Contact topology and holomorphic invariants via elementary combinatorics

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Outline

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- Much of it is quite involved, but *in the simplest cases* some of this structure reduces to some *elementary combinatorics and algebra* which is interesting in its own right.

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  - “String homology"

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  - Chord diagrams
  - “String homology"
- Explain how this elementary combinatorics arises from holomorphic invariants.

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Symplectic manifolds

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where

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Such M must be even-dimensional.

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Structure of Hamiltonian classical mechanics:

- Given a smooth function $H : M \longrightarrow \mathbb{R}$ (Hamiltonian) we obtain a 1-form dH and a dual vector field X_H via

$$\omega(X_H, \cdot) = dH$$

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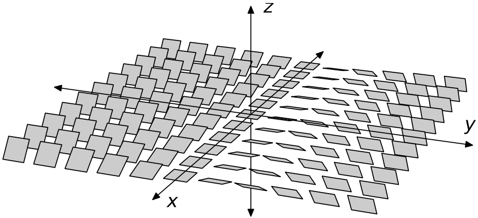

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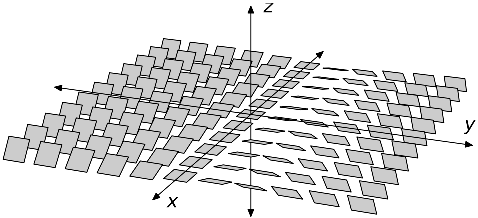

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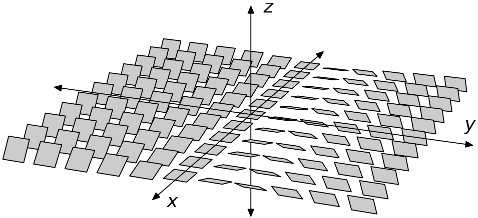

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- Non-integrability means $\alpha \wedge (d\alpha)^n \neq 0$ everywhere.
- (So $\alpha|_\xi$ is a symplectic form at each point.)

Contact geometry

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Today: $n = 1$, 3-dimensional contact geometry.

- Much of 3-dimensional contact geometry can be described combinatorially.

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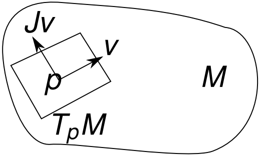
Definition

An almost complex structure on a smooth manifold is a map

$$J : TM \longrightarrow TM$$

preserving each fibre T_pM *and satisfying* $J^2 = -1$.



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Existence:

- Not every symplectic manifold has a complex structure...
- ... but every symplectic manifold has a compatible almost complex structure J, and all choices of J are homotopic.

(Compatible: J and ω behave in linear algebra like i and $dx \wedge dy$. $\omega(v, w) = \omega(Jv, Jw)$ and $\omega(v, Jv) > 0$)

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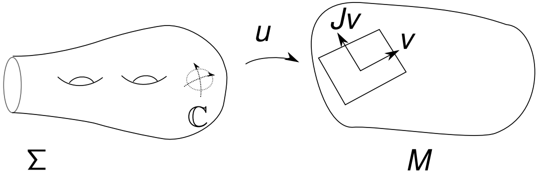
A holomorphic curve is a map $u : \Sigma \longrightarrow M$, where Σ is a Riemann surface, satisfying the Cauchy-Riemann equations

$$Du \circ i = J \circ Du.$$



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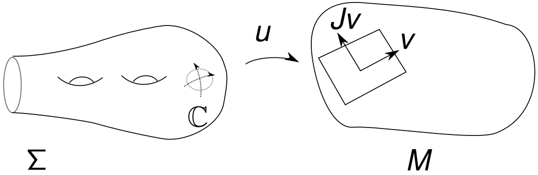

An *almost* complex structure is sufficient for the C-R equations.

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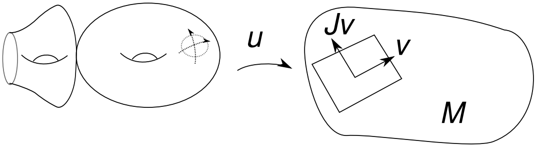

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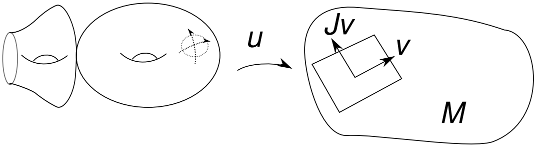

- $\mathcal{M}$ and $\overline{\mathcal{M}}$ encode a great deal of information about M.
- Some powerful invariants use only the *codimension-1 boundary* of $\overline{\mathcal{M}}$.

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Floer Homology theories (e.g. contact homology, Heegaard Floer homology), roughly...

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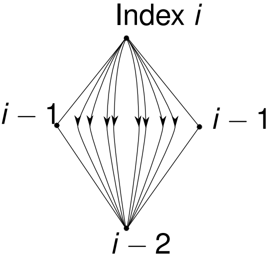
Analogy: singular homology via Morse complex.

- Complex generated by critical points of Morse function f.
- ∂ counts 0-dimensional families of trajectories of ∇f.



The power of holomorphic invariants

Floer homology theories give very powerful invariants of 3-manifolds, knots, etc...

- Related to Seiberg–Witten theory, Donaldson–Thomas theory, etc...
- E.g., *knot Floer homology* can compute the genus of a knot.

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- E.g., *knot Floer homology* can compute the genus of a knot.
- For a *less complicated* variant called *sutured Floer homology*, and a *simple class* of manifolds $M = \Sigma \times S^1$, we obtain all the combinatorial structure we are about to see, and more...

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Quantum Pawn Dynamics

- Pawns on a finite 1-dimensional chessboard.
- A state of the QPD universe:

$w =$ 

- Pawns move from left to right, one square at a time. (No capturing, no en passant, no double first moves.)

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Quantum pawns: "Inner product" $\langle \cdot | \cdot \rangle$ describes the possibility of pawn moves from one state to another.

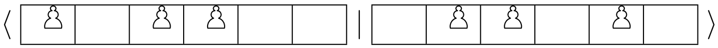
- Valued in $\mathbb{Z}_2$.

Definition (Pawn "inner product")

$$\langle w_0 | w_1 \rangle = \begin{cases} 1 & \textit{if it is possible for pawns to move from } w_0 \textit{ to } w_1 \\ & \quad \textit{(this includes the case } w_0 = w_1\textit{);} \\ 0 & \textit{if not.} \end{cases}$$

Quantum Pawn Dynamics

E.g.

