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Counting curves on surfaces

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We consider an elementary, and largely unexplored, combinatorial problem in low-dimensional topology: for a compact surface S, with a finite set of points F fixed on its boundary, how many configurations of disjoint arcs are there on S whose boundary is F? We find that this enumerative problem, counting curves on surfaces, has a rich structure. We show that such curve counts obey an effective recursion, in the general spirit of topological recursion, and exhibit quasi-polynomial behavior. This "elementary curve-counting" is in fact related to a more advanced notion of "curve-counting" from algebraic geometry or symplectic geometry. The asymptotics of this enumerative problem are closely related to the asymptotics of volumes of moduli spaces of curves, and the quasi-polynomials governing the enumerative problem encode intersection numbers on moduli spaces. Among several other results, we show that generating functions and differential forms for these curve counts exhibit structure that is reminiscent of the mathematical physics of free energies, partition functions and quantum curves.

Keywords: Topological recursion; Catalan numbers; curves on surfaces.

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1. Introduction

1.1. *Summary and motivation*

"Curve-counting" plays an important role in several areas of contemporary mathematics. For instance, moduli spaces of curves are central to Gromov–Witten theory, and zero-dimensional moduli spaces consist of a finite number of curves, which can be counted. Such curve counts are used to define boundary operators in Floer homology theories.

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In this paper, we "count curves" of a much simpler type. Consider a (real 2-dimensional) connected compact oriented surface S with boundary. We fix a finite set of boundary points $F \subset \partial S$ and count collections of curves on S — that is, embedded 1-manifolds — with boundary F, according to the following definitions.

Definition 1.1.

(i) An *arc diagram* on (S, F) is a properly embedded collection of arcs $C \subset S$ with boundary F.
(ii) Two arc diagrams C_1 and C_2 on (S, F) are *equivalent* if there is a homeomorphism $\phi : S \to S$, such that $\phi|_{\partial S}$ is the identity, and $\phi(C_1) = C_2$.
(iii) If S has genus g and n boundary components, and F contains $b_1, \ldots, b_n$ points on the n boundary components of S, then the number of equivalence classes of arc diagrams on (S, F) is denoted $G_{g,n}(b_1, \ldots, b_n)$.

Thus, an arc diagram simply consists of finitely many non-intersecting unoriented arcs connecting the points of F in pairs, as in Fig. 1.

In this paper, we present several results about the numbers $G_{g,n}(b_1, \ldots, b_n)$ — and related numbers counting collections of curves of various other types — including how they are related to "curve-counting" of the more advanced type. Roughly, our main results say the following.

- The curve counts on a surface S can be given recursively in terms of curve counts on surfaces of simpler topology.
- If we fix g and n, these curve counts exhibit *quasi-polynomial* behavior.
- The degrees of these quasi-polynomials, and their top-degree coefficients, are closely related to *moduli spaces of curves* and in fact recover the intersection numbers of ψ-classes.
- The counts can be encoded in generating functions and differential forms and in fact different types of counts can be obtained by expanding the same differential form in different coordinates.

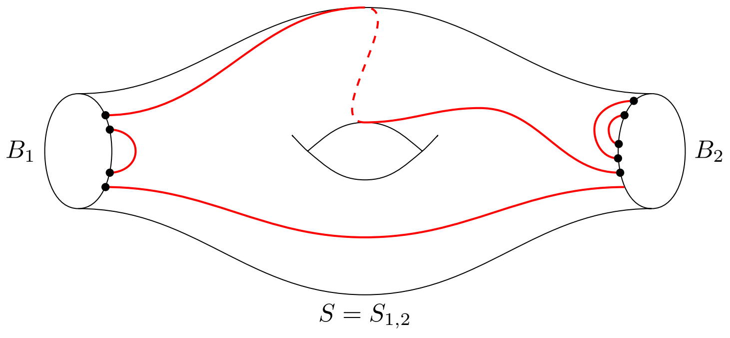

Fig. 1. An arc diagram on (S, F), with $S = S_{1,2}$, $F = F(4, 6)$ and four complementary regions.

- Various generating functions encoding these curve counts obey differential equations reminiscent of the mathematical physics of free energies and partition functions.

These results are similar in spirit to a wide range of results on the *topological recursion* of Chekhov, Eynard and Orantin [6, 15, 18]. There has been a great deal of recent work demonstrating that many enumerative problems formulated in terms of surfaces display similar phenomena: polynomiality, recursion and differential forms and generating functions obeying physically suggestive equations. Such problems arise, for instance, in matrix models [6], the theory of Hurwitz numbers [2, 4, 7, 17], moduli spaces of curves [9, 28, 31, 32], Gromov–Witten theory [3, 13, 16, 19, 33] and combinatorics [8, 12, 14, 23, 29].

We also note that the enumeration of isotopy classes of contact structures near a convex surface in a contact 3-manifold essentially reduces to a similar question, counting of arrangements of *dividing sets* on the surface (see e.g. [20, 22, 26]). The notions of dividing sets and arc diagrams are however distinct.

On a disc, our counting question leads immediately to the Catalan numbers. Our curve counts are thus an elementary generalization of the Catalan numbers from discs to surfaces of general topology. (Other generalizations also exist, see e.g. [12, 29].)

Despite being a straightforward combinatorial question that could have been asked well over a century ago, we have not found many results about these curve counts in the literature, beyond discs and annuli. Recently, Drube–Pongtanapaisan in [11] counted a slightly different notion of curves on annuli, and Kim in [24] counted non-crossing matchings and permutations on annuli.

In this introduction, we present an outline of the results in this paper.

1.2. *Counts of curves on surfaces*

As noted, $G_{0,1}(2m)$ is the mth Catalan number. For small g and n, explicit formulae for the $G_{g,n}(b_1, \ldots, b_n)$ can be given as follows. The formulae depend on the parity of the b_i, and so we write $b_i = 2m_i$ or $2m_i + 1$, with m_i a non-negative integer, accordingly.

Proposition 1.2. *For any integers $m_1, m_2, m_3 \geq 0$,*

$$G_{0,1}(2m) = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad \text{the } m\text{th Catalan number} \tag{1}$$

$$G_{0,2}(2m_1, 2m_2) = \frac{m_1 + m_2 + m_1 m_2}{m_1 + m_2}\binom{2m_1}{m_1}\binom{2m_2}{m_2} \tag{2}$$

$$G_{0,2}(2m_1 + 1, 2m_2 + 1) = \frac{(2m_1 + 1)(2m_2 + 1)}{m_1 + m_2 + 1}\binom{2m_1}{m_1}\binom{2m_2}{m_2} \tag{3}$$

$$G_{0,3}(2m_1, 2m_2, 2m_3) = (m_1+1)(m_2+1)(m_3+1)\binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} \tag{4}$$

$$G_{0,3}(2m_1+1, 2m_2+1, 2m_3) = (2m_1+1)(2m_2+1)(m_3+1)\binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} \tag{5}$$

$$G_{1,1}(2m) = \left(\frac{m^2}{12} + \frac{5m}{12} + 1\right)\binom{2m}{m}. \tag{6}$$

The result for $G_{0,1}(2m)$ is general knowledge. The special case $G_{0,2}(2n, 0) = \binom{2n}{n}$ appears in a paper of Przytycki [34]; we are unable to find it elsewhere in the literature. The result for $G_{0,2}$ was found by Kim [24, Theorem 6.2]; we were informed of this result after posting the initial version of this paper. The other formulae, so far as we know, are new. We prove the statements for annuli by direct combinatorial arguments, which we develop in Sec. 3.

In each case above, $G_{g,n}(b_1, \ldots, b_n)$ is given by a product of combinatorial factors of the form $\binom{2m}{m}$, multiplied by a symmetric rational function in the b_i (or equivalently m_i); these factors and rational functions depend on the parity of the b_i. We show that the $G_{g,n}$ have a similar structure for all (g, n). In fact, the cases $(g, n) = (0, 1)$ and $(0, 2)$ are exceptional: for any other (g, n), we obtain *polynomials* rather than rational functions.

Theorem 1.3. *For $(g, n) \neq (0, 1), (0, 2)$, $G_{g,n}(b_1, \ldots, b_n)$ is the product of*

(i) *a combinatorial factor $\binom{2m_i}{m_i}$ for each $i = 1, \ldots, n$, where $b_i = 2m_i$ if b_i is even and $b_i = 2m_i + 1$ if b_i is odd; and*
(ii) *a quasi-polynomial $P_{g,n}(b_1, \ldots, b_n)$, symmetric in the variables $b_1, \ldots, b_n$, depending on the parity of $b_1, \ldots, b_n$, with rational coefficients, of degree $3g - 3 + 2n$.*

(A *quasi-polynomial* $f(x_1, \ldots, x_n)$ is a family of polynomial functions depending on some congruence classes of the integers $x_1, \ldots, x_n$.)

Thus, for instance, if $(g, n) \neq (0, 1), (0, 2)$ and all b_i are even, $b_i = 2m_i$, then

$$G_{g,n}(2m_1, \ldots, 2m_n) = \binom{2m_1}{m_1}\cdots\binom{2m_n}{m_n} P_{g,n}(2m_1, \ldots, 2m_n),$$

where $P_{g,n}$ is a polynomial of degree $3g-3+2n$ with rational coefficients; there will be a similar expression (but with a different polynomial), for $G_{g,n}(2m_1+1, 2m_2+1, 2m_3, \ldots, 2m_n)$; and so on.

The proof of Theorem 1.3 is effective: it provides a method by which such formulae can be calculated for any (g, n).

The $G_{g,n}$ also satisfy a recursion, expressing the counts on a surface in terms of counts on surfaces with simpler topology.

Theorem 1.4. *For integers $g \geq 0$, $n \geq 1$ and $b_1, \ldots, b_n$ such that $b_1 > 0$ and $b_2, \ldots, b_n \geq 0$,*

$$\begin{aligned} G_{g,n}(b_1, \ldots, b_n) = & \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} G_{g-1,n+1}(i, j, b_2, \ldots, b_n) \\ & + \sum_{k=2}^{n} b_k G_{g,n-1}(b_1 + b_k - 2, b_2, \ldots, \widehat{b}_k, \ldots, b_n) \\ & + \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} \sum_{\substack{g_1,g_2 \geq 0 \\ g_1+g_2=g}} \sum_{I \sqcup J = \{2,\ldots,n\}} G_{g_1,|I|+1}(i, b_I) G_{g_2,|J|+1}(j, b_J). \end{aligned}$$

Any $G_{g,n}(b_1, \ldots, b_n)$ can be computed by this recursion with initial conditions $G_{g,n}(0, \ldots, 0) = 1$.

In particular, Theorem 1.4 implies that all $G_{g,n}$ are finite! The notation $\widehat{b}_k$ means that b_k is omitted from the list $b_2, \ldots, b_n$. We will discuss the details of this theorem, including the notation, in Sec. 6.1.

This recursion is not new: an identical recursion was written down by Walsh and Lehman to enumerate rooted maps [35]. This result was rediscovered in the context of the *generalized Catalan numbers* by Dumitrescu, Mulase, Safnuk, Sorkin and Sułkowski [12, 29]. Note however that our enumeration uses a different set of initial conditions. One wonders if two such enumerative problems satisfying the same recursion, but with different initial conditions, may be related in a more direct manner.

1.3. *Counts of non-boundary-parallel curves*

As we will see, it is natural also to count collections of curves satisfying an additional condition: that no curve be *boundary-parallel*. In other words, we require that no curve cut off a disc. Let $N_{g,n}(b_1, \ldots, b_n)$ be the number of such collections of curves. The relationship between $G_{g,n}$ and $N_{g,n}$ is analogous to the relationship between Hurwitz numbers and pruned Hurwitz numbers [10].

As with the $G_{g,n}$, we give some explicit formulae for the $N_{g,n}$.

Proposition 1.5. *For any integers $b_1, b_2, b_3, b_4 \geq 0$,*

$$N_{0,1}(b_1) = \delta_{b_1,0} \tag{7}$$

$$N_{0,2}(b_1, b_2) = \bar{b}_1 \delta_{b_1,b_2} \tag{8}$$

$$N_{0,3}(b_1, b_2, b_3) = \bar{b}_1 \bar{b}_2 \bar{b}_3 \quad \textit{provided } b_1 + b_2 + b_3 \textit{ is even}; 0 \textit{ otherwise.} \tag{9}$$

$$N_{0,4}(b_1, b_2, b_3, b_4) = \bar{b}_1 \bar{b}_2 \bar{b}_3 \bar{b}_4 \ \widehat{N}_{0,4}(b_1, b_2, b_3, b_4) \quad (\textit{provided not all } b_i = 0) \tag{10}$$

$$N_{1,1}(b_1) = \bar{b}_1 \left(\frac{b_1^2}{48} + \frac{5}{12} \right) \quad (\textit{provided } b_1 \neq 0 \textit{ even}), \tag{11}$$

where $\widehat{N}_{0,4}(b_1, b_2, b_3, b_4)$ *is the quasi-polynomial*

$$\widehat{N}_{0,4}(b_1, b_2, b_3, b_4) = \begin{cases} \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 2 & \textit{all } b_i \textit{ even} \\ \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + \frac{1}{2} & \textit{two } b_i \textit{ even, two } b_i \textit{ odd} \\ \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + 2 & \textit{all } b_i \textit{ odd} \\ 0 & \textit{otherwise} \end{cases}$$

Here, $\bar{n}$ is a convenient notation, defined as follows:

Definition 1.6. For an integer $n \geq 0$, we define

$$\bar{n} = n + \delta_{n,0} = \begin{cases} n & n > 0, \\ 1 & n = 0. \end{cases}$$

The pattern in the structure of $N_{g,n}$ continues, the cases $(g, n) = (0, 1)$ and $(0, 2)$ again being exceptional. We again obtain symmetric quasi-polynomials; in fact, they are all *even* symmetric polynomials.

Theorem 1.7. *For* $(g, n) \neq (0, 1), (0, 2)$ *and* $(b_1, \dots, b_n) \neq (0, \dots, 0)$,

$$N_{g,n}(b_1, \dots, b_n) = \bar{b}_1 \cdots \bar{b}_n \, \widehat{N}_{g,n}(b_1, \dots, b_n),$$

where $\widehat{N}_{g,n}(b_1, \dots, b_n)$ *is a symmetric quasi-polynomial over* $\mathbb{Q}$ *in* $b_1^2, \dots, b_n^2$ *of degree* $3g - 3 + n$, *depending on the parity of* $b_1, \dots, b_n$.

The proof is again effective: in principle, we can calculate the quasi-polynomials for any $N_{g,n}$.

The general curve count $G_{g,n}$ and the non-boundary-parallel curve count $N_{g,n}$ are related by the following result, for which we give a direct combinatorial proof in Sec. 4.

Theorem 1.8. *For* $(g, n) \neq (0, 1)$ *and integers* $b_1, \dots, b_n$,

$$G_{g,n}(b_1, \dots, b_n) = \sum_{a_1, \dots, a_n \in \mathbb{Z}} \binom{b_1}{\frac{b_1 - a_1}{2}} \cdots \binom{b_n}{\frac{b_n - a_n}{2}} N_{g,n}(a_1, \dots, a_n).$$

Here, we consider the binomial coefficient $\binom{M}{N}$ to be zero unless M, N are positive integers satisfying $0 \leq N \leq M$, and we regard $N_{g,n}(a_1, \dots, a_n)$ or $G_{g,n}(b_1, \dots, b_n)$ as zero if any $a_i < 0$ or $b_i < 0$.

The degree $3g - 3 + n$ of the quasi-polynomials $\widehat{N}_{g,n}$ is familiar as the (complex) dimension of the moduli space of curves $\mathcal{M}_{g,n}$. We will show, in fact, that the top-degree terms of these polynomials encode *intersection numbers* in the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ (see generally e.g. [21]).

Theorem 1.9. *For* $(g, n) \neq (0, 1), (0, 2)$, *the non-zero polynomials representing the quasi-polynomial* $\widehat{N}_{g,n}(b_1, \dots, b_n)$ *agree in their top-degree terms. For non-negative* *integers* $d_1, \dots, d_n$ *such that* $d_1 + \cdots + d_n = 3g - 3 + n$, *the coefficient* $c_{d_1,\dots,d_n}$ *of* $b_1^{d_1} \cdots b_n^{d_n}$ *satisfies*

$$c_{d_1,\dots,d_n} = \frac{1}{2^{5g-6+2n}\, d_1! \cdots d_n!} \langle \psi_1^{d_1} \cdots \psi_n^{d_n}, \overline{\mathcal{M}}_{g,n} \rangle.$$

Here, $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is the Chern class of the vector bundle over $\overline{\mathcal{M}}_{g,n}$ given by pulling back the cotangent bundle at the i'th marked point. We could also write

$$c_{d_1,\dots,d_n} = \frac{1}{2^{5g-6+2n}\, d_1! \cdots d_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

The top-degree coefficients $c_{d_1,\dots,d_n}$ in fact agree exactly with the *lattice count polynomials* of Norbury [31] and agree up to simple normalization constants with the *volume polynomials* of Kontsevich [25] and the *Weil–Petersson volume polynomials* calculated by Mirzakhani [27]. Hence, the asymptotics of the polynomials $\widehat{N}_{g,n}$ are equivalent to the asymptotics of volumes of moduli spaces of curves $\mathcal{M}_{g,n}$. Details are given in Sec. 7.3.

Thus, such a naive enterprise as counting curves on surfaces leads naturally to the topology of moduli spaces.

The $N_{g,n}$ and $\widehat{N}_{g,n}$ also obey a recursion, of a similar nature as for the $G_{g,n}$, given in Proposition 6.1.

1.4. *Curve-counting refined by regions*

When counting curves, we can also keep track of the number of *regions* into which they cut the surface.

Definition 1.10. A *complementary region* of an arc diagram C on (S, F) is a connected component of $S \backslash C$. The number of complementary components is denoted r.

We define $G_{g,n,r}(b_1, \dots, b_n)$ to be the number of collections of curves with r complementary regions; similarly, we can define $N_{g,n,r}(b_1, \dots, b_n)$. It turns out these counts, refined by the number of regions, obey many properties similar to unrefined curve counts. For instance, the $G_{g,n,r}$ obey a similar recursion to the $G_{g,n}$.

Theorem 1.11. *For any integers* $g \geq 0$, $n \geq 1$, $r \geq 1$, $b_1 > 0$ *and* $b_2, \dots, b_n \geq 0$,

$$\begin{aligned} &G_{g,n,r}(b_1, \dots, b_n) \\ &\quad = \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} G_{g-1,n+1,r}(i, j, b_2, \dots, b_n) \\ &\qquad + \sum_{k=2}^{n} b_k G_{g,n-1,r}(b_1 + b_k - 2, b_2, \dots, \widehat{b}_k, \dots, b_n) \\ &\qquad + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\dots,n\}}} \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} \sum_{\substack{r_1,r_2 \geq 1 \\ r_1+r_2=r}} G_{g_1,|I_1|+1,r_1}(i, b_{I_1}) G_{g_2,|I_2|+1,r_2}(j, b_{I_2}). \end{aligned}$$

A recursion of a similar nature is given for $N_{g,n,r}$ in Proposition 9.14.

Once $g, n, b_1, \ldots, b_n$ are fixed, the number of regions r into which S can be cut by a collection of arcs is clearly bounded. We prove various inequalities between these parameters in Sec. 9.5. In the process, we find that it is useful to introduce an alternative parameter to track the number of regions, which we call t. (Explicitly, $t = r - \chi(S) - \frac{1}{2}\sum b_i$.) As such, we have a second way of refining the curve counts, which we denote $G_{g,n}^t$ and $N_{g,n}^t$. The $G_{g,n}^t$ and $N_{g,n}^t$ obey polynomiality properties similar to, but more complicated than, $G_{g,n}$ and $N_{g,n}$. One result is the following.

Theorem 1.12. *For* $(g, n) \neq (0, 1), (0, 2)$, *positive integers* $b_1, \ldots, b_n$, *and setting* $t = 0$,

$$N_{g,n}^0(b_1, \ldots, b_n) = \bar{b}_1 \cdots \bar{b}_n \widehat{N}_{g,n}^0(b_1, \ldots, b_n),$$

where $\widehat{N}_{g,n}^0$ *is a symmetric quasi-polynomial over* $\mathbb{Q}$ *in* $b_1^2, \ldots, b_n^2$, *of degree* $3g-3+n$, *depending on the parity of* $b_1, \ldots, b_n$.

Precise and more detailed statements are given in Theorems 9.17 (polynomiality) and 9.19 (degree). A precise statement of polynomiality for the $G_{g,n}^t$ is Theorem 9.21. We compute several examples of refined counts explicitly in Sec. 9.1.

These refined polynomials also recover intersection numbers on moduli spaces.

Theorem 1.13. *For* $(g, n) \neq (0, 1), (0, 2)$, *positive integers* $b_1, \ldots, b_n$ *and* $t = 0$, *the non-zero polynomials representing the quasi-polynomial* $\widehat{N}_{g,n}^0(b_1, \ldots, b_n)$ *agree in their top-degree terms, and they agree with the top-degree terms of* $\widehat{N}_{g,n}(b_1, \ldots, b_n)$. *That is, the coefficient* $c_{d_1,\ldots,d_n}$ *of* $b_1^{d_1} \cdots b_n^{d_n}$ *is given by*

$$c_{d_1,\ldots,d_n} = \frac{1}{2^{5g-6+2n}\, d_1! \cdots d_n!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

A more general statement is proved in Theorem 9.19. The $\widehat{N}_{g,n}^t$ have similar properties for other values of t (not just $t = 0$). We can also set some of the variables b_i to zero. For different choices of t and choices of variables set to zero, we obtain *different* polynomials. It is as if $\widehat{N}_{g,n}(b_1, \ldots, b_n)$ is a quasi-polynomial depending on the "parity" of $b_1, \ldots, b_n$, where there are three possible "parities": even, odd and zero.

For each choice of k, the number of variables set to zero, and t, in an appropriate range, we obtain a separate quasi-polynomial in the b_i^2. We show that $\widehat{N}_{g,n}^t$ has degree at most $3g-3+n-t+k$ in general (Theorem 9.18); and if $k = t$, the degree is exactly $3g-3+n$, with top-degree coefficients agreeing with $\widehat{N}_{g,n}$ (Theorem 9.19).

In a certain sense, given a collection of curves, t is a measure of "how separating" the curves are. The above theorems say that it is sufficient to consider curves which are "as non-separating as possible" in order to recover the geometry of moduli spaces.

1.5. *Differential forms and free energies*

The curve counts $G_{g,n}$ and $N_{g,n}$ fit, at least to some extent, into the framework of the topological recursion of Chekhov, Eynard and Orantin, with its connections to enumerative geometry and mathematical physics [6, 15, 18].

Following this framework (e.g. [28, 29]), we define several generating functions based on the $N_{g,n}$ and $G_{g,n}$. Chief among these are multidifferentials in n variables $x_1, \ldots, x_n$ on $\mathbb{CP}^1$ defined by

$$\omega_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1 \geq 0} \cdots \sum_{\mu_n \geq 0} G_{g,n}(\mu_1, \ldots, \mu_n)\, x_1^{-\mu_1 - 1} \cdots x_n^{-\mu_n - 1}\, dx_1 \cdots dx_n.$$

(In the case $(g, n) = (0, 1)$, we have two distinct forms, which we denote $\omega_{0,1}^G$ and $\omega_{0,1}^N$; see Sec. 8.1.) Although defined as a formal power series, $\omega_{g,n}$ is in fact meromorphic (Proposition 8.3).

It turns out, if we rewrite $\omega_{g,n}$ with respect to new variables $z_1, \ldots, z_n$ defined by $x_i = z_i + \frac{1}{z_i}$, then the coefficients switch from the curve counts $G_{g,n}$, to the non-boundary-parallel curve counts $N_{g,n}$. A similar phenomenon occurs with pruned Hurwitz numbers [10].

Theorem 1.14. *For $(g, n) \neq (0, 1)$,*

$$\omega_{g,n} = \sum_{\nu_1 \geq 0} \cdots \sum_{\nu_n \geq 0} N_{g,n}(\nu_1, \ldots, \nu_n)\, z_1^{\nu_1 - 1} \cdots z_n^{\nu_n - 1}\, dz_1 \cdots dz_n.$$

We can then obtain *free energies* $F_{g,n}$ by integrating the $\omega_{g,n}$, i.e. finding functions $F_{g,n}(z_1, \ldots, z_n)$ such that

$$d_{z_1} \cdots d_{z_n} F_{g,n}(z_1, \ldots, z_n) = \omega_{g,n}(z_1, \ldots, z_n).$$

We compute several $\omega_{g,n}$ explicitly (Lemma 8.2) and also several free energies as follows:

Proposition 1.15. *The following functions are free energy functions.*

$$\begin{aligned}
F_{0,1}^N(z_1) &= \log z_1 \\
F_{0,1}^G(z_1) &= \frac{1}{2} z_1^2 - \log z_1 \\
F_{0,2}(z_1, z_2) &= \log z_1 \log z_2 - \log(1 - z_1 z_2) \\
F_{0,3}(z_1, z_2, z_3) &= \log z_1 \log z_2 \log z_3 + \frac{z_1 z_2 + z_2 z_3 + z_3 z_1 + 1}{(1 - z_1^2)(1 - z_2^2)(1 - z_3^2)} \\
&\quad + \sum_{\text{cyc}} \left(\frac{\log z_1 \log z_2}{1 - z_3^2} + \frac{(z_1 z_2 + 1) \log z_3}{(1 - z_1^2)(1 - z_2^2)} \right).
\end{aligned}$$

The differential forms $\omega_{g,n}$ can be refined according to number of regions. For each value of the number of regions r, and the related parameter t, we obtain meromorphic forms $\omega_{g,n,r}$ and $\omega_{g,n}^t$ (Proposition 10.5). We also show (Proposition 10.6) that changing coordinates from z_i to x_i, changes $\omega_{g,n}^t$ from a generating function for the $N_{g,n}^t$, into a generating function for the $G_{g,n}^t$. (However, such a statement does not hold for $\omega_{g,n,r}$.) In other words, Theorem 1.14 can be refined with respect to t.

Moreover, for given g, n, there are only finitely many possible values of t, so $\omega_{g,n}$ splits as a finite sum of $\omega_{g,n}^t$. We compute some $\omega_{g,n}^t$ explicitly in Sec. 10.2. We can similarly refine free energies $F_{g,n}$ into a finite sum of $F_{g,n}^t$. The following explicit computations can be compared with Proposition 1.15.

Proposition 1.16. *The following functions are free energy functions.*

$$\begin{aligned}
F_{0,1}^{N,0}(z_1) &= \log z_1 \\
F_{0,1}^{G,0}(x_1) &= \frac{1}{2} z_1^2 - \log z_1 \\
F_{0,2}^{0}(z_1, z_2) &= -\log(1 - z_1 z_2) \\
F_{0,2}^{1}(z_1, z_2) &= \log z_1 \log z_2 \\
F_{0,3}^{0}(z_1, z_2, z_3) &= \frac{z_1 z_2 + z_2 z_3 + z_3 z_1 + 1}{(1 - z_1^2)(1 - z_2^2)(1 - z_3^2)} \\
F_{0,3}^{1}(z_1, z_2, z_3) &= \frac{(z_2 z_3 + 1)\log z_1}{(1 - z_2^2)(1 - z_3^2)} + \frac{(z_3 z_1 + 1)\log z_2}{(1 - z_3^2)(1 - z_1^2)} + \frac{(z_1 z_2 + 1)\log z_3}{(1 - z_1^2)(1 - z_2^2)} \\
F_{0,3}^{2}(z_1, z_2, z_3) &= \log z_1 \log z_2 \log z_3 + \frac{\log z_1 \log z_2}{1 - z_3^2} + \frac{\log z_2 \log z_3}{1 - z_1^2} + \frac{\log z_3 \log z_1}{1 - z_2^2}.
\end{aligned}$$

1.6. *Differential equations and partition function*

The recursions on the curve counts $G_{g,n}$ and $N_{g,n}$ (and also their refined versions) translate into recursive differential equations on their generating functions.

The differential forms $\omega_{g,n}$ can be written as $f_{g,n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$, where

$$f_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1, \ldots, \mu_n \geq 0} G_{g,n}(\mu_1, \ldots, \mu_n) x_1^{-\mu_1 - 1} \cdots x_n^{-\mu_n - 1}$$

is a function of n variables. To form a recursive differential equation on the $f_{g,n}$, we take the recursion in Theorem 1.4, multiply by $x_1^{-\mu_1-1} \cdots x_n^{-\mu_n-1}$, and sum over $\mu_1, \ldots, \mu_n$. However, Theorem 1.4 does not apply when $b_1 = 0$, so certain terms are missing, corresponding to the initial conditions in the recursion. In other words, the obstacle to obtaining a recursive differential equation in the $f_{g,n}$ is not the recursion, but the initial conditions.

One way to deal with this issue is to "differentiate out" the initial terms; doing so, we obtain a differential equation given in Proposition 8.7.

A better way to deal with this issue is to use the *refined* counts of curves, keeping track of the number of regions. With refined counts, there is a simple way to express $G_{g,n,r}(0, b_2, \ldots, b_n)$ in terms of $G_{g,n-1,r}(b_2, \ldots, b_n)$ (Proposition 9.4). This is something like a "dilaton equation" for curve-counting.

Therefore, we define generating functions which keep track of the number of regions r, using a new variable α. We can define a generating function

$$\mathfrak{f}_{g,n}(x_1, \ldots, x_n; \alpha) = \sum_{r \geq 1} \sum_{\mu_1, \ldots, \mu_n \geq 0} G_{g,n,r}(\mu_1, \ldots, \mu_n)\, x_1^{-\mu_1 - 1} \cdots x_n^{-\mu_n - 1}\, \alpha^r.$$

(This function is also called $\mathfrak{f}^G_{g,n}$ in Sec. 10.5). In fact, in Sec. 10.5, we consider various generating functions and differential forms, which use the various refined counts $G_{g,n,r}$, $G^t_{g,n}$, $N_{g,n,r}$ and $N^t_{g,n}$. We find relations between them (Proposition 10.11) and show they are all meromorphic (Propositions 10.8 and 10.12). We also compute them in various small cases; the $\mathfrak{f}_{g,n}(x; \alpha)$ reduce to $f_{g,n}(x)$ upon setting $\alpha = 1$. We can then obtain a recursive differential equation in the $\mathfrak{f}_{g,n}$.

Proposition 1.17. *For any $g \geq 0$ and $n \geq 1$,*

$$\begin{aligned} x_1\, \mathfrak{f}_{g,n}(x_1, \ldots, x_n; \alpha) = {} & \mathfrak{f}_{g-1,n+1}(x_1, x_1, x_2, \ldots, x_n; \alpha) \\ & + \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \frac{1}{x_k - x_1} (\mathfrak{f}_{g,n-1}(x_2, \ldots, x_n; \alpha) \\ & - \mathfrak{f}_{g,n-1}(x_1, x_2, \ldots, \widehat{x}_k, \ldots, x_n; \alpha)) \\ & + \sum_{\substack{g_1 + g_2 = g \\ I_1 \sqcup I_2 = \{2, \ldots, n\}}} \mathfrak{f}_{g_1, |I_1|+1}(x_1, x_{I_1}; \alpha) \mathfrak{f}_{g_2, |I_2|+1}(x_1, x_{I_2}; \alpha) \\ & + \alpha \frac{\partial}{\partial \alpha} \mathfrak{f}_{g,n-1}(x_2, \ldots, x_n; \alpha). \end{aligned}$$

From this, we find a differential equation on free energies $\mathfrak{F}_{g,n}(x_1, \ldots, x_n; \alpha)$ defined by integrating the $\mathfrak{f}_{g,n}$.

Theorem 1.18. *There are free energies $\mathfrak{F}_{g,n}(x_1, \ldots, x_n; \alpha)$ such that*

$$\begin{aligned} & x_1 \frac{\partial}{\partial x_1} \mathfrak{F}_{g,n}(x_1, \ldots, x_n; \alpha) \\ & \quad = \frac{\partial^2}{\partial u \partial v} \mathfrak{F}_{g-1,n+1}(u, v, x_2, \ldots, x_n; \alpha)\Big|_{u=v=x_1} \\ & \qquad + \sum_{k=2}^{n} \frac{1}{x_k - x_1} \\ & \qquad \times \left(\frac{\partial}{\partial x_k} \mathfrak{F}_{g,n-1}(x_2, \ldots, x_n; \alpha) - \frac{\partial}{\partial x_1} \mathfrak{F}_{g,n-1}(x_1, \ldots, \widehat{x_k}, \ldots, x_n; \alpha) \right) \end{aligned}$$

$$+ \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \frac{\partial}{\partial x_1} \mathfrak{F}_{g_1,|I_1|+1}(x_1, x_{I_1}; \alpha) \, \frac{\partial}{\partial x_1} \mathfrak{F}_{g_2,|I_2|+1}(x_1, x_{I_2}; \alpha)$$

$$+ \alpha \frac{\partial}{\partial \alpha} \mathfrak{F}_{g,n-1}(x_2, \ldots, x_n; \alpha).$$

This differential recursion on the free energies $\mathfrak{F}_{g,n}$ resembles the recursion on free energies of Mulase–Sułkowsi's "generalized Catalan numbers" [29]. An identical recursion applies in that case, but the harder initial conditions here require our recursion to have an extra term.

Combining the free energies into a so-called *partition function*

$$\mathbf{Z} = \exp\left[\sum_{m=0}^{\infty} \hbar^{m-1} \sum_{2g+n-1=m} \frac{1}{n!} \mathfrak{F}_{g,n}(x, \ldots, x; \alpha) \right]$$

we obtain a differential equation satisfied by $\mathbf{Z}$.

Theorem 1.19.

$$\left(\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar x \frac{\partial}{\partial x} + \hbar^2 \alpha \frac{\partial}{\partial \alpha} + \alpha \right) \mathbf{Z} = 0.$$

This differential equation provides something like a "*quantum curve*" result for the curve counts $G_{g,n}$, although the extra parameter α appears non-standard. There is a resemblance to the equation $x^2 - xz + \alpha = 0$, which is obtained from setting $z = \mathfrak{f}^G_{0,1}(x_1; \alpha) = \frac{x_1 - \sqrt{x_1^2 - 4\alpha}}{2}$.

It would be interesting to know whether our enumeration is governed by the topological recursion of Chekhov, Eynard and Orantin [6, 15, 18]. The resemblance of the recursion of Theorem 1.4 to the recursion for the enumeration of ribbon graphs [23, 12], though with different initial conditions, suggests that our enumeration may relate instead to the so-called *blobbed topological recursion* of Borot [1]. Furthermore, the work of Kazarian and Zograf demonstrates that enumerative problems governed by recursions such as that of Theorem 1.4 may often be encapsulated in the form of Virasoro constraints [23]. A similar analysis in the context of counting curves on surfaces may hold, which would then lead to a relation between our enumeration and integrable hierarchies.

1.7. *Structure of paper*

This paper is organized as follows. In Sec. 2, we set up our framework for counting curves, and make some elementary observations. In Sec. 3, we count curves on discs and annuli, giving formulae for curve counts by elementary combinatorial arguments.

We then turn to the relationship between the curve counts $G_{g,n}$ and $N_{g,n}$. In Sec. 4, we show that any collection of curves on a surface can be decomposed in an essentially unique way into a part "local to the boundary", and a "core" (Sec. 4.1). We use this "local decomposition" to express $G_{g,n}$ in terms of $N_{g,n}$ (Sec. 4.2).

We are then able to count curves on pants in Sec. 5. After establishing some terminology (Sec. 5.1), we directly compute $N_{0,3}$ (Sec. 5.2) and $G_{0,3}$ (Sec. 5.3).

In Sec. 6, we turn to recursion. We establish recursions for $G_{g,n}$ (Sec. 6.1) and $N_{g,n}$ (Sec. 6.2), and use these to make some computations (Sec. 6.3).

We then turn to polynomiality. After some preliminary work (sec. 7.1), we establish polynomiality of the $N_{g,n}$ (Sec. 7.2). Reflecting on this proof establishes the agreement of top-degree terms with Norbury's lattice count, giving us results about moduli spaces and intersection numbers (Sec. 7.3). We can then prove polynomiality for the $G_{g,n}$ (Sec. 7.4).

Next, we consider generating functions and differential forms. After defining (Sec. 8.1) and computing some small cases (Sec. 8.2) of these generating functions, we show they are meromorphic (Sec. 8.3). We can then show that the expansion of $\omega_{g,n}$ in x and z coordinates yields the $G_{g,n}$ and $N_{g,n}$ (Sec. 8.4). Free energies can then be defined and computed in small cases (Sec. 8.5), and we can make some initial observations about recursions and differential equations for the generating functions (Sec. 8.6).

In Sec. 9, we introduce the refinement of counts by regions. After making definitions and compute refined counts on discs and annuli (Sec. 9.1), we prove a sort of "dilaton equation" (Sec. 9.2). We discuss how the concept of local decomposition (Sec. 9.3) can be refined, and then use it to compute refined counts on pants (Sec. 9.4). We consider bounds on the number of regions (Secs. 9.5 and 9.6), refine the recursion (Sec. 9.7) and then use these results to prove polynomiality for general refined curve counts (Secs. 9.8–9.11). Along the way, we obtain relations between the refined polynomials and intersection numbers on moduli spaces of curves (Sec. 9.10).

Section 10 is devoted to refining the results obtained in Sec. 8 according to the number of complementary regions (Secs. 10.1–10.6). Finally, we obtain differential equations for the free energies and use this to determine an equation satisfied by the partition function that is reminiscent of the notion of quantum curve (Sec. 10.7).

2. Which Curves to Count?

2.1. *Arc diagrams and equivalence*

Throughout, we assume all surfaces are compact, connected and oriented, unless specified otherwise. We will write $S_{g,n}$ to denote a surface of genus g with n boundary components; when g and n are clear, we simply write S.

We consider finite sets of *marked points* $F \subset \partial S_{g,n}$. We label the boundary components $B_1, \ldots, B_n$ and write $b_i = |F \cap B_i|$ and $\mathbf{b} = (b_1, \ldots, b_n)$. We allow $b_i = 0$ and indeed, we allow $\mathbf{b} = \mathbf{0}$. We will write $F(b_1, \ldots, b_n) = F(\mathbf{b})$ to denote such a finite set, and when $\mathbf{b}$ is clear, we simply write F.

