2\langle \mathcal{S}_C(\Sigma, F)\rangle}.$$

I.e. Formal sums of string diagrams, setting contractible curves to zero.

Curves on surfaces

The differential on $\widehat{CS}(\Sigma, F)$ resolves intersections:



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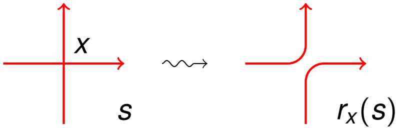

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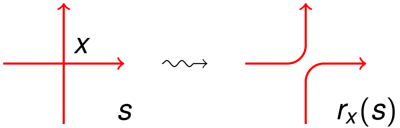

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Some questions immediately arise:

1. Is ∂ well defined?
2. Is $(\widehat{CS}(\Sigma, F), \partial)$ a chain complex?
3. If so, what is the string homology

$$\widehat{HS}(\Sigma, F) = \frac{\ker \partial}{\operatorname{Im} \partial} \; ?$$

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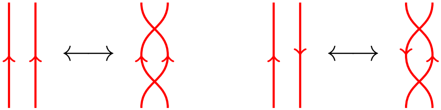

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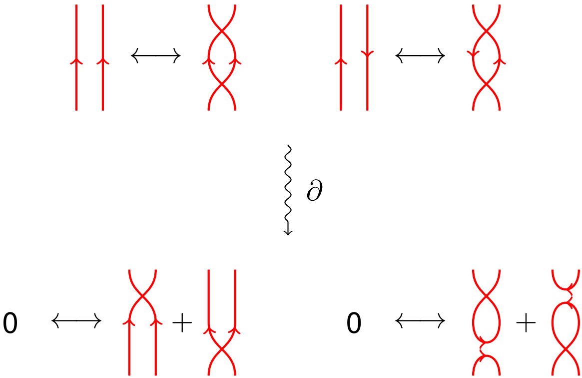

This shows why mod 2 is useful...

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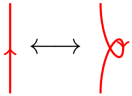

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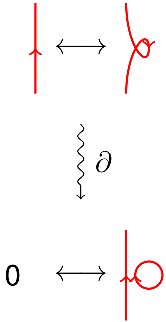

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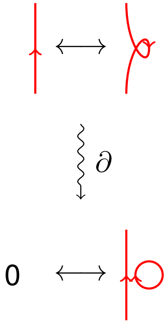

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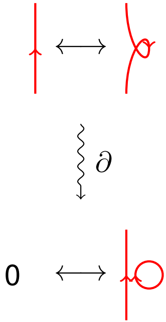

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Before discussing homology...

- Why this chain complex?

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Length thus gives a map

$$\zeta : \pi \longrightarrow C^\infty\left(\mathcal{T}_g, \mathbb{R}\right), \quad \alpha \mapsto l_\alpha.$$

Goldman Lie bracket and Turaev cobracket

Theorem (Goldman 1986)

There is a Lie bracket on $\mathbb{Z}\widehat{\pi}$ so that $\zeta : \mathbb{Z}\widehat{\pi} \longrightarrow C^\infty(\mathcal{T}_g, \mathbb{R})$ is a Lie algebra homomorphism.

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$$[\alpha, \beta] = \sum_{x \in \alpha \cap \beta} \operatorname{sgn}(x) r_x(\alpha, \beta) \quad \text{(resolving intersections)}$$

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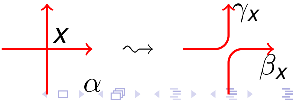
The Turaev cobracket $\nu : \mathbb{Z}\widehat{\pi} \longrightarrow \mathbb{Z}\widehat{\pi} \otimes \mathbb{Z}\widehat{\pi}$ is defined by

$$\nu(\alpha) = \sum_{x \text{ crossing of } \alpha} \beta_x \otimes \gamma_x - \gamma_x \otimes \beta_x$$



Multiple curves and symmetric algebra

From any abelian group $\mathfrak{g}$ we can form the symmetric algebra

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Goldman bracket extends to $S(\mathbb{Z}\widehat{\pi})^{\otimes 2} \longrightarrow S(\mathbb{Z}\widehat{\pi})$ and behaves *like a derivation.*

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Theorem (Turaev 1991)

$S(\mathfrak{g})$ forms a Poisson algebra and there is a Poisson algebra homomorphism $S(\mathbb{Z}\hat{\pi}) \longrightarrow C^\infty(\mathcal{T}_g, \mathbb{R})$.

Back to the chain complex

Return to consider our chain complex on marked (Σ, F):

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So ∂:

- reduces to Goldman bracket for two simple curves
- reduces to Turaev cobracket for a single curve

but generalises to multiple curves and arcs.