 $= 1$

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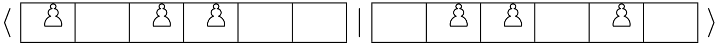

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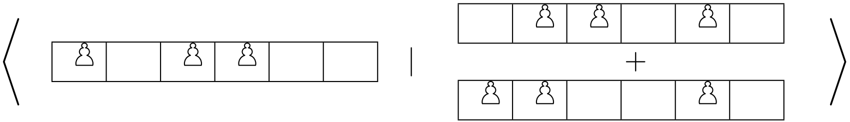
Also, entangled chessboards.



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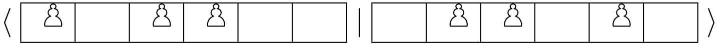

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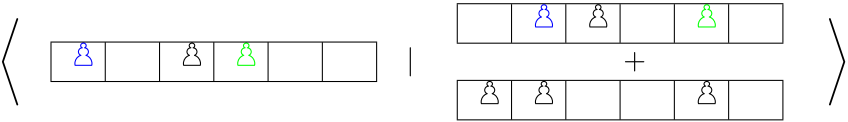

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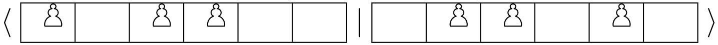

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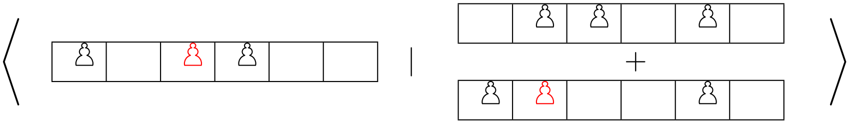

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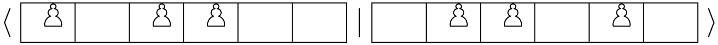

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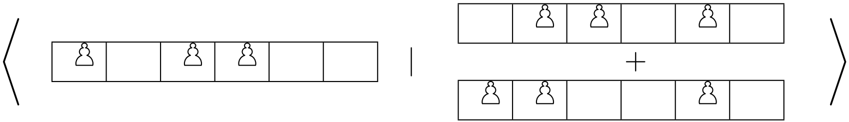

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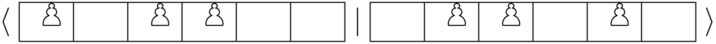

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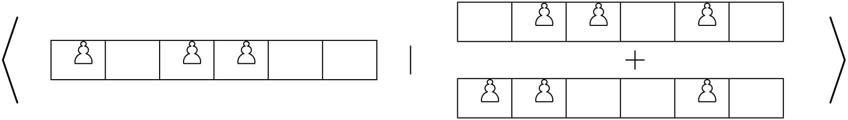

$= 1 + 0 = 1.$

Note asymmetry of $\langle\cdot|\cdot\rangle$.
A "booleanized" partial order. (Complete lattice.)

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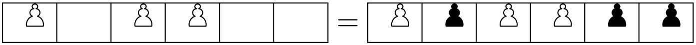

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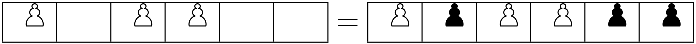

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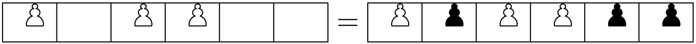

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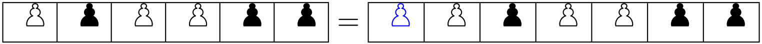
- Let
  - n_p = number of pawns
  - n_q = numbers of anti-pawns
  - n = number of squares on board $= n_p + n_q$

Creation and annihilation operators

The *initial pawn creation operator* $a^*_{p,0}$ adjoins a new *initial* (leftmost) square to the chessboard, containing a pawn.

$a^*_{p,0}$ 

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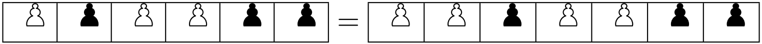
$a^*_{p,0}$ 

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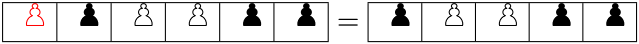
$a_{p,0}$ 

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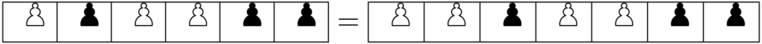
$a^*_{p,0}$ 

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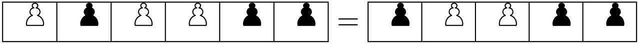
$a_{p,0}$ 

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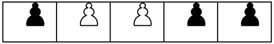
$a_{p,0}$  $= 0$

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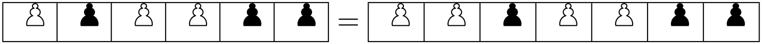
$a^*_{p,0}$ 

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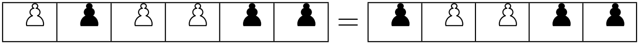
$a_{p,0}$ 

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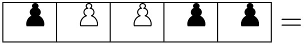
$a_{p,0}$  $= 0$

Similar initial anti-pawn annihilation $a_{q,0}$ and creation $a^\dagger_{q,0}$.

Creation of chessboards

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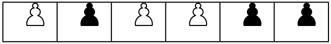

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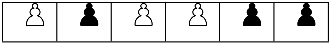

- The $*$ and $\dagger$ refer to *adjoints* — just as in quantum field theory.
  (Actually they form a *Galois connection* on partial orders.)

Adjoints

- Recall an adjoint f^* of an operator f usually means that

$$\langle fx|y\rangle = \langle x|f^*y\rangle, \quad \langle x|fy\rangle = \langle f^*x|y\rangle.$$

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$$\langle fx|y\rangle = \langle x|f^* y\rangle, \quad \langle x|fy\rangle = \langle f^\dagger x|y\rangle.$$

So $f^{*\dagger} = f^{\dagger *} = f$.

Initial creation and annihilation are adjoint

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Similarly, initial anti-pawn creation/annihilation †-adjoint.

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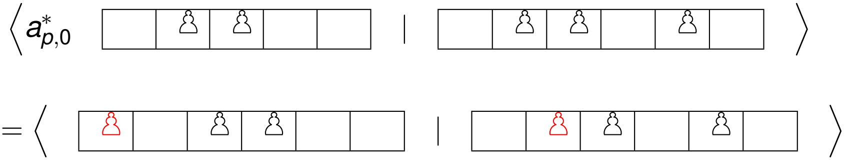

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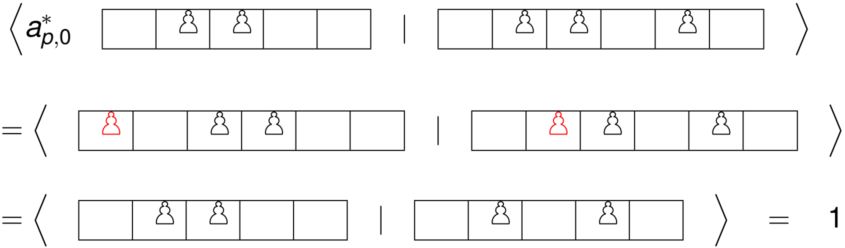

Call this operator $a_{p,1}$.