In Definition 1.1, we defined arc diagrams on $(S, F) = (S_{g,n}, F(\mathbf{b}))$ and their equivalences, and the numbers $G_{g,n}(b_1, \ldots, b_n)$ which count equivalence classes of arc diagrams. Since an arc diagram is a properly embedded collection of arcs $C \subset S$, it consists of finitely many unoriented arcs connecting points of F. Proper embedding requires that precisely one arc of C emanate from each point of F, and that arcs do not intersect. See Fig. 1. By prohibiting crossings, the topology of S restricts the possible arrangements of curves.

(Strictly speaking, one should distinguish between the embedding of a disjoint union of intervals into S, and the image of this map. Pre-composing such an embedding with a self-homeomorphism of these intervals gives an equivalent embedding. In practice, we abuse notation and conflate the embedding with its image, regarding C as a subset of S. It should not cause any confusion.)

Note that several other reasonable definitions of collections of curves on (S, F) are possible: for instance, one might allow certain closed curves, require curves to be oriented, or consider dividing sets or sutures.

While the set of arc diagrams on a given (S, F) is infinite, we will show that up to the notion of equivalence in Definition 1.1, the number of arc diagrams on (S, F) is finite (as mentioned in the introduction, this follows from the recursion in Theorem 1.4, which is proved in Sec. 6.1).

Our notion of equivalence, i.e. up to a homeomorphism of the surface fixing the boundary pointwise, is stronger than isotopy. But in general there are infinitely many isotopy classes of arc diagrams on a given (S, F). For instance, for an arc diagram C which essentially intersects a homologically non-trivial simple closed curve γ, applying Dehn twists about γ to C yields infinitely many non-isotopic arc diagrams. Arguably, then, the simplest way to count curves on surfaces is to count equivalence classes of arc diagrams as we have defined them.

As our notion of equivalence involves homeomorphisms fixing the boundary pointwise, the labels $1, \ldots, n$ on the boundary components, and the numbers $b_1, \ldots, b_n$ are fixed (they are not permuted) as we count curves. The number of equivalence classes only depends on the numbers $g, n, b_1, \ldots, b_n$. Hence, the following definition makes sense.

Definition 2.1. The set of equivalence classes of arc diagrams on $(S_{g,n}, F(\mathbf{b}))$ is denoted $\mathcal{G}_{g,n}(\mathbf{b})$.

Thus, $G_{g,n}(\mathbf{b}) = |\mathcal{G}_{g,n}(\mathbf{b})|$. Our notion of arc diagram includes the empty arc diagram. Thus, for all g and n, $G_{g,n}(\mathbf{0}) = 1$.

If ϕ is homeomorphism of S providing an equivalence between arc diagrams C_1, C_2, then as ϕ fixes ∂S pointwise, and so takes arcs and complementary regions of C_1 to those of C_2 in a canonical fashion. Hence, we may refer to an *arc* or *complementary region* of an equivalence class without ambiguity. In practice, we often drop the phrase "equivalence classes of" for convenience, and refer only to counting arc diagrams; we hope that the meaning is clear.

As discussed in the introduction, it will be useful to consider arc diagrams without boundary-parallel arcs. An embedded arc in S is boundary-parallel if it is homotopic (relative to endpoints) to an arc lying entirely in ∂S.

Definition 2.2. The set of equivalence classes of arc diagrams on $(S_{g,n}, F(\mathbf{b}))$ without boundary-parallel arcs is denoted $\mathcal{N}_{g,n}(\mathbf{b})$.

Thus $N_{g,n}(\mathbf{b}) = |\mathcal{N}_{g,n}(\mathbf{b})|$.

Definition 2.3. For $g \geq 0$, $n \geq 1$ and $b_1, \ldots, b_n \geq 0$, we define

$$\widehat{N}_{g,n}(\mathbf{b}) = \frac{N_{g,n}(\mathbf{b})}{\bar{b}_1 \cdots \bar{b}_n}.$$

2.2. *First considerations*

Some initial observations about $G_{g,n}(\mathbf{b})$ are clear.

Lemma 2.4. *For any $g \geq 0$ and $n \geq 1$, if $b_1 + \cdots + b_n$ is odd, then $G_{g,n}(\mathbf{b}) = 0$.*

Proof. Every arc has two endpoints, and the number of endpoints is $b_1 + \cdots + b_n$. $\square$

We may regard $G_{g,n}$ as a function $\mathbb{N}_0^n \to \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. That is, $G_{g,n}$ takes an n-tuple of non-negative integers $(b_1, \ldots, b_n)$ and returns a non-negative integer.

Lemma 2.5. *The function $G_{g,n}(b_1, \ldots, b_n)$ is a symmetric function of $b_1, \ldots, b_n$.*

Proof. For any permutation $\sigma \in S_n$, there is a homeomorphism $\phi : S \to S$ permuting the boundary components according to σ, $\phi(B_i) = B_{\sigma(i)}$. So $G_{g,n}(b_1, \ldots, b_n) = G_{g,n}(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$. $\square$

3. Counting Curves on Annuli and Discs

3.1. *Definitions and statements*

We begin counting curves on some simple surfaces, starting with annuli. As it turns out, along the way, we will be able to count curves on discs. So let $S = S_{0,2}$ and $F = F(b_1, b_2)$, and let $b_i = 2m_i$ or $2m_i + 1$ accordingly as b_i is even or odd.

Definition 3.1.

(i) A properly embedded arc on an annulus is *traversing* if its endpoints lie on distinct boundary components, and *insular* if its endpoints lie on the same boundary component.
(ii) An arc diagram on an annulus is *traversing* if it contains a traversing arc, and *insular* if all its arcs are insular.

(iii) The number of equivalence classes of traversing arc diagrams on (S, F) is denoted $T(b_1, b_2)$, and the number of equivalence classes of insular arc diagrams is denoted $I(b_1, b_2)$.

Note that "insular arc" is synonymous with "boundary-parallel arc", and "traversing arc" with "non-boundary-parallel arc". In an insular arc diagram, all arcs stay close to their home boundary component, but in a traversing arc diagram, some brave arc traverses the annulus from one side to the other. The empty arc diagram is vacuously insular.

We will give $I(b_1, b_2)$ and $T(b_1, b_2)$ explicitly. By Lemma 2.4, we only need to consider b_1, b_2 both even or both odd. And it is clear that in an insular arc diagram both b_1, b_2 must be even.

Proposition 3.2. *For integers $m_1, m_2 \geq 0$,*

$$I(2m_1, 2m_2) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}.$$

Proposition 3.3. *For integers $m_1, m_2 \geq 0$,*

$$T(2m_1, 2m_2) = \frac{m_1 m_2}{m_1 + m_2}\binom{2m_1}{m_1}\binom{2m_2}{m_2}$$

$$T(2m_1 + 1, 2m_2 + 1) = \frac{(2m_1 + 1)(2m_2 + 1)}{m_1 + m_2 + 1}\binom{2m_1}{m_1}\binom{2m_2}{m_2}.$$

Clearly $G_{0,2}(b_1, b_2) = I(b_1, b_2) + T(b_1, b_2)$, so Proposition 1.2(2)–(3) follows from these two propositions.

We prove these propositions by bijective combinatorial arguments, in Secs. 3.2 and 3.3, respectively. Proposition 3.3 is identical in content to [24, Theorem 6.2], which in fact contains a more general result with cyclic sieving (Lemma 3.3).

First, however, we calculate r, the number of complementary regions. This depends on whether the diagram is insular or traversing.

Lemma 3.4. *Let C be an arc diagram on an annulus.*

(i) *If C is insular then $r = \frac{1}{2}(b_1 + b_2) + 1$. One complementary region is an annulus; the rest are discs.*
(ii) *If C is traversing then $r = \frac{1}{2}(b_1 + b_2)$. All complementary regions are discs.*

Proof. First, note that C has precisely $\frac{1}{2}(b_1 + b_2)$ arcs.

If γ is a traversing arc, then cutting along γ cuts S into a disc. Cutting further along the other $\frac{1}{2}(b_1 + b_2) - 1$ arcs of C, each cut slices off an extra disc. So $S \backslash C$ consists of $\frac{1}{2}(b_1 + b_2)$ discs.

If C is insular then, successively cutting along outermost arcs, each cut slices off a disc. At the end, we have $\frac{1}{2}(b_1 + b_2)$ discs and an annulus. $\square$

3.2. *Insular diagrams*

We will draw annuli in a standard way, as the region between two concentric circles in the plane. We can naturally then speak of "clockwise" and "anticlockwise" directions.

An oriented insular arc γ is isotopic to a properly embedded arc consisting of a radial arc, followed by an "angular" arc at constant radius from the center, followed by another radial arc. We say γ is clockwise or anticlockwise according to the direction of the angular arc.

Given an insular arc diagram C on (S, F), we may orient each arc anticlockwise. Each arc then points into S at one endpoint, and out at the other end. The points of F can be labeled *in* and *out* accordingly; we can represent these labels by arrows pointing into or out of S. This leads us to the following definition.

Definition 3.5. An *arrow diagram* on (S, F) is a labeling of points of F either "in" or "out" so that on each boundary component, exactly half the points are labeled "in" and half are labeled "out".

The set of all arrow diagrams on (S, F), is denoted $A(b_1, b_2)$.

If an arrow diagram exists on (S, F), then b_1, b_2 must both be even, $(b_1, b_2) = (2m_1, 2m_2)$, and we have

$$|A(2m_1, 2m_2)| = \binom{2m_1}{m_1}\binom{2m_2}{m_2}.$$

First, suppose one boundary component has no marked points, $(b_1, b_2) = (2m, 0)$. Let $\Phi : \mathcal{G}_{0,2}(2m, 0) \to A(2m, 0)$ be the function which takes a diagram (necessarily insular) to the arrow diagram obtained by orienting each arc anticlockwise.

Lemma 3.6. *The map Φ is a bijection. Hence*

$$G_{0,2}(2m, 0) = |\mathcal{G}_{0,2}(2m, 0)| = |A(2m, 0)| = \binom{2m}{m}.$$

The idea is that an arc diagram $C \in \mathcal{G}_{0,2}(2m, 0)$ can be constructed from $a \in A(2m, 0)$ by starting at a point of F (any point will do) and proceeding anticlockwise around the annulus. Each time, we arrive at a point of F labeled "in", we start drawing a new arc, proceeding anticlockwise around the annulus. Each time, we arrive at a point of F labeled "out", we end an arc there (if possible). This process produces a unique arc diagram. See Fig. 2.

Proof. Proof by induction on m. When $m = 0$, there is nothing to prove. When $m = 1$, the construction is clear: draw an arc anticlockwise from the "in" to the "out" point of F. This is clearly unique up to equivalence of arc diagrams.

For general m, note that in the arrow diagram a, as we proceed anticlockwise around the $2m$ boundary points of F, there must be at least one point f_{in} labeled "in" followed immediately by another point f_{out} labeled "out". Any (equivalence

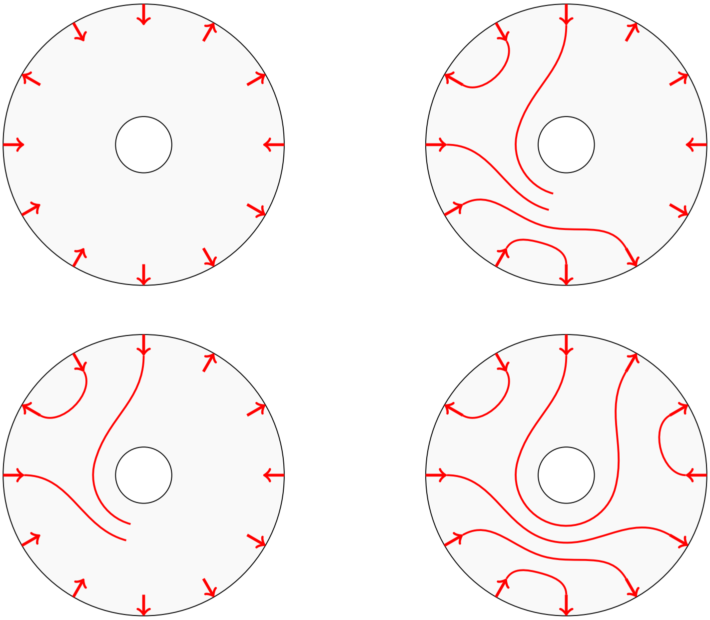

Fig. 2. Constructing an arc diagram from an arrow diagram.

class of) arc diagram C such that $\Phi(C) = a$ must contain a "short" boundary-parallel arc γ anticlockwise from f_{in} to f_{out}. The remaining $2m-2$ points of a form an arrow diagram $a' \in A(2m-2, 0)$. By induction, there exists a unique arc diagram C' with $\Phi(C') = a'$; taking C' together with γ gives an arc diagram C with $\Phi(c) = a$. Moreover, since γ must be included, the uniqueness of C' implies uniqueness of C. $\square$

The idea of the proof above appears in Przytycki [34] and is due to him, so far as we know. This argument is then used to give a formula for the Catalan numbers, as we show now.

Proposition 3.7 (Przytycki). *For any integer $m \geq 0$,*

$$G_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m} = C_m.$$

Proof. Consider the annulus $(S, F) = (S_{0,2}, F(2m, 0))$ and the disc $(D, F(2m))$. There is a map $\Psi : \mathcal{G}_{0,2}(2m, 0) \to \mathcal{G}_{0,1}(2m)$ given by gluing a disc to the boundary component B_2 of (S, F).

Given an arc diagram C on $(D, F(2m))$, we can remove a small disc D' from the interior of D, not intersecting any arcs, and obtain an arc diagram on $(S, F(2m, 0))$. There are $m+1$ complementary regions of the m arcs of C, and removing D' from these distinct regions produces $m+1$ distinct arc diagrams on $(S, F(2m, 0))$. These arc diagrams form $\Psi^{-1}(C)$ precisely, so Ψ is $(m+1)$-to-1, and

$$G_{0,1}(m) = |\mathcal{G}_{0,1}(m)| = \frac{|\mathcal{G}_{0,2}(m,0)|}{m+1} = \frac{G_{0,2}(m,0)}{m+1} = \frac{1}{m+1}\binom{2m}{m} = C_m. \qquad \square$$

This also proves Proposition 1.2(1).

Proof of Proposition 3.2. An insular diagram in $\mathcal{G}_{0,2}(2m_1, 2m_2)$ can be cut along a core curve into two insular arc diagrams, on $(S, F(2m_1, 0))$ and $(S, F(2m_2, 0))$, respectively. This gives a bijection between insular arc diagrams on $(S_{0,2}, F(2m_1, 2m_2))$ and $\mathcal{G}_{0,2}(2m_1, 0) \times \mathcal{G}_{0,2}(2m_2, 0)$, so

$$I(2m_1, 2m_2) = G_{0,2}(2m_1, 0)G_{0,2}(2m_2, 0) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}. \qquad \square$$

3.3. *Traversing diagrams*

We now turn to traversing diagrams on $(S, F) = (S_{0,2}, F(b_1, b_2))$. We draw our annuli within the plane with B_1 as "outer" and B_2 as "inner" boundary.

Since each insular arc connects two points on the same boundary component, the number of endpoints of traversing arcs on B_i has the same parity as b_i, yielding the following observation.

Lemma 3.8. *The number of traversing arcs in an arc diagram on (S, F) has the same parity as b_1 and b_2.* $\square$

Our computation of $T(b_1, b_2)$ involves bijections between certain combinatorial sets, which we now define.

Definition 3.9. A *decorated arc diagram* on (S, F) is a pair (C, R), where C is an arc diagram on (S, F), and R is a complementary region of C. The set of equivalence classes of decorated traversing arc diagrams on $(S, F) = (S_{0,2}, F(b_1, b_2))$ is denoted $\mathrm{DT}(b_1, b_2)$.

By Lemma 3.4, a traversing arc diagram has $\frac{1}{2}(b_1 + b_2)$ complementary regions, so

$$|\mathrm{DT}(b_1, b_2)| = \frac{1}{2}(b_1 + b_2)\, T(b_1, b_2).$$

To count $\mathrm{DT}(b_1, b_2)$, we find a bijection with objects, we call *special arrow diagrams*. We will deal with the even case $(b_1, b_2) = (2m_1, 2m_2)$ and the odd case $(b_1, b_2) = (2m_1 + 1, 2m_2 + 1)$ separately. First, we consider the even case.

Definition 3.10. A *special arrow diagram* on $(S, F(2m_1, 2m_2))$ is an arrow diagram together with a choice of one inward arrow on B_2 (the *special inward arrow*), and an outward arrow on B_1 (the *special outward arrow*). The set of such arrow diagrams is denoted $\mathrm{SA}(m_1, m_2)$.

We have seen there are $\binom{2m_1}{m_1}\binom{2m_2}{m_2}$ arrow diagrams on (S, F); hence $|\mathrm{SA}(2m_1, 2m_2)| = m_1 m_2 \binom{2m_1}{m_1}\binom{2m_2}{m_2}$.

Next, we define a map $\Psi : \mathrm{SA}(2m_1, 2m_2) \to \mathrm{DT}(2m_1, 2m_2)$ as follows. Let a be a special arrow diagram with special inward arrow i on the inner boundary B_2 and special outward arrow o on the outer boundary B_1.

(i) Join special arrows i to o by an oriented arc γ. (There are many choices for γ, but they are all related by homeomorphisms of S fixing the boundary, hence lead to equivalent arc diagrams.)
(ii) Cut S along γ to obtain a disc D. The boundary of D, starting from o and proceeding anticlockwise around B_1, consists of B_1 (traversed anticlockwise), followed by γ (traversed backwards), followed by B_2 (traversed clockwise), followed by γ (traversed forwards). The remaining (unconnected, non-special) arrows on (S, F) provide D with $2(m_1 + m_2 - 1)$ marked boundary points, half labeled "in" and half labeled "out".
(iii) Choose a point p in the interior of D, and remove a small neighborhood of p. We then have an annulus with $2(m_1 + m_2 - 1)$ marked boundary points on one boundary component, half "in" and half "out", and no points on the other boundary component. That is, we have an arrow diagram in $A(2m_1 + 2m_2 - 2, 0)$.
(iv) By the bijection Φ of Sec. 3.2, we obtain a unique (equivalence class of) arc diagram $\widetilde{C}$ on this annulus. This $\widetilde{C}$ has the property that if its arcs are oriented anticlockwise around the annulus, then the orientations agree with the arrows at boundary points.
(V) Gluing back the neighborhood of p which was previously removed gives the disc D, which now has an arc diagram $\overline{C}$. The point p and its removed-and-returned neighborhood now lie in a complementary region $\overline{R}$ of $\overline{C}$.
(vi) Recall that ∂D contains two copies of the oriented arc γ, along which we originally cut. We now glue these two copies of γ back together, reconstructing the original annulus S. Combining $\overline{C}$ and γ gives an (equivalence class of) arc diagram C on (S, F), and the complementary region $\overline{R}$ of $\overline{C}$ becomes a complementary region R of C. We define $\Psi(a) = (C, R)$.

At each step, all choices are unique up to homeomorphisms of the surface fixing the boundary. Thus, the equivalence class of the arc diagram C, and the region R, are well-defined; so Ψ is well-defined.

This construction is perhaps more natural than it seems. The special arrows show us how to cut the annulus into a disc. The remaining arrows around a disc show us how to draw the remaining arcs. But arrows around the boundary are equivalent to an arc diagram on an *annulus* obtained by removing a small sub-disc. This is equivalent to an arc diagram on the annulus together with a choice of region.

Suppose a is special arrow diagram, and $\Psi(a) = (C, R)$. The arc diagram C consists of $m_1 + m_2$ arcs, which are all given an orientation in the construction, agreeing with the arrows in a. From Lemma 3.8, the number of traversing arcs is even; let this number be $2k$.

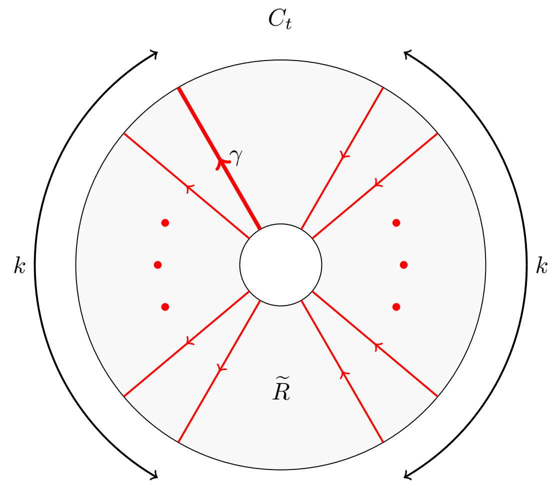

Fig. 3. The arc diagram C_t in the case $(b_1, b_2) = (2m_1, 2m_2)$.

We claim that k traversing arcs run "inward" from B_1 to B_2, and k traversing arcs run "outward" from B_2 to B_1. To see this, note that the arrow diagram a consists of m_1 inward and m_1 outward arrows on B_1, and insular arcs connect some of the inward to outward arrows in pairs. Thus, the remaining $2k$ arrows, which are the endpoints on B_1 of traversing arcs, contain the same number k of inward and outward arrows.

Let C_t denote the oriented arc diagram C with insular arcs removed. So C_t consists of k inward and k outward traversing arcs. These arcs cut the annulus S into $2k$ complementary disc regions, and we may speak of proceeding clockwise or anticlockwise around the annulus from one traversing arc to the next, or from one region to the next. One of the traversing arcs is the arc γ connecting the special arrows; by construction γ points outward. And one of the regions $\tilde{R}$ of C_t contains the region R and the point p in the construction. (As C_t is obtained from C by removing arcs, the complementary regions of C are subsets of the complementary regions of C_t.) See Fig. 3.

Lemma 3.11. *Starting from γ and proceeding anticlockwise, the first k traversing arcs of C_t (including γ) are oriented outward; then we pass through the region $\tilde{R}$; and the final k traversing arcs of C_t are oriented inward.*

Proof. Recall that in the construction of C, we first draw γ, oriented outward; then we cut along γ, remove a neighborhood of p, and construct an oriented arc diagram on the resulting annulus. These arcs are oriented so as to run anticlockwise around this annulus; that is, they run anticlockwise around the point p. Hence, once the traversing arcs are constructed, we see that those traversing arcs that lie anticlockwise of $\tilde{R}$ and clockwise of γ must be oriented inward. Similarly, the traversing arcs that lie clockwise of $\tilde{R}$ and anticlockwise of γ must be oriented outward. Since there are k inward and k outward traversing arcs, they must be arranged as claimed. $\square$

In particular, proceeding clockwise through C_t from $\tilde{R}$, the arc γ is the kth traversing arc encountered. (Similarly, proceeding anticlockwise through C_t from $\tilde{R}$, the arc γ is the $(k+1)$th traversing arc encountered.)

Lemma 3.12. *Given* $(C, R) \in \mathrm{DT}(2m_1, 2m_2)$, *there is a unique special arrow diagram* $a \in \mathrm{SA}(2m_1, 2m_2)$ *such that* $\Psi(a) = (C, R)$.

Proof. Let C_t be the arc diagram obtained by removing all insular arcs from C; as C is traversing, C_t is non-empty, with $2k > 0$ arcs. The complementary regions of C are subsets of the complementary regions of C_t; denote the complementary region of C_t containing R as $\widetilde{R}$.

Proceed clockwise through C_t from $\widetilde{R}$; denote the kth traversing arc encountered as γ, orient it outward, and draw a special inward and outward arrow at its endpoints. By the preceding remark, if a is an arrow diagram such that $\Psi(a) = (C, R)$, then the special arrows must be located at these points.

Now return to the original diagram C, cut along γ, and remove a small neighborhood of some point $p \in R$. Then we have an annulus $(S', F(2m_1 + 2m_2 - 2, 0))$, containing an arc diagram C'. By Lemma 3.6, there exists a unique arrow diagram a' on $(S', F(2m_1 + 2m_2 - 2, 0))$ such that $\Phi(a') = C'$.

We now take the special arrow diagram a to consist of the arrows of a', together with the special arrows constructed above. By construction, applying Ψ to a first reconstructs γ; then cuts along γ and removes a point p; then reconstructs the arc diagram C' on $(S', F(2m_1 + 2m_2 - 2, 0))$; and finally fills in the hole and selects the region containing the filled-in hole. This region is R, since the construction removes a point from R to create the annulus S'. Thus, $\Psi(a) = (C, R)$.

To show uniqueness, suppose we have a special arrow diagram $\widetilde{a}$ satisfying $\Psi(\widetilde{a}) = (C, R)$; we will show $\widetilde{a} = a$. This $\widetilde{a}$ must first contain the special arrows of a, as remarked above. Cutting along the arc γ between them, and removing a neighborhood of a point, we obtain an arrow diagram $\widetilde{a}'$ on $(S', F(2m_1 + 2m_2 - 2, 0))$. Now applying Φ to $\widetilde{a}'$ produces an arc diagram on S' such that, after filling in the hole and labeling the region R and re-gluing along γ, we obtain (C, R). Thus, $\Phi(\widetilde{a}') = \Phi(a')$. By Lemma 3.6, Φ is bijective, so $\widetilde{a}' = a'$, and combined with the special arrows, which agree, we have $\widetilde{a} = a$. $\square$

We have now shown Ψ is bijective, so

$$|\mathrm{DT}(2m_1, 2m_2)| = |\mathrm{SA}(2m_1, 2m_2)| = m_1 m_2 \binom{2m_1}{m_2} \binom{2m_2}{m_2}$$

and since above we showed $|\mathrm{DT}(b_1, b_2)| = \frac{b_1+b_2}{2} T(b_1, b_2)$, we have

$$T(2m_1, 2m_2) = \frac{|\mathrm{DT}(2m_1, 2m_2)|}{m_1 + m_2} = \frac{m_1 m_2}{m_1 + m_2} \binom{2m_1}{m_1} \binom{2m_2}{m_2}$$

proving Proposition 3.3 in the even case.

The argument in the odd case is similar, but with slightly different diagrams.

Definition 3.13. A *special arrow diagram* on $(S, F(2m_1+1, 2m_2+1))$ consists of a triple (f_1, f_2, a), where $f_i \in F \cap B_i$ is an *exceptional marked point* on each boundary component, and a is an arrow diagram on $(S, F \backslash \{f_1, f_2\})$. The set of special arrow diagrams on (S, F) is denoted $\mathrm{SA}(2m_1 + 1, 2m_2 + 1)$.

After removing f_1, f_2, each boundary component contains an even number of marked points, so that an arrow diagram exists. We have

$$|\mathrm{SA}(2m_1 + 1, 2m_2 + 1)| = (2m_1 + 1)(2m_2 + 1) \binom{2m_1}{m_1} \binom{2m_2}{m_2}.$$

We now define a map $\mathrm{SA}(2m_1 + 1, 2m_2 + 1) \to \mathrm{DT}(2m_1 + 1, 2m_2 + 1)$, which we will show to be a bijection. The definition is similar to the map $\mathrm{SA}(2m_1, .2m_2) \to \mathrm{DT}(2m_1, 2m_2)$. We call both maps Ψ, so that we will have bijections $\Psi : \mathrm{SA}(b_1, b_2) \to \mathrm{DT}(b_1, b_2)$ for all b_1, b_2. Let $(f_1, f_2, a) \in \mathrm{SA}(2m_1+1, 2m_2+1)$.

(i) Join the exceptional points f_1 and f_2 by a traversing arc γ. (There are many choices for γ, but they are all related by homeomorphisms of S fixing the boundary.)
(ii) Cut along γ to obtain a disc D. The arrow diagram a gives an arrow diagram on D with $2m_1 + 2m_2$ arrows, half in and half out.
(iii) Choose a point p in the interior of D and remove a small neighborhood of p. We then have an arrow diagram in $A(2m_1 + 2m_2, 0)$.
(iv) Using the bijection Φ of Sec. 3.2, we obtain a unique arc diagram $\widetilde{C}$ on this annulus; if the arcs of $\widetilde{C}$ are oriented anticlockwise around the annulus, then the orientations agree with the arrows.
(v) Glue back the neighborhood of p, which now lies in a complementary region $\widetilde{R}$ of the arc diagram $\overline{C}$ on D.
(vi) Glue the two copies of γ on ∂D back together to reconstruct the original annulus. Combining $\overline{C}$ and γ gives an arc diagram C on (S, F), and the complementary region $\widetilde{R}$ of $\overline{C}$ becomes a complementary region R of C. We define $\Psi(f_1, f_2, a) = (C, R)$.

As in the even case, the construction at each stage is unique up to equivalence, so Ψ is well-defined.

The arc diagram $\widetilde{C}$ in this construction can be regarded as an oriented arc diagram; the arcs are oriented so as to agree with the arrows. Hence, the arc diagram C resulting from the construction can be regarded as having one "exceptional" arc γ, and all other arcs oriented.

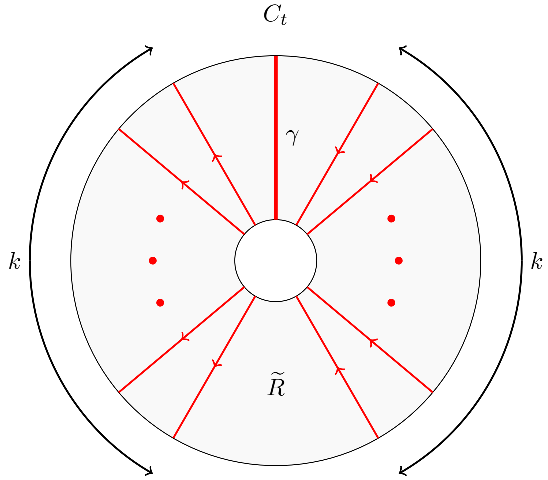

Fig. 4. The arc diagram C_t in the case $(b_1, b_2) = (2m_1 + 1, 2m_2 + 1)$.

Now consider the traversing arcs of C on (S, F). By Lemma 3.8, the number of traversing arcs is odd; let the number be $2k+1$. One of these is the exceptional arc γ, which we leave unoriented; the other $2k$ traversing arcs are oriented. As in the even case, exactly k of the oriented traversing arcs run inward, and k run outward.

Considering C_t, the arc diagram with insular arcs removed, we have $2k + 1$ traversing arcs, consisting of the exceptional arc γ, together with k inward arcs and k outward arcs. These cut the annulus S into $2k + 1$ complementary regions, which are naturally in a cyclic order, so we can proceed clockwise or anticlockwise through them. One of these regions $\widetilde{R}$ contains the decorated region R from the construction.

Again, just as in the even case, starting from γ and proceeding anticlockwise, the first k traversing arcs of C_t after γ are oriented outward; then we pass through the region $\widetilde{R}$; then the final k traversing arcs of C_t are oriented inward. For in the construction of C, after cutting along γ and removing a neighborhood of p, we construct an oriented arc diagram on the resulting annulus where arcs run anticlockwise. So those traversing arcs in S which are anticlockwise of $\widetilde{R}$ and clockwise of γ are oriented inward, and those clockwise of $\widetilde{R}$ and anticlockwise of γ are oriented outward. See Fig. 4.

Hence, proceeding clockwise through C_t from $\widetilde{R}$, the arc γ is the $(k + 1)$th traversing arc encountered. (Similarly, proceeding anticlockwise through C_t from $\widetilde{R}$, the arc γ is the $(k + 1)$th traversing arc encountered.)

We can now prove Ψ is bijective; again the proof is similar to the even case.

Lemma 3.14. *Given a pair* $(C, R) \in \mathrm{DT}(2m_1+1, 2m_2+1)$, *there is a unique special arrow diagram* $(f_1, f_2, a) \in \mathrm{SA}(2m_1 + 1, 2m_2 + 1)$ *such that* $\Psi(f_1, f_2, a) = (C, R)$.

Proof. First consider C_t, the arc diagram obtained from C by removing insular arcs. Let C_t contain $2k+1>0$ arcs, and let the complementary region of C_t containing R be $\widetilde{R}$. Proceed clockwise through C_t from $\widetilde{R}$; let the $(k+1)$th traversing arc encountered be γ. Let its endpoints on B_1 and B_2 be f_1 and f_2, respectively.

Now return to the original diagram C, cut along γ, and remove a small neighborhood of a point $p \in R$. Then we have an annulus S' with boundary conditions $F(2m_1+2m_2,0)$ and an arc diagram C'. By Lemma 3.6, there exists a unique arrow diagram a' on $(S', F(2m_1+2m_2,0))$ such that $\Phi(a')=C'$.

After gluing back along γ, the arrow diagram a' gives an arrow diagram a on the original annulus with the points f_1, f_2 removed. We take (f_1, f_2, a) as our special arrow diagram.

We claim that $\Psi(f_1, f_2, a) = (C, R)$. First, we connect f_1 to f_2, constructing γ, up to equivalence. Then we cut along γ and remove a point p; then we reconstruct C' on $(S', F(2m_1+2m_2,0))$; and finally, we fill the hole, select the region containing the filled-in hole, and glue back together along γ. The diagram obtained is C, and the region is R, since the construction removes a point from R to create the annulus S'. Thus, $\Psi(f_1, f_2, a) = (C, R)$.

To show uniqueness, suppose, we have an exceptional arrow diagram $(\widetilde{f}_1, \widetilde{f}_2, \widetilde{a})$ satisfying $\Psi(\widetilde{f}_1, \widetilde{f}_2, \widetilde{a}) = (C, R)$. This $\widetilde{a}$ must first have the same exceptional points as constructed, and the same arc γ (up to equivalence). Cutting along γ and removing a neighborhood of a point, we obtain an arrow diagram $\widetilde{a}'$ on $(S', F(2m_1+2m_2,0))$; applying Φ to $\widetilde{a}'$ produces the same arc diagram as a', so by bijectivity of Φ, we have $\widetilde{a}' = a'$, and hence $\widetilde{a} = a$. □

We have now shown Ψ is a bijection $\mathrm{SA}(2m_1+1, 2m_2+1) \to \mathrm{DT}(2m_1+1, 2m_2+1)$. Comparing the sizes of these sets, we have $(2m_1+1)(2m_2+1)\binom{2m_1}{m_1}\binom{2m_2}{m_2} = (m_1+m_2+1)T(2m_1+1, 2m_2+1)$, and we conclude

$$T(2m_1+1, 2m_2+1) = \frac{(2m_1+1)(2m_2+1)}{m_1+m_2+1}\binom{2m_1}{m_1}\binom{2m_2}{m_2}.$$

This proves the second half of Proposition 3.3. Putting together our counts of insular and traversing arc diagrams, we can compute $G_{0,2}(b_1, b_2)$ as $I(b_1, b_2) + T(b_1, b_2)$. We have now proved Proposition 1.2(2)–(3).

3.4. *Non-boundary-parallel diagrams*

It is straightforward to compute $N_{0,1}(b_1)$ and $N_{0,2}(b_1, b_2)$. (Recall the notation $\bar{b}$ from Definition 1.6.)

Lemma 3.15. *For any integer $b \geq 0$,*

$$N_{0,1}(0) = 1$$

$$N_{0,2}(b, b) = \bar{b}.$$

All other $N_{0,1}(b_1)$ and $N_{0,2}(b_1, b_2)$ are zero.

Proof. On a disc, every arc is boundary-parallel, so $N_{0,1}(0) = 1$, and all other $N_{0,1}(b) = 0$.

On an annulus, if there are no boundary-parallel arcs, then every arc must be traversing. It follows that $b_1 = b_2 = b$, and once one arc is drawn the others are determined up to equivalence. If $b > 0$, then this gives b equivalence classes of arc diagrams; if $b = 0$, then there is one equivalence class, namely that of the empty diagram. $\square$

This establishes Eqs. (7)–(8) in Proposition 1.5.

4. Decomposing Arc Diagrams

4.1. *Canonical decomposition*

We now show how to decompose an arc diagram C on $S = S_{g,n}$ into arc diagrams on annular neighborhoods $A_1, \ldots, A_n$ of the boundary components $B_1, \ldots, B_n$ ("local" to the boundary components), together with an arc diagram on the remaining surface $S' = S \backslash (\bigcup_{i=1}^n A_i)$ (the "core"). The annuli A_i contain all boundary-parallel curves, and S' contains no boundary-parallel arcs.

Definition 4.1. Let $S = S_{0,2}$ be an annulus with boundary components B, B', and let $F = F(b, b')$ consist of b points on B and b' points on B'. Let C be an arc diagram on (S, F).

(i) If every arc of C intersecting B' is traversing, then C is called *B-local*, or *b-local with b' legs*.
(ii) The set of equivalence classes of b-local arc diagrams with b' legs is denoted $L(b, b')$.

Note that in a b-local arc diagram with b' legs, we must have $b' \leq b$ and $b \equiv b'$ (mod 2).

Definition 4.2. Let $S = S_{g,n}$ have boundary components $B_1, \ldots, B_n$ and let C be an arc diagram on $(S, F(b_1, \ldots, b_n))$. A *local decomposition* of C consists of a set of disjoint simple closed curves $B'_1, \ldots, B'_n$ on S, such that the following conditions hold.

(i) Cutting S along $\bigcup_{i=1}^n B'_i$ produces a collection of annuli $A_1, \ldots, A_n$, where each annulus A_i has boundary $\partial A_i = B_i \cup B'_i$, and a surface S' (the *core*) homeomorphic to S.
(ii) The restriction C_i of the arc diagram C to each annulus A_i is B_i-local.
(iii) The restriction C' of the arc diagram C to S' contains no boundary-parallel arcs.

See Fig. 5. We will show that a local decomposition of an arc diagram exists and is unique up to a natural form of equivalence.

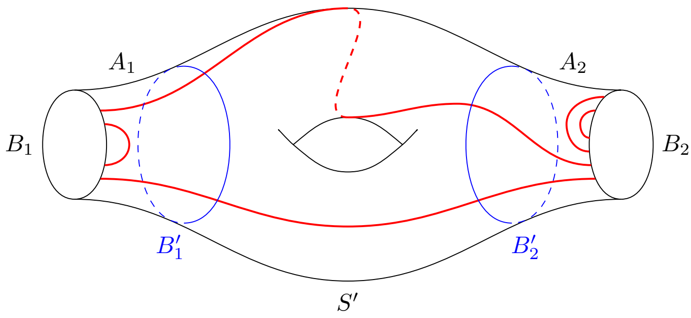

Fig. 5. Local decomposition of an arc diagram.

