Counting Curves on Surfaces

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(joint with Norman Do and Musashi Koyama)

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Australian Mathematics Society Meeting
Flinders University
30 September 2015

“Counting curves" in mathematics

“Counting curves" is prominent in mathematics today, e.g.:

- *Moduli spaces* are families of complex curves (Riemann surfaces) satisfying certain conditions.

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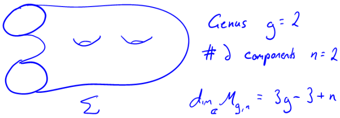

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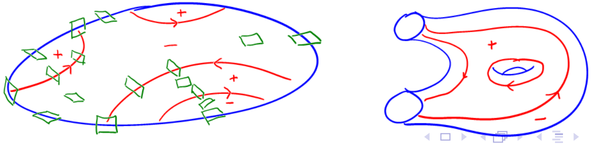
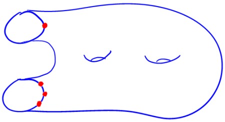
- *Contact structures* are enumerated by counting configurations of curves (*dividing sets*) on a surface.



A naive problem

(Joint work with N. Do & M. Koyama, in progress...)
Consider a compact orientable surface S with some points marked on the boundary ∂S.



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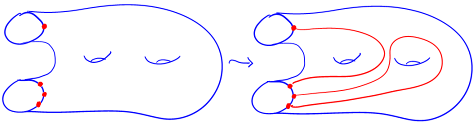

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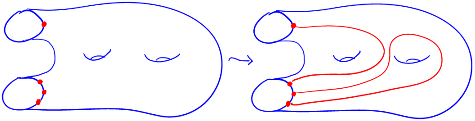

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How many ways are there to join these points with curves?

Many variants possible:

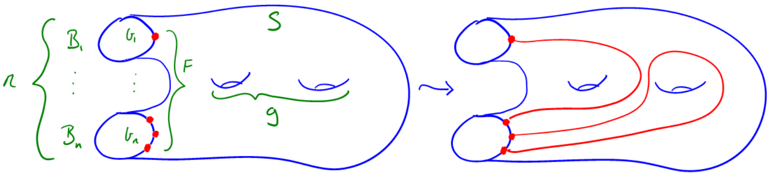
- Closed curves allowed? If so, which ones?
- What sets of curves are equivalent?
- Points signed? Curves oriented?

(Avoid answers of ∞.)

The problem

One way to set up the problem precisely.

- Let S be a compact orientable surface of genus g with n boundary components
- $B_1, \ldots, B_n$ be the boundary components of S
- $F \subset \partial S$ be a finite set with b_i points on B_i.



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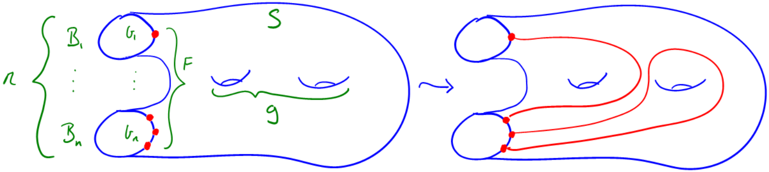

Definition

An *arc diagram* on (S, F) is a properly embedded collection of arcs on S with boundary F.

The problem

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Arc diagrams C_1, C_2 are *equivalent* if $\exists$ homeomorphism $\phi : S \to S$ such that $\phi|_{\partial S} = 1_{\partial S}$ and $\phi(C_1) = C_2$.

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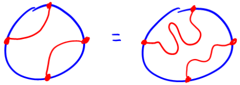

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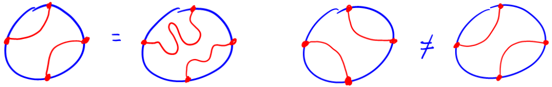

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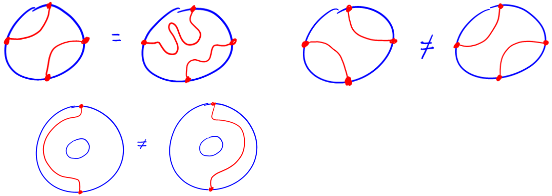