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Keep going. What is $a_{p,0}^{***} = a_{p,1}^{*}$? What operator g satisfies

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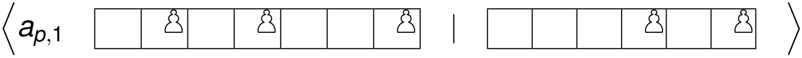

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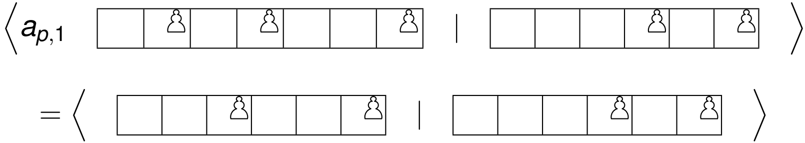

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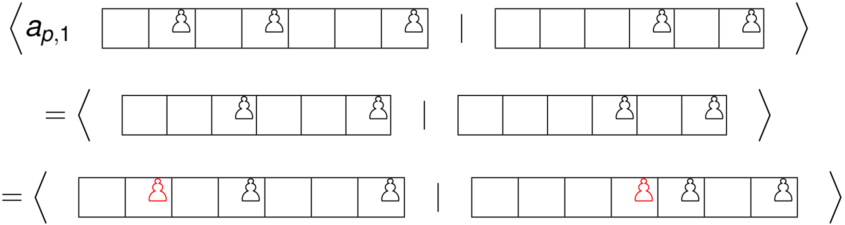

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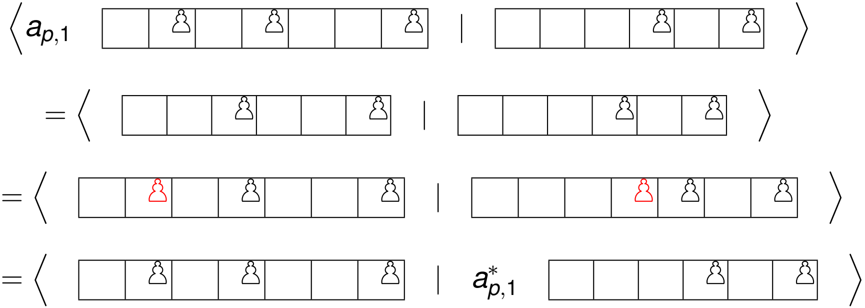

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The iterated adjoints of $a_{p,0}$ are

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Similarly for anti-pawns in the opposite direction.

$$a^\dagger_{q,\Omega} \to a_{q,\Omega} \to \cdots a_{q,2} \to a^\dagger_{q,1} \to a_{q,1} \to a^\dagger_{q,0} \to a_{q,0}$$

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(A *simplicial structure*.)

Adjoint periodicity

Hence

$$a_{p,0}^{*^{2n_p+2}} = a_{p,\Omega}$$

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One can also show that the *duality* operator defined by

$$\langle u|v\rangle = \langle v|Hu\rangle$$

satisfies

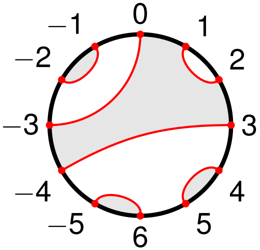
Theorem (M.)

$$H^{2n+2} = 1.$$

Outline

Chord diagrams

Consider a disc D with some $2n+2$ points F marked on ∂D.
A *chord diagram* is a collection of non-intersecting curves on D joining points of F.
E.g.



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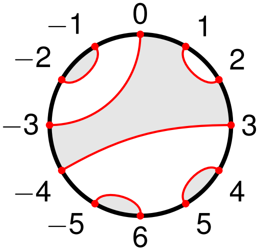

- Curves join points of opposite parity, so shade as shown.
- 0 is a basepoint.
- Label points mod $2n+2$.

Creation and annihilation of chords

Define *creation operators* $a^*_{p,i}$, $a^\dagger_{q,i}$ to insert a new chord in a specific place in a chord diagram as shown.

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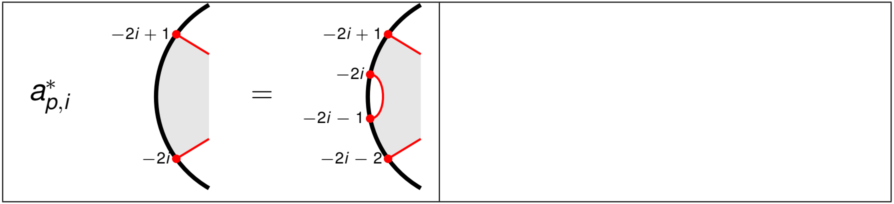

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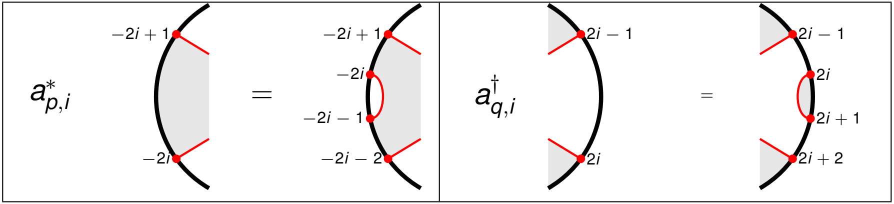

$a^*_{p,i}$ creates a *white* region i spots down on the left.
$a^\dagger_{q,i}$ creates a *black* region i spots down on the right.