Lemma 4.3. *Let S be an oriented connected compact surface with boundary, other than a disc. Any arc diagram C on S has a local decomposition $B_1', \ldots, B_n'$. If $B_1', \ldots, B_n'$ and $B_1'', \ldots, B_n''$ are two local decompositions of C, then there is a homeomorphism $\phi : S \to S$, fixing ∂S pointwise, such that $\phi(B_i) = \phi(B_i')$ and $\phi(C) = C$.*

(On a disc, a local decomposition is obtained by drawing B_1' inside a single complementary region; drawing B_1' in distinct complementary regions leads to inequivalent local decompositions.)

Note that the homeomorphism ϕ of the proposition takes each annulus A_i of the first decomposition to the corresponding annulus A_i'' of the second decomposition, while fixing their common boundary B_i pointwise, so that the arc diagrams on A_i and A_i'' are homeomorphic. The fact that A_i, A_i'' are B_i-local, then implies that ϕ identifies the points of $B_i' \cap C$ and $B_i'' \cap C$ in a canonical way. The core S' of the first decomposition is taken to the core S'' of the second decomposition, with boundary points identified, so that the arc diagrams on S' and S'' are homeomorphic.

Proof. First, we show a local decomposition exists. Consider an annulus A_i obtained by taking a small collar neighborhood of the boundary component B_i, enlarged to contain neighborhoods of each arc of C parallel to B_i. We can take such A_i to be disjoint. Let the boundary components of A_i be B_i and B_i', and let $S' = S \backslash \bigcup_{i=1}^n A_i$. Then C_i $(= C \cap A_i)$ consists of arcs parallel to B_i, and traversing arcs, so is B_i-local. Moreover, C' $(= C \cap S')$ contains no boundary-parallel arcs: if γ' were such an arc, then γ' would lie in a boundary-parallel arc γ of C, so would be contained in A_i, not in S'.

To demonstrate uniqueness, we show any local decomposition must look like the one just described. Consider a local decomposition $B_1', \ldots, B_n'$ of C, and an arc γ of C with an endpoint on B_i of S. Either γ is boundary-parallel to B_i, or γ is not boundary-parallel.

If γ is boundary-parallel to B_i, then in any local decomposition, the annulus containing B_i must contain γ: if γ took any other route, then it would create a boundary-parallel arc in S', or an arc in some A_j boundary-parallel to B'_j, violating the definition of local decomposition.

Similarly, if γ is not boundary-parallel, let γ have endpoints on B_i and B_j (possibly $i = j$). Then in any local decomposition, γ must proceed from B_i across annulus A_i via a traversing arc, across the core S' to the annulus A_j, and then across A_j via a traversing arc to B_j. If γ took any other route, then it would create a boundary-parallel arc in S' or some A_j violating the local decomposition.

Thus, in any local decomposition of C, each A_i contains precisely the arcs of C boundary-parallel to B_i, and traversing arcs from the remaining points of $F \cap B_i$. Hence, there is a homeomorphism taking the local annuli of any decomposition to the local annuli of any other decomposition, and extending to the desired equivalence. □

4.2. *Counting arc diagrams via local decomposition*

We now take advantage of local decomposition to count arc diagrams.

Let C be an arc diagram on $(S = S_{g,n}, F(b_1, \ldots, b_n))$, with a local decomposition $B'_1, \ldots, B'_n$, local annuli A_i and core S'. Let $|C \cap B'_i| = a_i$. On each A_i, we have a B_i-local arc diagram in $L(b_i, a_i)$, for some integer a_i satisfying $0 \le a_i \le b_i$ and $a_i \equiv b_i \pmod 2$. On the core S', the arc diagram has no boundary-parallel arcs, hence lies in $\mathcal{N}_{g,n}(a_1, \ldots, a_n)$.

Conversely, arc diagrams in $L(b_i, a_i)$ and $\mathcal{N}_{g,n}(a_1, \ldots, a_n)$ can be glued together to construct an arc diagram on S in locally-decomposed form, giving a map

$$L(b_1, a_1) \times L(b_2, a_2) \times \cdots \times L(b_n, a_n) \times \mathcal{N}_{g,n}(a_1, \ldots, a_n) \to \mathcal{G}_{g,n}(b_1, \ldots, b_n).$$

However, this map is not injective: in defining an element of $L(b_i, a_i)$ or $\mathcal{N}_{g,n}(a_1, \ldots, a_n)$, we need to label the marked points; and on each curve B'_i of the local decomposition, the a_i points could be labeled in distinct ways, starting from distinct basepoints. If $a_i > 0$, then there are precisely a_i ways to label the points; indeed there is a $\mathbb{Z}_{a_i}$ action on $L(b_i, a_i)$ and $\mathcal{N}_{g,n}(a_1, \ldots, a_n)$. If $a_i = 0$, then there is no basepoint to choose; effectively there is precisely one choice.

Thus, there is a $\mathbb{Z}_{\overline{a}_i}$ action on each $L(b_i, a_i)$ and $\mathcal{N}_{g,n}(a_1, \ldots, a_n)$, cyclically relabeling the points on B'_i. The orbits of the induced action of $\mathbb{Z}_{\overline{a}_1} \times \cdots \times \mathbb{Z}_{\overline{a}_n}$ on $L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}_{g,n}(\mathbf{a})$ correspond to equivalence classes of arc diagrams on (S, F). So, we obtain an injective quotient map

$$\frac{L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}_{g,n}(a_1, \ldots, a_n)}{\mathbb{Z}_{\overline{a}_1} \times \cdots \times \mathbb{Z}_{\overline{a}_n}} \to \mathcal{G}_{g,n}(b_1, \ldots, b_n).$$

Taken over all a_i satisfying $0 \le a_i \le b_i$ and $a_i \equiv b_i \pmod 2$, we obtain a bijection

$$\mathcal{G}_{g,n}(b_1, \ldots, b_n) \to \bigsqcup_{\substack{0 \le a_i \le b_i \\ a_i \equiv b_i \pmod 2}} \frac{L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}_{g,n}(a_1, \ldots, a_n)}{\mathbb{Z}_{\overline{a}_1} \times \cdots \times \mathbb{Z}_{\overline{a}_n}}$$

giving a precise correspondence between an arc diagram and its local decomposition.

The action of $\mathbb{Z}_{\overline{a}_1} \times \cdots \times \mathbb{Z}_{\overline{a}_n}$ on $L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}_{g,n}(\mathbf{a})$ is faithful; indeed, the stabilizer of each element of $L(b_i, a_i)$ under the action of $\mathbb{Z}_{\overline{a}_i}$ is trivial. Thus, we obtain the following proposition.

Lemma 4.4. *For any* $g \geq 0$, $n \geq 1$ *other than* $(g, n) = (0, 1)$, *and any* $b_1, \ldots, b_n \geq 0$, *we have*

$$G_{g,n}(b_1, \ldots, b_n) = \sum_{\substack{0 \leq a_i \leq b_i \\ a_i \equiv b_i \pmod 2}} \frac{|L(b_1, a_1)| \ \cdots \ |L(b_n, a_n)|}{\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n} N_{g,n}(a_1, \ldots, a_n). \qquad \square$$

We now calculate $L(b, a)$, using a bijection similar to Sec. 3.2. So, let $(S, F) = (S_{0,2}, F(b, a))$ now denote an annulus with boundary components B, B' containing b and a points, respectively, with B as "outer" and B' as "inner" boundary. Consider b-local arc diagrams C with a legs. Such an arc diagram C must have $\frac{1}{2}(b - a)$ boundary-parallel arcs.

Definition 4.5. A *local arrow diagram* on (S, F) is a labeling of $\frac{1}{2}(b - a)$ points of $F \cap B$ as "in"; other points of F remain unlabeled.

Lemma 4.6. *For any integers* $0 \leq a \leq b$ *of the same parity*,

$$|L(b, a)| = \binom{b}{\frac{1}{2}(b - a)} \bar{a}.$$

Proof. From the data of a local arrow diagram, we attempt to construct b-local arc diagrams with a legs as follows. Start at a marked point on B and proceed anticlockwise around B. Each time, we arrive at a point of F labeled "in", we start drawing a new arc anticlockwise. Each time, we arrive at a point of F that is unlabeled, we end an arc there if possible. This process produces a partial arc diagram on the annulus, consisting only of anticlockwise-oriented insular arcs, with a remaining points on each boundary component which are unlabeled and not yet joined by arcs.

After boundary-parallel arcs of a b-local arc diagram are drawn, the remaining points are connected by traversing arcs. If $a > 0$, then these remaining points can be connected in a ways: the first point on B' can be connected to any remaining point on B, and then the remaining points can only be connected by traversing arcs in one way. If $a = 0$, however all points are connected, and we already have a complete arc diagram.

Thus, a local arrow diagram uniquely determines the boundary-parallel arcs of a b-local arc diagram. Uniqueness can be proved by an inductive argument similar to Sec. 3.2. And conversely, the boundary-parallel arcs of a local arc diagram immediately provide a local arrow diagram, by orienting them anticlockwise. So, specifying the boundary-parallel arcs is equivalent to specifying a local arrow diagram.

Once boundary-parallel arcs are drawn, the a traversing arcs can be drawn in $\bar{a}$ ways, up to equivalence, giving the desired result. $\square$

Lemmas 4.4 and 4.6 now immediately provide a proof of Theorem 1.8. Note the results holds even when some or all of the b_i are zero. We can even regard them as holding when some b_i is negative, if we regard such $G_{g,n}(b_1, \ldots, b_n)$ as zero. (Recall, we regard a binomial coefficient $\binom{N}{M}$ as zero except when $0 \leq M \leq N$ are integers.)

5. Counting Curves on Pants

5.1. *Approach*

We now turn our attention to pairs of pants. Throughout this section, let $(S, F) = (S_{0,3}, F(b_1, b_2, b_3))$. We will first compute $N_{0,3}(b_1, b_2, b_3)$, then use local decomposition to compute $G_{0,3}(b_1, b_2, b_3)$.

We set some conventions. We draw pants as twice-punctured discs in the plane, with one outer boundary B_1 and two inner boundaries, B_2 (on the left) and B_3 (on the right). The orientation on the plane induces an orientation on the pants, hence on boundary components: B_1 is oriented anticlockwise, and B_2, B_3 are oriented clockwise. See Fig. 6.

We also establish some terminology, extending terminology from the annulus case. See Fig. 7.

Definition 5.1. An arc on a pair of pants (S, F) is

(i) *traversing* if its endpoints lie on distinct boundary components;
(ii) *prodigal* if its endpoints lie on the same boundary component, but it is not boundary-parallel;
(iii) *insular* if it is boundary-parallel.

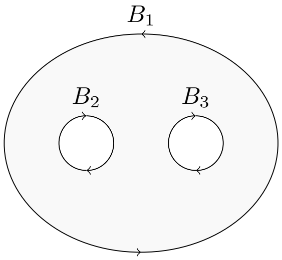

Fig. 6. Orientations on boundary components of pants.

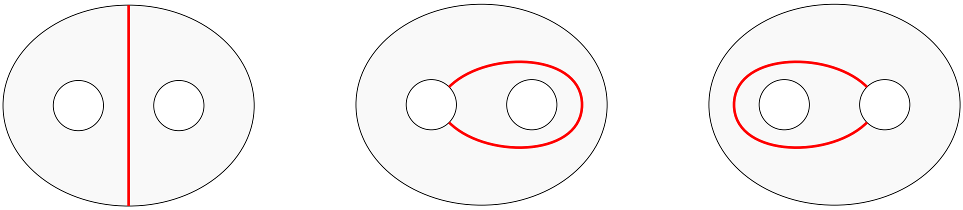

Fig. 7. Three prodigal arcs.

Thus, a prodigal arc travels far from home but eventually returns; an insular arc never goes far from home. In a local decomposition, insular arcs are contained in local annuli, while prodigal and traversing arcs pass through the core.

Definition 5.2. In an arc diagram on a pair of pants, let the number of

(i) prodigal arcs with endpoints on B_j be p_j;
(ii) traversing arcs with endpoints on B_i and B_j be t_{ij}.

5.2. *Non-boundary-parallel arc diagrams*

We now compute $N_{0,3}$, given by Eq. (9) in Proposition 1.5: for any integers $b_1, b_2, b_3 \geq 0$ such that $b_1 + b_2 + b_3$ is even,

$$N_{0,3}(b_1, b_2, b_3) = \bar{b}_1 \bar{b}_2 \bar{b}_3;$$

and if $b_1+b_2+b_3$ is odd, then $N_{0,3}(b_1, b_2, b_3) = 0$. The odd case is clear (Lemma 2.4), so we assume $b_1 + b_2 + b_3$ is even.

Now an arc diagram in $\mathcal{N}_{0,3}(b_1, b_2, b_3)$ contains only prodigal and traversing arcs. Counting the endpoints of prodigal and traversing arcs we have

$$b_1 = 2p_1 + t_{12} + t_{31}, \quad b_2 = 2p_2 + t_{23} + t_{12}, \quad b_3 = 2p_3 + t_{31} + t_{12}.$$

A prodigal arc cuts the pants into two annuli. If $p_1 > 0$, then a prodigal arc with endpoints on B_1 separates B_2 from B_3, so that there cannot be any traversing arc from B_2 to B_3, nor any prodigal arcs from these components; hence $p_2 = p_3 = t_{23} = 0$. Similarly, if $p_2 > 0$, then $p_3 = p_1 = t_{31} = 0$; and if $p_3 > 0$, then $p_1 = p_2 = t_{12} = 0$. In fact, such conditions are also sufficient to be able to draw an arc diagram. We can state this precisely.

Lemma 5.3. *There exists an arc diagram without boundary-parallel arcs on a pair of pants if and only if $t_{12}, t_{23}, t_{31}, p_1, p_2, p_3$ satisfy the following conditions*:

(i) *If $p_1 > 0$, then $p_2 = p_3 = t_{23} = 0$.*
(ii) *If $p_2 > 0$, then $p_3 = p_1 = t_{31} = 0$.*
(iii) *If $p_3 > 0$, then $p_1 = p_2 = t_{12} = 0$.*

(Note that if $p_1 = p_2 = p_3 = 0$, these conditions are all satisfied.)

Proof. The discussion above shows that the conditions are necessary. Now suppose, we have p_i and t_{ij} satisfying these conditions. If all $p_i = 0$, then the only possible non-zero parameters are t_{12}, t_{23}, t_{31} and such traversing arcs can easily be drawn. If some p_i is non-zero, say p_1, then the only possible non-zero parameters are t_{12} and t_{31}. After drawing p_1 parallel prodigal arcs with endpoints on B_1, there remain two complementary annuli on which any number of traversing arcs from B_1 to B_2, and from B_3 to B_1, can be drawn. $\square$

We have seen above three equations expressing the b_i in terms of the p_i and t_{ij}. The p_i and t_{ij} can also be expressed in terms of the b_i, although the situation is a little more complicated, as the next lemma shows.

Lemma 5.4. *Let $b_1, b_2, b_3 \geq 0$ be integers such that $b_1 + b_2 + b_3$ is even. Then there are unique non-negative integers $t_{12}, t_{23}, t_{31}, p_1, p_2, p_3$ satisfying the following conditions*:

(i) (a) $b_1 = t_{12} + t_{31} + 2p_1$
    (b) $b_2 = t_{23} + t_{12} + 2p_2$
    (c) $b_3 = t_{31} + t_{23} + 2p_3$
(ii) (a) *If $p_1 > 0$, then $p_2 = p_3 = t_{23} = 0$.*
    (b) *If $p_2 > 0$, then $p_3 = p_1 = t_{31} = 0$.*
    (c) *If $p_3 > 0$, then $p_1 = p_2 = t_{12} = 0$.*

Explicitly, such $t_{12}, t_{23}, t_{31}, p_1, p_2, p_3$ are given as follows. Let $\{i, j, k\} = \{1, 2, 3\}$ such that $b_i \leq b_j \leq b_k$.

(i) *If $b_i + b_j \geq b_k$, then $p_1 = p_2 = p_3 = 0$ and*

$$t_{12} = \frac{1}{2}(b_1 + b_2 - b_3), \quad t_{23} = \frac{1}{2}(b_2 + b_3 - b_1), \quad t_{31} = \frac{1}{2}(b_3 + b_1 - b_2).$$

(ii) *If $b_i + b_j < b_k$, then $p_i = p_j = t_{ij} = 0$ and*

$$p_k = \frac{1}{2}(b_k - b_i - b_j), \quad t_{ik} = b_i, \quad t_{jk} = b_j.$$

The two cases above correspond to whether or not b_1, b_2, b_3 obey the *triangle inequality* — that is, when any two of the b_i sum to at least the third.

Proof. First, we note that the triangle inequality is satisfied if and only if all $p_i = 0$. For if some p_i, say p_1, is positive, then $p_2 = p_3 = t_{23} = 0$ so $b_1 = 2p_1 + t_{12} + t_{31} > t_{12} + t_{31} = b_2 + b_3$ and the triangle inequality is violated. And if all $p_i = 0$, then we have $b_1 = t_{12} + t_{31}$, $b_2 = t_{23} + t_{12}$ and $b_3 = t_{31} + t_{12}$ so, for instance, $b_1 + b_2 = 2t_{12} + t_{23} + t_{31} \geq t_{23} + t_{31} = b_3$ and the triangle inequality holds.

Now if the triangle inequality holds, then all $p_i = 0$ so the b_i are given by $b_1 = t_{12} + t_{31}$, $b_2 = t_{23} + t_{12}$ and $b_3 = t_{31} + t_{23}$. This system of linear equations can be inverted to give the unique solution claimed for t_{12}, t_{23}, t_{31}, which are all non-negative by the triangle inequality.

If the triangle inequality fails, then some $p_i > 0$, say $p_1 > 0$, so $p_2 = p_3 = t_{23} = 0$ and we have $b_1 = t_{12} + t_{31} + 2p_1$, $b_2 = t_{12}$ and $b_3 = t_{31}$. So b_2, b_3 are as claimed and we immediately obtain $p_1 = \frac{1}{2}(b_1 - b_2 - b_3)$. $\square$

Proof of (9) in Proposition 1.5. Given $b_1, b_2, b_3 \geq 0$ with even sum, Lemma 5.4 shows that there exist unique t_{ij} and p_i which satisfy the conditions of Lemma 5.3, and hence give the numbers of traversing and prodigal arcs in any arc diagram in $\mathcal{N}_{0,3}(b_1, b_2, b_3)$. With the numbers of each type of arc determined, the arc diagram is uniquely determined, up to labeling of points on the boundary. There are $\bar{b}_i$ ways to choose a basepoint from the b_i points on the boundary component B_i, which determines the arc diagram up to equivalence. Hence, $N_{0,3}(b_1, b_2, b_3) = \bar{b}_1\bar{b}_2\bar{b}_3$ as claimed. □

5.3. *General arc diagrams*

From Theorem 1.8, we can now express $G_{0,3}$ in terms of $N_{0,3}$:

$$G_{0,3}(b_1, b_2, b_3) = \sum_{a_1,a_2,a_3\in\mathbb{Z}} \begin{pmatrix} b_1 \\ \frac{b_1 - a_1}{2} \end{pmatrix} \begin{pmatrix} b_2 \\ \frac{b_2 - a_2}{2} \end{pmatrix} \begin{pmatrix} b_3 \\ \frac{b_3 - a_3}{2} \end{pmatrix} \bar{a}_1\bar{a}_2\bar{a}_3 \tag{12}$$

so it remains to calculate the sum

$$\sum_{a\in\mathbb{Z}} \begin{pmatrix} b \\ \frac{b-a}{2} \end{pmatrix} = \sum_{\substack{0\le a\le b \\ a\equiv b \pmod 2}} \begin{pmatrix} b \\ \frac{b-a}{2} \end{pmatrix} \bar{a} = \sum_{\substack{0\le a\le b \\ a\equiv b \pmod 2}} L(b,a).$$

In fact, we will calculate some more general sums, which will prove useful in the sequel, applying ideas from [33]. Several of the following definitions come from that paper.

Definition 5.5. For an integer $\alpha \geq 0$, define the functions $\tilde{p}_\alpha(n), \tilde{q}_\alpha(n), \tilde{P}_\alpha(n)$, $\tilde{Q}_\alpha(n)$ as follows.

$$\tilde{P}_\alpha(n) = \sum_{l=0}^{n} \binom{2n}{n-l} \overline{(2l)}\ (2l)^{2\alpha},$$

$$\tilde{p}_\alpha(n) = \sum_{l=0}^{n} \binom{2n}{n-l} (2l)^{2\alpha+1},$$

$$\tilde{Q}_\alpha(n) = \tilde{q}_\alpha(n) = \sum_{l=0}^{n} \binom{2n+1}{n-l} \overline{(2l+1)}(2l+1)^{2\alpha} = \sum_{l=0}^{n} \binom{2n+1}{n-l} (2l+1)^{2\alpha+1}.$$

(Defining separate $\tilde{Q}_\alpha$ and $\tilde{q}_\alpha$ is useful for our notation in the sequel.) Clearly $\tilde{P}_\alpha(n)$ differs from $\tilde{p}_\alpha$ only in the $l = 0$ term, and this only when $\alpha = 0$, so

$$\tilde{P}_\alpha(n) = \tilde{p}_\alpha(n) + \binom{2n}{n} \delta_{\alpha,0}.$$

Norbury–Scott in [33] show that $\tilde{p}_\alpha(n), \tilde{q}_\alpha(n)$ are closely related to the following polynomials $p_\alpha(n), q_\alpha(n)$.

Definition 5.6. For integers $\alpha \geq 0$, the integer polynomials $p_\alpha(n), q_\alpha(n)$ are defined recursively by

$$p_0(n) = 1, \quad p_{\alpha+1}(n) = 4n^2(p_\alpha(n) - p_\alpha(n-1)) + 4np_\alpha(n-1)$$

$$q_0(n) = 1, \quad q_{\alpha+1}(n) = 4n^2(q_\alpha(n) - q_\alpha(n-1)) + (4n+1)q_\alpha(n).$$

(Equation (15) in [33] appears to have a typo; the $(4n+1)q_\alpha(n-1)$ should be $(4n+1)q_\alpha(n)$.)

Proposition 5.7 (Norbury–Scott [33]). *Let $\alpha \geq 0$ be an integer. Then p_α, q_α are integer polynomials of degree α with positive leading coefficients. Moreover,*

$$\tilde{p}_\alpha(n) = \binom{2n}{n} n\, p_\alpha(n) \quad \textit{and} \quad \tilde{q}_\alpha(n) = \binom{2n}{n}(2n+1)q_\alpha(n).$$

Further,

$$\tilde{P}_\alpha(n) = \binom{2n}{n} P_\alpha(n) \quad \textit{and} \quad \tilde{Q}_\alpha(n) = \binom{2n}{n} Q_\alpha(n),$$

where $P_\alpha(n) = np_\alpha(n) + \delta_{\alpha,0}$ and $Q_\alpha(n) = (2n+1)q_\alpha(n)$ are integer polynomials of degree $\alpha+1$ with positive leading coefficients.

Proof. Norbury–Scott [33] show that $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$ are as claimed, and p_α, q_α have degree α. It is clear from the recurrence that the coefficients are integers and the leading coefficients are positive. The claims for $\tilde{P}_\alpha$ and $\tilde{Q}_\alpha$, then follow immediately from $\tilde{P}_\alpha(n) = \tilde{p}_\alpha(n) + \binom{2n}{n}\delta_{\alpha,0}$ and $\tilde{Q}_\alpha(n) = \tilde{q}_\alpha(n)$. $\square$

We compute the first few of the sums $\tilde{P}_\alpha(n)$ and $\tilde{Q}_\alpha(n)$.

$$\begin{aligned}
\tilde{P}_0(n) &= \binom{2n}{n}(n+1) & \tilde{Q}_0(n) &= \binom{2n}{n}(2n+1)\\
\tilde{P}_1(n) &= \binom{2n}{n} n\, 4n & \tilde{Q}_1(n) &= \binom{2n}{n}(2n+1)\,(4n+1)\\
\tilde{P}_2(n) &= \binom{2n}{n} n\, 16n(2n-1) & \tilde{Q}_2(n) &= \binom{2n}{n}(2n+1)\,(32n^2+8n+1).
\end{aligned}$$

Observe that, for $\alpha \geq 0$ an integer, then

$$\sum_{\substack{0 \leq a \leq b \\ a \equiv b \pmod 2}} \binom{b}{\frac{b-a}{2}} \overline{a}\, a^{2\alpha} = \begin{Bmatrix} \tilde{P}_\alpha(m) \\ \tilde{Q}_\alpha(m) \end{Bmatrix} = \begin{cases} \binom{2m}{m} P_\alpha(m) & b \text{ even, } b = 2m \\ \binom{2m}{m} Q_\alpha(m) & b \text{ odd, } b = 2m+1 \end{cases}$$

$$\begin{aligned}
\sum_{\substack{0 \leq a \leq b \\ a \equiv b \pmod 2}} \binom{b}{\frac{b-a}{2}} a^{2\alpha+1} &= \begin{Bmatrix} \tilde{p}_\alpha(m) \\ \tilde{q}_\alpha(m) \end{Bmatrix}\\
&= \begin{cases} \binom{2m}{m} m p_\alpha(m) & b \text{ even, } b = 2m \\ \binom{2m}{m}(2m+1) q_\alpha(m) & b \text{ odd, } b = 2m+1. \end{cases}
\end{aligned}$$

We have now computed the sums arising in $G_{0,3}(b_1, b_2, b_3)$ and we have the following.

$$G_{0,3}(b_1, b_2, b_3) = \prod_{i=1}^{3} \sum_{\substack{0 \le a_i \le b_i \\ a_i \equiv b_i \pmod 2}} \binom{b_i}{\frac{b_i - a_i}{2}} \bar{a}_i$$

$$= \prod_{i=1}^{3} \begin{Bmatrix} \binom{2m_i}{m_i}(m_i + 1) & b_i \text{ even, } b_i = 2m_i \\ \binom{2m_i}{m_i}(2m_i + 1) & b_i \text{ odd, } b_i = 2m_i + 1 \end{Bmatrix}.$$

This immediately gives the formulae in Proposition 1.2(4)–(5).

6. Combinatorial Topological Recursion

6.1. *For curve counts*

We now show that the $G_{g,n}(b_1, \dots, b_n)$ obey the recursion of Theorem 1.4: for $b_1 > 0$ and $b_2, \dots, b_n \ge 0$,

$$\begin{aligned} G_{g,n}(b_1, \dots, b_n) = &\sum_{\substack{i,j \ge 0 \\ i+j = b_1 - 2}} G_{g-1,n+1}(i, j, b_2, \dots, b_n) \\ &+ \sum_{k=2}^{n} b_k G_{g,n-1}(b_1 + b_k - 2, b_2, \dots, \widehat{b_k}, \dots, b_n) \\ &+ \sum_{\substack{i,j \ge 0 \\ i+j = b_1 - 2}} \sum_{\substack{g_1, g_2 \ge 0 \\ g_1 + g_2 = g}} \sum_{I \sqcup J = \{2, \dots, n\}} G_{g_1, |I|+1}(i, b_I) G_{g_2, |J|+1}(j, b_J). \end{aligned}$$

Here, the first term is a sum over integers $i, j \ge 0$ summing to $b_1 - 2$; if $b_1 = 1$ this sum is empty. In the second term, the notation $\widehat{b_k}$ means that b_k is omitted from the list $b_2, \dots, b_n$. In the third term, the sum over I, J is a sum over all pairs of (possibly empty) disjoint sets (I, J) whose union is $\{2, \dots, n\}$. The notation b_I is shorthand for the set of all b_k, where $k \in I$; and similarly for b_J. As $G_{g,n}(b_1, \dots, b_n)$ is a symmetric function of the b_i, it is sufficient to give b_I as a set rather than a sequence.

Note that when $b_1 + \cdots + b_n$ is odd, all terms are zero; in each term the inputs to each $G_{g,n}$ have the same parity sum.

This recursion expresses $G_{g,n}$ in terms of curve counts on "simpler" surfaces. We regard the complexity of a surface as given by $-\chi$, where $\chi = 2 - 2g - n$ is the Euler characteristic. All terms on the right involve surfaces with complexity $\le -\chi(S_{g,n}) = 2g + n - 2$. The first two terms involve surfaces with complexity strictly less than $2g + n - 2$, but the third term may involve surfaces homeomorphic to S, for instance when $g_1 = g$ and $I = \{2, \dots, n\}$; however, in this case the number of marked points $b_1 + \cdots + b_n$ decreases. Thus, repeatedly applying the recursion, (and permuting the b_i if necessary, to avoid $b_1 = 0$, as $G_{g,n}$ is a symmetric function), one eventually arrives at terms of the form $G_{g,n}(\mathbf{0}) = 1$.

Proof of Theorem 1.4. Label the boundary components of $S_{g,n}$ as $B_1, \ldots, B_n$ as usual, and label the marked points on B_j by $1, \ldots, b_j$ according to the boundary orientation of B_j. Let p be the first marked point on B_1, which exists since $b_1 > 0$.

Given an arc diagram C on $(S_{g,n}, F(\mathbf{b}))$, consider the arc γ with an endpoint at p. Cutting along γ yields a surface S' with an arc diagram C' and one less arc; γ becomes two arcs on $\partial S'$. We consider the various cases for γ and show how, in each case, we can give a standard numbering on the boundary components and points, so that the arc diagrams so obtained are counted in $G_{g',n'}(\mathbf{b}')$ for "simpler" $g', n', \mathbf{b}'$. In each case, the location of γ on $\partial S'$ after cutting can be determined, so that C can be uniquely reconstructed from S' and C' by gluing these two boundary arcs together. In this way, we obtain a bijection between $\mathcal{G}_{g,n}(\mathbf{b})$ and various sets involving simpler $\mathcal{G}_{g',n'}(\mathbf{b}')$ and establish the desired recursion.

We deal with the cases as follows.

(i) *γ has both endpoints on B_1 and is non-separating.* In this case, cutting along γ gives S' of genus $g-1$ with $n+1$ boundary components. Let the endpoints of γ have labels 1 and $i+2$, for some integer i with $2 \leq i+2 \leq b_1$, i.e. $0 \leq i \leq b_1 - 2$. We name the boundary components of S' as $B'_1, \ldots, B'_{n+1}$, and their marked points, as follows. The original boundary component B_1 splits into B'_1 and B'_2 so that B'_1 contains the points originally labeled $2, \ldots, i+1$; we now number these points $1, \ldots, i$. The other boundary component contains points originally labeled $i+3, \ldots, b_1$. Letting $j = b_1 - i - 2$, we now number them $1, \ldots, j$. We obtain an element of $\mathcal{G}_{g-1,n+1}(i, j, b_2, \ldots, b_n)$, where $i, j \geq 0$ satisfy $i + j = b_1 - 2$. And given such i, j and an element of $\mathcal{G}_{g-1,n+1}(i, j, b_2, \ldots, b_n)$, we can reconstruct the original arc diagram in $\mathcal{G}_{g,n}(\mathbf{b})$, giving a bijective correspondence between arc diagrams in $\mathcal{G}_{g,n}(\mathbf{b})$ of this type, and elements of $\mathcal{G}_{g-1,n+1}(i, j, b_2, \ldots, b_n)$ for a choice of $i, j \geq 0$ with $i + j = b_1 - 2$.

(ii) *γ has endpoints on distinct boundary components.* In this case, cutting along γ gives S' of genus g with $n-1$ boundary components. Let the endpoints of γ lie on boundary components B_1 and B_k. We name the boundary components of S' as $B'_1, \ldots, B'_{n-1}$, where B'_1 is a union of B_1, B_k and the two copies of γ, and then number $B'_2, \ldots, B'_{n-1}$ in order as $B_2, \ldots, \widehat{B}_k, \ldots, B_n$. The marked points, in order around B'_1, consist of $b_1 - 1$ points of B_1, followed by $b_k - 1$ points of B_k, so that B'_1 has $b_1 + b_k - 2$ boundary points. Numbering marked points on other boundary components as on S, we obtain an element of $\mathcal{G}_{g,n-1}(b_1 + b_k - 2, b_2, \ldots, \widehat{b}_k, \ldots, b_n)$; and we also keep track of which of the b_k points on B_k was an endpoint of γ. Conversely, given an arc diagram of genus g with $n-1$ boundary components, with one of the points not marked 1 on B_1 marked, we can reconstruct an arc diagram in $\mathcal{G}_{n-1}(b_1 + b_k - 2, b_2, \ldots, \widehat{b}_k, \ldots, b_n)$ by gluing two arcs on the boundary together. This gives a bijective correspondence between arc diagrams in $\mathcal{G}_{g,n}(\mathbf{b})$ of this type with elements of $\mathcal{G}_{g,n-1}(b_1+b_k-2, b_2, \ldots, \widehat{b}_k, \ldots, b_n)$ with a special marked point on B_k.

(iii) *γ has both endpoints on B_1 and is separating.* In this case, γ cuts S into two surfaces, S' and S''; for definiteness, we say that as we proceed along γ from the point marked 1, S' is on the left and S'' is on the right. Let S' have genus g_1 and S'' have genus g_2. The boundary components of S_1 are then B_1', which contains some of B_1 and γ, as well as other boundary components B_k for $k \in I \subset \{2, \ldots, n\}$. Similarly, the boundary components of S_2 are B_1'', which contains some of B_1 and γ, as well as other boundary components B_k for $k \in J \subset \{2, \ldots, n\}$. Here, I and J are disjoint and $I \sqcup J = \{2, \ldots, n\}$. Let B_1' and B_1'' contain i and j marked points, respectively; then $i, j \geq 0$ and $i+j = b_1 - 2$. As in the previous cases, we obtain a bijection between arc diagrams in $\mathcal{G}_{g,n}(\mathbf{b})$ of this type, and elements of $\mathcal{G}_{g_1,|I|+1}(i, b_I) \times \mathcal{G}_{g_2,|J|+1}(j, b_J)$ over the various possible i, j, g_1, g_2, I, J. Since B_1 is split into two boundary components, we can number the marked points on the pair of smaller surfaces so as to indicate how they can be glued back together. □

6.2. *For non-boundary-parallel curve counts*

The $N_{g,n}(\mathbf{b})$ and $\widehat{N}_{g,n}(\mathbf{b})$ also obey a recursion, slightly more complicated than the $G_{g,n}(\mathbf{b})$ case.

Proposition 6.1. *For $(g,n) \neq (0,1), (0,2), (0,3)$ and integers $b_1, \ldots, b_n$ such that $b_1 > 0$, $b_2, \ldots, b_n \geq 0$,*

$$\begin{aligned}
N_{g,n}(\mathbf{b}) = & \sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{m}{2}\, N_{g-1,n+1}(i, j, b_2, \ldots, b_n) \\
& + \sum_{j=2}^{n} \left(\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b}_j, \ldots, b_n) \right. \\
& \left. + \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b}_j, \ldots, b_n) \right) \\
& + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\} \\ \text{No discs or annuli}}} \sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{m}{2}\, N_{g_1,|I|+1}(i, b_I)\, N_{g_2,|J|+1}(j, b_J)
\end{aligned}$$

$$
\begin{aligned}
b_1 \widehat{N}_{g,n}(\mathbf{b}) = & \sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{1}{2} \, \bar{i} \, \bar{j} \, m \, \widehat{N}_{g-1,n+1}(i,j,b_2,\ldots,b_n) \\
& + \sum_{j=2}^{n} \frac{1}{2} \left(\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \bar{i} \, m \, \widehat{N}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right. \\
& \left. + \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}}} \bar{i} \, m \widehat{N}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right) \\
& + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\} \\ \text{No discs or annuli}}} \sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{1}{2} \, \bar{i} \, \bar{j} \, m \, \widehat{N}_{g_1,|I|+1}(i,b_I) \, \widehat{N}_{g_2,|J|+1}(j,b_J).
\end{aligned}
$$

We explain the notation on the right-hand side of each equation. The tilde over the second summation in brackets is interpreted as follows. If $b_1 - b_j \geq 0$, then read the sum as is: it is a sum over non-negative integers i, m such that $i + m = b_1 - b_j$ and m is even. If $b_1 - b_j \leq 0$, then replace $b_1 - b_j$ with $b_j - b_1$ and make the sum negative: i.e. the term in the first equation becomes

$$
- \sum_{\substack{i,m \geq 0 \\ i+m=b_j-b_1 \\ m \text{ even}}} \frac{m}{2} \, \bar{b}_j \, N_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n)
$$

the "no discs or annuli" condition means that we exclude terms in which $(g_1, |I|+1)$ or $(g_2, |J|+1)$ is equal to $(0,1)$ or $(0,2)$. This idea of splitting the sum this way appears in [31]; indeed these recursions are very similar to the recursions appearing in that paper. (In fact, if we drop the bars over i's and j's, the recursion on $\widehat{N}_{g,n}$ should be identical. Norbury does not explicitly specify the parity requirements, but they are implicit. His Eq. (5) also has a typographical error, since there should be factors of $1/2$ in each term.)

Again, the terms are only non-zero when $b_1 + \cdots + b_n$ is even; but the result holds even when this sum is odd, as all terms are then zero.

The proof is based on a similar analysis as the recursion for $G_{g,n}(\mathbf{b})$, but there are significantly more subtleties arising from the lack of boundary-parallel arcs, and the argument fails for annuli and pants. Hence, we exclude $(g,n) = (0,1), (0,2), (0,3)$.

To illustrate some of the difficulties, let C be a non-boundary-parallel arc diagram on $(S_{g,n}, F(\mathbf{b}))$, and let γ be an arc of C starting at the base point on B_1. After cutting along γ, we obtain a less complex surface S' (possibly disconnected),

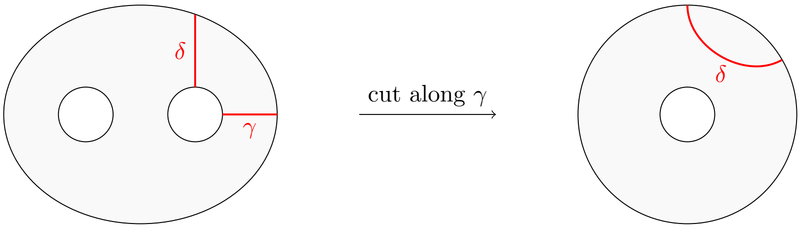

Fig. 8. Arc δ parallel to γ becomes boundary-parallel after cutting along γ.