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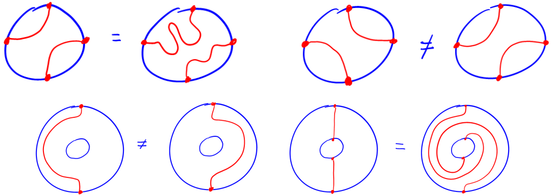

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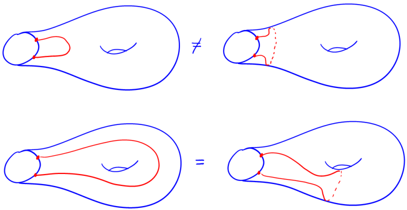

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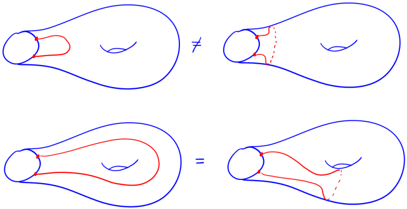

Let $G_{g,n}(b_1, \ldots, b_n)$ denote the number of equivalence classes of arc diagrams on (S, F). This gives a finite count!

Question

What is $G_{g,n}(b_1, \ldots, b_n)$?

Small cases

Some preliminary observations:

- For any g and n, $G_{g,n}(0, 0, \ldots, 0) = 1$
- If $b_1 + \cdots + b_n$ is odd, then $G_{g,n}(b_1, \ldots, b_n) = 0$

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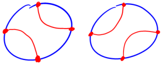
Discs

$(g,n) = (0,1)$

- $G_{0,1}(4) = 2$



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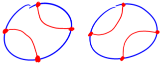

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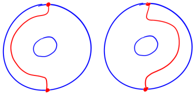

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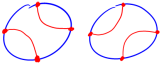

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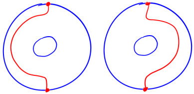

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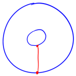

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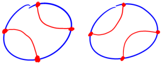

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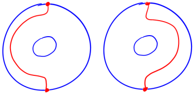

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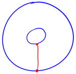
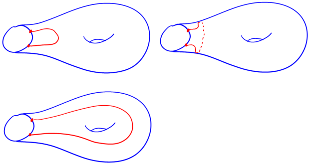


- $G_{1,1}(2) = 3$



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One can continue in this vein...

$$\text{(M.?)} \quad G_{0,3}(2n_1,2n_2,2n_3) = \binom{2n_1}{n_1}\binom{2n_2}{n_2}\binom{2n_3}{n_3}(n_1+1)(n_2+1)(n_3+1)$$

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$$G_{0,4}(2n_1,2n_2,2n_3,2n_4) = \binom{2n_1}{n_1}\binom{2n_2}{n_2}\binom{2n_3}{n_3}\binom{2n_4}{n_4}\Big(1+2\sum n_i + \sum n_i^2$$

A combinatorial argument

Proof that $G_{0,2}(2n,0) = \binom{2n}{n}$ and $G_{0,1}(2n) = \frac{1}{n+1}\binom{2n}{n}$ (Przytycki?)

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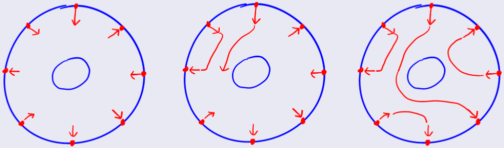

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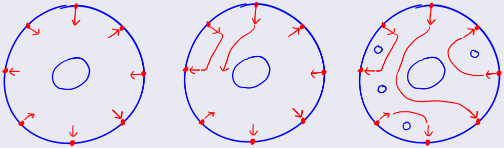

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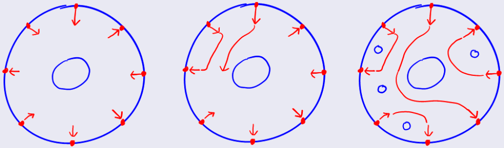

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A more elaborate but similar style argument can be used to prove the general $G_{0,2}$ formula.

A general result

A *quasi-polynomial* $f(x_1, \ldots, x_n)$ is a family of polynomial functions depending on the congruence classes of $x_1, \ldots, x_n$.

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Theorem (Do–M.)