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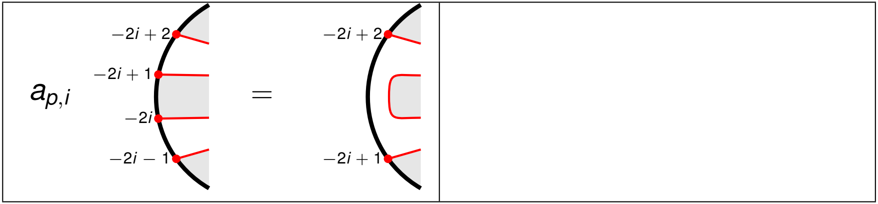

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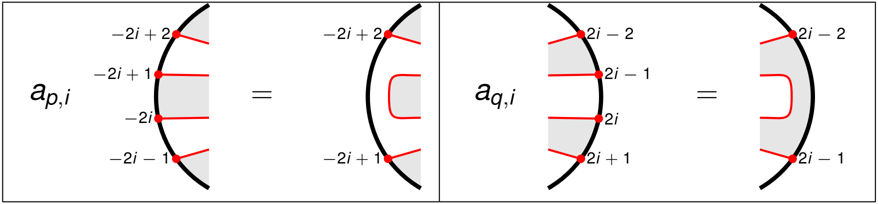

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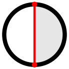
Diagrams of chessboards

The simplest chord diagram is called the *vacuum* $\Gamma_\emptyset$.



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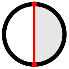

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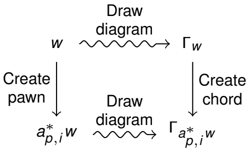
For any chessboard w, there is a chord diagram Γ_w (called a slalom chord diagram) such that creation and annihilation operators agree (are equivariant): $\Gamma_{a^*_{p,i}w} = a^*_{p,i}\Gamma_w$.



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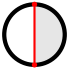

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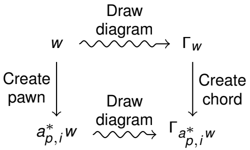

- If chessboard has n squares, i.e. $|w| = n$, then Γ_w has $n+1$ chords.

Ski slopes

Construction of the *slalom skiing* chord diagram of a chessboard.

$qpqq \quad \leftrightarrow$ 

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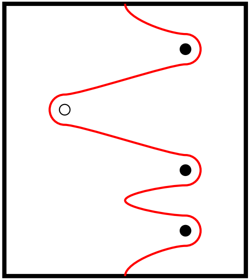
$\leftrightarrow$ 

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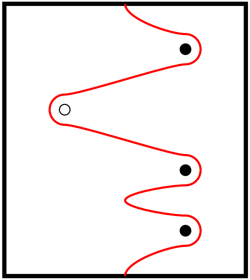
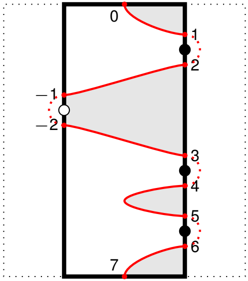
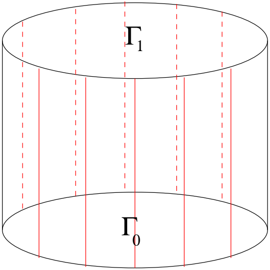
$\leftrightarrow$  $\leftrightarrow$ 

$= \Gamma_{qpqq}$

An “Inner product” on chord diagrams

There’s a bilinear form on chord diagrams defind by *entering into a cylinder*.



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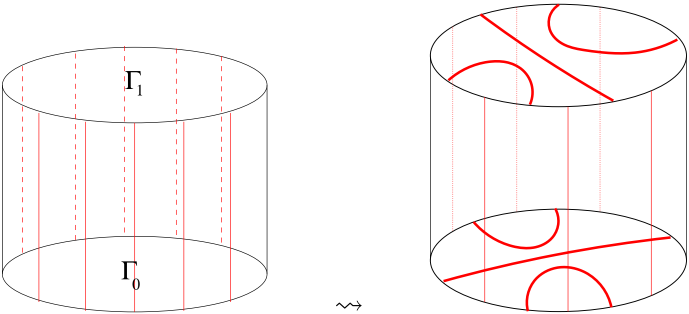

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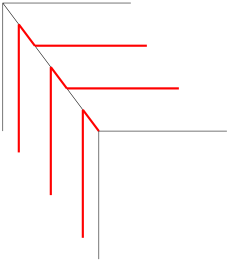

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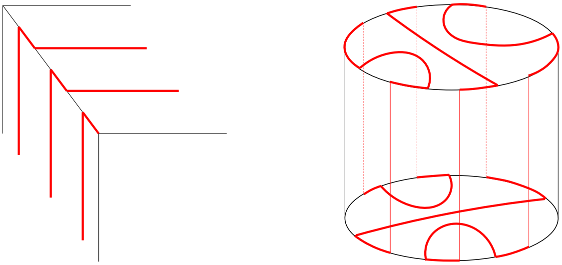

Definition

$$\langle \Gamma_0 | \Gamma_1 \rangle = \begin{cases} 1 & \textit{if the resulting curves on the cylinder form a single connected curve;} \\ 0 & \textit{if the result is disconnected.} \end{cases}$$

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For any two chessboards w_0, w_1,

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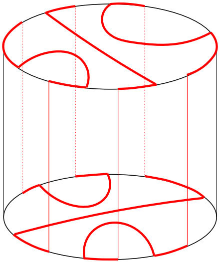
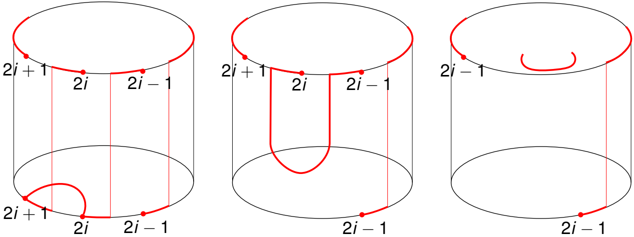


Adjoints

Adjoint relations can be seen *topologically* as “finger moves”.

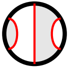


$$\langle a^\dagger_{q,i}\,\Gamma_0 \mid \Gamma_1\rangle \qquad = \qquad \langle \Gamma_0 \mid a_{q,i}\,\Gamma_1\rangle$$

Now perhaps believable that adjoint is periodic.

Bypass surgery

In a chord diagram on disc D, consider a sub-disc B as shown:



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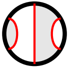
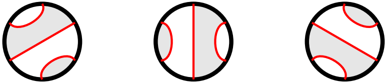


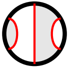
Two natural ways to adjust this chord diagram, consistent with the colours: *bypass surgeries*.



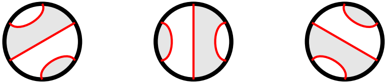
Γ' Γ Γ''

Bypass surgery

In a chord diagram on disc D, consider a sub-disc B as shown:



Two natural ways to adjust this chord diagram, consistent with the colours: *bypass surgeries*.