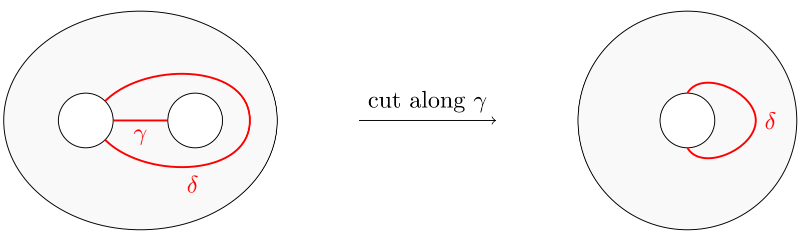

Fig. 9. Arc δ running from B_1 around B_2 becomes boundary-parallel after cutting along γ from B_1 to B_2.

with a simpler arc diagram C'. However, the arc diagram C' may contain boundary-parallel arcs: it is possible that arcs of C, which were not boundary-parallel, might become boundary-parallel after cutting along γ.

For instance, suppose, we have an arc δ which is parallel to γ. After cutting along γ, δ becomes boundary-parallel: see Fig. 8. For another example, suppose γ connects two distinct boundary components B_1 and B_2, and δ is an arc which runs from B_1, around B_2, back to B_1: see Fig. 9. Again δ is not boundary-parallel, but after cutting along γ, δ becomes boundary-parallel. We now establish that these are the only cases in which arcs can become boundary-parallel.

Lemma 6.2. *Let C be an arc diagram on $(S, F) = (S_{g,n}, F(\mathbf{b}))$ without boundary-parallel arcs. Let γ be an arc of C and let the result of cutting along γ be the arc diagram C' on (S', F'). If δ is an arc of C which is boundary-parallel in S', then exactly one of the following cases occurs*:

(i) *γ has endpoints on two distinct boundary components B_i, B_j of S, and*
   (a) *δ is parallel to γ (as in Fig.* 8)
   (b) *δ has both endpoints on B_i, and runs around B_j as in Fig.* 9;

(ii) *γ is non-separating with both endpoints on the same boundary component B_i, and δ is parallel to γ*;

(iii) *γ is separating, and δ is parallel to γ.*

Proof. Any arc γ of C falls into precisely one of the cases (i), (ii) or (iii). Clearly any arc δ of one of the types listed becomes boundary-parallel after cutting along γ.

Now suppose an arc δ becomes boundary-parallel after cutting along γ. Let the endpoints of γ lie on boundary components B_i and B_j (possibly $i = j$). Then δ must be homotopic, relative to endpoints, to an arc lying along $B_i \cup B_j \cup \gamma$. So δ is parallel to γ, or runs around B_j as claimed. □

Proof of Proposition 6.1. We first prove the recursion on $N_{g,n}$. Let p be the first marked point on B_1, and let γ be the arc of C with an endpoint at p. We consider three cases for γ; in each case, we cut along γ, and remove any arcs which become parallel (i.e. those described in Lemma 6.2) to obtain a simpler boundary-parallel arc diagram on a simpler surface. We can then construct bijections between arc diagrams on (S, F) and arc diagrams on simpler surfaces.

(i) *γ has both endpoints on B_1, and is non-separating.* Orient γ so, that p is the start point. Cutting along γ produces $S' = S_{g-1,n+1}$. So, B_1 is split into two boundary components B'_1, B'_2. Let the number of arcs parallel to γ, including γ, be $\frac{m}{2}$, so $m \geq 2$ is even. The $\frac{m}{2} - 1$ arcs parallel to γ, other than γ, are precisely the ones that become boundary-parallel in S'. Let the number of points remaining on B'_1 and B'_2 after removing these boundary-parallel arcs be i and j, respectively. Together γ and the arcs parallel to γ have m endpoints, all originally on B_1, so we have $i + j + m = b_1$. Labeling boundary points in a standard fashion, we obtain an equivalence class of arc diagram C' in $\mathcal{N}_{g-1,n+1}(i, j, b_2, \ldots, b_n)$. For any $i, j, m \geq 0$ such that $i + j + m = b_1$ and m is even, such arc diagrams C' exist. Moreover, from the data of C' and m, the original arc diagram C can be reconstructed: the boundary labeling on C' indicates which boundary segments are to be glued back together, and m parallel arcs are drawn there.

However, there may be several arc diagrams on (S, F) which lead to the same arc diagram C' and the same number m. In particular, this occurs if we take the same arc diagram C but shift the basepoint p on B_1 so that γ becomes another one of the $m/2$ arcs parallel to the original γ. All the arc diagrams on S which lead to C' and m are of this form. As γ starts at p, there are $m/2$ such possibilities for p. Hence, the number of arc diagrams in $\mathcal{N}_{G,n}(b_1, \ldots, b_n)$ for which γ has both endpoints on B_1 and is non-separating is given by the first summation in the recursion.

(ii) *γ has endpoints on distinct boundary components B_1 and B_j, or is separating* (*hence has both endpoints on B_1*) *and cuts an annulus off S.* In the latter case, let the boundary component around which γ loops be B_j, so that B_j is a boundary component of the annulus cut off by γ. (Note that as $(g, n) \neq (0, 3)$, γ cannot cut S into *two* annuli; if γ cuts off an annulus, then only one annulus appears. The possibility of two annuli causes the recursion to fail in the case $(g, n) = (0, 3)$.)

Let $m/2$ be the number of arcs "parallel" to γ, in the following sense. If γ runs from B_1 to B_j, we take the arcs parallel to γ, including γ; and also

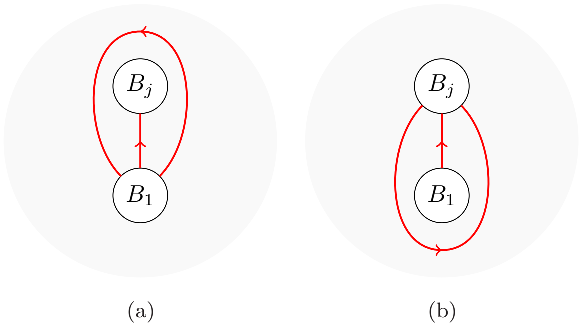

Fig. 10. Orientation on arcs running (a) from B_1 around B_j, and (b) from B_j around B_1.

those which run from B_1 around B_j and back to B_1; and also those which run from B_j around B_1 and back to B_j; these curves become boundary-parallel in $S' = S_{g,n-1}$. If γ cuts off an annulus around B_j, we take the arcs parallel to γ, including γ, and also those which run from B_1 to B_j. These $m/2$ arcs consist precisely of γ and the arcs which become boundary-parallel in S'. (Note that there cannot both be loops from B_1 around B_j, and loops from B_j around B_1. The former can only occur if $b_1 > b_j$, and the latter can only occur if $b_j > b_1$.)

For those arcs with endpoints on B_1 and B_j, orient them from B_1 to B_j. For those arcs with endpoints on B_1 which loop around B_j (i.e. those which cut off annuli), orient them as shown in Fig. 10(a) so that they run anticlockwise around B_j. For those arcs with endpoints on B_j which loop around B_1, orient them as shown in Fig. 10(b) so that they run anticlockwise around B_1.

After cutting along γ and removing all the "parallel" arcs described above — which are precisely the arcs that become boundary-parallel in S' — we obtain an arc diagram on S'. Boundary components B_1 and B_j are combined into a boundary component B'_1 of S'. Let i be the number of marked points on B'_1. Now γ and all the arcs "parallel" to it have m endpoints, so $i+m = b_1+b_j$, and m is even. Labeling boundary points in a standard fashion (starting near p, say, and proceeding around the boundary numbering points consecutively), we obtain an arc diagram C' in $\mathcal{N}_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n)$. For any integers $2 \le j \le n$ and $i, m \ge 0$ such that $i+m = b_1+b_j$ and m is even, the arc diagram C can be reconstructed from C' and m: again, the labeling on C' indicates which boundary segments of C' to glue to obtain two boundary components B_1, B_j with b_1, b_j marked points; and we draw m parallel arcs there, possibly including loops from B_1 around B_j or loops from B_j around B_1.

However, there may be several arc diagrams on S which lead to the same arc diagram C' on S' and the same m. For one thing, if we adjust the basepoint p on B_1 so that γ is replaced by any of the arcs "parallel" to the original γ, cutting and removing boundary-parallel arcs leads to the same C' and m. For another, the arcs from B_1 to B_j can be adjusted so as to meet B_j at different points; there are $\overline{b}_j$ ways to adjust any such diagram. (The effect is like a "fractional Dehn twist" about B_j. Equivalently, the labeling on B_j can be adjusted so as to place the basepoint at any position.) As S is not an annulus, these two types of adjustment are independent. (When S is an annulus, these two types of adjustment have the same result, and the claimed recursion fails.)

Suppose for now that $b_1 \geq b_j$. Then the $m/2$ arcs "parallel" to γ consist of arcs from B_1 to B_j, and possibly arcs from B_1 looping around B_j. (But there cannot be any arcs from B_j looping around B_1.) There are precisely $m/2$ positions for p on B_1 so that p is the *start* point of one of these arcs; and for each such choice, the points at which arcs meet B_j can be adjusted in $\bar{b}_j$ ways. Hence, the number of (equivalence classes of) arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which γ runs from B_1 to B_j, or runs from B_1 around B_j, and is oriented so that p is the start point of γ, is given by

$$\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n).$$

However, there is also the possibility that p is the *endpoint* of γ. Such a situation only arises when γ is an arc from B_1 which loops around B_j; in this case, as $b_j \leq b_1$, there must be b_j arcs connecting B_1 to B_j. Redefine $m/2$ to be the number of arcs from B_1 looping around B_j (i.e. cutting off an annulus with B_j as a boundary component). Still letting i denote the number of marked points of C' on B_1', these i points together with the m endpoints of the arcs looping around B_j and the b_j arcs from B_1 to B_j together make up all the marked points on B_1, so $i + m + b_j = b_1$. Again C can be reconstructed from C' and m. Again there are several arc diagrams on (S, F) with p as the endpoint of γ which lead to the same C' and m: we may rotate the arcs to meet B_j at different points in $\bar{b}_j$ ways; and we may adjust the basepoint p to be any of the $m/2$ points on B_1 at which an arc looping around B_j ends. Thus, the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which γ runs from B_1 around B_j, and is oriented so that p is the end point of γ, is given by

$$\sum_{\substack{i,m \geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n).$$

This covers all possibilities in the case $b_1 \geq b_j$.

Now suppose $b_1 \leq b_j$. Note that in this case the arcs "parallel" to γ consist of arcs from B_1 to B_j, and possibly arcs from B_j looping around B_1. Let $m/2$ denote the number of these "parallel" arcs. As in the case $b_1 \geq b_j$, the term

$$\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n)$$

gives the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which p is the start point of the arc γ along which we cut. However, it counts these diagrams regardless of whether p lies on B_1 or B_j! We must therefore subtract off the diagrams for which p lies on B_j.

In such cases, γ is an arc-based at B_j looping around B_1, and as $b_1 \leq b_j$, there must be b_1 arcs connecting B_1 to B_j. Redefine $m/2$ to be the number of arcs from B_j looping around B_1, so $i + m + b_1 = b_j$. By a similar argument as above, the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which γ runs from B_j around B_1, and is oriented so that p is the start point, is given by

$$\sum_{\substack{i,m \geq 0 \\ i+m=b_j-b_1 \\ m \text{ even}}} \frac{m}{2}\, \bar{b}_j\, N_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n).$$

Hence, the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which γ has endpoints on distinct boundary components, or is separating and cuts off an annulus, is given by the summations in the second line of the recursion.

(iii) *γ is separating but does not cut off an annulus.* As C has no boundary-parallel arcs, γ cannot cut off a disc either. Thus, it remains to consider separating γ where no discs or annuli arise. If we orient γ to start at p, as S is oriented, then cutting along γ there is a surface S_1 to the left of γ and a surface S_2 to its right. Let S_1 have genus g_1 and S_2 have genus g_2, so $g_1, g_2 \geq 0$ and $g_1 + g_2 = g$. After cutting along γ, boundary component B_1 contributes a boundary component B_1' to S_1 and B_1'' to S_2; the remaining boundary components of S_1 and S_2 come from the original boundary components $B_2, \ldots, B_n$ of S. Let S_1 contain boundary components whose numbers consist of $I \subset \{2, \ldots, n\}$, and let S_2 contain boundary components $J \subset \{2, \ldots, n\}$, so $I \sqcup J = \{2, \ldots, n\}$. There may be arcs which become boundary-parallel after cutting along C: such arcs will be parallel to γ; let there be $\frac{m}{2} - 1$ of them, so that γ and its parallel arcs together contain m endpoints. Let B_1', B_1'' contain i, j marked points respectively, so $i + j + m = b_1$ and m is even. Labeling boundary points in a standard fashion, we obtain an arc diagram C_1 in $\mathcal{N}_{g_1,|I|+1}(i, b_I)$ and an arc diagram $C_2 \in \mathcal{N}_{g_2,|J|+1}(j, b_J)$. For any $g_1, g_2, i, j, m \geq 0$ and I, J such that $g_1 + g_2 = g$, $I \sqcup J = \{2, \ldots, n\}$, $i + j + m = b_1$ and m is even, such arc diagrams C_1 and C_2 exist, and conversely, from C_1, C_2 and m, the original C can be reconstructed.

However, several arc diagrams on (S, F) could lead to the same C' and m: this occurs if we shift p so that γ is another one of the $m/2$ arcs parallel to the original γ. Since γ starts at p, there are $m/2$ such possibilities for p. Hence, the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ for which γ is separating, but does not cut off any discs or annuli, is given by the third line of the recursion.

Putting these cases together, the number of arc diagrams in $\mathcal{N}_{g,n}(b_1, \ldots, b_n)$ is as claimed. Dividing through by $\bar{b}_2 \cdots \bar{b}_n$ (and since $b_1 > 0$, so $\bar{b}_1 = b_1$) we obtain the recursion on $\widehat{N}_{g,n}$. □

6.3. *Applying the recursion*

So far, we have found $N_{0,1}$, $N_{0,2}$ and $N_{0,3}$ (Lemma 3.15 and Sec. 5.2), and hence also $\widehat{N}_{0,1}, \widehat{N}_{0,2}, \widehat{N}_{0,3}$. If we apply the recursion to $\widehat{N}_{1,1}(b)$ directly, then using $\widehat{N}_{0,2}(b_1, b_2) = \delta_{b_1,b_2}/\bar{b}_1$, we obtain

$$\bar{b}\widehat{N}_{1,1}(b) = \sum_{\substack{i,j,m\geq 0\\ i+j+m=b\\ m \text{ even}}} \frac{1}{2}\bar{i}\,\bar{j}\, m\, \widehat{N}_{0,2}(i,j) = \sum_{\substack{i,j,m\geq 0\\ i+j+m=b\\ m \text{ even}}} \frac{1}{2}\bar{i}\, m\delta_{i,j} = \sum_{\substack{i,m\geq 0\\ 2i+m=b\\ m \text{ even}}} \frac{1}{2}\bar{i}\, m.$$

When b is odd, there are no terms in the sum; when b is even, we obtain

$$\widehat{N}_{1,1}(b) = \frac{1}{2\bar{b}} \sum_{\substack{p,q\geq 0\\ p+q=b\\ q \text{ even}}} \overline{(p/2)}\, q = \frac{1}{4\bar{b}} \sum_{\substack{p,q\geq 0\\ p+q=b\\ q \text{ even}}} \bar{p}\, q + \frac{1}{4}. \tag{13}$$

In the last step, we used the fact that $\overline{p/2} = \bar{p}/2$, except when $p = 0$, in which case $\overline{p/2} = \bar{p}/2 + \frac{1}{2}$. In the next section, we compute this and more general sums.

7. Polynomiality Results

7.1. *Some useful sums*

We aim to show quasi-polynomiality of $\widehat{N}_{g,n}(b_1, \ldots, b_n)$ for $(g, n) \neq (0, 1), (0, 2)$. For this, it will be useful first to compute certain summations, following the techniques of Norbury in [31]. Several of the following definitions appear in that paper.

Definition 7.1. For integers $m \geq 0$, define the functions $A_m, S_m : \mathbb{N}_0 \to \mathbb{N}_0$ by the following sums:

$$A_m(k) = \sum_{\substack{p,q\geq 0\\ p+q=k\\ q \text{ even}}} \bar{p}p^{2m}q, \quad S_m(k) = \sum_{\substack{p,q\geq 0\\ p+q=k\\ q \text{ even}}} p^{2m+1}q.$$

Note that once the parity of k is given, the sum is over p and q of fixed parity: q is even, and p has the same parity as k. The functions A_m and S_m are clearly very similar; they only differ in their $p = 0$ terms, and then only when $m = 0$.

Definition 7.2. For integers $m, n \geq 0$, define the functions $B_{m,n}$, $B^0_{m,n}$, $B^1_{m,n}$, $R_{m,n}$, $R^0_{m,n}$, $R^1_{m,n} : \mathbb{N}_0 \to \mathbb{N}_0$ by the following sums.

$$B_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ r \text{ even}}} \bar{p}\,\bar{q}\,p^{2m}q^{2n}r \qquad R_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ r \text{ even}}} p^{2m+1}q^{2n+1}r$$

$$B^0_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ p \text{ even},\ r \text{ even}}} \bar{p}\,\bar{q}\,p^{2m}q^{2n}r \quad R^0_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ p \text{ even},\ r \text{ even}}} p^{2m+1}q^{2n+1}r$$

$$B^1_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ p \text{ odd},\ r \text{ even}}} \bar{p}\,\bar{q}\,p^{2m}q^{2n}r \quad R^1_{m,n}(k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ p \text{ odd},\ r \text{ even}}} p^{2m+1}q^{2n+1}r.$$

The summations in $B_{m,n}, R_{m,n}$ are over integers $p, q, r \geq 0$ such that $p+q+r = k$ and r is even. If the parity of k is given, then the parities of p and q in the sum are not fixed. For instance, if k is even, then the sum will be over triples (p, q, r), where $(p, q, r) \equiv (0, 0, 0)$ and $(1, 1, 0) \pmod 2$. When we split these sums into those terms for which p is even or odd, we obtain $B^0_{m,n}$ and $B^1_{m,n}$, respectively, so $B_{m,n} = B^0_{m,n} + B^1_{m,n}$. Similarly, $R_{m,n} = R^0_{m,n} + R^1_{m,n}$.

Clearly each B sum is very similar to the corresponding R sum; they differ only in terms where $p = 0$ or $q = 0$, and then only when $m = 0$ or $n = 0$.

The sums S_m, $R_{m,n}$ were defined by Norbury in [31]. He proved results about the ir polynomiality, and we will show similar results for $A_m, B_{m,n}, B^0_{m,n}, B^1_{m,n}, R^0_{m,n}$ and $R^1_{m,n}$. Our proof follows the methods of Norbury, which in turn rely on a result of Brion–Vergne [5] generalizing Ehrhart's theorem. By a *convex lattice polytope* in $\mathbb{R}^n$, we mean a polytope P in $\mathbb{R}^n$, with all vertices in the lattice $\mathbb{Z}^n$, i.e. the convex hull of a finite subset of $\mathbb{Z}^n$. We denote by P^0 the interior of P and by ∂P the boundary of P; if P^0 is non-empty, then P must be n-dimensional. For any non-negative integer k, the set $kP = \{kx : x \in P\}$ is again a convex lattice polytope. Given a function $\phi : \mathbb{R}^n \to \mathbb{R}$, we may sum it on the lattice points of P, P^0 or ∂P. We may in fact sum ϕ over the lattice points of kP or kP^0 or ∂P and see how this sum varies with k. Thus, we define

$$N_P(\phi, k) = \sum_{x \in \mathbb{Z}^n \cap kP} \phi(x),$$

$$N_{P^0}(\phi, k) = \sum_{x \in \mathbb{Z}^n \cap kP^0} \phi(x), \quad \text{and} \quad N_{\partial P}(\phi, k) = \sum_{x \in \mathbb{Z}^n \cap k\partial P} \phi(x).$$

Since $\partial P = P \backslash P^0$, and similarly $k\partial P = kP \backslash kP^0$, we have immediately

$$N_{\partial P}(\phi, k) = N_P(\phi, k) - N_{P^0}(\phi, k).$$

The result of Brion–Vergne says that under certain circumstances, these are *polynomials* obeying a surprising property.

Proposition 7.3 (Brion–Vergne [5, Proposition 4.1]). *Let P be a convex lattice polytope in $\mathbb{R}^n$ with non-empty interior P^0. Let $\phi(x_1, \ldots, x_n)$ be a homogeneous rational polynomial of degree d. Then $N_P(\phi, k)$ and $N_{P^0}(\phi, k)$ are rational polynomials in k of degree $n + d$. Moreover,*

$$N_{P^0}(\phi, k) = (-1)^{n+d} N_P(\phi, -k). \qquad \square$$

Note that while $N_P(\phi, -k)$ does not appear to be defined, when $-k$ is a negative integer, the notation means to substitute $-k$ for k in the polynomial function $N_P(\phi, k)$. Thus, the two polynomials are obtained from each other by replacing k with $-k$ and adjusting the overall sign.

Lemma 7.4 (Cf. Norbury [31, Lemma 1]).

(i) *Let $m \geq 0$ be an integer. Then $A_m(k)$ and $S_m(k)$ are rational odd quasi-polynomials of degree $2m+3$, depending on the parity of k, which differ by a lower-degree quasi-polynomial.*

(ii) *Let $m, n \geq 0$ be integers. Then $B_{m,n}(k), B^0_{m,n}(k)$, $B^1_{m,n}(k)$, $R_{m,n}(k)$, $R^0_{m,n}(k)$, $R^1_{m,n}(k)$ are all rational odd quasi-polynomials of degree $2m+2n+5$, depending on the parity of k. Each of $B_{m,n}, B^0_{m,n}, B^1_{m,n}$ differs from the respective $R_{m,n}, R^0_{m,n}, R^1_{m,n}$ by a lower-degree quasi-polynomial.*

In all cases, the leading coefficients are positive.

Proof. Norbury [31, Lemma 1] proved that $S_m(k)$ is an odd quasi-polynomial of degree $2m+3$, depending on the parity of k; it follows from the proof that the coefficients are rational. We then observe that

$$A_m(k) = \sum_{\substack{p,q \geq 0 \\ p+q=k \\ q \text{ even}}} \bar{p} p^{2m} q = \sum_{\substack{p,q \geq 0 \\ p+q=k \\ q \text{ even}}} p^{2m+1} q + \sum_{\substack{q \geq 0 \\ p=0,\ q=k \\ q \text{ even}}} p^{2m} q = S_m(k) + \delta_{m,0} \sum_{\substack{q=k \\ q \text{ even}}} q.$$

The second term here is zero, unless $m = 0$ and k is even, in which case it is k. Thus, $A_0(k) = S_0(k) + k$ for k even, and $A_0(k) = S_0(k)$ for k odd. Since $S_0(k)$ has degree 3, $A_0(k)$ is a rational odd quasi-polynomial of degree 3, depending on the parity of k. When $m > 0$ we have $A_m(k) = S_m(k)$ for all k. So for all m, $A_m(k)$ is a rational odd quasi-polynomial of degree $2m+3$, given by $S_m(k)$, plus a lower-degree quasi-polynomial.

We now turn to the various R functions. Consider the following convex lattice polytope in $\mathbb{R}^3$:

$$P = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ 2x + y + 2z \leq 2\}$$

which is the convex hull of $\{(0,0,0), (1,0,0), (0,2,0), (0,0,1)\}$. This P is a 3-simplex, with three of its four (2-dimensional) faces right-angled triangles in the xy, yz and zx planes, and the fourth face cut out by the plane $2x + y + 2z = 2$ in the positive octant.

Consider the polynomial function

$$\phi(x, y, z) = x^{2m+1} y^{2n+1} z$$

fixing m and n throughout this discussion, so $\deg \phi = 2m + 2n + 3$. Applying Proposition 7.3 to P and ϕ, $N_P(\phi, k)$ and $N_{P^0}(\phi, k)$ are rational polynomials in k of degree $2m + 2n + 6$, and $N_{P^0}(\phi, k) = N_P(\phi, -k)$. Hence

$$N_{\partial P}(\phi, k) = N_P(\phi, k) - N_{P^0}(\phi, k) = N_P(\phi, k) - N_P(\phi, -k)$$

is an odd rational polynomial function of k of degree $\leq 2m + 2n + 5$.

Now $\phi(0,y,z) = \phi(x,0,z) = \phi(x,y,0) = 0$, so ϕ vanishes in the xy, yz and zx planes, hence on three sides of P. We thus have

$$N_{\partial P}(\phi,k) = \sum_{\substack{x,y,z\geq 0\\ 2x+y+2z=2k}} x^{2m+1}y^{2n+1}z.$$

Let $p=2x$, $q=y$, $r=2z$. So x,y,z are non-negative integers such that $2x+y+2z = 2k$ if and only if p,q,r are non-negative integers such that $p+q+r=2k$ with p,r are even (hence q is also even). Hence

$$N_{\partial P}(\phi,k) = \sum_{\substack{p,q,r\geq 0\\ p+q+r=2k\\ p \text{ even, } r \text{ even}}} \left(\frac{p}{2}\right)^{2m+1} q^{2n+1}\left(\frac{r}{2}\right) = \frac{1}{2^{2m+2}}R^0_{m,n}(2k).$$

Thus, for even k, $R^0_{m,n}(k)$ is an odd polynomial of degree $\leq 2m+2n+5$. An elementary estimate gives us that the degree is exactly $2m+2n+5$. For instance, for any positive integer k and positive integers u,v,w with $u+v+w=k$, we have $(k+u, 2k+2v, k+w) \in 8k\,\partial P$. For given k there are $\binom{k-1}{2}$ such points, and for each we have

$$\phi(k+u,2k+2v,k+w) = (k+u)^{2m+1}(2k+2v)^{2n+1}(k+2) > k^{2m+2n+3}$$

so that

$$N_{\partial P}(\phi,8k) = \sum_{v\in\mathbb{Z}^3\cap 8k\partial P}\phi(v) \geq \sum_{u,v,w}\phi(k+u,2k+2v,k+w) > \binom{k-1}{2}k^{2m+2n+3}.$$

Hence, $\deg N_{\partial P}(\phi,k) \geq 2m+2n+5$ and thus the degree must be exactly $2m+2n+5$. We have proved that for even k, $R^0_{m,n}(k)$ is an odd polynomial of degree $2m+2n+5$.

Now we consider $R^0_{m,n}(k)$ for odd k. So let $k=2\kappa+1$ and consider (following Norbury)

$$N_{P^0}(\phi,\kappa+1) - N_P(\phi,\kappa) = N_{P^0}\left(\phi,\frac{k+1}{2}\right) - N_P\left(\phi,\frac{k-1}{2}\right).$$

The first sum is a sum of $\phi(x,y,z)$ over lattice points (x,y,z) in the interior of $(\kappa+1)P$, hence over all integers $x,y,z>0$ such that $2x+y+2z<2(\kappa+1)=k+1$. The second sum is a sum of $\phi(x,y,z)$ over lattice points (x,y,z) in κP, hence over all integers $x,y,z\geq 0$ such that $2x+y+2z \leq 2\kappa = k-1$. After subtracting (and recalling that ϕ vanishes when any of x,y,z are zero), we are only left with $x,y,z>0$ such that $2x+y+2z=k$. Thus

$$N_{P^0}\left(\phi,\frac{k+1}{2}\right) - N_P\left(\phi,\frac{k-1}{2}\right) = \sum_{\substack{x,y,z>0\\ 2x+y+2z=k}} x^{2m+1}y^{2n+1}z$$

$$= \sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ p \text{ even, } r \text{ even}}} \left(\frac{p}{2}\right)^{2m+1} y^{2n+1}\left(\frac{r}{2}\right) = \frac{1}{2^{2m+2}}R^0_{m,n}(k).$$

Applying Proposition 7.3, $N_P(\phi, t)$ and $N_{P^0}(\phi, t)$ are rational polynomials in t of degree $2m+2n+6$, and $N_{P^0}(\phi, t) = N_P(\phi, -t)$. Thus, still taking $k = 2\kappa+1$ to be odd,

$$\begin{aligned} \frac{1}{2^{2m+2}} R^0_{m,n}(k) &= N_{P^0}\left(\phi, \frac{k+1}{2}\right) - N_P\left(\phi, \frac{k-1}{2}\right) \\ &= N_P\left(\phi, \frac{-k-1}{2}\right) - N_P\left(\phi, \frac{k-1}{2}\right). \end{aligned}$$

This is evidently an odd function of k, and it is a polynomial in k of degree $\leq 2m+2n+5$. A similar estimate as above in the case k even shows that the degree is exactly $2m+2n+5$.

We have now shown that $R^0_{m,n}(k)$ is a rational odd quasi-polynomial of degree $2m+2n+5$. Norbury in [31, Lemma 1] showed that $R_{m,n}(k)$ has the same property. It is clear from his argument that the coefficients are rational. Thus $R^1_{m,n} = R_{m,n} - R^0_{m,n}$ is a rational odd quasi-polynomial of degree $\leq 2m+2n+5$, depending on the parity of k. An estimate of the sort used above shows that $R^1_{m,n}$ has degree exactly $2m+2n+5$.

Now consider the various B functions. For $B_{m,n}(k)$ we compute,

$$\begin{aligned} B_{m,n}(k) &= \sum_{\substack{p,q,r \geq 0 \\ p+q+r=k \\ r \text{ even}}} \bar{p}\,\bar{q}\,p^{2m} q^{2n} r \\ &= R_{m,n}(k) + \delta_{n,0} S_m(k) + \delta_{m,0} S_n(k) + \delta_{m,0}\delta_{n,0} \sum_{\substack{r=k \\ r \text{ even}}} r. \end{aligned}$$

The last sum is k, if k is even, and 0 if k is odd.

When $m = n = 0$ are zero, $B_{0,0}(k)$ is given by $R_{0,0}(k) + 2S_0(k) + k$ for k even and $R_{0,0}(k) + 2S_0(k)$ for k odd, where $\deg S_0 = 3 < 5 = \deg R_{0,0}$. When $m = 0$ and $n \neq 0$, we have $B_{0,n}(k) = R_{0,n}(k) + S_n(k)$, where $\deg S_n = 2n+3 < 2n+5 = \deg R_{0,n}$. The case $m \neq 0$ and $n = 0$ is similar. If m, n are both non-zero, then $B_{m,n}(k) = R_{m,n}(k)$.

In all cases, then $B_{m,n}(k)$ is given by $R_{m,n}(k)$, plus a lower-degree odd rational quasi-polynomial (possibly zero) depending on the parity of k. Hence, $B_{m,n}(k)$ is a rational odd quasi-polynomial of degree $2m+2n+5$.

We can perform a similar computation for $B^0_{m,n}(k)$, expressing it as $R^0_{m,n}(k)$ plus lower degree terms; and similarly again for $B^1_{m,n}(k)$. We conclude that both are also odd rational quasi-polynomials of degree $2m+2n+5$ depending on the parity of k.

As all the functions are defined by summing positive polynomials on positive integers, all highest degree coefficients must be positive. ◻

The first few polynomials among the A_m and $B_{m,n}$ are

$$A_0(k) = \begin{cases} \frac{1}{12}k^3 + \frac{2}{3}k & k \text{ even} \\ \frac{1}{12}k^3 - \frac{1}{12}k & k \text{ odd} \end{cases}$$

$$A_1(k) = \begin{cases} \frac{1}{40}k^5 - \frac{1}{6}k^3 + \frac{4}{15}k & k \text{ even} \\ \frac{1}{40}k^5 - \frac{1}{6}k^3 + \frac{17}{120}k & k \text{ odd} \end{cases}$$

$$A_2(k) = \begin{cases} \frac{1}{84}k^7 - \frac{1}{6}k^5 + \frac{2}{3}k^3 - \frac{16}{21}k & k \text{ even} \\ \frac{1}{84}k^7 - \frac{1}{6}k^5 + \frac{2}{3}k^3 - \frac{43}{84}k & k \text{ odd} \end{cases}$$

$$A_3(k) = \begin{cases} \frac{1}{144}k^9 - \frac{1}{6}k^7 + \frac{7}{5}k^5 - \frac{40}{9}k^3 + \frac{64}{15}k & k \text{ even} \\ \frac{1}{144}k^9 - \frac{1}{6}k^7 + \frac{7}{5}k^5 - \frac{40}{9}k^3 + \frac{769}{240}k & k \text{ odd} \end{cases}$$

$$B_{0,0}(k) = \begin{cases} \frac{1}{240}k^5 + \frac{1}{8}k^3 + \frac{13}{30}k & k \text{ even} \\ \frac{1}{240}k^5 + \frac{1}{8}k^3 - \frac{31}{240}k & k \text{ odd} \end{cases}$$

$$B_{0,1}(k) = \begin{cases} \frac{1}{1680}k^7 + \frac{7}{480}k^5 - \frac{7}{60}k^3 + \frac{41}{210}k & k \text{ even} \\ \frac{1}{1680}k^7 + \frac{7}{480}k^5 - \frac{7}{60}k^3 + \frac{341}{3360}k & k \text{ odd} \end{cases}$$

$$B_{0,2}(k) = \begin{cases} \frac{1}{6048}k^9 + \frac{1}{144}k^7 - \frac{169}{1440}k^5 + \frac{185}{378}k^3 - \frac{17}{30}k & k \text{ even} \\ \frac{1}{6048}k^9 + \frac{1}{144}k^7 - \frac{169}{1440}k^5 + \frac{185}{378}k^3 - \frac{91}{240}k & k \text{ odd} \end{cases}$$

$$B_{1,1}(k) = \begin{cases} \frac{1}{20160}k^9 - \frac{1}{840}k^7 + \frac{1}{96}k^5 - \frac{23}{630}k^3 + \frac{3}{70}k & k \text{ even} \\ \frac{1}{20160}k^9 - \frac{1}{840}k^7 + \frac{1}{96}k^5 - \frac{23}{630}k^3 + \frac{61}{2240}k & k \text{ odd} \end{cases}$$

Although $A_m(k)$ was originally defined as a function $\mathbb{N}_0 \to \mathbb{N}_0$, since we now know it is a quasi-polynomial it naturally extends to a function $\mathbb{Z} \to \mathbb{Z}$. We can similarly extend all the B, R, S functions.

Lemma 7.5. *For any non-negative integers m, n and any integer k,*

$$A_m(k) = \widetilde{\sum_{\substack{p,q\geq 0\\ p+q=k\\ q \text{ even}}}} \bar{p}\, p^{2m} q \quad \textit{and} \quad B_{m,n}(k) = \widetilde{\sum_{\substack{p,q,r\geq 0\\ p+q+r=k\\ r \text{ even}}}} \bar{p}\, \bar{q}\, p^{2m} q^{2m} r.$$

Recall from Sec. 6.2 that the tilde over the summation means that if $k \geq 0$, interpret the sum as is; if $k < 0$, then take the negative of the sum over $p+q=-k$ or $p+q+r=-k$.

Proof. We prove the result for $A_m(k)$; $B_{m,n}(k)$ is similar. When $k > 0$, this is true by definition. When $k = 0$, we know $A_m(k) = 0$ as A_m is odd, and the summation consists only of the term with $p = q = 0$. When $k < 0$, we have

$$\widetilde{\sum_{\substack{p,q\geq 0\\ p+q=k\\ q \text{ even}}}} \bar{p}\, p^{2m} q = -\sum_{\substack{p,q\geq 0\\ p+q=-k\\ q \text{ even}}} \bar{p}\, p^{2m} q = -A_m(-k)$$

which is equal to $A_m(k)$, as A_m is odd. $\square$

The following lemma now follows immediately from the properties of $A_m(k)$ in Lemma 7.4.

Lemma 7.6. *For any integer $m \geq 0$,*

$$A_m(x+y) + A_m(x-y)$$

is a quasi-polynomial function of x and y, depending on the parity of x and y, odd in x and even in y, of degree $2m+3$, with all coefficients of highest total degree being positive.

7.2. *Polynomiality for non-boundary-parallel counts*

We now prove polynomiality of $\widehat{N}_{g,n}$. Lemma 7.4 allows us to compute the summations in the recursion of Proposition 6.1 for $\widehat{N}_{g,n}$, and we have computed enough initial values.

For instance, in Eq. (13), we found an expression for $\widehat{N}_{1,1}(b)$ for b even, which we now recognize as

$$\widehat{N}_{1,1}(b) = \frac{1}{4\bar{b}} A_0(b) + \frac{1}{4}.$$

Since $A_0(b) = \frac{1}{12}b^3 + \frac{2}{3}b$ for even b, we immediately obtain a closed expression

$$\widehat{N}_{1,1}(b) = \frac{1}{48}b^2 + \frac{5}{12} \quad \text{for } b \neq 0 \text{ even}$$

proving Eq. (11) in Proposition 1.5.

Similarly, we can compute $\widehat{N}_{0,4}$. From the recursion (Proposition 6.1) for $\widehat{N}_{g,n}$, we have

$$b_1 \widehat{N}_{0,4}(\mathbf{b}) = \sum_{j=2}^{4} \frac{1}{2} \left(\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \bar{i}\, m\, \widehat{N}_{0,3}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_4) \right.$$

$$\left. + \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}}} \bar{i}\, m\, \widehat{N}_{0,3}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_4) \right).$$

When $\sum b_i$ is odd, all terms are zero, so assume $\sum b_i$ is even. In this case, all $\widehat{N}_{0,3}$ occurring have arguments with even sum, and hence each $\widehat{N}_{0,3}$ occurring is equal to 1. We then obtain (using Lemma 7.5)

$$2b_1 \widehat{N}_{0,4}(\mathbf{b}) = \sum_{j=2}^{4} \left(\sum_{\substack{i,m \geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \bar{i}\, m + \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}}} \bar{i}\, m \right) = \sum_{j=2}^{4} A(b_1 + b_j) + A(b_1 - b_j).$$

We will compute $\widehat{N}_{0,4}(b_1, b_2, b_3, b_4)$ assuming that b_1, b_2 are even and b_3, b_4 are odd; when b_1, b_2, b_3, b_4 have different parities a similar method will work. Recall $A_0(k) = \frac{1}{12}k^3 + \frac{2}{3}k$ when k is even and $\frac{1}{12}k^3 - \frac{1}{12}k$ when k is odd. By Lemma 7.6, each $A_0(b_1 + b_i) + A_0(b_1 - b_i)$ is odd in b_1 and even in b_i; explicitly

$$A_0(b_1 + b_2) + A_0(b_1 - b_2) = \frac{1}{6}b_1^3 + \frac{1}{2}b_1 b_2^2 + \frac{4}{3}b_1$$

and

$$A_0(b_1 + b_3) + A_0(b_1 - b_3) = \frac{1}{6}b_1^3 + \frac{1}{2}b_1 b_3^2 - \frac{1}{6}b_1$$

with a similar calculation for $A_0(b_1 \pm b_4)$. Putting these together, we obtain

$$\widehat{N}_{0,4} = \frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + \frac{1}{2}.$$

Completing the calculation for other parities of (b_1, b_2, b_3, b_4), we obtain Eq. (10) in Proposition 1.5.