For $(g, n) \neq (0, 1), (0, 2)$, $G_{g,n}(b_1, \ldots, b_n)$ is the product of

- *a combinatorial factor for each $i = 1, \ldots, n$, which is*

$$\binom{b_i}{b_i/2} \text{ if } b_i \text{ is even, or} \binom{b_i - 1}{(b_i - 1)/2} \text{ if } b_i \text{ is odd; and}$$

- *a quasi-polynomial $P_{g,n}(b_1, \ldots, b_n)$ over $\mathbb{Q}$ of degree $3g - 3 + 2n$ depending on $b_1, \ldots, b_n$ mod 2.*

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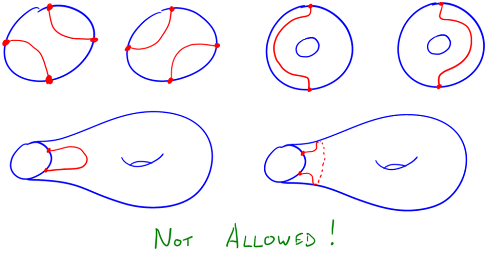
E.g. for any $(g, n) \neq (0, 1), (0, 2)$,

$$G_{g,n}(2m_1, \ldots, 2m_n) = \binom{2m_1}{m_1} \cdots \binom{2m_n}{m_n} P_{g,n}(2m_1, \ldots, 2m_n).$$

where $P_{g,n}$ is a polynomial of degree $3g - 3 + 2n$.

Non-boundary-parallel arcs

Turns out to be useful to consider arc diagrams *without boundary-parallel arcs* — i.e. no arc can cut off a disc.



Definition

Let $N_{g,n}(b_1, \ldots, b_n)$ be the number of (equivalence classes) of arc diagrams on (S, F) without boundary-parallel arcs.

Non-boundary-parallel arcs

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- $N_{0,1}(b_1) = \delta_{b_1,0}$

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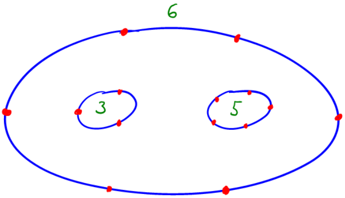
$N_{0,3}(b_1, b_2, b_3) = \cdots ?$ Turns out once b_1, b_2, b_3 are specified (and their sum is even), this specifies a *unique* way to join boundary components. Then rotate around each boundary.



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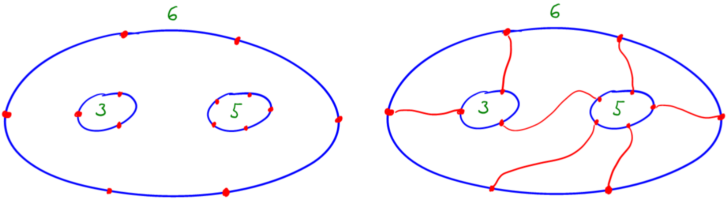

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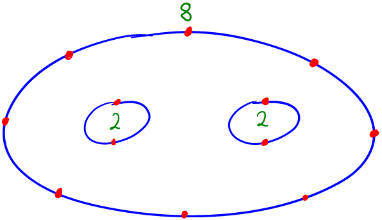

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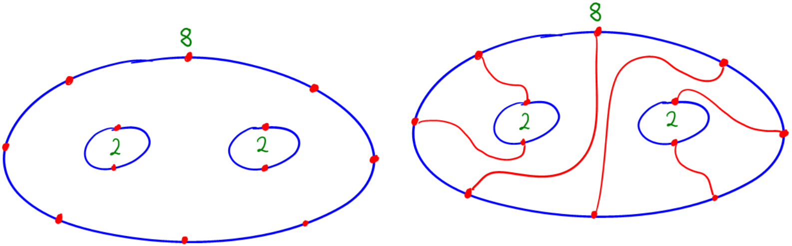

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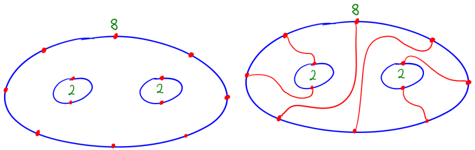

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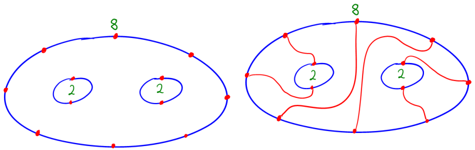

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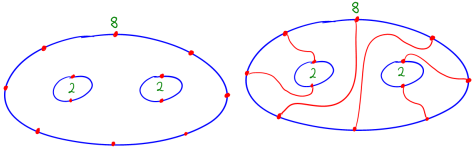

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More formulas

One can continue in this vein...