Γ' Γ Γ''

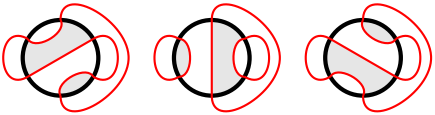
Proposition

With $\Gamma, \Gamma', \Gamma''$ *as above, for any* Γ_1*,*

$$\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma'' | \Gamma_1 \rangle = 0.$$

Bypass surgery

Idea of proof:



$$1 \quad + \quad 0 \quad + \quad 1 \quad = \quad 0$$

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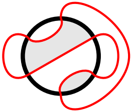 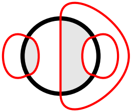 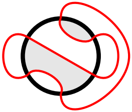

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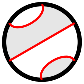
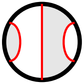
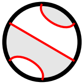
If $\langle \cdot | \cdot \rangle$ is to be nondegenerate, any three chord diagrams related by bypass surgery should sum to 0: *bypass relation*.

 +  +  = 0

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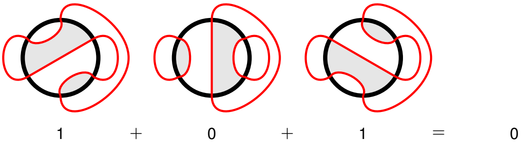

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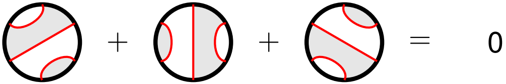

So we define a vector space

$$V_n = \frac{\mathbb{Z}_2\langle \text{Chord diagrams with } n+1 \text{ chords} \rangle}{\text{Bypass relation}}$$

A vector space of chord diagrams

Theorem (M.)

V_n has dimension 2^n and the slalom diagrams from chessboards of n squares form a basis.

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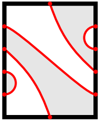

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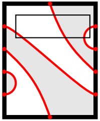

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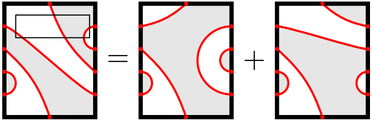

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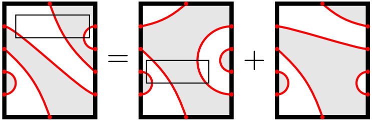

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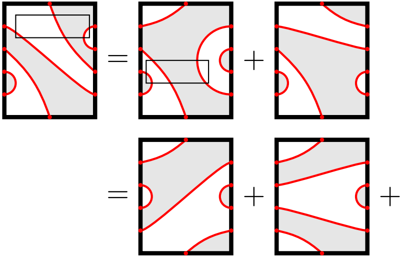

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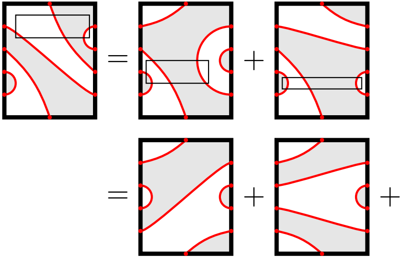

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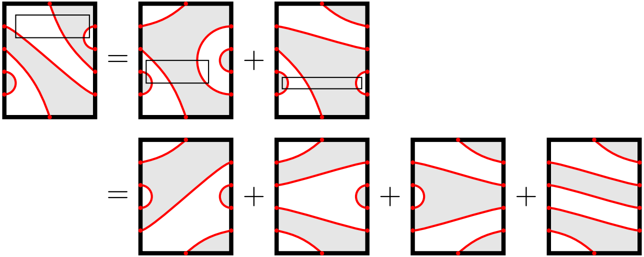

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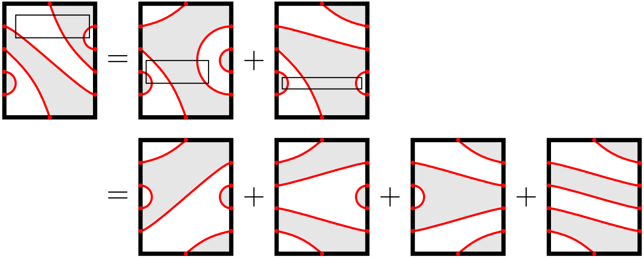
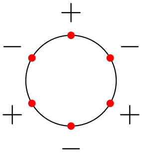


$$= \Gamma_{ppqq} + \Gamma_{pqqp} + \Gamma_{qppq} + \Gamma_{qpqp}$$

Outline

String diagrams

Return to a disc D with some $2n+2$ points F marked on ∂D.
The points of F are signed: half $+$, half $-$.



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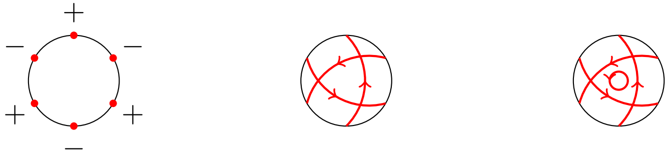

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Definition

A *string diagram* s *is a collection of oriented immersed curves on* D *with* $\partial s = F$.

The string complex

We define a chain complex based on string diagrams.

Definition

$$\widehat{CS}(D,F) = \frac{\mathbb{Z}_2\langle \textit{String diagrams on } (D,F)\rangle}{\mathbb{Z}_2\langle \textit{String diagrams with contractible curves}\rangle}$$

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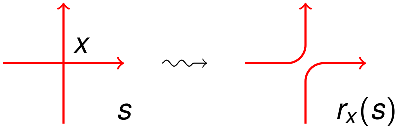

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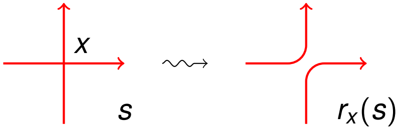

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$$\partial s = \sum_{x \textit{ crossing of } s} r_x(s).$$

Well-definition

Is ∂ well defined?
Show that ∂ is unchanged by “string Reidemeister moves". E.g.

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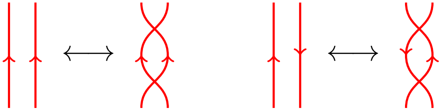

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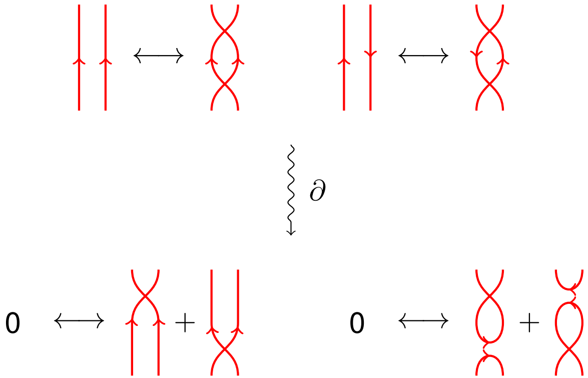
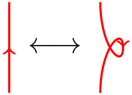


This shows why mod 2 is useful...