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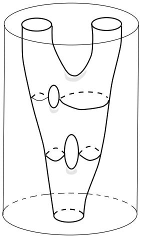

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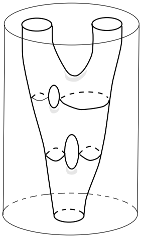

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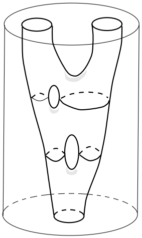

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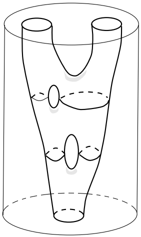

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Theorem (Eliashberg-Givental-Hofer)

Master equation: $e^{\mathbf{F}}\overleftarrow{\mathbf{H}^{+}} - \overrightarrow{\mathbf{H}^{-}}e^{\mathbf{F}} = 0.$

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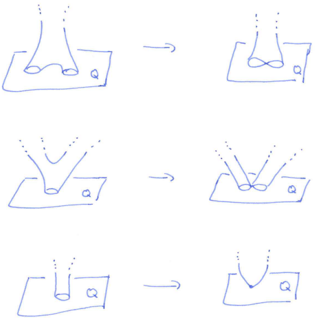

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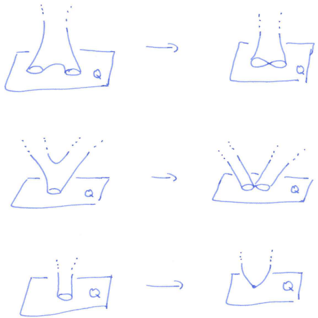
The $\widehat{CS}$ differential describes codimension-1 phenomena in moduli spaces of holomorphic curves.



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(Σ, F) is *alternating* *if the points of F alternate in, out, in out, ..., around each boundary component.*

Proposition (M.)

If F has two consecutive points of the same sign, then $\widehat{HS}(\Sigma, F) = 0$.

Proof

Consider the switching operation on string diagrams

$$W : \widehat{CS}(\Sigma, F) \longrightarrow \widehat{CS}(\Sigma, F).$$

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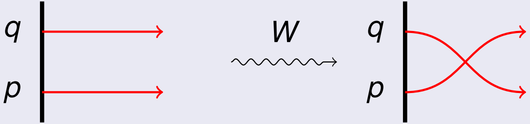

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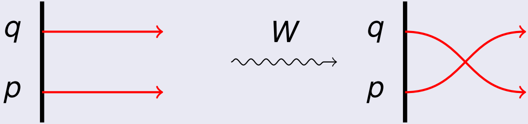

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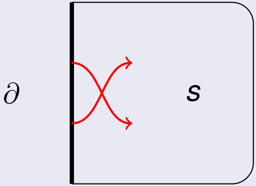

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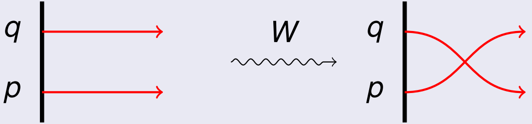

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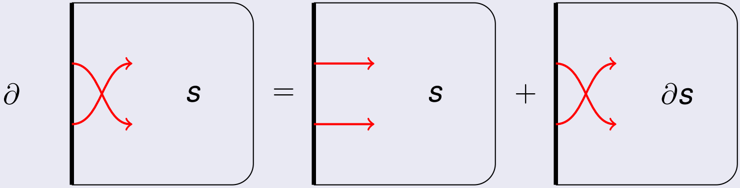

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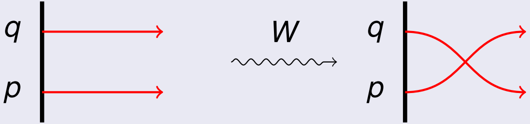

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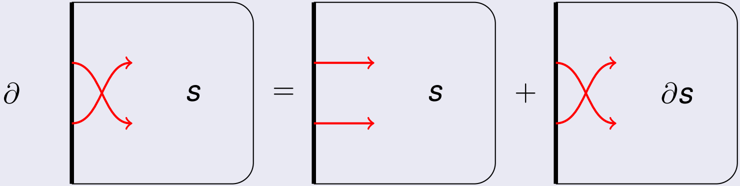

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Thus W is a chain homotopy from 1 to 0, and $\widehat{HS}(\Sigma, F) = 0$.