Theorem (Do–M.)

$$N_{0,4}(b_1, b_2, b_3, b_4) = \bar{b}_1 \bar{b}_2 \bar{b}_3 \bar{b}_4 P_{0,4}(b_1, b_2, b_3, b_4)$$

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Arc diagrams and moduli spaces

A similar formula holds for each g and n.

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- Top-degree terms give intersection numbers on $\mathcal{M}_{g,n}$: the coefficient $c_{d_1,\ldots,d_n}$ of $b_1^{d_1}\cdots b_n^{d_n}$ satisfies

$$c_{d_1,\ldots,d_n} = \frac{1}{2^{5g-6+2n} d_1! \cdots d_n!} \langle \psi_1^{d_1} \cdots \psi_n^{d_n}, \overline{\mathcal{M}}_{g,n} \rangle.$$

Idea of proofs

Proofs draw heavily on ideas of Do–Norbury, Eynard–Orantin, Mulase, Norbury, Norbury–Scott.

1. $G_{g,n}(b_1, \ldots, b_n)$ (also $N_{g,n}$, $\hat{N}_{g,N}$) satisfy a *topological recursion* of Eynard–Orantin & Mulase

$$G_{g,n}(b_1, \ldots, b_n) = \sum_{j=0}^{b_1-2} G_{g-1,n+1}(j, b_1 - j - 2, b_2, \ldots, b_n)$$
$$+ \sum_{k=2}^{n} b_k G_{g,n-1}(b_1 + b_k - 2, b_2, \ldots, \widehat{b_k}, \ldots, b_n)$$
$$+ \sum_{j=0}^{b_1-2} \sum_{g_1+g_2=g} \sum_{I_1 \sqcup I_2 = \{2,\ldots,n\}} G_{g_1,|I_1|+1}(j, g_{I_1})\, G_{g_2,|I_2|+1}(b_1 - j - 2, b_{I_2}).$$

2. Show $\widehat{N}_{g,n}$ are even polynomials in b_i (Norbury).
3. Arc diagrams decompose into "∂-parallel" and "guts":

$$G_{g,n}(b_1, \ldots, b_n) = \sum_{a_1=0}^{b_1} \cdots \sum_{a_n=0}^{b_n} \binom{b_1}{\frac{b_1-a_1}{2}} \cdots \binom{b_n}{\frac{b_n-a_n}{2}} N_{g,n}(a_1, \ldots, a_n)$$

4. Show $G_{g,n}$ have desired form (Norbury–Scott).

Summary

1. "Count curves" on a surface in the most naive way:
   - $G_{g,n}(b_1, \ldots, b_n)$ count embedded arcs with fixed boundary points, up to homeomorphism fixing boundary pointwise.
   - A generalisation of Catalan numbers.
2. Polynomiality results
   - For each g and n, $G_{g,n}(b_1, \ldots, b_n)$ is given by combinatorial factors $\binom{b_i}{b_i/2}$, multiplied by a quasipolynomial in $b_1, \ldots, b_n$.
   - Consider arc diagrams *without boundary-parallel arcs*; counted by $N_{g,n}(b_1, \ldots, b_n)$.
   - For each g and n, $N_{g,n}(b_1, \ldots, b_n)$ turns out to be given by $\bar{b}_1 \cdots \bar{b}_n$ multiplied by a quasipolynomial $\hat{N}_{g,n}$ in $b_1^2, \ldots, b_n^2$.
3. Arrive at more advanced forms of "curve-counting"
   - $\widehat{N}_{g,n}$ closely related to volume of moduli space $\mathcal{M}_{g,n}$
   - Encode intersection numbers of ψ-classes.

Many questions raised:

- More structure? Eynard–Orantin topological recursion... spectral curve, quantum curve, etc.
- Do other "curve-counting" problems have similar structure?

Thanks for listening!