Using a similar method, we now prove polynomiality of $\widehat{N}_{g,n}$ in general.

Proof of Theorem 1.7. We prove the result by induction on the complexity $-\chi = 2g + n - 2$ of the surface. For $-\chi = 1$, i.e. $(g, n) = (1, 1)$ and $(0, 3)$, we have proved the result directly; we consider $\widehat{N}_{g,n}$ with complexity at least 2, supposing the result holds for all surfaces of smaller positive complexity.

Fix a parity for each $b_1, \ldots, b_n$. We claim that $\widehat{N}_{g,n}(\mathbf{b})$ is given by a polynomial in $b_1, \ldots, b_n$ which is even in every variable and with total degree $3g-3+n$ in $b_1^2, \ldots, b_n^2$, with all top-degree coefficients positive.

Consider the recursion for $\widehat{N}_{g,n}(\mathbf{b})$ in Proposition 6.1. Each $\widehat{N}$ in the recursion is of the form $\widehat{N}_{g-1,n+1}$, $\widehat{N}_{g,n-1}$, or $\widehat{N}_{g',n'}$ where $g' \leq g$ and $n' \leq n-2$, but $(g', n') \neq (0,1)$ or $(0,2)$. So each term corresponds to a surface with strictly smaller complexity. Further, neither $\widehat{N}_{0,1}$ nor $\widehat{N}_{0,2}$ ever appears; every term appearing has positive complexity. So we know that every $\widehat{N}$ appearing in the recursion is given by a quasi-polynomial with all the claimed properties.

After expanding out the sum $\sum_{j=2}^{n}$ in the second line of the recursion, and the sum over $g_1 + g_2 = g$ and $I \sqcup J = \{2, \ldots, n\}$ in the third line, we can express $b_1 \widehat{N}_{g,n}(\mathbf{b})$ as a finite collection of sums, where each sum is either over $i, m \geq 0$ such that $i + m = b_1 \pm b_j$ (for some j) and m is even, or over $i, j, m \geq 0$ such that $i + j + m = b_1$ and m is even. In the first case, each sum is, up to a constant, of the form $\bar{i} m \widehat{N}_{g',n'}(i, b_I)$; and in the second case, of the form $\bar{i}\ \bar{j}\ m \widehat{N}_{g',n'}(i, j, b_I)$ or $\bar{i}\ \bar{j}\ m \widehat{N}_{g',n'}(i, b_I) \widehat{N}_{g'',n''}(j, b_J)$. In both cases, (g', n') or (g'', n'') has positive complexity that is lower than (g, n), and b_I denotes some subset of $b_2, \ldots, b_n$. Either way, the $\widehat{N}$ factors are all even functions of all their variables. Also, all sums are equal to the sums, we obtain by writing a tilde over them; and all top-degree coefficients are positive. Thus, each summation is of one of the following two types, for some function p that is a positive constant times an $\widehat{N}$ or a product of $\widehat{N}$'s:

$$\text{Type 1:} \quad \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=b_1 \pm b_j \\ m \text{ even}}}} \bar{i}\, m\, p(i, b_I), \qquad \text{Type 2:} \quad \widetilde{\sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}}} \bar{i}\, \bar{j}\, m\, p(i, j, b_I).$$

Having fixed the parity of $b_1, \ldots, b_n$, we now consider the possible parity of i and j occurring in the sums. In a summation of type 1, the parity of i is fixed. In a summation of type 2, the parity of $i + j$ is fixed; hence there are two possibilities for the parity of i and j, and we can split the summation into two separate summations where the parity of each variable is fixed.

In any case, we are able to express $b_1 \widehat{N}_{g,n}(\mathbf{b})$ as a finite sum of terms, where each term is a summation of type 1 or 2, with the parities of each variable fixed. In each summation $p(i, b_I)$ or $p(i, j, b_I)$ is a polynomial with top-degree coefficients positive and even in all its variables. Taking each term of each polynomial separately, and factoring out variables not involved in the summation, each term of type 1 becomes a finite collection of sums of the form

$$q(b_I) \widetilde{\sum_{\substack{i,m \geq 0 \\ i+m=k \\ i \equiv \epsilon \pmod 2,\ m \text{ even}}}} \bar{i}\, i^{2a} m = \begin{cases} q(b_I) A_a(k) & k \equiv \epsilon \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $q(b_I)$ is a constant (for terms of highest degree, a positive constant) times a product of even powers of b_i's. We determined in Lemma 7.4 that $A_a(k)$ is odd in k; and since the parity of k is fixed, $A_a(k)$ is a polynomial in k. Every time, we see A_a arising, it is from the second line of the recursion, hence appears in the form $A_a(b_1 \pm b_j)$, both terms appearing together; by Lemma 7.6, the result is odd in b_1, even in b_j (and indeed all other b_i), with top-degree coefficients positive.

Similarly, each term of type 2 becomes a finite collection of sums of the form

$$q(b_I) \quad \widetilde{\sum_{\substack{i,j,m\geq 0 \\ i+j+m=k \\ i\equiv\delta \pmod 2, \\ j\equiv\epsilon \pmod 2,\, m \text{ even}}}} \quad \bar{i}\,\bar{j}\, i^{2a} j^{2m} m = \begin{cases} q(b_I) B^{\delta}_{a,b}(k) & k \equiv \delta + \epsilon \pmod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta, \epsilon \in \{0,1\}$. Here, again $q(b_I)$ is a constant (positive for highest degree terms) times a product of even powers of b_i's. From Lemma 7.4, each $B_{a,b}(k)$ is odd in k, and since the parity of k is fixed, $B_{a,b}(k)$ is a polynomial in k. Every time, we see $B_{a,b}$ arising, it appears in the form $B_{a,b}(b_1)$, hence the result is odd in b_1 and even in all other b_i.

After collecting terms and simplifying all A_a's and $B_{a,b}$'s, the result for $b_1 \widehat{N}_{g,n}(\mathbf{b})$ is divisible by b_1, odd in b_1, and even in all the other variables. Hence, $\widehat{N}_{g,n}(\mathbf{b})$ is an even polynomial in all the variables as desired.

We can also keep track of degrees. Let us keep track of the degrees of the variables rather than their squares, so we will show $\widehat{N}_{g,n}$ has degree $6g-6+2n$. In the recursion, the first term has $\widehat{N}_{g-1,n+1}$, which has degree $6g+2n-10$: it is multiplied by $\bar{i}\,\bar{j}m$ and all summations are of $B_{a,b}$'s, leading to a total degree of $6g-5+2n$. The terms in the second line have $\widehat{N}_{g,n-1}$, which has degree $6g-8+2n$: it is multiplied by $\bar{i}\,m$ and the summations give A_a polynomials, leading to a total degree of $6g-5+2n$; the summation over j does not alter the degree. The terms in the third line have $\widehat{N}_{g_1,|I|+1}\widehat{N}_{g_2,|J|+1}$ which has degree $6(g_1+g_2)-12+2|I|+2|J|+4 = 6g-10+2n$; we then multiply by $\bar{i}\,\bar{j}\,m$ and sum, obtaining $B_{a,b}$ polynomials and a total degree of $6g-5+2n$. As all top-degree terms are positive, there can be no cancelation of terms and the right-hand side of the recursion is of degree $6g-5+2n$, with all highest-degree coefficients positive. Dividing by b_1 then gives the degree of $\widehat{N}_{g,n}$ as $6g-6+2n$. $\square$

7.3. *Lattice count polynomials and moduli spaces*

Norbury in [31] derives a recursion for counts of lattice points in the moduli space of curves, which correspond to ribbon graphs without degree 1 vertices. We denote the number of such ribbon graphs with prescribed genus, number of boundary components, and boundary lengths, by $\mathfrak{N}_{g,n}(b_1,\ldots,b_n)$. Writing Eq. (5) of [31] in our notation, these lattice point counts satisfy the recursion

$$\begin{aligned}
&b_1 \mathfrak{N}_{g,n}(b_1,\ldots,b_n)\\
&\quad = \sum_{\substack{i,j,m\geq 0\\ i+j+m=b_1\\ m \text{ even}}} \frac{1}{2}\, i\, j\, m\, \mathfrak{N}_{g-1,n+1}(i,j,b_2,\ldots,b_n)\\
&\qquad + \sum_{j=2}^{n} \frac{1}{2}\left(\sum_{\substack{i,j\geq 0\\ i+m=b_1+b_j\\ m \text{ even}}} i\, m\, \mathfrak{N}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n)\right.\\
&\qquad + \left. \widetilde{\sum_{\substack{i,m\geq 0\\ i+m=b_1-b_j\\ m \text{ even}}}} i\, m\, \mathfrak{N}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n)\right)\\
&\qquad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\ldots,n\}\\ \text{No discs or annuli}}} \sum_{\substack{i,j,m\geq 0\\ i+j+m=b_1\\ m \text{ even}}} \frac{1}{2}\, i\, j\, m\, \mathfrak{N}_{g_1,|I|+1}(i,b_I)\, \mathfrak{N}_{g_2,|J|+1}(j,b_J).
\end{aligned}$$

This recursion is identical to the recursion on $\widehat{N}_{g,n}$ (Proposition 6.1), with the bars dropped from i's and j's.

The initial conditions for the recursions on $\mathfrak{N}_{g,n}$ and $\widehat{N}_{g,n}$ are

$$\begin{aligned}
\mathfrak{N}_{0,3}(b_1,b_2,b_3) &= 1 & \widehat{N}_{0,3}(b_1,b_2,b_3) &= 1\\
\mathfrak{N}_{1,1}(b_1) &= \frac{1}{48}b_1^2 - \frac{1}{12} & \widehat{N}_{1,1}(b_1) &= \frac{1}{48}b_1^2 + \frac{5}{12}.
\end{aligned}$$

(Both $(g,n) = (1,1)$ expressions are for even b_1; they are both zero when b_1 is odd.) Norbury's proof that each $\mathfrak{N}_{g,n}(b_1,\ldots,b_n)$ is a quasi-polynomial, depending on the parity of $b_1,\ldots,b_n$, of degree $3g-3+n$ in $b_1^2,\ldots,b_n^2$, is analogous to our $\widehat{N}_{g,n}(b_1,\ldots,b_n)$; indeed, we adapted his proof above. Thus, $\mathfrak{N}$ and $\widehat{N}$ agree in initial cases in highest-degree terms. As their recursions are also similar, it is now not too surprising that they should have the same highest degree terms.

Proposition 7.7. *Let $(g,n) \neq (0,1)$ or $(0,2)$ and fix the parity of $b_1,\ldots,b_n$. Then the corresponding polynomials in the quasi-polynomials $\mathfrak{N}_{g,n}(b_1,\ldots,b_n)$ and $\widehat{N}_{g,n}(b_1,\ldots,b_n)$ have identical terms of highest total degree.*

Proof. We compare the proofs of quasi-polynomiality of $\widehat{N}_{g,n}$ and $\mathfrak{N}_{g,n}$.

Having fixed the parity of each $b_1,\ldots,b_n$, the expression for $b_1\widehat{N}_{g,n}(b_1,\ldots,b_n)$ in the recursion can be written as a sum of terms, each consisting of a positive constant, multiplied by a product of powers of b_i's, multiplied by some $A_a(b_1 \pm b_j)$ or $B^0_{a,b}(b_1)$ or $B^1_{a,b}(b_1)$. Each A_a term occurs in a pair where we can factor out

$A_a(b_1+b_j)+A_a(b_1-b_j)$; these terms are then collected together, and we obtain the desired polynomial.

Exactly the same applies to $b_1\mathfrak{N}_{g,n}$, replacing A_a and $B_{a,b}$ with S_a and R_{ab}. From Lemma 7.4, $A_a(k)$ and $S_a(k)$ agree in their leading terms; and similarly $B^0_{a,b}(k), B^1_{a,b}(k)$ and $R^0_{a,b}(k), R^1_{a,b}(k)$, respectively agree in their leading terms. So if all $\widehat{N}$ and $\mathfrak{N}$ of lower complexity have identical terms of highest degree, then their highest degree terms also agree at a given complexity, and by induction on complexity, we obtain the desired result. $\square$

In [31, Theorem 3], Norbury shows that

$$\mathfrak{N}_{g,n}(b_1,\ldots,b_n)=\frac{1}{2}V_{g,n}(b_1,\ldots,b_n)+\text{lower order terms},$$

where $V_{g,n}(b_1,\ldots,b_n)$ is the volume polynomial of Kontsevich [25]. So $\widehat{N}_{g,n}$, $\mathfrak{N}_{gn}$ and $\frac{1}{2}V_{g,n}$ all agree in highest degree terms.

In fact, the Kontsevich volumes also agree with the highest order terms in the Weil–Petersson volume polynomials of Mirzakhani up to a simple normalization [27]. Note $V_{g,n}$ is a *polynomial*, not quasi-polynomial. It immediately follows that the polynomials defining each quasi-polynomial $\widehat{N}_{g,n}(b_1,\ldots,b_n)$ all agree in highest degree. Moreover, the coefficients of $V_{g,n}$ are, up to a combinatorial factor, the intersection numbers on the moduli space of curves. In $\mathfrak{N}_{g,n}$, the coefficient of $b_1^{2d_1}\cdots b_n^{2d_n}$, for $\sum d_i=3g-3+n$, is given by

$$\frac{1}{2^{5g-6+2n}d_1!\cdots d_n!}\langle\psi_1^{d_1}\cdots\psi_n^{d_n},\overline{\mathcal{M}}_{g,n}\rangle.$$

We have thus proved Theorem 1.9.

7.4. *Polynomiality for general curve counts*

We now use the polynomiality of $\widehat{N}_{g,n}$ to prove polynomiality for $G_{g,n}$. It is now no more difficult than our computation of $G_{0,3}$ in Sec. 5.3; in fact, we developed all we need there.

Recall (Theorem 1.8) that $G_{g,n}(b_1,\ldots,b_n)$ can be written in terms of $\widehat{N}_{g,n}$:

$$G_{g,n}(b_1,\ldots,b_n)=\sum_{\substack{0\le a_i\le b_i\\ a_i\equiv b_i \pmod 2}} \times\begin{pmatrix} b_1\\ \frac{b_1-a_1}{2}\end{pmatrix}\cdots\begin{pmatrix} b_n\\ \frac{b_n-a_n}{2}\end{pmatrix}\bar{a}_1\cdots\bar{a}_n\,\widehat{N}_{g,n}(a_1,\ldots,a_n). \tag{14}$$

Recall from Definition 5.5 that, for integers $\alpha\ge 0$,

$$\tilde{P}_\alpha(n)=\sum_{l=0}^{n}\binom{2n}{n-l}\overline{(2l)}\,(2l)^{2\alpha},\quad \tilde{Q}_\alpha(n)=\sum_{l=0}^{n}\binom{2n+1}{n-l}\overline{(2l+1)}(2l+1)^{2\alpha}.$$

We also defined $\tilde{p}_\alpha(n)$ and $\tilde{q}_\alpha(n)$ "without the bars". We showed (Proposition 5.7) that $\tilde{P}_\alpha(n) = \binom{2n}{n} P_\alpha(n)$, $\tilde{Q}_\alpha(n) = \binom{2n}{n} Q_\alpha(n)$, where P_α, Q_α are integer polynomials of degree $\alpha + 1$; and similarly for $\tilde{p}_\alpha(n)$ and $\tilde{q}_\alpha(n)$.

In evaluating the summations in (14), we can write the even polynomial $\widehat{N}_{g,n}(b_1, \ldots, b_n)$ as a sum of even monomials, and factorize each term into sums of the form

$$\sum_{\substack{0 \le a \le b \\ a \equiv b \pmod 2}} \binom{b}{\frac{b-a}{2}} \bar{a}\, a^{2\alpha}.$$

When b is even, $b = 2n$, we only sum over even a, so with $a = 2l$ and the sum is $\tilde{P}_\alpha(n)$. When b is odd, $b = 2n+1$, we sum over odd $a = 2l+1$ and the sum is $\tilde{Q}_\alpha(n)$. When all a_i are set to zero however, $\widehat{N}_{g,n}(\mathbf{0}) = 1$, to which the quasi-polynomial for $\widehat{N}_{g,n}$ does not apply; separating out this term, we have a $\tilde{p}_\alpha(n)$ rather than a $\tilde{P}_\alpha(n)$.

Proof of Theorem 1.3. We may evaluate $G_{g,n}(b_1, \ldots, b_n)$ by simply replacing sums of the above type with functions $\tilde{P}_\alpha$, $\tilde{p}_\alpha$ and $\tilde{Q}_\alpha$. More precisely, each monomial in $\bar{a}_1 \cdots \bar{a}_n \widehat{N}_{g,n}(a_1, \ldots, a_n)$ is of the form $\bar{a}_1 \cdots \bar{a}_n a_1^{2\alpha_1} \cdots a_n^{2\alpha_n}$, and we replace each factor $\bar{a}_i a_i^{2\alpha_i}$ with $\tilde{P}_\alpha(m_i) = \tilde{P}_\alpha(b_i/2)$ or $\tilde{p}_\alpha(m_i)$, when $b_i = 2m_i$ is even, and with $\tilde{Q}_\alpha(m_i) = \tilde{Q}_\alpha(\frac{b_i - 1}{2})$ when $b_i = 2m_i + 1$ is odd. Each such substitution replaces a factor of degree $2\alpha_i + 1$ with an expression $\binom{2m_i}{m_i}$ multiplied by a polynomial of degree $\alpha_i + 1$ in b_i.

After performing this substitution over all a_i, each monomial becomes an expression of the form $\binom{2m_1}{m_1} \cdots \binom{2m_n}{m_n}$ multiplied by a product of $P_\alpha(m)$ and $Q_\alpha(m)$, which is a polynomial in $b_1, \ldots, b_n$. Since each monomial has $\sum 2\alpha_i = 6g - 6 + 2n$, we end up with a polynomial of degree $\sum(\alpha_i + 1) = 3g - 3 + 2n$.

Furthermore, it follows from the proof of Theorem 1.7 in Sec. 7.2 that each polynomial that appears in the quasi-polynomial $\widehat{N}_{g,n}(b_1, \ldots, b_n)$ had positive highest-degree coefficients. After making the substitutions described above, we still have positive leading coefficients. When we collect terms, the result then is of the form $\binom{2m_1}{m_1} \cdots \binom{2m_n}{m_n} P_{g,n}(b_1, \ldots, b_n)$, where $P_{g,n}$ has positive highest-order coefficients and degree $3g - 3 + 2n$. $\square$

We illustrate the technique with an example, computing $G_{1,1}(b)$; clearly, we need only consider b even, $b = 2m$. We computed in Sec. 7.2, Eq. (11) of Proposition 1.5,

$$\widehat{N}_{1,1}(b) = \frac{1}{48} b^2 + \frac{5}{12} \text{ for } b \neq 0 \text{ even}, \quad \widehat{N}_{1,1}(0) = 1.$$

Hence

$$\begin{aligned} G_{1,1}(b) &= \sum_{\substack{0\le a\le b \\ a\equiv b \pmod 2}} \bar{a}\, \widehat{N}_{1,1}(a) \\ &= \binom{b}{b/2}\widehat{N}_{1,1}(0) + \sum_{\substack{0< a\le b \\ a\equiv b \pmod 2}} \binom{b}{\frac{b-a}{2}} \bar{a} \left(\frac{1}{48}a^2 + \frac{5}{12}\right) \\ &= \binom{2m}{m} + \frac{1}{48}\tilde{p}_1(m) + \frac{5}{12}\tilde{p}_0(m) = \binom{2m}{m}\left(\frac{1}{12}m^2 + \frac{5}{12}m + 1\right). \end{aligned}$$

This gives Eq. (6) in Proposition 1.2.

8. Differentials and Generating Functions

8.1. *Definitions*

We now string the curve counts $N_{g,n}$ and $G_{g,n}$ out into generating functions and differentials.

Definition 8.1 (Generating functions and differentials). For any $g \ge 0$ and $n \ge 1$, let

$$\begin{aligned} f^G_{g,n}(x_1,\ldots,x_n) &= \sum_{\mu_1,\ldots,\mu_n\ge 0} G_{g,n}(\mu_1,\ldots,\mu_n) x_1^{-\mu_1-1}\cdots x_n^{-\mu_n-1} \\ f^N_{g,n}(z_1,\ldots,z_n) &= \sum_{\nu_1,\ldots,\nu_n\ge 0} N_{g,n}(\nu_1,\ldots,\nu_n) z_1^{\nu_1-1}\cdots z_n^{\nu_n-1} \\ \omega^G_{g,n}(x_1,\ldots,x_n) &= f^G_{g,n}(x_1,\ldots,x_n)\, dx_1\otimes\cdots\otimes dx_n \\ \omega^N_{g,n}(z_1,\ldots,z_n) &= f^N_{g,n}(z_1,\ldots,z_n)\, dz_1\otimes\cdots\otimes dz_n. \end{aligned}$$

Here, $x_1,\ldots,x_n$ are coordinates on $\mathbb{CP}^1$, as are $z_1,\ldots,z_n$. For now, we treat these as formal Laurent series. In Sec. 8.3, we show that they are all meromorphic functions and forms.

The differential forms can be regarded as sections of the product bundle

$$(T^*\mathbb{CP}^1)^{\boxtimes n} = \pi_1^*(T^*\mathbb{CP}^1)\otimes\pi_2^*(T^*\mathbb{CP}^1)\otimes\cdots\otimes(\pi_n^*T^*\mathbb{CP}^1).$$

This is a vector bundle over $(\mathbb{CP}^1)^n$, where $\pi_i : (\mathbb{CP}^1)^n \to \mathbb{CP}^1$ is projection onto the i'th coordinate. For convenience, we write $dz_1\cdots dz_n$ rather than $dz_1\otimes\cdots\otimes dz_n$. The $\omega_{g,n}$ are multidifferentials.

We will often regard the coordinates z and x as related by the equation $x = z+\frac{1}{z}$; indeed, as we will see in Sec. 8.4, $\omega^G_{g,n}$ and $\omega^N_{g,n}$ are essentially *equal*, with this change of coordinates.

8.2. *Small cases*

We can compute the generating functions $f^G_{g,n}, f^N_{g,n}$ and differential forms $\omega^G_{g,n}, \omega^N_{g,n}$ directly in the cases $(g,n) = (0,1)$ or $(0,2)$.

Let $x_i = z_i + \frac{1}{z_i}$ and $z_i = \frac{x_i - \sqrt{x_i^2-4}}{2}$.

Lemma 8.2.

$$\omega^N_{0,1}(z_1) = z_1^{-1} dz_1$$

$$\omega^G_{0,1}(x_1) = \frac{x_1 - \sqrt{x_1^2 - 4}}{2}\, dx_1 = z_1\, dx_1 = (z_1 - z_1^{-1})\, dz_1$$

$$\omega^N_{0,2}(z_1, z_2) = \left(\frac{1}{z_1 z_2} + \frac{1}{(1 - z_1 z_2)^2} \right) dz_1 dz_2$$

$$\omega^N_{0,3}(z_1, z_2, z_3) = \frac{\begin{array}{c} 1 + z_1^4 z_2^4 z_3^4 + \sum_{\text{cyc}} (z_1^4 + z_1 z_2 + z_1^3 z_2^3 + z_1^4 z_2^4) \\ + \sum_{\text{sym}} (z_1^3 z_2 + z_1^4 z_2^3 z_3 + z_1^4 z_2 z_3) \end{array}}{z_1 z_2 z_3 (1 - z_1^2)^2 (1 - z_2^2)^2 (1 - z_3^2)^2}\, dz_1 dz_2 dz_3.$$

The $\sum_{\text{cyc}}$ refers to a sum over cyclic permutations of z_1, z_2, z_3 (i.e. $(1,2,3) \mapsto (2,3,1), (3,1,2)$, 3 terms), and $\sum_{\text{sym}}$ to a sum over all permutations (6 terms).

Proof. For $(g,n) = (0,1)$, we have $N_{0,1}(0) = 1$ and all other $N_{0,1}(\nu) = 0$, so $f^N_{0,1}(z_1) = z_1^{-1}$ and $\omega^N_{0,1}$ is as claimed. We also have $G_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m}$ and $G_{0,1}(\mu) = 0$ for odd μ, so $f^G_{0,1}(x_1)$ is a generating function for the Catalan numbers:

$$\begin{aligned} f^G_{0,1}(x_1) &= \sum_{m=0}^{\infty} G_{0,1}(2m) x_1^{-2m-1} \\ &= \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} x_1^{-2m-1} = \frac{x_1 - \sqrt{x_1^2 - 4}}{2} = z_1. \end{aligned}$$

Since $dx_i = (1 - z_i^{-2}) dz_i$, then $\omega^G_{0,1}$ is as claimed.

Turning to $(g,n) = (0,2)$, recall $N_{0,2}(\nu_1, \nu_2) = \delta_{\nu_1,\nu_2} \overline{\nu_1}$ (Lemma 3.15). Noting that $\sum_{\nu=0}^{\infty} \nu z^{\nu-1} = \frac{1}{(1-z)^2}$, we compute

$$f^N_{0,2}(z_1, z_2) = \sum_{\nu=0}^{\infty} \overline{\nu}\, (z_1 z_2)^{\nu-1} = z_1^{-1} z_2^{-1} + \sum_{\nu=0}^{\infty} \nu (z_1 z_2)^{\nu-1} = z_1^{-1} z_2^{-1} + \frac{1}{(1 - z_1 z_2)^2}.$$

Thus, $\omega^N_{0,2}$ is as desired.

Turning to $(g,n) = (0,3)$, from Sec. 5.2, we have $N_{0,3}(b_1, b_2, b_3) = \bar{b}_1 \bar{b}_2 \bar{b}_3$ if $b_1 + b_2 + b_3$ is even, and 0 otherwise. Thus

$$\begin{aligned} &f^N_{0,3}(z_1, z_2, z_3) \\ &\quad = \sum_{\substack{\nu_1, \nu_2, \nu_3 \geq 0 \\ \nu_1 + \nu_2 + \nu_3 \text{ even}}} \overline{\nu_1}\, \overline{\nu_2}\, \overline{\nu_3}\, z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} z_3^{\nu_3 - 1} \end{aligned}$$

$$= \left(\sum_{\nu_1,\nu_2,\nu_3 \text{ even}} + \sum_{\substack{\nu_1 \text{ even} \\ \nu_2,\nu_3 \text{ odd}}} + \sum_{\substack{\nu_2 \text{ even} \\ \nu_3,\nu_1 \text{ odd}}} + \sum_{\substack{\nu_3 \text{ even} \\ \nu_1,\nu_2 \text{ odd}}} \right)$$

$$\times \overline{\nu_1}\, \overline{\nu_2}\, \overline{\nu_3}\, z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} z_3^{\nu_3 - 1}.$$

If we define

$$\rho(z) = \sum_{\substack{\nu \geq 0 \\ \nu \text{ even}}} \overline{\nu}\, z^{\nu-1}, \quad \sigma(z) = \sum_{\substack{\nu \geq 0 \\ \nu \text{ odd}}} \overline{\nu}\, z^{\nu-1}$$

then we have

$$f_{0,3}^N(z_1, z_2, z_3) = \rho(z_1)\rho(z_2)\rho(z_3) + \rho(z_1)\sigma(z_2)\sigma(z_3) \\ + \rho(z_2)\sigma(z_3)\sigma(z_1) + \rho(z_3)\sigma(z_1)\sigma(z_2).$$

We can compute $\rho(z), \sigma(z)$ directly (say by differentiating the geometric series $\frac{1}{1-z^2} = \sum_{m \geq 0} z^{2m}$):

$$\rho(z) = \left(z^{-1} + \frac{2z}{(1-z^2)^2} \right) \quad \text{and} \quad \sigma(z) = \frac{1+z^2}{(1-z^2)^2}.$$

Writing out $f_{0,3}^N$ in terms of z_1, z_2, z_3, we obtain the claimed expression. $\square$

Observe that all the functions and forms computed above are meromorphic; we next show this is true in general.

8.3. *Meromorphicity*

Proposition 8.3. *For all $g \geq 0$ and $n \geq 1$, $f_{g,n}^N(z_1, \ldots, z_n)$ is a meromorphic function and $\omega_{g,n}^N(z_1, \ldots, z_n)$ is a meromorphic differential form.*

Proof. In Sec. 8.2 above, we computed $\omega_{0,1}^N(z_1)$ and $\omega_{0,2}^N(z_1, z_2)$, seeing directly that they are meromorphic. And $\omega_{g,n}^N = f_{g,n}^N dz_1 \cdots dz_n$. So, it suffices to show $f_{g,n}^N(z_1, \ldots, z_n)$ is a meromorphic function, for $(g, n) \neq (0, 1), (0, 2)$.

By Theorem 1.7, for $(g, n) \neq (0, 1), (0, 2)$, each $N_{g,n}(\nu_1, \ldots, \nu_n)$ is $\bar{\nu}_1 \bar{\nu}_2 \cdots \bar{\nu}_n$ times a quasi-polynomial function of $\nu_1, \ldots, \nu_n$, depending on the parity of each ν_i. Letting $\nu_i \equiv \epsilon_i \pmod 2$, where $\epsilon_i \in \{0, 1\}$, we split $f_{g,n}^N$ into 2^n sums of the form

$$\sum_{\substack{\nu_1 \geq 0 \\ \nu_1 \equiv \epsilon_1 \pmod 2}} \cdots \sum_{\substack{\nu_n \geq 0 \\ \nu_n \equiv \epsilon_n \pmod 2}} \overline{\nu_1} \cdots \overline{\nu_n}\, P(\nu_1, \ldots, \nu_n) z_1^{\nu_1 - 1} \cdots z_n^{\nu_n - 1},$$

where $P(\nu_1, \ldots, \nu_n)$ is a polynomial. Splitting each such polynomial into monomials, we can write $f_{g,n}^N$ as a finite sum of terms of the form of a constant times

$$\sum_{\substack{\nu_1 \geq 0 \\ \nu_1 \equiv \epsilon_1 \pmod 2}} \cdots \sum_{\substack{\nu_n \geq 0 \\ \nu_n \equiv \epsilon_n \pmod 2}} \overline{\nu_1} \cdots \overline{\nu_n}\, \nu_1^{a_1} \cdots \nu_n^{a_n} z_1^{\nu_1 - 1} \cdots z_n^{\nu_n - 1}$$

$$= \prod_{i=1}^n \left(\sum_{\substack{\nu_i \geq 0 \\ \nu_i \equiv \epsilon_i \pmod 2}} \overline{\nu_i}\, \nu_i^{a_i} z_i^{\nu_i - 1} \right),$$

where $a_1, \ldots, a_n$ are non-negative integers. Thus, it suffices to show that for $a \geq 0$ and $\epsilon \in \{0, 1\}$,

$$\sum_{\substack{\nu \geq 0 \\ \nu \equiv \epsilon \pmod 2}} \overline{\nu}\, \nu^a\, z^{\nu - 1} = \delta_{a,0} z^{-1} + \sum_{\substack{\nu \geq 0 \\ \nu \equiv \epsilon \pmod 2}} \nu^{a+1} z^{\nu - 1}$$

is meromorphic. Now we have

$$\sum_{\substack{\nu \geq 0 \\ \nu \equiv \epsilon \pmod 2}} \nu^a z^\nu = \left(z \frac{d}{dz} \right)^a \sum_{\substack{\nu \geq 0 \\ \nu \equiv \epsilon \pmod 2}} z^\nu$$

so it suffices to show that $\sum_{\nu \equiv \epsilon \pmod 2} z^\nu$ is meromorphic. Accordingly as $\epsilon = 0$ or 1, we have

$$\sum_{\substack{\nu \geq 0 \\ \nu \text{ even}}} z^\nu = \sum_{m \geq 0} z^{2m} = \frac{1}{1 - z^2}, \quad \text{or} \quad \sum_{\substack{\nu \geq 0 \\ \nu \text{ odd}}} z^\nu = \sum_{m \geq 0} z^{2m+1} = \frac{z}{1 - z^2},$$

both of which are clearly meromorphic. □

In fact, since $z \frac{d}{dz}$ introduces no new poles, we note that for all $(g, n) \neq (0, 2)$, $\omega_{g,n}^N$ has poles only at $z_i = -1, 0, 1$.

8.4. *Change of coordinates between non-boundary-parallel and general curve counts*

Recall $\omega_{g,n}^N$ is a generating function for the $N_{g,n}$, while $\omega_{g,n}^G$ is a generating function for the $G_{g,n}$. It turns out that after the change of variable $x_i = z_i + \frac{1}{z_i}$ (so that $dx_i = (1 - z_i^{-2}) dz_i$), these two formal differential forms are *equal*.

So define $\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ by $\phi(z) = z + \frac{1}{z} = x$, and consider pulling back $\omega_{g,n}^G(x_1, \ldots, x_n)$ under ϕ. We may therefore express Theorem 1.14 more precisely by saying that for any $(g, n) \neq (0, 1)$,

$$\phi^* \omega_{g,n}^G(x_1, \ldots, x_n) = \omega_{g,n}^N(z_1, \ldots, z_n).$$

Thus, if we regard x and z as alternative coordinates on $\mathbb{CP}^1$ and ϕ as a change of coordinate, then $\omega_{g,n}^G$ and $\omega_{g,n}^N$ give the same differential form, which we simply denote $\omega_{g,n}$. We can also express this theorem as

$$\omega_{g,n}^N(z) = \sum_{\nu=\mathbf{0}}^{\infty} N_{g,n}(\nu) z^{\nu-1} dz = \sum_{\mu=\mathbf{0}}^{\infty} G_{g,n}(\mu) x^{-\mu-1} dx = \omega_{g,n}^G(x).$$

Our explicit computations of $\omega_{0,1}^G$ and $\omega_{0,1}^N$ show that the theorem fails for $(g,n) = (0,1)$.

The proof is by a residue argument, following ideas of Do–Norbury in [10].

Proof of Theorem 1.14. Let $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$. From Theorem 1.8, we have

$$G_{g,n}(\mu) = \sum_{\nu_1,\ldots,\nu_n \geq 0} N_{g,n}(\nu) \prod_{i=1}^n \binom{\mu_i}{\frac{\mu_i - \nu_i}{2}}.$$

Now we note that, for any integers μ, ν (even if negative, even if $\nu > \mu$),

$$\begin{aligned}\binom{\mu}{\frac{\mu-\nu}{2}} &= \operatorname*{Res}_{z=0}\ z^{\nu-\mu-1} \sum_{m=0}^{\infty} \binom{\mu}{m} z^{2m}\ dz \\ &= \operatorname*{Res}_{z=0}\ z^{\nu-\mu-1}(1+z^2)^{\mu}\ dz = \operatorname*{Res}_{z=0}\ z^{\nu-1}\ dz\ x^{\mu}.\end{aligned}$$

Substituting this residue expression for $\binom{\mu}{\frac{\mu-\nu}{2}}$ and recalling that $\omega_{g,n}^N$ is meromorphic, we obtain

$$G_{g,n}(\mu) = \operatorname*{Res}_{(z_1,\ldots,z_n)=(0,\ldots,0)} \omega_{g,n}^N(z_1,\ldots,z_n) \prod_{i=1}^n x_i^{\mu_i}.$$

Now suppose, we rewrite $\omega_{g,n}^N(z_1,\ldots,z_n)$ in terms of $x_1,\ldots,x_n$; as $\omega_{g,n}^N$ is meromorphic this form is determined by its Laurent series. Let $a_{g,n}(\lambda_1,\ldots,\lambda_n)$ be the coefficient of $x_1^{-\lambda_1-1}\cdots x_n^{-\lambda_n-1}\, dx_1\cdots dx_n$, so

$$\omega_{g,n}^N = \sum_{\lambda_1,\ldots,\lambda_n} a_{g,n}(\lambda_1,\ldots,\lambda_n) x_1^{-\lambda_1-1}\cdots x_n^{-\lambda_n-1}\ dx_1\cdots dx_n.$$

The residue at $(z_1,\ldots,z_n) = (0,\ldots,0)$ corresponds to the residue at $(x_1,\ldots,x_n) = (\infty,\ldots,\infty)$; if we substitute $y_i = x_i^{-1}$, this corresponds to the residue at $(y_1,\ldots,y_n) = (0,\ldots,0)$. Since $dx_i = -y_i^{-2}\ dy_i$ and $x_i^{-\lambda_i-1}dx_i = -y_i^{\lambda_i-1}\ dy_i$, we have

$$\begin{aligned}G_{g,n}(\mu) = &\operatorname*{Res}_{(x_1,\ldots,x_n)=(\infty,\ldots,\infty)} \\ &\times \sum_{\lambda_1,\ldots,\lambda_n} a_{g,n}(\lambda_1,\ldots,\lambda_n) x_1^{-\lambda_1-1}\cdots x_n^{-\lambda_n-1}\ dx_1\cdots dx_n \prod_{i=1}^n x_i^{\mu_i}\end{aligned}$$

$$= (-1)^{2n} \operatorname*{Res}_{(y_1,\ldots,y_n)=(0,\ldots,0)}$$

$$\times \sum_{\lambda_1,\ldots,\lambda_n} a_{g,n}(\lambda_1,\ldots,\lambda_n) y_1^{\lambda_1-1-\mu_1} \cdots y_n^{\lambda_n-1-\mu_n}\, dy_1 \cdots dy_n$$

$$= a_{g,n}(\mu_1,\ldots,\mu_n).$$

Hence $G_{g,n}(\mu) = a_{g,n}(\mu)$, and $\omega^N_{g,n}$, expressed in terms of the x_i, is actually a generating function for the $G_{g,n}(\mu)$, as desired. □

We illustrate the use of this theorem by calculating $f^G_{0,2}(x_1,x_2)$, the generating function for all $G_{0,2}(\mu_1,\mu_2)$, given by the complicated formulae in Eqs. (2) and (3).