String homology of discs

Definition

A *set of sutures* Γ *on* (Σ, F) *is an* *embedded* *string diagram that splits* Σ *into alternating* *positive* *and* *negative* *regions.*

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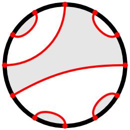

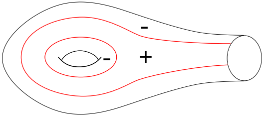

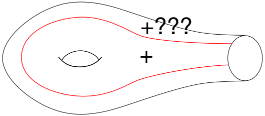

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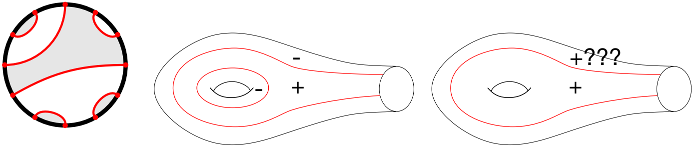

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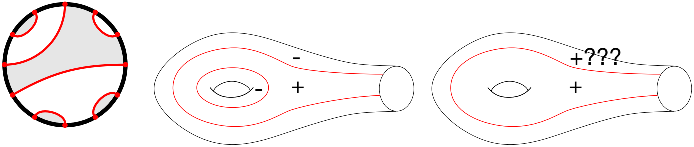

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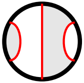

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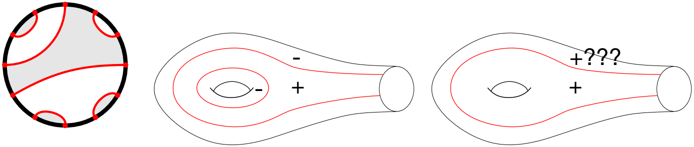

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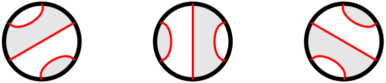

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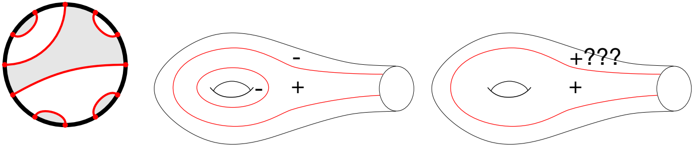

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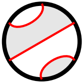
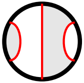
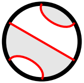
 $+$  $+$  $= 0$

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The bypass relation says bypass triples sum to zero.

String homology of discs

Theorem (M.–Schoenfeld)

For alternating F, $\widehat{HS}(D^2, F) \cong \dfrac{\mathbb{Z}_2\langle \textit{Sutures on } (D^2, F)\rangle}{\textit{Bypass relation}}$

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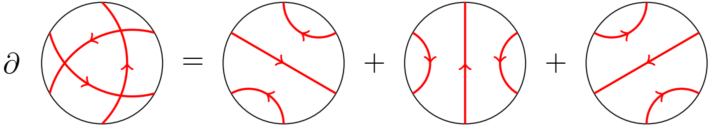

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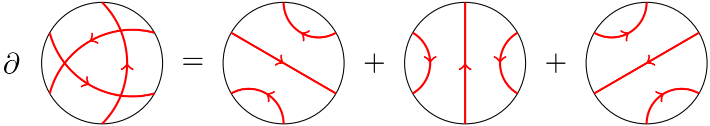

Also... another relationship to holomorphic curves...

Theorem (M.–Schoenfeld)

$\widehat{HS}(D^2, F) \cong SFH(D^2 \times S^1, F \times S^1)$

SFH = sutured Floer homology, an invariant of sutured 3-manifolds defined by counting holomorphic curves.

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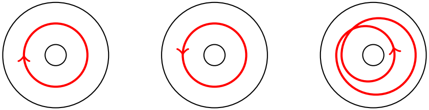

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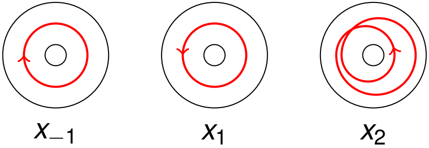

We see

$$\widehat{CS}(\mathbb{A}, \emptyset) = \mathbb{Z}_2[\dots, x_{-2}, x_{-1}, x_1, x_2, \dots]$$
$$\partial x_n = x_1 x_{n-1} + x_2 x_{n-2} + \cdots + x_{n-1} x_1 .$$

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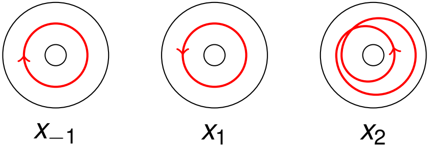

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and Leibniz rule holds.
Over $\mathbb{Z}_2$ most terms cancel:

$$\partial x_{2k} = x_k^2, \quad \partial x_{2k+1} = 0.$$

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Theorem (M.)

$$\widehat{HS}(\mathbb{A}, \emptyset) = \frac{\mathbb{Z}_2[\ldots, \bar{x}_{-3}, \bar{x}_{-1}, \bar{x}_1, \bar{x}_3, \ldots]}{(\ldots, \bar{x}_{-3}^2, \bar{x}_{-1}^2, \bar{x}_1^2, \bar{x}_3^2, \ldots)}.$$

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This ring is very “fermionic":

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$$\widehat{HS}(\mathbb{A}, F_{2m+2,2n+2}) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\otimes(m+n)} \otimes_{\mathbb{Z}_2} \widehat{HS}(\mathbb{A}, F_{2,2}).$$

- Adding alternating marked points corresponds to tensor product with $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

Thanks for listening!

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