Well-definition

Also...



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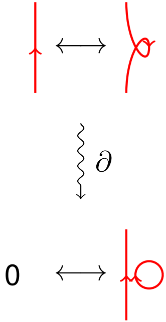

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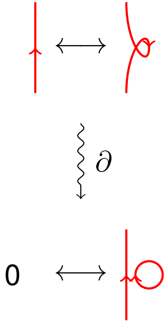

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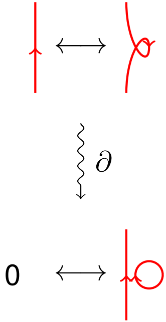

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What is the homology $\widehat{HS}(D, F)$?

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For alternating F, $\widehat{HS}(D, F)$ is generated by chord diagrams, and the bypass relation is satisfied. In fact,

$$\widehat{HS}(D^2, F) \cong \frac{\mathbb{Z}_2 \langle \text{Chord diagrams on } (D, F) \rangle}{\text{Bypass relation}} \cong V_n.$$

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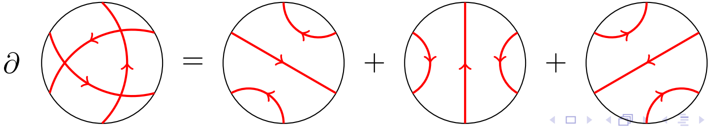
A “reason" why the theorem is plausible:



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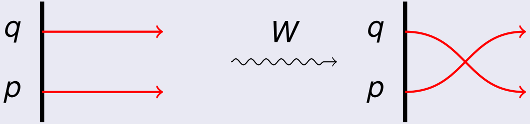

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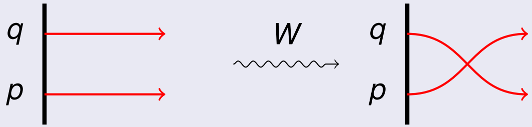

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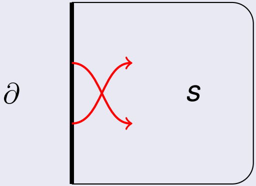

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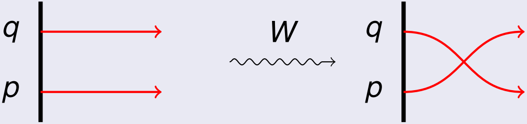

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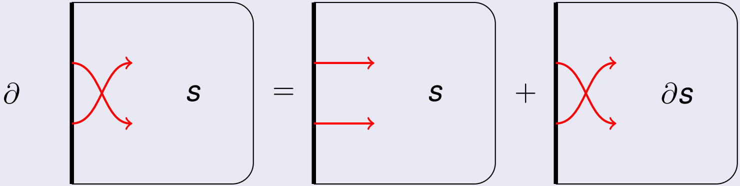

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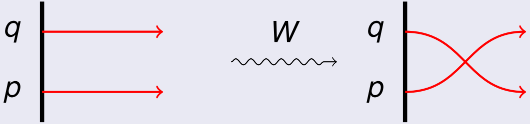
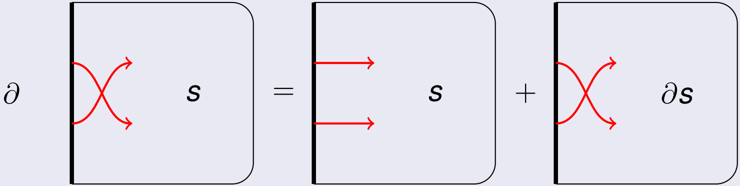


Now consider $\partial W s$:



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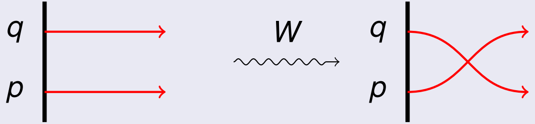

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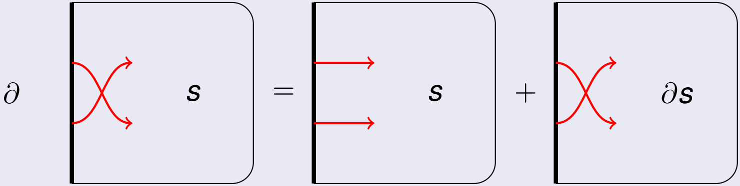

$$\partial Ws = s + W\partial s$$

Thus W is a chain homotopy from 1 to 0, and $\widehat{HS}(\Sigma, F) = 0$.

Outline

Sutured Floer homology

Actually all the above comes from *sutured Floer homology*, a holomorphic invariant of sutured manifolds.
Very roughly... (Ozsváth–Szabó 2004, Juhasz 2006)

- A *sutured manifold* is a 3-manifold M with boundary, and some curves Γ on ∂M dividing ∂M into alternating positive and negative regions.

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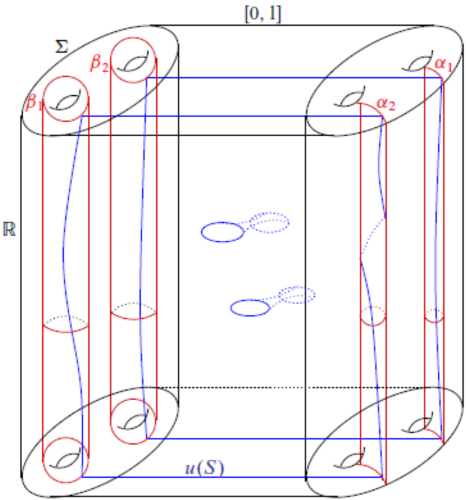
- Consider $\Sigma \times I \times \mathbb{R}$ as a symplectic manifold with an almost complex structure and consider holomorphic curves

$$u : S \longrightarrow \Sigma \times I \times \mathbb{R}$$

where S is a Riemann surface.
- Boundary conditions based on Heegaard curves α_i and β_i.