Lemma 8.4.

$$f^G_{0,2}(x_1,x_2) = \frac{1}{2(x_1-x_2)^2}\left(1 + \frac{2x_1^2 - 3x_1x_2 + 2x_2^2 - 4}{\sqrt{(x_1^2-4)(x_2^2-4)}}\right).$$

Proof. Substitute $z_i = \frac{x_i - \sqrt{x_i^2-4}}{2}$ into the expression $\omega_{0,2} = (\frac{1}{z_1z_2} + \frac{1}{(1-z_1z_2)^2})dz_1dz_2$ from Lemma 8.2. □

8.5. *Free energies*

Each $\omega_{g,n}(z_1,\ldots,z_n)$ is a meromorphic section of the vector bundle $(T^*\mathbb{CP}^1)^{\boxtimes n}$ over $(\mathbb{CP}^1)^n$. A form of this type may be obtained by taking a function $F : (\mathbb{CP}^1)^n \to \mathbb{CP}^1$ and the exterior differential d_{z_i} in each coordinate z_i. Then $d_{z_1}d_{z_2}\cdots d_{z_n}F$ is a section of $(T^*\mathbb{CP}^1)^{\boxtimes n}$.

Definition 8.5. A function $F_{g,n} : (\mathbb{CP}^1)^n \to \mathbb{CP}^1$ such that

$$d_{z_1}\cdots d_{z_n}F_{g,n}(z_1,\ldots,z_n) = \omega_{g,n}(z_1,\ldots,z_n)$$

is called a *free energy.*

Given $\omega_{g,n}$, there are many free energies: $F_{g,n} = \int^{z_1}\cdots\int^{z_n}\omega_{g,n}$; each integral introduces a constant of integration. We have

$$f^G_{g,n}(x_1,\ldots,x_n) = \frac{\partial^n F_{g,n}}{\partial x_1\,\partial x_2\,\cdots\,\partial x_n} \quad \text{and} \quad f^N_{g,n}(z_1,\ldots,z_n) = \frac{\partial^n F_{g,n}}{\partial z_1\,\partial z_2\,\cdots\,\partial z_n}.$$

In the case $(g,n)=(0,1)$, we can integrate $\omega^G_{0,1}$ and $\omega^N_{0,1}$; in this case, we can obtain two distinct free energy functions $F^G_{0,1}(z_1)$ and $F^N_{0,1}(x_1)$.

Proof of Proposition 1.15. It is straightforward (if a little tedious in the $(0,3)$ case) to check that differentiating the claimed free energy functions yields the forms $\omega^G_{0,1}, \omega^N_{0,1}, \omega_{0,2}, \omega_{0,3}$ calculated in Lemma 8.2. □

8.6. *Recursion and generating functions*

We now make a first attempt to turn the recursion on $G_{g,n}$ into a recursion on generating functions $f^G_{g,n}$. Throughout this section, we write $f_{g,n}$ rather than $f^G_{g,n}$, for convenience. (No $f^N_{g,n}$'s arise, so there is no possible ambiguity.)

The recursion on $G_{g,n}$ (Theorem 1.4), as noted in Sec. 1.2, is identical to the recursion obeyed by the "generalized Catalan numbers", but has different initial conditions. Since generating functions for the "generalized Catalan numbers" obey a recursive differential equation [29], we might expect the $f_{g,n}$ to obey a similar differential equation. However the different initial conditions lead to some difficulties. The recursion on $G_{g,n}$ fails when $b_1 = 0$; the generalized Catalan numbers avoid this issue, as $b_1 = 0$ implies that the corresponding generalized Catalan number is zero.

In our first attempt now, we postpone the issue and only consider $b_1 > 0$, and prove the following.

Lemma 8.6. *For any $g \geq 0$ and $n \geq 1$, we have*

$$\sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0}} G_{g,n}(b_1,\ldots,b_n) x_1^{-b_1-1} \cdots x_n^{-b_n-1}$$

$$\begin{aligned} &= x_1^{-1} f_{g-1,n+1}(x_1,x_1,x_2,\ldots,x_n) \\ &\quad + x_1^{-1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \frac{1}{x_k - x_1} (f_{g,n-1}(x_2,\ldots,x_n) - f_{g,n-1}(x_1,x_2,\ldots,\widehat{x}_k,\ldots,x_n)) \\ &\quad + x_1^{-1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} f_{g_1,|I_1|+1}(x_1,x_{I_1})\, f_{g_2,|I_2|+1}(x_1,x_{I_2}). \end{aligned}$$

Proof. Take the recursion on $G_{g,n}$ (Theorem 1.4), multiply by $x_1^{-b_1-1} \cdots x_n^{-b_n-1}$, and sum over all $b_1 \geq 1$ and $b_2,\ldots,b_n \geq 0$. We obtain, on the right-hand side, the three terms

$$I = \sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0}} \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} G_{g-1,n+1}(i,j,b_2,\ldots,b_n)\, x_1^{-b_1-1} \cdots x_n^{-b_n-1}$$

$$II = \sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0}} \sum_{k=2}^{n} b_k G_{g,n-1}(b_1+b_k-2,b_2,\ldots,\widehat{b}_k,\ldots,b_n)\, x_1^{-b_1-1} \cdots x_n^{-b_n-1}$$

$$\begin{aligned} III = &\sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0}} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \\ &\times \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} G_{g_1,|I_1|+1}(i,b_{I_1}) G_{g_2,|I_2|+1}(j,b_{I_2})\, x_1^{-b_1-1} \cdots x_n^{-b_n-1}. \end{aligned}$$

Each of I, II, III can be written in terms of the generating functions $f_{g,n}$. We show that these correspond to the three terms in the claimed equation.

Considering I, we note that $i+j = b_1 - 2$ implies $x_1^{-b_1-1} = x_1^{-i-1}x_1^{-j-1}x_1^{-1}$; and we note that a sum over $b_1 \geq 1$, followed by a sum over $i, j \geq 0$ with $i+j = b_1 - 2$, is simply a sum over $i, j \geq 0$. Thus

$$\begin{aligned} I &= x_1^{-1} \sum_{b_2,\ldots,b_n \geq 0} \sum_{i,j\geq 0} G_{g-1,n+1}(i,j,b_2,\ldots,b_n)\, x_1^{-i-1}x_1^{-j-1}x_2^{-b_2-1}\cdots x_n^{-b_n-1} \\ &= x_1^{-1} f^G_{g-1,n+1}(x_1,x_1,x_2,\ldots,x_n). \end{aligned}$$

To simplify II, let $m = b_1 + b_k - 2$, and replace the sum over b_1 and b_k with a sum over m, followed by a sum over $b_1 \geq 1$, $b_k \geq 0$ satisfying $b_1 + b_k - 2 = m$.

$$\begin{aligned} II = \sum_{k=2}^{n} \sum_{\substack{b_2,\ldots,\widehat{b}_k,\ldots,b_n\geq 0\\ m\geq 0}} & G_{g,n-1}(m,b_2,\ldots,\widehat{b}_k,\ldots,b_n)x_2^{-b_2-1}\cdots \widehat{x_k^{-b_k-1}}\cdots x_n^{-b_n-1} \\ & \times \sum_{\substack{b_1\geq 1,\ b_k\geq 0\\ b_1+b_k-2=m}} b_k\, x_1^{-b_1-1}x_k^{-b_k-1} \end{aligned}$$

Now we note the sum over b_1 and b_k is

$$\begin{aligned} \sum_{\substack{b_1\geq 1,\ b_k\geq 0\\ b_1+b_k-2=m}} b_k\, x_1^{-b_1-1}x_k^{-b_k-1} &= -x_1^{-1}\frac{\partial}{\partial x_k}\left(\frac{x_1^{-m-1}-x_k^{-m-1}}{x_k-x_1}\right) \\ &= \frac{-x_1^{-1}}{x_k-x_1}(m+1)x_k^{-m-2} + \frac{x_1^{-1}}{(x_k-x_1)^2}\left(x_1^{-m-1}-x_k^{-m-1}\right). \end{aligned}$$

Hence, we obtain an expression for II, which is a sum over k and the parameters $b_2, \ldots, \widehat{b}_k, \ldots, b_n, m$ appearing in the $G_{g,n}$:

$$\begin{aligned} II = \sum_{k=2}^{n} & \sum_{b_2,\ldots,\widehat{b}_k,\ldots,b_n,m\geq 0} G_{g,n-1}(m,b_2,\ldots,\widehat{b}_k,\ldots,b_n)x_2^{-b_2-1}\cdots \widehat{x_k^{-b_k-1}}\cdots x_n^{-b_n-1} \\ & \left[\frac{-x_1^{-1}}{x_k-x_1}(m+1)x_k^{-m-2} + \frac{x_1^{-1}}{(x_k-x_1)^2}(x_1^{-m-1}-x_k^{-m-1})\right] \\ = \sum_{k=2}^{n} & \sum_{b_2,\ldots,\widehat{b}_k,\ldots,b_n,m\geq 0} \frac{-x_1^{-1}}{x_k-x_1} G_{g,n-1}(m,b_2,\ldots,\widehat{b}_k,\ldots,b_n)x_2^{-b_2-1}\cdots \widehat{x_k^{-b_k-1}} \\ & \cdots x_n^{-b_n-1}\ (m+1)x_k^{-m-2} + \frac{x_1^{-1}}{(x_k-x_1)^2} G_{g,n-1}(m,b_2,\ldots,\widehat{b}_k,\ldots,b_n)x_2^{-b_2-1} \\ & \cdots \widehat{x_k^{-b_k-1}}\cdots x_n^{-b_n-1}(x_1^{-m-1}-x_k^{-m-1}) \end{aligned}$$

$$= \sum_{k=2}^{n} \frac{x_1^{-1}}{x_k - x_1} \frac{\partial}{\partial x_k} f_{g,n-1}(x_2, \ldots, x_n)$$

$$+ \frac{x_1^{-1}}{(x_k - x_1)^2} (f_{g,n-1}(x_1, x_2, \ldots, \widehat{x}_k, \ldots, x_n) - f_{g,n-1}(x_2, \ldots, x_n))$$

$$= x_1^{-1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \frac{1}{x_k - x_1} (f_{g,n-1}(x_2, \ldots, x_n) - f_{g,n-1}(x_1, x_2, \ldots, \widehat{x}_k, \ldots, x_n)).$$

Finally, we turn to III. As with I, a sum over $b_1 \geq 1$ and then over $i, j \geq 0$ with $i + j = b_1 - 2$ is equivalent to a sum over $i, j \geq 0$, so we obtain

$$III = x_1^{-1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \sum_{i, b_I \geq 0} G_{g_1,|I_1|+1}(i, b_{I_1}) x_1^{-i-1} \, x_{I_1}^{-b_{I_1}-1}$$

$$\times \sum_{j, b_J \geq 0} G_{g_2,|I_2|+1}(j, b_{I_2}) x_1^{-j-1} \, x_{I_2}^{-b_{I_2}-1}$$

$$= x_1^{-1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} f_{g_1,|I_1|+1}(x_1, x_{I_1}) \, f_{g_2,|I_2|+1}(x_1, x_{I_2}).$$

This gives the desired result. □

From the above lemma, we can obtain a differential equation for $f_{g,n}$ by arranging all the terms with $b_1 = 0$ to be constant terms, and differentiating them away.

Proposition 8.7. *For any $g \geq 0$ and $n \geq 1$,*

$$\frac{\partial}{\partial x_1} (x_1 f_{g,n}(x_1, \ldots, x_n))$$

$$= \frac{\partial}{\partial x_1} f_{g-1,n+1}(x_1, x_1, x_2, \ldots, x_n)$$

$$+ \frac{\partial}{\partial x_1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \frac{1}{x_k - x_1} (f_{g,n-1}(x_2, \ldots, x_n) - f_{g,n-1}(x_1, x_2, \ldots, \widehat{x}_k, \ldots, x_n))$$

$$+ \frac{\partial}{\partial x_1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} f_{g_1,|I_1|+1}(x_1, x_{I_1}) f_{g_2,|I_2|+1}(x_1, x_{I_2}).$$

□

Proof. Consider the equation in Lemma 8.6 above. Multiplying the left-hand side by x_1 and then differentiating with respect to x_1, we obtain $\frac{\partial}{\partial x_1}(x_1 f_{g,n}(x_1, \ldots, x_n))$, since terms with $b_1 = 0$ are annihilated by the differentiation. Doing the same to the right-hand side yields the result. □

We return to the search for a differential equation in Sec. 10.6. An alternative method to obtain a differential equation is to find a simple way to compute

$G_{g,n}(0, b_2, \ldots, b_n)$. There is a straightforward way to do this, if we keep track of the number of *complementary regions* in the arc diagram. This is the subject of the next section.

9. Keeping Track of Regions

9.1. *Refining curve counts*

As it turns out, many of the results already proved about $G_{g,n}$ and $N_{g,n}$, can be refined by keeping track of the number r of complementary regions (Definition 1.10) in arc diagrams. It will turn out to be very useful to use a related parameter t. We begin by making the following definitions.

Definition 9.1.

(i) The set of equivalence classes of arc diagrams on $(S_{g,n}, F(\mathbf{b}))$ with r complementary regions is denoted $\mathcal{G}_{g,n,r}(\mathbf{b})$. The number of such equivalence classes is denoted $G_{g,n,r}(\mathbf{b})$.

(ii) The subset of $\mathcal{G}_{g,n,r}(\mathbf{b})$ without boundary-parallel arcs is denoted $\mathcal{N}_{g,n,r}(\mathbf{b})$. The number of such equivalence classes is denoted $N_{g,n,r}(\mathbf{b})$. We also define

$$\widehat{N}_{g,n,r}(b_1, \ldots, b_n) = \frac{N_{g,n,r}(b_1, \ldots, b_n)}{\overline{b_1}\,\overline{b_2}\,\cdots\,\overline{b_n}}.$$

(iii) For $g \geq 0$ and $n, r \geq 1$ and $b_1, \ldots, b_n \geq 0$, define

$$t = r - (2 - 2g - n) - \frac{1}{2}\sum_{i=1}^{n} b_i = r - \chi - \frac{1}{2}\sum_{i=1}^{n} b_i.$$

(iv) For $g \geq 0$, $n \geq 1$, and $b_1, \ldots, b_n \geq 0$, define $\mathcal{G}^t_{g,n}(\mathbf{b}) = \mathcal{G}_{g,n,r}(\mathbf{b})$, $\mathcal{N}^t_{g,n}(\mathbf{b}) = \mathcal{N}_{g,n,r}(\mathbf{b})$ and

$$G^t_{g,n}(\mathbf{b}) = G_{g,n,r}(\mathbf{b}), \quad N^t_{g,n}(\mathbf{b}) = N_{g,n,r}(\mathbf{b}), \quad \widehat{N}^t_{g,n}(\mathbf{b}) = \widehat{N}_{g,n,r}(\mathbf{b}).$$

Clearly $\mathcal{G}_{g,n}(\mathbf{b}) = \sqcup_{r \geq 0}\mathcal{G}_{g,n,r}(\mathbf{b}) = \sqcup_t \mathcal{G}^t_{g,n}(\mathbf{b})$ and $G_{g,n}(\mathbf{b}) = \sum_{r\geq 0} G_{g,n,r}(\mathbf{b})$, so we have $G_{g,n}(\mathbf{b}) = \sum_{r\geq 0} G_{g,n,r}(\mathbf{b}) = \sum_t G^t_{g,n}(\mathbf{b})$, $N_{g,n}(\mathbf{b}) = \sum_{r\geq 0} N_{g,n,r}(\mathbf{b}) = \sum_t N^t_{g,n}(\mathbf{b})$ and $\widehat{N}_{g,n}(\mathbf{b}) = \sum_{r\geq 0} \widehat{N}_{g,n,r}(\mathbf{b}) = \sum_t \widehat{N}^t_{g,n}(\mathbf{b})$. We will discuss how many non-zero terms are in these sums, i.e. bounds on r and t, in Sec. 9.5.

We can easily compute the refined counts $G_{g,n,r}$ and $N_{g,n,r}$ explicitly for $(g, n) = (0, 1), (0, 2)$.

Lemma 9.2.

(i) *For any integer* $m \geq 0$,

$$G_{0,1,m+1}(2m) = G^0_{0,1}(2m) = \frac{1}{m+1}\binom{2m}{m}.$$

(ii) *For integers* $m_1, m_2 \geq 0$,

$$G^0_{0,2}(2m_1, 2m_2) = G_{0,2,m_1+m_2}(2m_1, 2m_2) = \frac{m_1 m_2}{m_1 + m_2}\binom{2m_1}{m_1}\binom{2m_2}{m_2}$$

$$\begin{aligned} G^0_{0,2}(2m_1+1, 2m_2+1) &= G_{0,2,m_1+m_2}(2m_1+1, 2m_2+1) \\ &= \frac{(2m_1+1)(2m_2+1)}{m_1+m_2+1}\binom{2m_1}{m_1}\binom{2m_2}{m_2} \end{aligned}$$

$$G^1_{0,2}(2m_1, 2m_2) = G_{0,2,m_1+m_2+1}(2m_1, 2m_2) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}.$$

All other $G_{0,1,r}(b_1), G^t_{0,1}(b_1), G_{0,2,r}(b_1, b_2)$ *and* $G^t_{0,2}(b_1, b_2)$ *are zero.*

Proof. An a disc $(S_{0,1}, F(2m))$, an arc diagram has m arcs, which divide the disc into $m+1$ complementary regions. Thus $G_{0,1,r}(2m) = G_{0,1}(2m)$ if $r = m+1$, and is zero otherwise. This value of r corresponds to $t = 0$.

Now consider annuli $(S_{0,2}, F(b_1, b_2))$. For an arc diagram to exist, we need $b_1 + b_2 \equiv 0 \pmod 2$. From Lemma 3.4, a traversing arc diagram has $r = \frac{1}{2}(b_1 + b_2)$, and an insular diagram has $r = \frac{1}{2}(b_1 + b_2) + 1$; these correspond to $t = 0$ and $t = 1$, respectively. Propositions 3.2 and 3.3 then give the result. □

Lemma 9.3.

(i) $N_{0,1,1}(0) = N^0_{0,1}(0) = 1$, *and all other* $N_{0,1,r}(b_1)$ *and* $N^t_{0,1}(b_1)$ *are zero.*
(ii) $N_{0,2,1}(0,0) = 1$, $N_{0,2,b}(b,b) = b$ *for* $b > 0$, *and all other* $N_{0,2,r}(b_1, b_2)$ *are zero. Equivalently,* $N^1_{0,2}(0,0) = 1$, $N^0_{0,2}(b,b) = b$ *for* $b > 0$, *and all other* $N^t_{0,2}(b_1, b_2)$ *are zero.* □

Proof. As discussed in Sec. 3.4, the only arc diagram without boundary-parallel arcs on a disc is the empty diagram, for which $r = 1$ and $t = 0$. On an annulus, such a diagram must consist entirely of parallel traversing arcs, so $b_1 = b_2 = b$; there are $\bar{b}$ such diagrams, which have $\bar{b}$ complementary regions, so $r = \bar{b}$ and $t = \bar{b} - b = \delta_{b,0}$. □

9.2. *Counting arc diagrams with punctures*

When $b_1 = 0$, the first boundary component of $S_{g,n}$ has no points marked on it; we may regard the boundary component as a *puncture* in $S_{g,n-1}$. Filling in the puncture gives arc diagrams on $S_{g,n-1}$; we already saw this idea in Proposition 3.7. We now show precisely how keeping track of regions allows us to compute $G_{g,n,r}(0, b_2, \ldots, b_n)$.

Proposition 9.4. *For any* $g \geq 0$, $n \geq 2$ *and* $b_2, \ldots, b_n \geq 0$,

$$G_{g,n,r}(0, b_2, \ldots, b_n) = r\, G_{g,n-1,r}(b_2, \ldots, b_n).$$

In the case of enumerating lattice points in moduli spaces of curves [32], the evaluation $b_1 = 0$ is related to the dilaton equation that appears in the general theory of the topological recursion [18]. Thus, the equation above can be regarded as a kind of dilaton equation for curve counts.

Proof. Filling in the first boundary component with a disc gives a well-defined map

$$\mathcal{G}_{g,n,r}(0, b_2, \ldots, b_n) \to \mathcal{G}_{g,n-1,r}(b_2, \ldots, b_n).$$

Conversely, removing a disc from any complementary region of an arc diagram in $\mathcal{G}_{g,n-1,r}(b_2, \ldots, b_n)$ gives an arc diagram in $\mathcal{G}_{g,n,r}(0, b_2, \ldots, b_n)$. Two arc diagrams obtained on $(S_{g,n}, F(0, b_2, \ldots, b_n))$ by removing discs from a given arc diagram on $(S_{g,n-1}, F(b_2, \ldots, b_n))$ are equivalent if and only if the discs were removed from the same complementary region. Thus, the map above is surjective and r-to-1, giving the claimed equality. □

9.3. *Refining local decomposition*

In the local decomposition of an arc diagram C on $(S_{g,n}, F(b_1, \ldots, b_n))$, we obtain a B_i-local arc diagram C_i on an annulus neighborhood of each boundary component B_i, lying in $L(b_i, a_i)$, and a diagram C' without boundary-parallel arcs on the core S'. So C_i has a_i traversing arcs and $(b_i - a_i)/2$ insular arcs.

Let C and C' have r, r' complementary regions respectively, and corresponding parameters t, t'. Now C' can be obtained by successively removing from C outermost boundary-parallel arcs, at each stage cutting off a disc complementary region. There are $\sum_{i=1}^{n}(b_i - a_i)/2$ such boundary-parallel arcs, so

$$r' = r - \frac{1}{2}\sum_{i=1}^{n}(b_i - a_i).$$

Since S and S' have the same Euler characteristic χ, we have

$$t' = r' - \chi - \frac{1}{2}\sum_{i=1}^{n} a_i = r - \chi - \frac{1}{2}\sum_{i=1}^{n} b_i = t.$$

In other words, C and C' have the same t-parameter. There is thus a map

$$L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}^t_{g,n}(a_1, \ldots, a_n) \to \mathcal{G}^t_{g,n}(b_1, \ldots, b_n)$$

which glues local decompositions into a general arc diagram. Taking the quotient by the action of $\mathbb{Z}_{\bar{a}_1} \times \cdots \times \mathbb{Z}_{\bar{a}_n}$ by rotations, and a union over a_i as in Sec. 4.2, we obtain a bijection

$$\Delta : \mathcal{G}^t_{g,n}(b_1, \ldots, b_n) \to \bigsqcup_{\substack{0 \le a_i \le b_i \\ a_i \equiv b_i \pmod 2}} \frac{L(b_1, a_1) \times \cdots \times L(b_n, a_n) \times \mathcal{N}^t_{g,n}(a_1, \ldots, a_n)}{\mathbb{Z}_{\bar{a}_1} \times \cdots \times \mathbb{Z}_{\bar{a}_n}}$$

and hence, using Lemma 4.6, we have the folowing refinement of Theorem 1.8.

Proposition 9.5. *For $(g,n) \neq (0,1)$ and integers $b_1, \ldots, b_n$ and t, we have*

$$G^t_{g,n}(b_1, \ldots, b_n) = \sum_{\substack{a_i \in \mathbb{Z} \\ i=1,\ldots,n}} \begin{pmatrix} b_1 \\ \dfrac{b_1 - a_1}{2} \end{pmatrix} \cdots \begin{pmatrix} b_n \\ \dfrac{b_n - a_n}{2} \end{pmatrix} N^t_{g,n}(a_1, \ldots, a_n).$$

□

Because of this proposition, for many purposes, it is more convenient to use the parameter t rather than r.

9.4. *Refined curve counts on pants*

We now compute refined curve counts on pants, so let $(S,F) = (S_{0,3}, F(b_1, b_2, b_3))$. Recall the notation of Sec. 5.1: let p_i be the number of prodigal arcs with endpoints on B_i, and t_{ij} the number of traversing arcs with endpoints on B_i and B_j.

Consider an arc diagram on (S,F) without boundary-parallel arcs. In Sec. 5.2, we showed that b_1, b_2, b_3 determine the p_i and t_{ij} uniquely, so that there is a unique arc diagram in $\mathcal{N}_{0,3}(b_1, b_2, b_3)$, up to rotations around boundary components. The next lemma shows that b_1, b_2, b_3 also determine r and t.

Lemma 9.6. *Let $b_1, b_2, b_3 \geq 0$ be integers such that $b_1 + b_2 + b_3 \equiv 0 \pmod 2$. Then for any arc diagram C without boundary-parallel curves on (S,F), r and t are given by*

$$\begin{aligned}
&r = 1, && t = 2 && \text{if all } b_i = 0 \\
&r = \frac{1}{2}(b_1 + b_2 + b_3) + 1, && t = 2 && \text{if two } b_i \text{ are zero and one is non-zero} \\
&r = \frac{1}{2}(b_1 + b_2 + b_3), && t = 1 && \text{if one } b_i \text{ is zero and two are non-zero} \\
&r = \frac{1}{2}(b_1 + b_2 + b_3) - 1, && t = 0 && \text{if all } b_i \text{ are non-zero}
\end{aligned}$$

Proof. We repeatedly apply Lemma 5.4, which gives the number of arcs of each type. It suffices to compute r, since $t = r + 1 - \frac{1}{2}\sum b_i$. Without loss of generality suppose $b_1 \geq b_2 \geq b_3$.

If all $b_i = 0$, then clearly $r = 1$. If $b_2 = b_3 = 0$, then C consists of $b_1/2$ parallel prodigal arcs, which cut S into $\frac{b_1}{2} + 1 = \frac{1}{2}(b_1 + b_2 + b_3) + 1$ regions.

If $b_3 = 0$ and $b_1, b_2 \neq 0$, then we have $p_1 = \frac{1}{2}(b_1 - b_2)$ and $t_{12} = b_2$. Cutting along the first traversing arc leaves a connected surface; cutting along every subsequent arc increases the number of components by 1, so $r = \frac{1}{2}(b_1 + b_2 + b_3)$.

If all b_i are non-zero, then at least two of t_{12}, t_{23}, t_{31} are non-zero. Cutting along traversing arcs of two different types cuts S into a disc; each subsequent arc increases the number of components by 1. So r is one less than the number of arcs in C. □

Proposition 9.7. *For integers $b_1, b_2, b_3 > 0$,*

$$\widehat{N}^0_{0,3}(b_1, b_2, b_3) = 1 \quad \textit{provided } b_1 + b_2 + b_3 \equiv 0 \pmod 2,$$
$$\widehat{N}^1_{0,3}(b_1, b_2, 0) = 1 \quad \textit{provided } b_1 + b_2 \equiv 0 \pmod 2,$$
$$\widehat{N}^2_{0,3}(b_1, 0, 0) = 1 \quad \textit{provided } b_1 \equiv 0 \pmod 2,$$
$$\widehat{N}^2_{0,3}(0, 0, 0) = 1.$$

All other $\widehat{N}^t_{0,3}(b_1, b_2, b_3)$ are zero.

Proof. By Lemma 9.6, b_1, b_2, b_3 determine t, and for this value of t, we have $\mathcal{N}^t_{g,n}(b_1, b_2, b_3) = \mathcal{N}_{g,n}(b_1, b_2, b_3)$. The result now follows from (9) in Proposition 1.5. $\square$

Letting k denote the number of b_i equal to zero, we can tabulate the $\widehat{N}^t_{0,3}$ as follows.

k \ t	0	1	2
0	1		
1		1	
2			1
3			1

Proposition 9.8. *For integers $m_1, m_2, m_3 \geq 0$,*

$$G^0_{0,3}(2m_1, 2m_2, 2m_3) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} m_1 m_2 m_3$$
$$G^0_{0,3}(2m_1+1, 2m_2+1, 2m_3) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}(2m_1+1)(2m_2+1)m_3$$
$$G^1_{0,3}(2m_1, 2m_2, 2m_3) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}(m_1m_2 + m_2m_3 + m_3m_1)$$
$$G^1_{0,3}(2m_1+1, 2m_2+1, 2m_3) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}(2m_1+1)(2m_2+1)$$
$$G^2_{0,3}(2m_1, 2m_2, 2m_3) = \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}(m_1 + m_2 + m_3 + 1).$$

For any other t and b_1, b_2, b_3 not covered by these cases, $G^t_{0,3}(b_1, b_2, b_3) = 0$.

Proof. Proposition 9.5 expresses $G^t_{0,3}(b_1, b_2, b_3)$ as a linear combination of $\widehat{N}^t_{0,3}$; by Proposition 9.7, then $G^t_{0,3}(b_1, b_2, b_3)$ is zero unless $t \in \{0, 1, 2\}$. We consider each value of t separately.

If $t = 0$, we have $\widehat{N}^0_{0,3}(a_1, a_2, a_3) = 1$ when each $a_i > 0$ and $\sum a_i \equiv 0 \pmod 2$, and 0 otherwise, so

$$G^0_{0,3}(b_1, b_2, b_3) = \sum_{a_i > 0} \binom{b_1}{\frac{b_1 - a_1}{2}} \binom{b_2}{\frac{b_2 - a_2}{2}} \binom{b_3}{\frac{b_3 - a_3}{2}} \bar{a}_1 \bar{a}_2 \bar{a}_3$$

$$= \prod_{i=1}^{3} \sum_{a_i > 0} \binom{b_i}{\frac{b_i - a_i}{2}} \bar{a}_i.$$

Write $b_i = 2m_i$ if b_i is even and $b_i = 2m_i + 1$ if b_i is odd. If all b_i are even, then all the a_i must also be even and the above expression is $\tilde{p}_0(m_1)\tilde{p}_0(m_2)\tilde{p}_0(m_3)$. If two b_i are odd and one is even, say b_1, b_2, and b_3 are even, then the expression is $\tilde{q}_0(m_1)\tilde{q}_0(m_2)\tilde{p}_0(m_3)$. In Sec. 5.3, we found $\tilde{p}_0(m) = \binom{2m}{m} m$ and $\tilde{q}_0(m) = \binom{2m}{m}(2m + 1)$, giving $G^0_{0,3}$ as claimed.

Now suppose $t = 1$. For $\widehat{N}^1_{0,3}(a_1, a_2, a_3)$ to be non-zero, we require exactly one of the a_i to be zero.

$$G^1_{0,3}(b_1, b_2, b_3) = \left(\sum_{\substack{a_1 = 0 \\ a_2, a_3 > 0}} + \sum_{\substack{a_2 = 0 \\ a_3, a_1 > 0}} + \sum_{\substack{a_3 = 0 \\ a_1, a_2 > 0}} \right) \binom{b_1}{\frac{b_1 - a_1}{2}} \binom{b_2}{\frac{b_2 - a_2}{2}}$$

$$\times \binom{b_3}{\frac{b_3 - a_3}{2}} \bar{a}_1 \bar{a}_2 \bar{a}_3.$$

If all $b_i = 2m_i$ are even, we obtain $G^1_{0,3}(2m_1, 2m_2, 2m_3) = \binom{2m_1}{m_1}\tilde{p}_0(m_2)\tilde{p}_0(m_3) + \binom{2m_2}{m_2}\tilde{p}_0(m_3)\tilde{p}_0(m_1) + \binom{2m_3}{m_3}\tilde{p}_0(m_1)\tilde{p}_0(m_2)$, and if b_1, b_2 are odd and b_3 even, we have $G^1_{0,3}(2m_1 + 1, 2m_2 + 1, 2m_3) = \binom{2m_3}{m_3}\tilde{q}_0(m_1)\tilde{q}_0(m_2)$, so $G^1_{0,3}$ is as claimed.

Finally, let $t = 2$. Now for $\widehat{N}^2_{0,3}(a_1, a_2, a_3)$ to be non-zero, at least two of the a_i to be zero; hence for $G^2_{0,3}(b_1, b_2, b_3)$ to be non-zero all b_i must be even. We then have

$$G^2_{0,3}(b_1, b_2, b_3) = \left(\sum_{\substack{a_1 = a_2 = 0 \\ a_3 > 0}} + \sum_{\substack{a_2 = a_3 = 0 \\ a_1 > 0}} + \sum_{\substack{a_3 = a_1 = 0 \\ a_2 > 0}} + \sum_{a_1 = a_2 = a_3 = 0} \right)$$

$$\times \binom{b_1}{\frac{b_1 - a_1}{2}} \binom{b_2}{\frac{b_2 - a_2}{2}} \binom{b_3}{\frac{b_3 - a_3}{2}} \bar{a}_1 \bar{a}_2 \bar{a}_3$$

which is equal to $\binom{2m_1}{m_1}\binom{2m_2}{m_2}\tilde{p}_0(m_3) + \binom{2m_2}{m_2}\binom{2m_3}{m_3}\tilde{p}_0(m_1) + \binom{2m_3}{m_3}\binom{2m_1}{m_1}\tilde{p}_0(m_2) + \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}$, giving the claimed expression for $G^2_{0,3}$. □

9.5. *Inequalities on regions*

Clearly, if g, n and $b_1, \ldots, b_n$ are fixed, the number of regions r is bounded. We now establish some precise bounds.

Lemma 9.9. *Suppose an arc diagram on $(S_{g,n}, F(b_1, \ldots, b_n))$ has r complementary regions, and k of the numbers $b_1, \ldots, b_n$ are zero. Then the following statements hold.*

(i) $r \leq 1 + \frac{1}{2}\sum_{i=1}^n b_i$
(ii) *If* $0 \leq k \leq n-1$, *then* $r \geq \chi(S) + k + \frac{1}{2}\sum_{i=1}^n b_i$
(iii) *If* $k = n$, *then* $r = 1$.

These inequalities are necessary, but not sufficient, conditions for the existence of an arc diagram. For instance $G_{0,2,2}(1,1) = 0$, despite satisfying all the inequalities above.

Proof. The diagram has $\frac{1}{2}\sum b_i$ arcs, and cutting along each arc of C can increase the number of components by at most 1, establishing (i). If $k = n$, then there are no arcs, so $r = 1$, establishing (iii).

To see (ii), fill the k boundary components of S with $b_i = 0$ by gluing discs; this increases χ by k. Then cut along the curves of C; each of these $\frac{1}{2}\sum b_i$ cuts increases χ by 1. The resulting surface has r components, all with non-empty boundary, hence has Euler characteristic at most r. Thus $\chi(S) + k + \frac{1}{2}\sum b_i \leq r$. □

We give a further bound when there are no boundary-parallel curves.

Lemma 9.10. *Suppose* $(g,n) \neq (0,1),(0,2)$. *Let C be an arc diagram on* $(S_{g,n}, F(b_1, \ldots, b_n))$ *with no boundary-parallel arcs, r complementary regions, and with k of the numbers $b_1, \ldots, b_n$ zero, where* $0 \leq k \leq n-1$. *Then*

$$r \leq g + k - 1 + \frac{1}{2}\sum_{i=1}^n b_i.$$

Note that $g + k \geq 2$ is equivalent to $g + k - 1 + \frac{1}{2}\sum_{i=1}^n b_i \geq 1 + \frac{1}{2}\sum_{i=1}^n b_i$. So if $g + k \geq 2$, then Lemma 9.9 immediately implies this result; this upper bound thus only gives new information when $g + k \leq 1$.

Proof. As discussed, we can assume $g + k \leq 1$.

First, suppose $g = 0$ and $0 \leq k \leq 1$. We prove $r \leq k - 1 + \frac{1}{2}\sum b_i$ for all $n \geq 3$ by induction on n.

If $n = 3$, then by Lemma 9.6, $r = k - 1 + \frac{1}{2}\sum_{i=1}^n b_i$, so the desired inequality holds (in fact is an equality).

Now consider general $n \geq 4$, and an arc γ in the arc diagram. If γ connects two distinct boundary components, then cutting along γ gives an arc diagram with the same r, with k increased by at most 1, and $\frac{1}{2}\sum_{i=1}^n b_i - 1$ arcs, on an $(n-1)$-holed sphere, so by induction $r \leq (k+1) - 1 + (\frac{1}{2}\sum_{i=1}^n b_i - 1)$ and the result holds. If γ has both endpoints on the same boundary component, then γ is separating. Let the number of arcs parallel to γ (including γ) be p. Cutting along γ, and removing these parallel arcs, yields two surfaces S', S'' with n', n'' boundary components,

r', r'' complementary regions, and numbers of marked points given by b'_i and b''_i, of which k', k'' are zero respectively. We have $n', n'' \geq 2$ (as γ is not boundary parallel), $\frac{1}{2}\sum b'_i + \frac{1}{2}\sum b''_i + p = \frac{1}{2}\sum b_i$, $k' + k'' \leq k + 2$, and $r' + r'' + p - 1 = r$. If both $n', n'' \geq 3$, then the inequality holds for both S' and S'', and we have

$$r = r' + r'' + p - 1 \leq k' + k'' + p - 3 + \frac{1}{2}\sum b'_i + \frac{1}{2}\sum b''_i \leq k - 1 + \frac{1}{2}\sum_{i=1}^{n} b_i$$

as desired. So now suppose, we obtain an annulus. As $n \geq 4$, S' and S'' cannot both be annuli. So, we may assume S' is an annulus (i.e. $n' = 2$), and S'' is not (i.e. $n'' \geq 3$). Then the inequality holds for S''. If the annulus S' has no arcs, then actually the inequality holds for S' too ($r = g + k - 1 + \frac{1}{2}\sum b_i = 0 + 2 - 1 + 0 = 1$), so we are done. If the annulus S' has non-empty arc diagram, then we have $r' = \frac{1}{2}\sum b'_i$ and $k' = 0$; and since we do not obtain any extra boundary components with zero marked points on S', we must have $k'' \leq k + 1$. Then we obtain

$$r = r' + r'' + p - 1 \leq \frac{1}{2}\sum b'_i + \frac{1}{2}\sum b''_i + p + k'' - 2 \leq k - 1 + \frac{1}{2}\sum_{i=1}^{n} b_i$$

and the inequality holds. This completes the proof in the case $g = 0$.

Now suppose $g = 1$ and $k = 0$, and we prove $r \leq \frac{1}{2}\sum_{i=1}^{n} b_i$. We proceed by induction on $n \geq 1$. If $n = 1$, then take an arc γ in the arc diagram; as γ is not boundary-parallel, it cuts S into an annulus. Suppose there are p arcs parallel to γ (including γ), so cutting along γ and removing these parallel arcs gives a diagram on the annulus without boundary-parallel curves, with $r - p + 1$ regions and $\frac{1}{2}b_1 - p$ arcs. If there are no arcs on the annulus, then we have one region, so $r - p + 1 = 1$ and $\frac{1}{2}b_1 - p = 0$, and hence $r = p = \frac{1}{2}b_1$. If there are arcs on the annulus, then the number of arcs and regions are equal, so $r = -1 + \frac{1}{2}b_1$. Either way, we have $r \leq \frac{1}{2}b_1 = \frac{1}{2}\sum_{i=1}^{n} b_i$.

Now take a general $n \geq 2$. Take an arc γ on S with p parallel copies (including γ). If γ is non-separating, then cutting along γ and removing its parallel arcs gives a surface S' of genus g', with n' boundary components, with b'_i marked points on boundary components, k' of which are zero, and an arc diagram with $r' = r - p + 1$ complementary regions and $\frac{1}{2}\sum b'_i = \frac{1}{2}\sum b_i - p$ arcs. Since we originally had all $b_i > 0$, after cutting along γ and removing parallel copies, we can make at most one $b'_i = 0$, so $k' \leq 1$. Now S' either has genus zero and $n' = n + 1 \geq 3$ boundary components, in which case the result holds by our previous arguments; or S' has genus 1 and $n' = n - 1$ boundary components, in which case the result holds by inductive assumption (if $k' = 0$) or previous argument (if $k' = 1$). Either way, $r' \leq g' + k' - 1 + \frac{1}{2}\sum_{i=1}^{n'} b'_i$ and $g' + k' \leq 2$, and hence

$$r = r' + p - 1 \leq p + g' + k' - 2 + \frac{1}{2}\sum_{i=1}^{n'} b'_i = g' + k' - 2 + \frac{1}{2}\sum_{i=1}^{n} b_i \leq \frac{1}{2}\sum_{i=1}^{n} b_i.$$

On the other hand, if γ is separating, with p parallel copies, then cutting along γ and removing parallel arcs gives two surfaces S', S'', with genera $g' + g'' = 1$;

say $g' = 0$ and $g'' = 1$. Let them have n', n'' boundary components, with b'_i, b''_i marked points, of which k', k'' are zero, and arc diagrams with r', r'' complementary regions. So we have $n' + n'' = n + 1$. As there are no boundary-parallel curves, $n', n'' \geq 2$ and hence $n', n'' \leq n - 1$. We also have $r' + r'' + p - 1 = r$ and $\frac{1}{2}\sum b'_i + \frac{1}{2}\sum b''_i + p = \frac{1}{2}\sum b_i$. The only way to have $b'_i = 0$ or $b''_i = 0$ is from the boundary components involving γ, so $k', k'' \leq 1$. The inductive assumption applies to S'', and we obtain $r'' \leq k'' + \frac{1}{2}\sum b''_i \leq 1 + \frac{1}{2}\sum b''_i$. If S' is an annulus, then as $k' \leq 1$, the arc diagram is non-empty and $r' = \frac{1}{2}\sum b'_i$. If S' is not an annulus, then the inductive hypothesis (if $k' = 0$) or the above argument applies (if $k' = 1$), so the inequality holds for S' giving $r' \leq k' - 1 + \frac{1}{2}\sum b'_i \leq \frac{1}{2}\sum b'_i$. Either way, $r' \leq \frac{1}{2}\sum b'_i$. Putting this together yields

$$r = r' + r'' + p - 1 \leq \frac{1}{2}\sum b'_i + \frac{1}{2}\sum b''_i + p = \frac{1}{2}\sum_{i=1}^{n} b_i.$$

This completes the proof. $\square$

Putting together Lemmas 9.9 and 9.10 immediately gives the following result.

Proposition 9.11. *Let $(g, n) \neq (0, 1), (0, 2)$. Suppose an arc diagram on $(S_{g,n}, F(b_1, \ldots, b_n))$ has no boundary-parallel arcs, r complementary regions and k boundary components without marked points. If $0 \leq k \leq n - 1$, then*

$$\max\left(1,\ k + 2 - 2g - n + \frac{1}{2}\sum_{i=1}^{n} b_i\right) \leq r$$

$$\leq \min\left(1 + \frac{1}{2}\sum_{i=1}^{n} b_i,\ g + k - 1 + \frac{1}{2}\sum_{i=1}^{n} b_i\right)$$

and

$$\max\left(k,\ 2g + n - 1 - \frac{1}{2}\sum_{i=1}^{n} b_i\right) \leq t \leq \min\left(2g + n - 1,\ k + 3g - 3 + n\right).$$

If $k = n$, then $r = 1$ and $t = 2g + n - 1$. $\square$

The above inequalities are necessary for the existence of an arc diagram without boundary-parallel arcs, but not sufficient. For instance, $\widehat{N}^7_{3,2}(2n + 1, 1) = N_{3,2,n+2}(2n + 1, 1) = 0$, but $\max(0, 6 - n) = \max(k, 2g + n - 1 - \frac{1}{2}\sum b_i) \leq 7 = t \leq \min(3g - 3 + n, 2g + n - 1) = 7$. To see why, suppose there were such an arc diagram; then there must be an arc connecting the two boundary components. Cutting along this arc gives an arc diagram in $\mathcal{G}_{3,1,n+2}(2n)$, hence with n arcs. But cutting along n arcs can only create $n + 1$ regions, not the required $n + 2$. In the particular case $t = k$, we can give necessary and sufficient conditions in the next section.

When $k = 0$, so that all boundary components have marked points, we have $0 \leq t \leq 1 - \chi$. So t is roughly a measure of how "separating" an arc diagram is: when $t = 0$ it is as non-separating as possible, and as t increases, it is more and more separating.

9.6. *Existence of certain arc diagrams*

We now give some results guaranteeing the existence of arc diagrams in certain circumstances.

Lemma 9.12. *Suppose* $(g,n) \neq (0,1),(0,2)$, *and* $0 \leq k \leq n-1$. *Let* $b_1,\ldots,b_{n-k} > 0$ *be positive integers such that* $b_1+\cdots+b_{n-k}$ *is even, and suppose* $b_{n-k+1} = \cdots = b_n = 0$.

(i) *If* $\frac{1}{2}\sum_{i=1}^n b_i < 2g+n-1-k$, *then* $N_{g,n}^k(b_1,\ldots,b_{n-k},0,\ldots,0) = 0$.
(ii) *If* $\frac{1}{2}\sum_{i=1}^n b_i \geq 2g+n-1-k$, *then* $N_{g,n}^k(b_1,\ldots,b_{n-k},0,\ldots,0) > 0$.

(Here the notation $N_{g,n}^k$ means that we set $t = k$.)

Proof. If $t = k$, then $r = k + \chi + \frac{1}{2}\sum_{i=1}^n b_i = k + 2 - 2g - n + \frac{1}{2}\sum_{i=1}^n b_i$. If $\frac{1}{2}\sum_{i=1}^n b_i < 2g+n-1-k$, then $r < 1$, so no arc diagram exists, proving (i).

It remains to prove (ii); we first prove it under the assumption $k = 0$. So suppose all $b_i > 0$, $\frac{1}{2}\sum_{i=1}^n b_i \geq 2g+n-1 = 1-\chi$, and we will construct an arc diagram with the desired parameters.

First, suppose $g = 0$, so $n \geq 3$. Then, as $k = 0$, we may successively draw arcs joining distinct boundary components and cut along them, in order to reduce the number of boundary components. (Provided at each stage, we do not join two boundary components each with one marked point, we retain at least one point on each boundary component. And this is certainly possible since $\sum_{i=1}^n b_i \geq 4g+2n-2 \geq 2n-2$.) We proceed until, we have a pair of pants, with a non-zero number of points on each boundary component. Since each cut increases Euler characteristic by 1, at this stage, we have drawn and cut along $-1-\chi$ arcs; so we have $\frac{1}{2}\sum_{i=1}^n b_i + 1 + \chi$ remaining arcs to draw. From Sec. 5.2 above, there is an arc diagram on the pants, with the required number of points on each boundary component, without boundary-parallel curves, and from Lemma 9.6, the number of regions into which they cut the pants is one less than the number of arcs drawn. So, drawing these arcs and cutting, we obtain $\frac{1}{2}\sum_{i=1}^n b_i + \chi$ components. This corresponds to an arc diagram on the original surface without boundary-parallel arcs, and with $r = \frac{1}{2}\sum_{i=1}^n b_i + \chi$ complementary regions, hence with $t = 0$.

Now suppose $g \geq 1$. Use a similar method to join boundary components until, we obtain a single boundary component, with an even number of points on it. At this stage, we have a genus g surface with a single boundary component, hence Euler characteristic has increased from χ to $1-2g$, so we have drawn and cut along $1-2g-\chi$ non-boundary-parallel arcs. There are $\frac{1}{2}\sum_{i=1}^n b_i + \chi + 2g - 1 = \frac{1}{2}\sum_{i=1}^n b_i - n + 1 \geq 2g$ remaining arcs to draw.

Now we can draw $2g$ curves to cut the genus g surface into a disc. We draw these curves, along with some parallel copies of them, so that there are $\frac{1}{2}\sum_{i=1}^n b_i + \chi + 2g - 1$ arcs drawn in total, none of them boundary-parallel. Cutting along all these curves, including the parallel copies splits the surface into $\frac{1}{2}\sum_{i=1}^n b_i + \chi$ components.

This corresponds to a diagram on the original surface, without boundary-parallel arcs, and with $r = \frac{1}{2}\sum_{i=1}^{n} b_i + \chi$ complementary regions, so $t = 0$.

This proves the result in the case $k = 0$. For general k, fill in the k boundary components with no marked points with discs, to obtain a surface of genus g with $n-k$ boundary components. Provided, we do not end up with a disc or annulus, the $k = 0$ argument applies, and we obtain an arc diagram with $\frac{1}{2}\sum_{i=1}^{n} b_i + 2 - 2g - n + k$ regions, with no boundary-parallel arcs. Then removing the k discs gives an arc diagram on the original surface, still with no boundary-parallel arcs, and with the same number of complementary regions, hence with $t = k$.

If this argument fails, ending up with a disc or annulus, then we must have $g = 0$, $n \geq 3$, and $k \geq n - 2$. In this case, we fill in $n - 3$ of the k boundary components without marked points, to obtain a pair of pants, on which $k' = k - n + 3$ boundary components have no marked points. Note $1 \leq k' \leq 2$. Using Sec. 5.2 again, there is an arc diagram on the pants with no boundary-parallel arcs, and with the required number of points on each boundary component. Using Lemma 9.6, the number of complementary regions of this arc diagram on the pants is $\frac{1}{2}\sum b_i + k' - 1 = \frac{1}{2}\sum b_i + 2 - n + k$. Removing the $n - 3$ discs gives an arc diagram on the original surface with no boundary-parallel arcs and with the same number of regions, hence with $t = k$. $\square$

For general arc diagrams, we have the following easier result, which can be proved by similar methods.

Lemma 9.13. *Suppose $g \geq 0$ and $n \geq 1$, and $0 \leq k \leq n - 1$. Let $b_1, \ldots, b_{n-k} > 0$ be positive integers such that $b_1 + \cdots + b_{n-k}$ is even, and suppose $b_{n-k+1} = \cdots = b_n = 0$, so that k of $b_1, \ldots, b_n$ are zero.*

(i) *If $\frac{1}{2}\sum_{i=1}^{n} b_i < 2g + n - 1 - k$, then $G^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0) = 0$.*
(ii) *If $\frac{1}{2}\sum_{i=1}^{n} b_i \geq 2g + n - 1 - k$, then $G^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0) > 0$.*

So, fixing g, n and setting $t = k$ (and hence fixing $r - \frac{1}{2}\sum_{i=1}^{n} b_i$), provided we have sufficiently many marked points, we can find an arc diagram with these parameters — and, provided $(g, n) \neq (0, 1)$ or $(0, 2)$, one without any boundary-parallel arcs.

9.7. *Refining recursion*

Now we prove the recursion on $G_{g,n,r}$ in Theorem 1.11, refining Theorem 1.4.

Proof of Theorem 1.11. The proof is essentially the same as that of Theorem 1.4. Given an arc diagram C in $\mathcal{G}_{g,n,r}(\mathbf{b})$, take the arc γ at the first marked point on the first boundary component. Cutting along γ gives a surface S' with an arc diagram C'. Consider the various cases for the topology of γ and S', as in Theorem 1.4. In each case, C can be reconstructed by gluing together two boundary arcs on S' in a specified way.

Cutting along γ does not change the number of complementary regions; all the arc diagrams considered have r complementary regions. Hence, enumerating the various cases, the arc diagrams in $\mathcal{G}_{g,n,r}(\mathbf{b})$ are in bijection with the various arc diagrams enumerated on the right-hand side of the equation. □

Turning to the $N_{g,n}$, we now obtain the following, refining Proposition 6.1.

Proposition 9.14. *For $(g,n) \neq (0,1),(0,2),(0,3)$, $r \geq 1$ and $b_1,\ldots,b_n$ such that $b_1 > 0$, $b_2,\ldots,b_n \geq 0$,*

$$
\begin{aligned}
N_{g,n,r}(\mathbf{b}) = {} & \sum_{\substack{i,j,m\geq 0\\ i+j+m=b_1\\ m \text{ even}}} \frac{m}{2} N_{g-1,n+1,r-\frac{m}{2}+1}(i,j,b_2,\ldots,b_n) \\
& + \sum_{j=2}^{n} \left(\sum_{\substack{i,m\geq 0\\ i+m=b_1+b_j\\ m \text{ even}}} \frac{m}{2} \bar{b}_j N_{g,n-1,r-\frac{m}{2}+1-\delta_{b_j,0}}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right. \\
& \left. + \widetilde{\sum_{\substack{i,m\geq 0\\ i+m=b_1-b_j\\ m \text{ even}}}} \frac{m}{2} \bar{b}_j N_{g,n-1,r-\frac{m}{2}-\overline{\min(b_1,b_j)}+1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right) \\
& + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\ldots,n\}\\ \text{No discs or annuli}}} \sum_{\substack{i,j,m\geq 0\\ i+j+m=b_1\\ m \text{ even}}} \\
& \times \sum_{\substack{r_1,r_2\geq 0\\ r_1+r_2=r-\frac{m}{2}+1}} \frac{m}{2} N_{g_1,|I|+1,r_1}(i,b_I) N_{g_2,|J|+1,r_2}(j,b_J)
\end{aligned}
$$

$$
\begin{aligned}
b_1 \widehat{N}^t_{g,n}(\mathbf{b}) = {} & \sum_{\substack{i,j,m\geq 0\\ i+j+m=b_1\\ m \text{ even}}} \frac{1}{2}\, \bar{i}\, \bar{j}\, m\, \widehat{N}^t_{g-1,n+1}(i,j,b_2,\ldots,b_n) \\
& + \sum_{j=2}^{n} \frac{1}{2} \left(\sum_{\substack{i,m\geq 0\\ i+m=b_1+b_j\\ m \text{ even}}} \bar{i}\, m\, \widehat{N}^{t-\delta_{b_j,0}}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right. \\
& \left. + \widetilde{\sum_{\substack{i,m\geq 0\\ i+m=b_1-b_j\\ m \text{ even}}}} \bar{i}\, m\, \widehat{N}^{t-\delta_{b_j,0}}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n) \right)
\end{aligned}
$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\} \\ \text{No discs or annuli}}} \sum_{\substack{i,j,m\geq 0 \\ i+j+m=b_1 \\ m \text{ even}}}$$

$$\times \sum_{\substack{t_1,t_2\geq 0 \\ t_1+t_2=t}} \frac{1}{2}\, \bar{i}\, \bar{j}\, m\, \widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\, \widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J).$$

Proof. We proceed similarly to the proof of Proposition 6.1. First, we prove the recursion on $N_{g,n}$. Let C be a non-boundary-parallel arc diagram on $(S_{g,n}, F(\mathbf{b}))$, with arc γ at the first marked point p on the first boundary component B_1. We consider the same three cases for γ as in the proof of Proposition 6.1.

First, suppose γ has both endpoints on B_1 and is non-separating. There are $\frac{m}{2}$ arcs (including γ) parallel to γ. Between the $\frac{m}{2}$ parallel arcs there are $\frac{m}{2} - 1$ complementary regions. Cutting along γ and removing arcs which become boundary-parallel produces an $S_{g-1,n+1}$ with $\frac{m}{2} - 1$ fewer complementary regions. So all diagrams considered in this case have $r - \frac{m}{2} - 1$ regions, and following the argument in the proof of Proposition 6.1, the number of arc diagrams so obtained is given by the first term in the recursion.

The second case is when γ has endpoints on distinct boundary components B_1 and B_j, or is separating and cuts off an annulus with B_j as a boundary component. This corresponds to the second and third lines above.

Let $m/2$ be the number of arcs which are "parallel" to γ, in the extended sense of the argument of Proposition 6.1: if γ runs from B_1 to B_j, then we take as "parallel" all arcs parallel to γ, and those which run from from B_1 around B_j back to B_1, and those which run from B_j around B_1 back to B_j; while if γ cuts off an annulus around B_j, we take as "parallel" all arcs parallel to γ, and those which run from B_1 to B_j. These $m/2$ arcs consist precisely of γ and those arcs which become boundary-parallel after cutting along γ.

Assuming that $b_j > 0$, these $m/2$ arcs, running from B_1 to B_j, or from one of these boundary components around the other and back to itself, enclose $\frac{m}{2} - 1$ regions within an annular region which is effectively removed from S: see Fig. 11. If $b_j = 0$, then we only have $m/2$ arcs running around B_j, which enclose precisely $m/2$ regions. Thus the number of regions effectively removed from S is $\frac{m}{2} - 1 + \delta_{b_j,0}$. We again orient these arcs so that they run from B_1 to B_j, or run anticlockwise around B_1 or B_j, and again the number of arc diagrams for which γ runs from B_1 to B_j, or runs from B_1 around B_j, and is oriented so that p is the start point of γ, is given by the summation in the second line above: all diagrams obtained by cutting along such γ and removing boundary-parallel arcs have $r - \frac{m}{2} + 1 - \delta_{b_j,0}$ complementary regions.

If $b_1 \geq b_j$, then as in Proposition 6.1, we need to count arc diagrams, where p is the endpoint of γ. We redefine m so that the number of arcs from B_1 looping around B_j is $m/2$. If $b_j = 0$, then these $m/2$ arcs looping around B_j enclose $m/2$ regions within an annular region which is effectively removed from S, so resulting diagrams

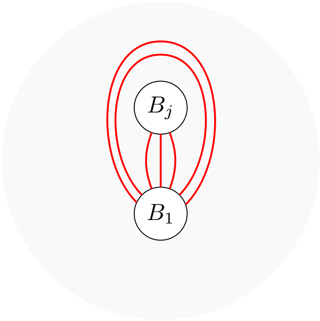

Fig. 11. The $m/2$ "parallel" arcs enclose $m/2-1$ regions.

have $r-\frac{m}{2}$ complementary regions. If $b_j>0$, then the $m/2$ arcs looping around B_j also enclose b_j arcs running from B_1 to B_j and the annular region has $\frac{m}{2}-b_j+1$ regions. Thus, the resulting diagrams have $r-\frac{m}{2}+b_j-1$ complementary regions. Either way, the resulting diagrams have $r-\frac{m}{2}+\bar{b}_j-1$ regions and, following the proof of Proposition 6.1 (and noting $\overline{\min(b_1,b_j)}=\bar{b}_j$), we obtain the summation in the third line.

If $b_1\leq b_j$, then we have overcounted, and as in Proposition 6.1 need to subtract off diagrams where p lies on B_j. We redefine m so that the number of arcs from B_j looping around B_1 is $m/2$. As in the previous paragraph, these arcs enclose an annular region with $\frac{m}{2}-\bar{b}_1+1$ complementary regions, so that diagrams obtained after removing this annulus have $r-\frac{m}{2}+\bar{b}_1-1$ complementary regions. (Note $b_1>0$, so that $\bar{b}_1=b_1$ in any case; but we write $\bar{b}_1$ for consistency.) Since we have $\overline{\min(b_1,b_j)}=\bar{b}_1$, we obtain the summation in the third line again.

The third and final case is when γ is separating but does not cut off an annulus. There are $m/2$ arcs (including γ) parallel to γ. As in the first case, there are $\frac{m}{2}-1$ complementary regions between the $\frac{m}{2}$ parallel arcs. Cutting along γ and removing arcs which become boundary-parallel, we obtain a surface with $r-\frac{m}{2}+1$ complementary regions. This surface is disconnected, with two components having r_1,r_2 complementary regions satisfying $r_1+r_2=r-\frac{m}{2}+1$. The number of arc diagrams is then given by the final line in the recursion.

This completes the proof of the recursion for $N_{g,n,r}$. We now rewrite it in terms of t-parameters. Let t be the parameter for the left-hand side, and t' for the term in the first line of the right-hand side. Then

$$\begin{aligned}t&=r-(2-2g-n)-\frac{1}{2}\sum_{i=1}^{n}b_i\\&=r-\frac{m}{2}+1-(2-2(g-1)-(n+1))-\frac{1}{2}\left(i+j+\sum_{i=2}^{n}b_i\right)=t',\end{aligned}$$

where we used $i+j+m=b_1$. If we write t'' for the parameter for the term in the second line, we similarly obtain $t''=t-\delta_{b_j,0}$. In the third line, if $b_1\geq b_j$ then

$\min(b_1, b_j) = b_j$, so writing t''' for the parameter, we have

$$t''' = r - \frac{m}{2} - \overline{b_j} + 1 - (2 - 2g - (n-1)) - \frac{1}{2}(i + b_2 + \cdots + \widehat{b_j} + \cdots + b_n)$$
$$= r - (2 - 2g - n) - \frac{1}{2}\sum_{i=1}^{n} b_i + \frac{1}{2}(b_1 - b_j - i - m) + b_j - \bar{b}_j = t - \delta_{b_j,0}.$$

Here, we used the fact that $i+m = b_1 - b_j$ in the summation. If alternatively $b_1 \leq b_j$, then we obtain $t''' = t - \delta_{b_1,0}$. Either way, we have $t''' = t - \delta_{\min(b_1,b_j),0}$. But we are assuming $b_1 > 0$, so $\delta_{\min(b_1,b_j),0} = \delta_{b_j,0}$. In the final term, if the two factors have parameters t_1, t_2, the condition $r_1 + r_2 = r - \frac{m}{2} + 1$ translates to $t_1 + t_2 = t$. Dividing through by $\bar{b}_2 \cdots \bar{b}_n$ immediately gives the desired recursion on $\widehat{N}^t_{g,n}$. □

9.8. *Polynomiality in small cases*

We can now use the recursion of Proposition 9.14 to find $\widehat{N}^t_{g,n}$ for $(g,n) = (1,1)$ and $(0,4)$.

Proposition 9.15. *For b_1 even and non-zero,*

$$\widehat{N}^0_{1,1}(b_1) = \frac{1}{48}b_1^2 - \frac{1}{12}, \quad \widehat{N}^1_{1,1}(b_1) = \frac{1}{2}, \quad \widehat{N}^2_{1,1}(0) = 1.$$

All other $\widehat{N}^t_{1,1}(b_1)$ are zero.

Proof. Consider $\widehat{N}^t_{1,1}(b_1)$. We assume b_1 is even. We have, for $b_1 > 0$,

$$b_1 \widehat{N}^t_{1,1}(b_1) = \sum_{\substack{i,j,m \geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{1}{2}\bar{i}\,\bar{j}\, m\, \widehat{N}^t_{0,2}(i,j).$$

Lemma 9.3 found that $\widehat{N}^0_{0,2}(b,b) = \frac{1}{b}$ for $b > 0$, $\widehat{N}^1_{0,2}(0,0) = 1$, and all other $\widehat{N}^t_{0,2}(b_1,b_2) = 0$.

Thus, we only need consider the cases $t = 0, 1$. In the $t = 0$ case, we obtain

$$b_1 \widehat{N}^0_{1,1}(b_1) = \sum_{\substack{i>0,\ m\geq 0 \\ 2i+m=b_1 \\ m \text{ even}}} \frac{1}{2} i^2\, m\, \frac{1}{i} = \sum_{\substack{i>0,\ m\geq 0 \\ 2i+m=b_1 \\ m \text{ even}}} \frac{1}{2} i\, m$$
$$= \frac{1}{4}\sum_{\substack{\iota,m\geq 0 \\ \iota+m=b_1 \\ m \text{ even}}} \iota\, m = \frac{1}{4}S_0(b_1) = \frac{1}{48}b_1^3 - \frac{1}{12}b_1,$$

where we let $2i = \iota$, and S_0 is the sum studied in Sec. 7.1.

For $t = 1$, we have a non-zero term only when $i = j = 0$:

$$b_1 \widehat{N}^1_{1,1}(b_1) = \sum_{\substack{i,j,m\geq 0 \\ i+j+m=b_1 \\ m \text{ even}}} \frac{1}{2}\bar{i}\,\bar{j}\, m\, \widehat{N}^1_{0,2}(i,j) = \frac{1}{2}b_1.$$

The above assumes that $b_1 > 0$. When $b_1 = 0$, the only non-zero count is $N_{1,1,1}(0) = N^2_{1,1}(0) = 1$. $\square$

We can summarize $\widehat{N}^t_{1,1}$ in a table of k and t; we present the result for $(g, n) = (0, 4)$ in a similar fashion.

$k \backslash t$	0	1	2
0	$\frac{1}{48}b_1^2 - \frac{1}{12}$	$\frac{1}{2}$	
1			1

Proposition 9.16. *For the various possible values of t, k, with $b_1, \ldots, b_{n-k} > 0$ and $b_{n-k+1} = \cdots = b_n = 0$, $\widehat{N}^t_{0,4}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ is given by the following tables.*

(i) *If all b_i are even*:

$k \backslash t$	0	1	2	3
0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 1$	3		
1		$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2) - 1$	3	
2			$\frac{1}{4}(b_1^2 + b_2^2)$	2
3				$\frac{1}{4}b_1^2 + 2$
4				1

(ii) *If two b_i are odd*:

$k \backslash t$	0	1	2	3
0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - \frac{1}{2}$	1		
1		$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2) - \frac{1}{2}$	1	
2			$\frac{1}{4}(b_1^2 + b_2^2) + \frac{1}{2}$	0
3				0
4				0

(iii) *If four b_i are odd*:

$k \backslash t$	0	1	2	3
0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 1$	3		
1		0	0	
2			0	0
3				0
4				0

Proof. Proposition 9.14 gives, for $b_1 > 0$,

$$b_1 \widehat{N}^t_{0,4}(\mathbf{b}) = \sum_{j=2}^{4} \frac{1}{2} \left(\sum_{\substack{i,m\geq 0 \\ i+m=b_1+b_j \\ m \text{ even}}} \bar{i}m \widehat{N}_{0,3}^{t-\delta_{b_j,0}}(i, b_2, \dots, \widehat{b_j}, \dots, b_n) \right.$$
$$\left. + \widetilde{\sum_{\substack{i,m\geq 0 \\ i+m=b_1-b_j \\ m \text{ even}}}} \bar{i}m \widehat{N}_{0,3}^{t-\delta_{b_j,0}}(i, b_2, \dots, \widehat{b_j}, \dots, b_n) \right). \tag{15}$$

Proposition 9.11 gives us bounds on k and t. Either $0 \leq k \leq 3$ and $\max(k, 3 - \frac{1}{2}\sum_{i=1}^n b_i) \leq t \leq \min(3, 1+k)$, or $k = 4$ and $t = 3$. Since b_i may become large, we first consider $0 \leq k \leq 3$ and $k \leq t \leq \min(k+1, 3)$. This gives eight cases to consider: $(k,t) = (0,0), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3), (4,3)$. We present the cases $(0,0)$ and $(0,1)$; the remaining cases can be handled in a similar fashion.

If $(k,t) = (0,0)$, then (15) expresses $\widehat{N}^0_{0,4}(b_1, b_2, b_3, b_4)$ in terms of $\widehat{N}^0_{0,3}$. From Proposition 9.7, $\widehat{N}^0_{0,3}(b_1, b_2, b_3) = 1$ provided $b_1 + b_2 + b_3$ is even, and all b_i are non-zero. Hence, every $\widehat{N}^0_{0,3}(i, b_2, \dots, \widehat{b_j}, \dots, b_n) = 1$, except when $i = 0$. We see sums $S_0(b_1 \pm b_j)$, and obtain

$$\begin{aligned} 2b_1 \widehat{N}^0_{0,4}(b_1, b_2, b_3, b_4) &= S_0(b_1 + b_2) + S_0(b_1 - b_2) + S_0(b_1 + b_3) \\ &\quad + S_0(b_3 - b_3) + S_0(b_1 + b_4) + S_0(b_1 - b_4). \end{aligned}$$

We have $S_0(k) = \frac{k^3}{12} - \frac{k}{3}$ when k is even, and $\frac{k^3}{12} - \frac{k}{12}$ when k is odd. Thus, we obtain

$$\widehat{N}^0_{0,4}(b_1, b_2, b_3, b_4) = \begin{cases} \dfrac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 1 & \text{all } b_i \text{ even,} \\[2ex] \dfrac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - \dfrac{1}{2} & \text{two } b_i \text{ even, two odd,} \\[2ex] \dfrac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2) - 1 & \text{all } b_i \text{ odd.} \end{cases}$$

If $(k,t) = (0,1)$ then 15 expresses $\widehat{N}^1_{0,4}(\mathbf{b})$ in terms of $\widehat{N}^1_{0,3}$. Proposition 9.7 says that $\widehat{N}^1_{0,3}(b_1, b_2, b_3) = 1$ provided that precisely one of the b_i is zero, and $b_1 + b_2 + b_3$ is even. As $k = 0$, all $b_i > 0$ so only setting $i = 0$ (hence $m = b_1 \pm b_j$) can provide the zero. But $i \equiv b_1 \pm b_j \pmod 2$, so only those j for which $b_j \equiv b_1$ provide a non-zero term. If all b_i are even, or all b_i are odd, then all j provide a non-zero term, and we obtain

$$\begin{aligned} 2b_1 \widehat{N}^1_{0,4}(b_1, b_2, b_3, b_4) &= (b_1 + b_2) + (b_1 - b_2) + (b_1 + b_3) \\ &\quad + (b_1 - b_3) + (b_1 + b_4) + (b_1 - b_4) = 6b_1 \end{aligned}$$

and hence $\widehat{N}^1_{0,4}(b_1, b_2, b_3, b_4) = 3$. But if two of the b_i are even and two of the b_i are odd, then we obtain $2b_1 \widehat{N}_{0,4}(b_1, b_2, b_3, b_4) = 2b_1$, so $\widehat{N}^1_{0,4}(b_1, b_2, b_3, b_4) = 1$. □

In these examples, within the range $0 \le k \le n-1$ and $k \le t \le \min(2g+n-1, k+3g-3+n)$, the degrees of the polynomials decrease as t increases, and increase as k increases. When all b_i are even, these polynomials are all non-zero and their degrees in the b_i^2 precisely decrease by 1 at each step. However, when the b_i are not all even, sometimes the polynomials drop abruptly to zero. Sometimes this is forced: for instance if k of the b_i are zero, then we can have at most $n-k$ of the b_i being odd. But even when the value of k does not force $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ to be zero for parity reasons, the polynomial may drop to zero anyway, as seen above for $\widehat{N}^3_{0,4}(b_1, b_2, 0, 0)$ with b_1, b_2 odd.

We will prove that such behavior always occurs in the next section.

9.9. *Polynomiality of refined non-boundary-parallel counts*

Theorem 9.17. *Suppose that $(g, n) \ne (0, 1), (0, 2)$. Let k, t be non-negative integers and $b_1, \ldots, b_{n-k}$ be positive integers.*

(i) *If $0 \le k \le n-1$ and $k \le t \le \min(2g+n-1, k+3g-3+n)$, then $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ is a symmetric quasi-polynomial over $\mathbb{Q}$ in $b_1^2, \ldots, b_{n-k}^2$, depending on the parity of $b_1, \ldots, b_{n-k}$.*
(ii) *If $k = n$ and $t = 2g+n-1$, then $\widehat{N}^t_{g,n}(0, \ldots, 0) = 1$.*
(iii) *For any other values of k and t, $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0) = 0$.*

Note that the inequalities on t in (i) are from Proposition 9.11, and are necessary conditions for the existence of non-boundary-parallel arc diagrams.

When $k = t = 0$, this theorem reduces to Theorem 1.12 (apart from the statement about degree). The proof is essentially a refinement of the proof of Theorem 1.7. The computations above have established the theorem for $(g, n) = (0, 3), (0, 4)$ and $(1, 1)$. However, because of the inequalities on g, k, n, t, establishing that various terms are non-zero is a more technical exercise.

Proof. We first dispose of parts (ii) and (iii). When $k = n$, we have all $b_i = 0$, so the only possible arc diagram is the empty one, which has $t = 2g+n-1$, so $\widehat{N}^t_{g,n}(0, \ldots, 0) = 1$ as claimed, proving (ii).

To see (iii), suppose k, t are not covered by parts (i) or (ii). As k is the number of zero boundary components, $0 \le k \le n$. If $0 \le k \le n-1$, then we must have $t < k$ or $t > 2g+n-1$ or $t > k+3g-3+n$; and if $k = n$, then we must have $t \ne 1-\chi$. In any of these cases, the conditions of Proposition 9.11 are violated, so $\widehat{N}^t_{g,n} = 0$, proving (iii).

It remains to prove (i). The proof is by induction on the complexity $-\chi = 2g+n-2$; we have computed the $-\chi = 1$ cases $(g, n) = (0, 3)$ and $(1, 1)$ explicitly. We now take (g, n) with complexity ≥ 2, assuming the theorem holds for any smaller complexity. We also take k, t such that $0 \le k \le n-1$ and $k \le t \le \min(2g+n-1, k+3g-3+n)$. Take k of the b_i to be zero; without loss of generality assume $b_1, \ldots, b_{n-k} > 0$ and $b_{n-k+1} = \cdots = b_n = 0$. Further, fix the parity of $b_1, \ldots, b_{n-k}$; we must show that $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ is a polynomial with the required properties.

The recursion in Proposition 9.14 expresses $\widehat{N}^t_{g,n}(\mathbf{b})$ in terms of $\widehat{N}^{t'}_{g',n'}$ where (g', n') is of smaller complexity (but neither $(g', n') = (0,1)$ nor $(0,2)$ are ever seen), hence for which the result holds. Explicitly, the following $\widehat{N}$s occur:

- $\widehat{N}^t_{g-1,n+1}(i, j, b_2, \ldots, b_n)$, where $i, j \ge 0$, $i + j \le b_1$ and $i + j \equiv b_1 \pmod 2$;
- $\widehat{N}^{t-\delta_{b_j,0}}_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_n)$, where $i \ge 0$, $i \le b_1 \pm b_j$ and $i \equiv b_1 \pm b_j \pmod 2$;
- $\widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\, \widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J)$, where $g_1, g_2, i, j, t_1, t_2 \ge 0$, $g_1 + g_2 = g$, $i + j \le b_1$, $i + j \equiv b_1 \pmod 2$, $t_1 + t_2 = t$, $|I|, |J| \ge 2$, and $|I| + |J| = n - 1$.

Expanding out the $\sum_{j=2}^n$ sum in the second line, and the sums over $g_1 + g_2 = g$, $I \sqcup J = \{2, \ldots, n\}$, $t_1 + t_2 = t$ in the third line, we express $b_1 \widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ as a finite collection of sums of the types

$$\text{Type 1:} \quad \widetilde{\sum_{\substack{i,m \ge 0 \\ i+m = b_1 \pm b_j \\ m \text{ even}}}} \bar{i}\, m \,\cdots \quad \text{or} \quad \text{Type 2:} \quad \widetilde{\sum_{\substack{i,j,m \ge 0 \\ i+j+m = b_1 \\ m \text{ even}}}} \bar{i}\, \bar{j}\, m \,\cdots.$$

Here, the $\cdots$ represents some constant times an $\widehat{N}^{\cdot}_{\cdot,\cdot}(b_I, 0, \ldots, 0)$, or a product of two such terms. As discussed in the proof of Proposition 9.14, having fixed the parity of $b_1, \ldots, b_{n-k}$, the parity of i in a sum of type 1 is determined, but in a sum of type 2 only the parity of $i + j$ is fixed; so there are two possibilities for $(i, j) \pmod 2$. Fixing the parity of all variables, every $\widehat{N}$ occurring has inputs which are all fixed in parity. We further need to distinguish between zero and non-zero inputs to each $\widehat{N}$. So, we split sums of type 1 into the $i = 0$ term and the sum over $i > 0$ terms. And we split sums of type 2 into the $i = 0, j = 0$ term, a sum over $i = 0, j > 0$ terms, a sum over $i > 0, j = 0$ terms, and a sum over $i > 0, j > 0$ terms.

Each term of type 1 becomes a finite collection of monomial terms, or sums, of one of the forms

$$q(b_I)(b_1 \pm b_j) \quad \text{or} \quad q(b_I) \widetilde{\sum_{\substack{i>0,\ m \ge 0 \\ i+m = b_1 \pm b_j \\ i \equiv \epsilon \bmod 2,\ m \text{ even}}}} \bar{i}\, i^{2a} m$$

$$= \begin{cases} q(b_I) S_a(b_1 \pm b_j) & b_1 \pm b_j \equiv \epsilon \pmod 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $\epsilon \in \{0, 1\}$. (Note we obtain $S_a(b_1 \pm b_j)$, not $A_a(b_1 \pm b_j)$, because in the sum $i > 0$.) Here, each $q(b_I)$ is a constant multiplied by a monomial in the b_i^2 other than b_1^2 and b_j^2. We have seen (Lemma 7.4) that $S_a(k)$, with fixed parity of k, is an odd polynomial. Every time, we see an S_a, it appears in a pair $S_a(b_1+b_j)+S_a(b_1-b_j)$, which is odd in b_1 and even in b_j.