Sutured Floer homology

Cylindrical picture of Lipshitz (2006):



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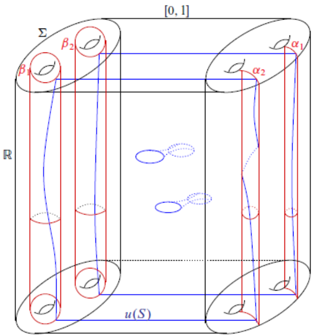


$$\operatorname{ind}(D\bar{\partial}) = k - \chi(S) + \sum_{i=1}^{k} \mu(a_i) - \sum_{i=1}^{k} \mu(b_i).$$

Sutured Floer homology

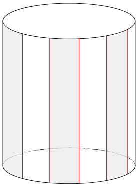
- Chain complex generated by boundary conditions, which are *intersections* of boundary curves.

$$z_1 \in \alpha_1 \cap \beta_{\sigma(1)}, \; z_2 \in \alpha_2 \cap \beta_{\sigma(2)}, \; \ldots, \; z_k \in \alpha_k \cap \beta_{\sigma(k)}.$$

- Differential counting index-1 holomorphic curves between boundary conditions.
- Resulting homology is $SFH(M, \Gamma)$.
- Etnyre–Honda (2009): Any *contact structure* ξ on (M, Γ) defines a natural *element* $c(\xi) \in SFH(M, \Gamma)$.

Solid tori

We consider the *sutured solid torus* $D^2 \times S^1$ with $2n+2$ longitudinal curves $F \times S^1$. ($|F| = 2n+2$)



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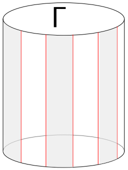
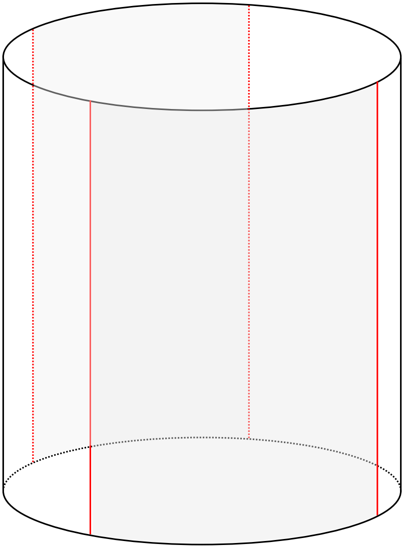


Theorem (M.)

$$SFH(D^2 \times S^1, F \times S^1) \cong V_n = \frac{\mathbb{Z}_2\langle \text{Chord diagrams w/ } n+1 \text{ chords}\rangle}{\text{Bypass relation}}$$
$$\cong \widehat{HS}(D^2, F)$$

Any chord diagram Γ *in* V_n *or* $\widehat{HS}(D^2, F)$ *corresponds to a a contact structure* ξ_Γ *on* $D^2 \times S^1$ *and maps to* $c(\xi_\Gamma)$.

A “computation” of Sutured Floer homology



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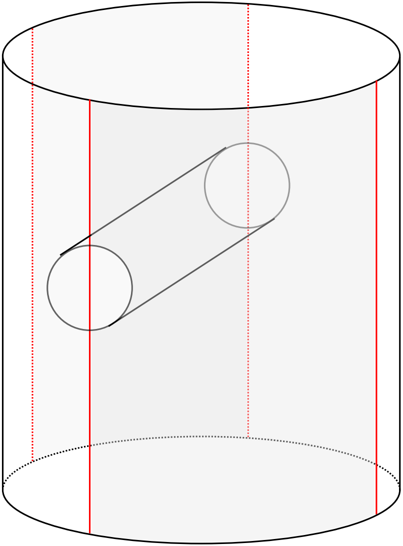

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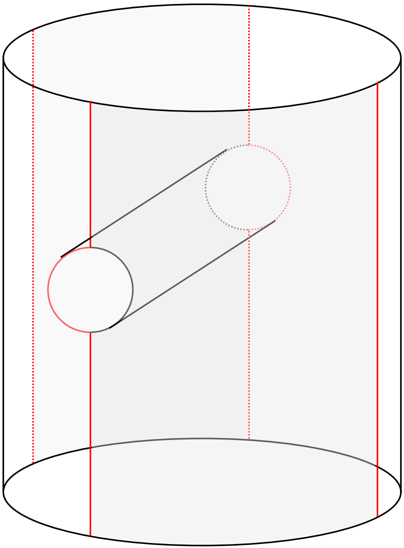

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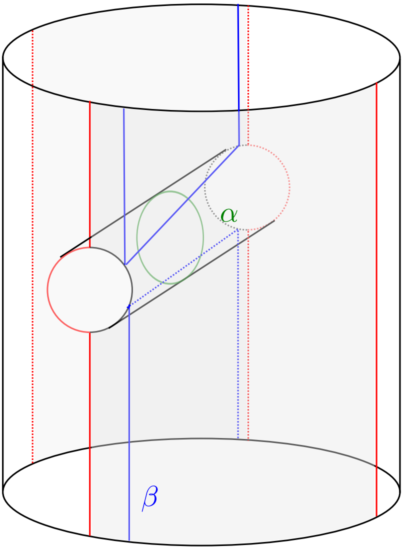

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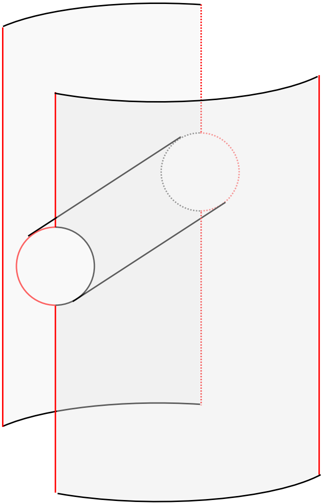

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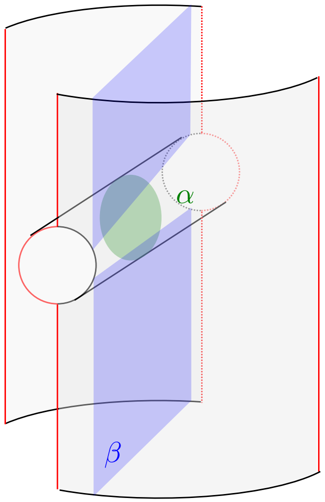

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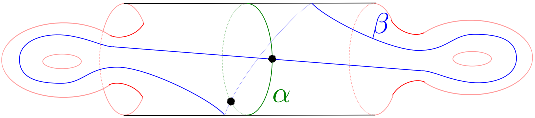

Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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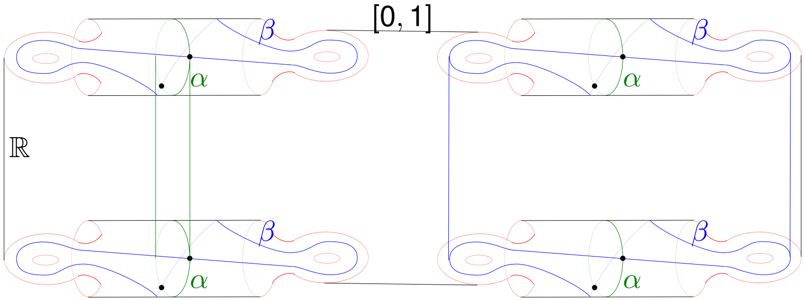

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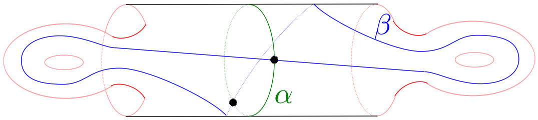

Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
Where do holomorphic curves go?
Nowhere for holomorphic curves to go! $\partial = 0$.

$$SFH = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = V_1$$

Chord diagrams and contact structures

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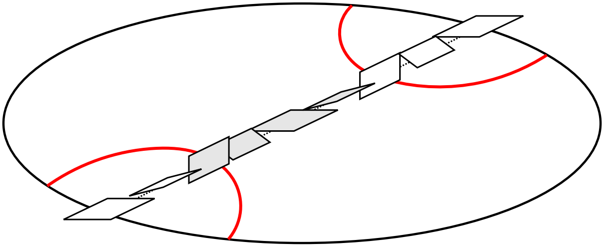

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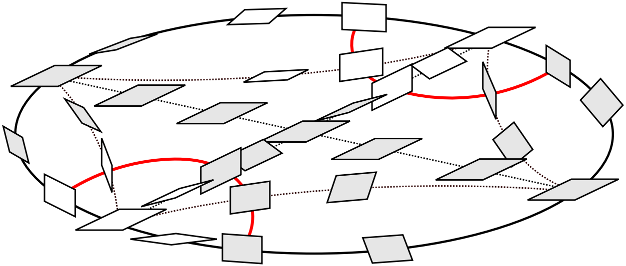

Colours in chord diagram = visible side of contact plane.

Overtwisted contact structures

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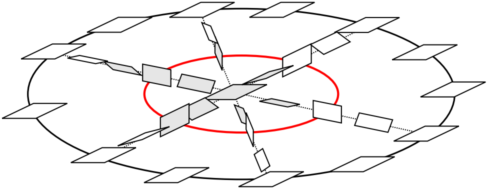

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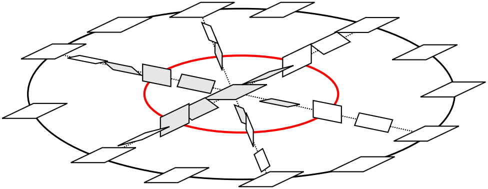

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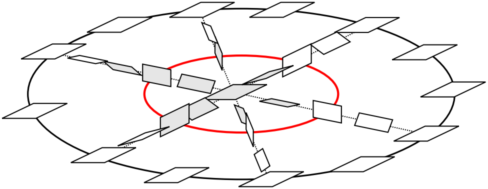
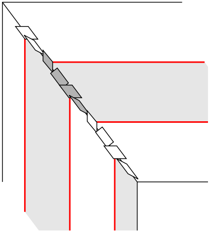


- Overtwisted contact geometry reduces to (well-understood) homotopy theory. Tight contact structures offer important topological information.
- Eliashberg (1992): contact structure near an S^2 is tight iff dividing set is *connected*. If so, contact structure extends uniquely (up to isotopy) to a tight contact structure on B^3.

Contact corners

When two convex surfaces meet along a boundary, contact planes are arranged as shown.



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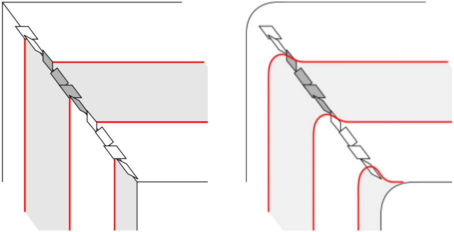


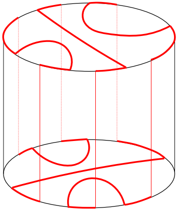
Proposition

Let Γ_0, Γ_1 be chord diagrams. The following are equivalent:

- $\langle \Gamma_0 | \Gamma_1 \rangle = 1$.
- *The solid cylinder with dividing set Γ_0 on the bottom and Γ_1 on the top has a tight contact structure.*

Bypasses

Honda (2000's): any 3-manifold can be built up from a surface and dividing set by adding *bypasses*.



Effect on dividing set is "bypass surgery" as defined earlier.

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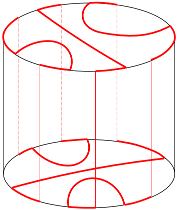
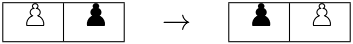


Effect on dividing set is “bypass surgery” as defined earlier. Corresponds to

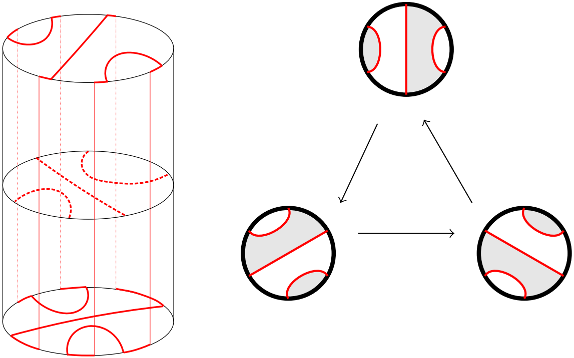
$$\langle \Gamma_{pq} | \Gamma_{qp} \rangle = 1$$

or



Bypasses

Stacking two bypasses on top of each other produces an overtwisted contact structure!



Can build something like a *triangulated category* out of dividing sets and contact structures (Honda, M.). V_n is the *Grothiendick group*.

Contact TQFT = Quantum pawn dynamics

These definitions give many of the properties of a (2+1)-dimensional *topological quantum field theory*.

- Contact structure near disc (2-dim) $\rightsquigarrow$ "states" in V_n
- Contact structure over cylinder (2+1-dim) $\rightsquigarrow$ element of $\mathbb{Z}_2$.
- "Possibility of a tight contact structure from one state to another" $\rightsquigarrow$ inner product $\langle \cdot | \cdot \rangle : V_n \otimes V_n \longrightarrow \mathbb{Z}_2$.

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Theorem (M.)

"Contact TQFT is isomorphic to quantum pawn dynamics."

Thanks for listening!

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