Each term of type 2, similarly, becomes a finite collection of sums of one of the forms

$$q(b_I)b_1 \quad \text{or}$$

$$q(b_I) \widetilde{\sum_{\substack{i>0, m\geq 0\\ i+m=b_1\\ i\equiv\epsilon \bmod 2,\ m \text{ even}}}} \bar{i}\, i^{2a}\, m = \begin{cases} q(b_I)S_a(b_1) & b_1 \equiv \epsilon \pmod 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{or}$$

$$q(b_I) \widetilde{\sum_{\substack{i,j>0,\ m\geq 0\\ i+j+m=b_1\\ i\equiv\delta \pmod 2,\\ j\equiv\epsilon \pmod 2,\ m \text{ even}}}} \bar{i}\,\bar{j}\, i^{2a} j^{2b} m = \begin{cases} q(b_I)R^{\delta}_{a,b}(b_1) & b_1 \equiv \delta+\epsilon \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Each $S(b_1)$ and $R_{a,b}(b_1)$ is an odd polynomial in b_1 (Lemma 7.4). Collecting all these terms together, we obtain on the right-hand side a polynomial which is odd in b_1 and even in all other variables. Hence, $\widehat{N}^t_{g,n}(b_1,\ldots,b_{n-k},0,\ldots,0)$ is an even polynomial in all variables. $\square$

Theorem 9.18. *Suppose that* $(g,n)\neq(0,1),(0,2)$. *Let* k,t *be non-negative integers satisfying* $0\leq k\leq n-1$ *and* $k\leq t\leq \min(2g+n-1,k+3g-3+n)$. *Let* $b_1,\ldots,b_{n-k}$ *be positive integers. Fixing the parity of* $b_1,\ldots,b_{n-k}$, *the degree of the polynomial* $\widehat{N}^t_{g,n}(b_1,\ldots,b_{n-k},0,\ldots,0)$ *in the* b_i^2 *is at most* $3g-3+n-t+k$.

We will see in Theorem 9.19 that when $0\leq k\leq n-1$ and $t=k$, the degree is in fact exactly $3g-3+n-t+k$. Note that the bounds $k\leq t\leq k+3g-3+n$ provide "just enough room" in t for the degrees of the polynomials $\widehat{N}^t_{g,n}$ to decrease from $3g-3+n$ (when $t=k$) to 0 (when $t=k+3g-3+n$). However, as we have seen, it is possible for the polynomials obtained to have degree less than $k+3g-3+n$.

Proof. From the previous theorem, $\widehat{N}^t_{g,n}(b_1,\ldots,b_{n-k},0,\ldots,0)$ there are polynomials as claimed; we only need to check degrees. To do this, we consider each term of the recursion separately, and consider the possible $\widehat{N}^{t'}_{g',n'}(b_1,\ldots,b_{n-k'},0,\ldots,0)$ which can occur, keeping track of the possible genera g', numbers of boundary components n', complementary region parameter t', and number of boundary components with no marked points k'.

In the first line of the recursion (case 1), we see terms involving $\widehat{N}^t_{g-1,n+1}(i,j,b_2,\ldots,b_{n-k},0,\ldots,0)$. So $g'=g-1$, $n'=n+1$ and $t'=t$. The variables i and j can be zero or non-zero, hence $k'=k,k+1$ or $k+2$. We refer to these cases as 1a, 1b, 1c, respectively.

In the second line of the recursion (case 2), we have $\widehat{N}^{t-\delta_{b_j,0}}_{g,n-1}(i,b_2,\ldots,\widehat{b_j},\ldots,b_n)$. Here, i and b_j can be zero or non-zero. We refer to the cases $(\operatorname{Sgn} i,\operatorname{Sgn} b_j)=(0,0),(0,1),(1,0),(1,1)$ as 2a, 2b, 2c, 2d, respectively.

In the third line of the recursion (case 3), we have $\widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J)$. Let k_1, k_2 be the number of zeroes in (i, b_I) and (j, b_J), respectively. We deal with the two $\widehat{N}$ terms separately. There are many possibilities for $g_1, g_2, |I|, |J|, t_1, t_2, k_1 k_2$, subject to the constraints in the summations. There are also the further possibilities that i, j may be zero or non-zero. We refer to the cases $(\mathrm{Sgn}\, i, \mathrm{Sgn}\, j) = (0, 0), (0, 1), (1, 0), (1, 1)$ as 3a, 3b, 3c, 3d, respectively.

In cases 1a–2d, we calculate the maximum degree $3g' - 3 + n' - t' + k'$ of the corresponding quasi-polynomials $\widehat{N}^{t'}_{g',n'}$, as shown, always assuming that $0 \leq k \leq n - 1$ (since $b_1 > 0$ in Proposition 9.14). These are as shown.

Case	g'	n'	t'	k'	$3g' - 3 + n' - t' + k'$
1a	$g-1$	$n+1$	t	k	$3g-5+n-t+k$
1b	$g-1$	$n+1$	t	$k+1$	$3g-4+n-t+k$
1c	$g-1$	$n+1$	t	$k+2$	$3g-3+n-t+k$
2a	g	$n-1$	$t-1$	k	$3g-3+n-t+k$
2b	g	$n-1$	t	$k+1$	$3g-3+n-t+k$
2c	g	$n-1$	$t-1$	$k-1$	$3g-4+n-t+k$
2d	g	$n-1$	t	k	$3g-4+n-t+k$

In case 1a, we have $k' = k$, so we sum $\frac{1}{2}ij\widehat{N}^{t'}_{g',n'}$ over $i, j > 0$, subject to $i + j + m = b_1$, where $m \geq 0$ is even. By induction, after fixing the parity of all entries, $\widehat{N}^{t'}_{g',n'}(i, j, b_2, \ldots, b_{n-k}, 0, \ldots, 0)$ has degree at most $6g-10+2n-2t+2k$ in i, j and the b_i. (If $i+j$ and b_1 have distinct parity then the polynomial is zero, but the degree condition is still satisfied.) After multiplying by ijm and performing the summation, obtaining a $R^{0 \text{ or } 1}_{a,b}(b_1)$ in the process, we have a polynomial of degree at most $6g - 5 + 2n - 2t + 2k$ which is odd in b_1 and even in all other variables; dividing by b_1, we obtain a polynomial of degree at most $6g - 6 + 2n - 2t + 2k$, hence of degree $\leq 3g - 3 + n - t + k$ in the b_i^2.

Cases 1b and 1c follow a similar analysis.

In case 2, we consider the sum of $\frac{1}{2}\bar{i}m\widehat{N}^{t'}_{g,n-1}(i, b_2, \ldots, \widehat{b_j}, \ldots, b_{n-k}, 0, \ldots, 0)$, over i and $m \geq 0$ satisfying $i+m = b_1 \pm b_j$ with m even. There are two summations, one with $b_1 + b_j$ and one with $b_1 - b_j$, and we add them.

In case 2a, we have $i = 0$ and $b_j = 0$, and the sums both reduce to the same single term $\frac{1}{2}b_1\widehat{N}^{t'}_{g',n'}(0, b_2, \ldots, b_{n-k}, 0, \ldots, 0)$ with $t' = t-1$ and $k' = k$. (This term is zero if b_1 is odd.) Fixing the parity of the variables and dividing out by b_1, we have a polynomial of degree $\leq 3g - 3 + n - t + k$ in the b_i^2.

In case 2b, we have $i = 0$ again, so the sums reduce to single terms, but now $b_j \neq 0$, so the single terms are $\frac{1}{2}(b_1 \pm b_j)\widehat{N}^{t'}_{g',n'}(0, b_2, \ldots, \widehat{b_j}, \ldots, b_{n-k}, 0, \ldots, 0)$, where $t' = t$ and $k' = k + 1$. These sum to $b_1\widehat{N}^{t'}_{g',n'}(0, b_2, \ldots, \widehat{b_j}, \ldots, b_{n-k}, 0, \ldots, 0)$. (This is zero unless $i \equiv b_1 \pm b_j \pmod 2$.) Fixing parity and dividing out by b_1, again we have a polynomial of degree $\leq 3g - 3 + n - t + k$ in the b_i^2.

Cases 2c and 2d follow a similar analysis.

We turn next to case 3. Here, each $\widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)$ and $\widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J)$ by induction satisfies the conditions of the theorem; so once, we fix parity of the non-zero variables, and recalling that $g_1 + g_2 = g$, $t_1 + t_2 = t$ and $|I| + |J| = n - 1$, we obtain polynomials in the b_i^2, with degree

$$\begin{aligned}
&\deg \widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\ \widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J)\\
&\quad \leq\ (3g_1 - 3 + (|I| + 1) - t_1 + k_1) + (3g_2 - 3 + (|J| + 1) - t_2 + k_2)\\
&\quad = 3g - 5 + n - t + (k_1 + k_2).
\end{aligned}$$

In case 3a, we have $i = j = 0$, so $k_1 + k_2 = k + 2$ and the sum reduces to a single term $\frac{1}{2} b_1 \widehat{N}^{t_1}_{g_1,|I|+1}(0, b_I)\ \widehat{N}^{t_2}_{g_2,|J|+1}(0, b_J)$. (This term is zero if b_1 is odd, since the sum is over $i + j \equiv b_1 \pmod 2$.) Dividing out by b_1 yields a polynomial of degree $\leq 3g - 5 + n - t + (k_1 + k_2) = 3g - 3 + n - t + k$.

In cases 3b and 3c, we have one of i, j being zero and the other non-zero; without loss of generality suppose $j = 0$ and $i > 0$. Then the sum reduces to a sum over $i > 0$ and $m \geq 0$ with $i + m = b_1$ and m even. We have $k_1 + k_2 = k + 1$, so fixing parities, $\widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\widehat{N}^{t_2}_{g_2,|J|+1}(0, b_J)$ has degree $\leq 6g - 8 + 2n - 2t + 2k$ in its variables. (If i and b_1 have distinct parity, it is zero.) Multiplying by $\frac{1}{2} im$ and summing, we see an $S_a(b_1)$, and obtain a polynomial of degree $\leq 6g - 5 + 2n - 2t + 2k$ which is odd in b_1 and even in all other b_i. Dividing by b_1, we obtain a polynomial of degree $\leq 3g - 3 + n - t + k$ in the b_i^2.

Finally, in case 3d, we sum over $i, j > 0$. We have $k_1 + k_2 = k$, so fixing parities, the product $\widehat{N}^{t_1}_{g_1,|I|+1}(i, b_I)\widehat{N}^{t_2}_{g_2,|J|+1}(j, b_J)$ has degree $\leq 6g - 10 + 2n - 2t + 2k$ (zero unless $i + j \equiv b_1 \pmod 2$). Multiplying by $\frac{1}{2} ijm$ and summing, we see a $R^{0 \text{ or } 1}_{a,b}(b_1)$ and obtain a polynomial of degree $\leq 6g - 5 + 2n - 2t + 2k$; dividing by b_1 gives a polynomial of degree $\leq 3g - 3 + n - t + k$ in the b_i^2.

We have now shown that using the recursion, we can take $b_1 \widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$, express it as a finite collection of sums, and, fixing the parity of $b_1, \ldots, b_{n-k}$, each non-zero sum yields a polynomial of degree $\leq 3g - 3 + n - t + k$ in the b_i^2 with positive coefficients of highest degree. Summing them, the result is a polynomial of degree at most $3g - 3 + n - t + k$. □

9.10. *Relations between polynomials, volumes and moduli spaces*

Summing $\widehat{N}^t_{g,n}(\mathbf{b})$ over t gives $\widehat{N}_{g,n}(\mathbf{b})$, regardless of whether some b_i are zero. Thus, for any $g \geq 0$, $n \geq 1$ and $0 \leq k \leq n$,

$$\widehat{N}_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0) = \sum_t \widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0). \tag{16}$$

This sum is finite and when $(g, n) \neq (0, 1), (0, 2)$, these functions are quasi-polynomials.

Obviously, $\widehat{N}_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ can be obtained from $\widehat{N}_{g,n}(b_1, \ldots, b_{n-k}, b_{n-k+1}, \ldots, b_n)$ by setting $b_{n-k+1} = \cdots = b_n = 0$. But it is not true that setting $b_{n-k+1} = \cdots = b_n = 0$ in $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, b_{n-k+1}, \ldots, b_n)$ gives $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$, since t depends on the (sum of the) b_i. Indeed, as seen from the examples in Sec. 9.8, for distinct values of k, the sequences (in t) of quasi-polynomials $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ may be quite distinct, and cannot be obtained from each other simply by setting some variables to zero (or even by setting variables of designated even parity equal to zero).

Nonetheless, fix k in the range $0 \leq k \leq n-1$, and consider the sequence (in t) of quasi-polynomials $\widehat{N}^t_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$. These quasi-polynomials can only be non-zero for t in the range $k \leq t \leq \min(2g+n-1, k+3g-3+n)$, by Theorem 9.17. By Theorem 9.18, for such k and t, these quasi-polynomials have degree at most $3g-3+n-t+k$. And since $k \leq t$, we have $3g-3+n-t+k \leq 3g-3+n$. However, we know from Theorem 1.7 that the quasi-polynomials $\widehat{N}_{g,n}(b_1, \ldots, b_n)$ have degree $3g-3+n$. This leads us to the following.

Theorem 9.19. *Let $g \geq 0$ and $n \geq 1$ satisfy $(g,n) \neq (0,1), (0,2)$. Let $0 \leq k \leq n-1$.*

(i) *Each polynomial defining the quasi-polynomial $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ has degree $3g-3+n$ in $b_1^2, \ldots, b_{n-k}^2$.*

(ii) *Fix parities for $b_1, \ldots, b_{n-k}$. Then the highest degree $(3g-3+n)$ terms of the polynomial $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ in $b_1^2, \ldots, b_{n-k}^2$ agree with the highest degree terms of the following three polynomials*:

$$\widehat{N}_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0),$$

$$\mathfrak{N}_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0), \quad \frac{1}{2} V_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0).$$

(iii) *The polynomials defining the quasi-polynomial $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ all agree in their terms of highest degree, and for non-negative integers $d_1, \ldots, d_{n-k}$ satisfying $d_1 + \cdots + d_{n-k} = 3g-3+n$, the coefficient $c_{d_1, \ldots, d_{n-k}}$ of $b_1^{2d_1} \cdots b_{n-k}^{2d_{n-k}}$ in each of these polynomials is given by*

$$c_{d_1, \ldots, d_{n-k}} = \frac{1}{2^{5g-6+2n}\, d_1! \cdots d_{n-k}!} \left\langle \psi_1^{d_1} \cdots \psi_{n-k}^{d_{n-k}}, \overline{\mathcal{M}}_{g,n} \right\rangle.$$

Recall that $\mathfrak{N}_{g,n}(b_1, \ldots, b_n)$ denotes the lattice count quasi-polynomials of [31], and $V_{g,n}(b_1, \ldots, b_n)$ denotes the volume polynomials of [25].

When $k = 0$, we obtain Theorem 1.13, and (ii) gives the degree statement in Theorem 1.12, completing the proof of that theorem.

Proof. Fix parities of $b_1, \ldots, b_{n-k}$. By Theorem 1.9, for any $d_1, \ldots, d_n \geq 0$ such that $d_1 + \cdots + d_n = 3g-3+n$, the coefficient of $b_1^{2d_1} \cdots b_n^{2d_n}$ is non-zero. In particular, setting $b_{n-k+1}, \ldots, b_n$ to zero, the degree of $\widehat{N}_{g,n}$ remains $3g-3+n$. This proves (i).

Still fixing parities of the b_i, as discussed above, Eq. 16 expresses a polynomial of degree $3g-3+n$ as a sum of polynomials of degree at most $3g-3+n-t+k$; moreover in this sum we always have $t \geq k$, so $3g-3+n-t+k \leq 3g-3+n$, with equality if and only if $t = k$. Thus the polynomial with $t = k$ has degree exactly $3g-3+n$. Moreover, the the terms of degree $3g-3+n$ in $\widehat{N}_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ and $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ agree. Theorem 1.9, Proposition 7.7 and subsequent discussion then immediately imply the rest of the conclusions. $\square$

Thus, we can recover the full set of intersection numbers of ψ-classes on the moduli space of curves from $\widehat{N}^0_{g,n}(b_1, \ldots, b_n)$, restricting the number of regions in arc diagrams by $k = t = 0$. The constraints $k = t = 0$ mean topologically that each boundary component has at least one arc endpoint, and that the arcs cut the surface into the minimum number of regions possible.

When $k = t$, the quasi-polynomials $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$, in addition to recovering intersection numbers on moduli spaces, have an interesting set of zeroes and positivity constraints. Indeed, Lemma 9.12 immediately implies that these quasi-polynomials must be zero, or positive, for certain values of $b_1, \ldots, b_{n-k}$, giving the following result.

Proposition 9.20. *Consider the quasi-polynomials* $\widehat{N}^k_{g,n}(b_1, \ldots, b_{n-k}, 0, \ldots, 0)$, *for* $(g, n) \neq (0, 1), (0, 2)$, $0 \leq k \leq n-1$ *and* $b_1, \ldots, b_{n-k} > 0$.

(i) *Any integer point* $\mathbf{b} = (b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ *satisfying* $\frac{1}{2}(b_1 + \cdots + b_{n-k}) < 2g + n - 1 - k$ *is a zero of* $\widehat{N}^k_{g,n}$, i.e. $\widehat{N}^k_{g,n}(\mathbf{b}) = 0$.

(ii) *At any integer point* $\mathbf{b} = (b_1, \ldots, b_{n-k}, 0, \ldots, 0)$ *satisfying* $\frac{1}{2}(b_1 + \cdots + b_{n-k}) \geq 2g + n - 1 - k$, $\widehat{N}^k_{g,n}$ *is positive*, i.e. $\widehat{N}^k_{g,n}(\mathbf{b}) > 0$.

9.11. *Polynomiality for general refined curve counts*

It is now not difficult to show that the $G^t_{g,n}$ obey a polynomiality result similar to the $G_{g,n}$, using a method similar to Sec. 7.4, refining it as in the proof of Proposition 9.8.

Theorem 9.21. *Let* $(g, n) \neq (0, 1), (0, 2)$, *let* t *be an integer satisfying* $0 \leq t \leq \min(2g + n - 1, 3g - 3 + n)$, *and let* $b_1, \ldots, b_n$ *be non-negative integers. Then* $G^t_{g,n}(b_1, \ldots, b_n)$ *is given by a product of*

(i) *a combinatorial factor* $\binom{2m_i}{m_i}$ *for each* $1 \leq i \leq n$, *where* $b_i = 2m_i$ *if* b_i *is even and* $b_i = 2m_i + 1$ *if* b_i *is odd, and*

(ii) *a symmetric rational quasi-polynomial* $P^t_{g,n}(b_1, \ldots, b_n)$, *depending on the parity of each* b_i, *of degree* $\leq 3g - 3 + 2n - t$.

If we fix the parity of each b_i *so that at least* t *of the* b_i *are even, then the degree of the corresponding polynomial in* $P^t_{g,n}(b_1, \ldots, b_n)$ *is exactly* $3g - 3 + 2n - t$.

Proof. Fix the parity of $b_1, \ldots, b_n$, and write $b_i = 2m_i$ or $b_i = 2m_i + 1$ accordingly. Using Proposition 9.5, we express $G^t_{g,n}(b_1, \ldots, b_n)$ as a sum over $a_1, \ldots, a_n$, where $0 \leq a_i \leq b_i$ and $a_i \equiv b_i \pmod 2$. For those b_i which are even, we split the sum over a_i into the $a_i = 0$ term and the $a_i > 0$ terms. This expresses $G^t_{g,n}$ as a sum of terms of the form

$$\prod_{i\in K}\binom{2m_i}{m_i}\sum_{\substack{1\le a_j\le b_j\\ a_j\equiv b_j \pmod 2\\ j\in J}}\left(\prod_{j\in J}\binom{b_j}{\frac{b_j-a_j}{2}}a_j\right)\widehat{N}^t_{g,n}(a_1,\ldots,a_n).$$

In each such term, each a_i is fixed to be even or odd, and each of even a_i is fixed to be zero or non-zero; J and K denote the set of i for which a_i has been set to non-zero or zero, respectively, so $K \sqcup J = \{1,\ldots,n\}$. Hence, we can write $\widehat{N}^t_{g,n}(a_1,\ldots,a_n) = \widehat{N}^t_{g,n}(a_J, 0)$. In fact, $G^t_{g,n}(b_1,\ldots,b_n)$ is the sum of all such expressions, where K runs over subsets of I, the set of indices i for which b_i has been chosen to be even. We write $|K| = k$ as per previous notation, so $|J| = n-k$.

If $k = n$, we have $\widehat{N}^t_{g,n}(0,\ldots,0) = \delta_{t,2g+n-1}$. Depending on t, this is also a symmetric quasi-polynomial of degree 0, or is zero; it gives a term of the form $\binom{2m_1}{m_1}\cdots\binom{2m_n}{m_n}$ times a constant in $G^t_{g,n}$, when $t = 2g+n-1$. Thus, it remains to consider the terms with $0 \le k \le n-1$.

By Theorem 9.17, when $0 \le k \le n-1$, $\widehat{N}^t_{g,n}(a_J,0)$ is either zero, or t lies in the range specified in the theorem (in particular, $k \le t$), and $\widehat{N}^t_{g,n}(a_J,0)$ is a symmetric quasi-polynomial in the a_j^2, of degree $\le 3g-3+n-t+k \le 3g-3+n$. Splitting up $\widehat{N}^t_{g,n}(a_J,0)$ as a sum of monomials $c_\alpha \prod_{j\in J} a_j^{2\alpha_j}$, we can write the corresponding terms of $G^t_{g,n}$ as a finite sum of terms of the form

$$\prod_{i\in K}\binom{2m_i}{m_i}\sum_{\alpha} c_\alpha \prod_{j\in J}\sum_{\substack{1\le a_j\le b_j\\ a_j\equiv b_j \pmod 2}}\binom{b_j}{\frac{b_j-a_j}{2}}a_j^{2\alpha_j}.$$

As $\widehat{N}^t_{g,n}(a_J,0)$ has degree $\le 3g-3+n-t+k$, we always have $\alpha_1+\cdots+\alpha_n \le 3g-3+n-t+k$. When $t=k$, by Theorem 9.19, the degree of $\widehat{N}^t_{g,n}(a_J,0)$ is exactly $3g-3+n-t+k = 3g-3+n$, so in this case there are terms with $\alpha_1+\cdots+\alpha_n = 3g-3+n$. Each $\sum_{a_j}\binom{b_j}{\frac{b_j-a_j}{2}}a^{2\alpha_j}$ is either $\tilde{p}_{\alpha_j}(m_j) = \binom{2m_j}{m_j}m_j p_{\alpha_j}(m_j)$, if a_j is even, or $\tilde{q}_{\alpha_j}(m_j) = \binom{2m_j}{m_j}(2m_j+1)q_{\alpha_j}(m_j)$, if a_j is odd. Each $m_j p_{\alpha_j}(m_j)$ and $(2m_j+1)q_{\alpha_j}(m_j)$ is a polynomial of degree α_j+1, so in each term, the degree of their product is $\sum_{j\in J}(\alpha_j+1) = \left(\sum_{j\in J}\alpha_j\right)+n-k \le 3g-3+2n-t$, and when $t=k$ there are terms where equality holds.

Thus, $G^t_{g,n}(b_1,\ldots,b_n)$ can be expressed as a finite sum of terms, where each term is a constant multiplied by $\binom{2m_1}{m_1}\cdots\binom{2m_n}{m_n}$, multiplied by a polynomial in $m_1,\ldots,m_n$. This polynomial is either a constant (in the case $k=n$ and $t = 2g+n-1$), or is a product of $m_j p_{\alpha_j}(m_j)$ and $(2m_j+1)q_{\alpha_j}(m_j)$, over $j \in J$. Terms where the number k of variables set to zero satisfies $0 \le k \le n-1$ contribute polynomials of degree $\le 3g-3+2n-t$. When $k=n$, the polynomial contribution is a constant. Thus, $G^t_{g,n}$ has properties (i) and (ii) claimed.

If at least t of the b_i are even, then it is possible to set t of the variables to zero, so there is a term with $k = t$ which contributes to the polynomial $P^t_{g,n}(b_1, \ldots, b_n)$. As discussed above, there are then monomials appearing with $\sum_{j\in J} \alpha_j = 3g - 3 + n - t + k$, and when we perform summations, we obtain products of $m_j p_{\alpha_j}(m_j)$ and $(2m_j + 1)q_{\alpha_j}$, contributing a polynomial of degree exactly $3g - 3 + 2n - t$ to $P^t_{g,n}$. As all the polynomials involved have positive highest degree terms, the resulting polynomial $P^t_{g,n}(b_1, \ldots, b_n)$ must have degree exactly $3g - 3 + 2n - t$. □

10. Differential Equations and Partition Functions

10.1. *Refined differential forms and generating functions*

Refining by regions, we now develop generating functions $f^G_{g,n}, f^N_{g,n}$ and differential forms $\omega_{g,n}$ by regions which will satisfy differential equations.

Definition 10.1 (Refined generating functions and differentials). For integers $g \geq 0$, $n \geq 1$ and $r \geq 1$, and t, let

$$f^G_{g,n,r}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n \geq 0} G_{g,n,r}(\mu_1, \ldots, \mu_n) x_1^{-\mu_1-1} \cdots x_n^{-\mu_n-1}$$

$$f^N_{g,n,r}(z_1, \ldots, z_n) = \sum_{\nu_1,\ldots,\nu_n \geq 0} N_{g,n,r}(\nu_1, \ldots, \nu_n) z_1^{\nu_1-1} \cdots z_n^{\nu_n-1}$$

$$f^{G,t}_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n \geq 0} G^t_{g,n}(\mu_1, \ldots, \mu_n) x_1^{-\mu_1-1} \cdots x_n^{-\mu_n-1},$$

$$f^{N,t}_{g,n}(z_1, \ldots, z_n) = \sum_{\nu_1,\ldots,\nu_n \geq 0} N^t_{g,n}(\nu_1, \ldots, \nu_n) z_1^{\nu_1-1} \cdots z_n^{\nu_n-1}$$

$$\omega^G_{g,n,r}(x_1, \ldots, x_n) = f^G_{g,n,r}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$\omega^N_{g,n,r}(z_1, \ldots, z_n) = f^N_{g,n,r}(z_1, \ldots, z_n)\, dz_1 \cdots dz_n$$

$$\omega^{G,t}_{g,n}(x_1, \ldots, x_n) = f^{G,t}_{g,n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$\omega^{N,t}_{g,n}(z_1, \ldots, z_n) = f^{N,t}_{g,n}(z_1, \ldots, z_n)\, dz_1 \cdots dz_n.$$

Since $G_{g,n}(\mu) = \sum_r G_{g,n,r}(\mu) = \sum_t G^t_{g,n}(\mu)$ and $N_{g,n}(\nu) = \sum_r N_{g,n,r}(\nu) = \sum_t N^t_{g,n}(\nu)$, we immediately have $f^G_{g,n} = \sum_{r\geq 1} f^G_{g,n,r} = \sum_t f^{G,t}_{g,n}$ and $f^N_{g,n} = \sum_{r\geq 1} f^N_{g,n,r} = \sum_t f^{N,t}_{g,n}$.

10.2. *Small cases of refined generating functions and differential forms*

We can compute these generating functions directly in small cases $(g, n) = (0, 1), (0, 2), (0, 3)$.

Proposition 10.2. *The generating functions* $f^G_{0,1,r}(x_1)$, $f^N_{0,1,r}(z_1)$, $f^{G,t}_{0,1}(x_1)$ *and* $f^{N,t}_{0,1}(z_1)$ *are all meromorphic, given by*

$$f^G_{0,1,r}(x_1) = \frac{1}{r}\binom{2r-2}{r-1}x_1^{-2r+1}, \quad f^{G,t}_{0,1}(x_1) = \begin{cases} \dfrac{x_1 - \sqrt{x_1^2-4}}{2} & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$f^N_{0,1,r}(z_1) = \begin{cases} z_1^{-1} & \text{if } r = 1, \\ 0 & \text{otherwise,} \end{cases} \quad f^{N,t}_{0,1}(z_1) = \begin{cases} z_1^{-1} & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. An arc diagram on the disc with r complementary regions has $r-1$ arcs, so the only non-zero $G_{0,1,r}(\mu)$ is $G_{0,1,r}(2r-2) = \frac{1}{r}\binom{2r-2}{r-1}$, yielding $f^G_{0,1,r}(x_1)$. The only arc diagram without boundary-parallel arcs is the empty diagram, so the only contribution to $f^N_{0,1,r}$ is a z_1^{-1}, when $r = 1$.

All arc diagrams on the disc have $t = 0$, hence $f^{G,t}_{0,1}(x_1)$ is equal to $f^G_{0,1}(x_1)$ when $t = 0$, and zero otherwise; similarly for $f^{N,t}_{0,1}$. □

Proposition 10.3. *The function* $f^{N,t}_{0,2}$ *is meromorphic and is given by*

$$f^{N,t}_{0,2}(z_1, z_2) = \begin{cases} \dfrac{1}{(1-z_1z_2)^2} & \text{if } t = 0, \\ \dfrac{1}{z_1z_2} & \text{if } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We calculated in Sec. 8.2 that $f^N_{0,2}(z_1, z_2) = \frac{1}{z_1z_2} + \frac{1}{(1-z_1z_2)^2}$; we now see that the two terms in this sum correspond precisely to $t = 0$ and $t = 1$.

Proof. From Lemma 9.3, we have $N^0_{0,2}(b_1, b_2) = b_1$ for $b_1 = b_2 > 0$, $N^1_{0,2}(0,0) = 1$, and all other $N^t_{0,2}(b_1, b_2) = 0$. Thus, $f^{N,0}_{0,2}(z_1, z_2) = \sum_{\nu=1}^{\infty} \nu(z_1z_2)^{\nu-1} = \frac{1}{(1-z_1z_2)^2}$ and $f^{N,1}_{0,2}(z_1, z_2) = z_1^{-1}z_2^{-1}$. □

Proposition 10.4. *The function* $f^{N,t}_{0,3}$ *is meromorphic and is given by*

$$f^{N,0}_{0,3}(z_1, z_2, z_3) = \frac{2(z_1+z_2+z_3+z_1z_2z_3)(1+z_1z_2+z_2z_3+z_3z_1)}{(1-z_1^2)^2(1-z_2^2)^2(1-z_3^2)^2}$$

$$f^{N,1}_{0,3}(z_1, z_2, z_3) = \frac{1+4z_1z_2+z_1^2+z_2^2+z_1^2z_2^2}{(1-z_1^2)^2(1-z_2^2)^2z_3} + \frac{1+4z_2z_3+z_2^2+z_3^2+z_2^2z_3^2}{(1-z_2^2)^2(1-z_3^2)^2z_1}$$
$$+ \frac{1+4z_3z_1+z_3^2+z_1^2+z_3^2z_1^2}{(1-z_3^2)^2(1-z_1^2)^2z_2}$$

$$f^{N,2}_{0,3}(z_1, z_2, z_3) = \frac{1+16z_1^2z_2^2z_3^2+z_1^4z_2^4z_3^4+\sum_{\text{cyc}}(z_1^4-4z_1^2z_2^2+z_1^4z_2^4-4z_1^4z_2^2z_3^2)}{z_1z_2z_3(1-z_1^2)^2(1-z_2^2)^2(1-z_3^2)^2}$$

and $f^{N,t}_{0,3}(z_1, z_2, z_3) = 0$ *otherwise.*

One can check that these three $f_{0,3}^{N,t}$ sum to the $f_{0,3}^{N}$ calculated in Lemma 8.2.

Proof. From Proposition 9.7, we have $N_{0,3}^0(b_1,b_2,b_3)=b_1b_2b_3$, for positive b_i with even sum; $N_{0,3}^1(b_1,b_2,0)=b_1b_2$ for positive b_i with even sum; $N_{0,3}^2(b_1,0,0)=b_1$ for positive even b_1; and $N_{0,3}^2(0,0,0)=1$. All other $N_{0,3}^t$ are zero. Thus, following a similar method to Lemma 8.2, we may define

$$\rho(z)=\sum_{\substack{\nu\geq 1\\ \nu\text{ even}}}\nu\, z^{\nu-1}=\frac{2z}{(1-z^2)^2},\quad \sigma(z)=\sum_{\substack{\nu\geq 1\\ \nu\text{ odd}}}\nu\, z^{\nu-1}=\frac{1+z^2}{(1-z^2)^2}.$$

(note ρ here is slightly different from Lemma 8.2) and then

$$\begin{aligned}
f_{0,3}^{N,0}(z_1,z_2,z_3)&=\rho(z_1)\rho(z_2)\rho(z_3)+\rho(z_1)\sigma(z_2)\sigma(z_3)\\
&\quad+\rho(z_2)\sigma(z_3)\sigma(z_1)+\rho(z_3)\sigma(z_1)\sigma(z_2)\\
f_{0,3}^{N,1}(z_1,z_2,z_3)&=\sum_{\text{cyc}}z_1^{-1}(\rho(z_2)\rho(z_3)+\sigma(z_2)\sigma(z_3))\\
f_{0,3}^{N,2}(z_1,z_2,z_3)&=z_1^{-1}z_2^{-1}z_3^{-1}+z_2^{-1}z_3^{-1}\rho(z_1)+z_3^{-1}z_1^{-1}\rho(z_2)+z_1^{-1}z_2^{-1}\rho(z_3).
\end{aligned}$$

Expanding these out gives the claimed expressions. □

10.3. *Meromorphicity and change of coordinates*

A similar method to Sec. 8.3 shows that we have meromorphicity in many cases.

Proposition 10.5. *For all integers $g\geq 0$, $n\geq 1$ and t, the functions $f_{g,n}^{N,t}$, $f_{g,n,r}^{G}$, $f_{g,n,r}^{N}$ and the differential forms $\omega_{g,n}^{N,t}$, $\omega_{g,n,r}^{G}$, $\omega_{g,n,r}^{N}$ are all meromorphic.*

Proof. First, we deal with $f_{g,n,r}^{G}$ and $f_{g,n,r}^{N}$. Once g,n and r are given, Lemma 9.9 says that if $G_{g,n,r}(\mu_1,\ldots,\mu_n)>0$, then

$$\frac{1}{2}\sum_{i=1}^{n}\mu_i\leq r+2g+n-2.$$

Thus, only finitely many $(\mu_1,\ldots,\mu_n)$ contribute to the sum for $f_{g,n,r}^{G}(x_1,\ldots,x_n)$. Similarly, the sum for $f_{g,n,r}^{N}(z_1,\ldots,z_n)$ is finite. Thus $f_{g,n,r}^{G}$ and $f_{g,n,r}^{N}$ are Laurent polynomials in $x_1,\ldots,x_n$, hence meromorphic; and hence $\omega_{g,n,r}^{G}(x_1,\ldots,x_n)$ and $\omega_{g,n,r}^{N}(z_1,\ldots,z_n)$ are meromorphic.

Turning to $f_{g,n}^{N,t}$, the proof follows Proposition 8.3. Propositions 10.2 and 10.3 show that $f_{0,1}^{N,t}(z_1)$ and $f_{0,2}^{N,t}(z_1,z_2)$ are meromorphic; and hence $\omega_{0,1}^{N,t}(z_1)$ and $\omega_{0,2}^{N,t}(z_1,z_2)$ are meromorphic forms.

For $(g,n)\neq(0,1),(0,2)$, we proved in Theorem 9.17 that $\widehat{N}_{g,n}^{t}(\nu_1,\ldots,\nu_{n-k},0,\ldots,0)$ is a rational symmetric quasi-polynomial in $\nu_1^2,\ldots,\nu_{n-k}^2$. Hence, if we fix each ν to be zero, positive odd, or positive even, then we obtain a polynomial.

Let $\{1, 2, \ldots, n\} = K \sqcup J$, where K is the set of i for which $\nu_i = 0$, and J is the set of j for which $\nu_j > 0$. When $j \in J$, we can set $\nu_j \equiv \epsilon_j \pmod 2$, where $\epsilon_j \in \{0, 1\}$. Thus, we can split the sum for $f_{g,n}^{N,t}$ into 3^n sums of the form

$$\sum_{\substack{\nu_j \geq 1 \\ \nu_j \equiv \epsilon_j \pmod 2 \\ j \in J}} \left(\prod_{j \in J} \nu_j \right) P(\nu_1, \ldots, \nu_n)|_{\nu_K = 0} \, z_1^{\nu_1 - 1} \cdots z_n^{\nu_n - 1},$$

where $P(\nu_1, \ldots, \nu_n)$ is a polynomial, and $P(\nu_1, \ldots, \nu_n)|_{\nu_K=0}$ means we set all $\nu_i = 0$ for $i \in K$. This is a polynomial in the ν_j for $j \in J$. Splitting each such polynomial into monomials, we can write $f_{g,n}^{N,t}$ as a finite sum of terms of the form of a constant times

$$\left(\prod_{i \in K} z_i^{-1} \right) \sum_{\substack{\nu_j \geq 1 \\ \nu_j \equiv \epsilon_j \pmod 2 \\ j \in J}} \left(\prod_{j \in J} \nu_j^{a_j} z_j^{\nu_j - 1} \right).$$

Now we know from the proof of Proposition 8.3 that for any positive integer a and $\epsilon \in \{0, 1\}$,

$$\sum_{\substack{\nu \geq 0 \\ \nu \equiv \epsilon \pmod 2}} \nu^a z^\nu = \sum_{\substack{\nu \geq 1 \\ \nu \equiv \epsilon \pmod 2}} \nu^a z^\nu$$

is meromorphic. Hence, $f_{g,n}^{N,t}$ is a finite sum of meromorphic functions, hence is meromorphic, as is $\omega_{g,n}^{N,t}$. □

We have now shown all generating functions and differential forms are meromorphic, except $f_{g,n}^{G,t}$ and $\omega_{g,n}^{G,t}$, to which we now turn. Just as Theorem 1.8 expresses $G_{g,n}(b_1, \ldots, b_n)$ in terms of $N_{g,n}(a_1, \ldots, a_n)$, Proposition 9.5 expresses $G_{g,n}^t(b_1, \ldots, b_n)$ in terms of $N_{g,n}^t(a_1, \ldots, a_n)$. The proof of Theorem 1.14, then applies verbatim, replacing $G_{g,n}, N_{g,n}$ with $G_{g,n}^t, N_{g,n}^t$, and we obtain the following.

Proposition 10.6. *For any $g \geq 0$, $n \geq 1$ other than $(g, n) = (0, 1)$ and integer t,*

$$\phi^* \omega_{g,n}^{G,t}(x_1, \ldots, x_n) = \omega_{g,n}^{N,t}(z_1, \ldots, z_n),$$

where $\phi(z_1, \ldots, z_n) = (z_1 + \frac{1}{z_1}, \ldots, z_n + \frac{1}{z_n})$. In particular, $f_{g,n}^{G,t}$ and $\omega_{g,n}^{G,t}$ are meromorphic.

As in the unrefined case, we can regard $x_i \leftrightarrow z_i$ as a change of coordinates and simply write $\omega_{g,n}^t$, rather than $\omega_{g,n}^{G,t}$ or $\omega_{g,n}^{N,t}$, for $(g, n) \neq (0, 1)$. Proposition 10.6 indicates again that the parameter t is more natural than r for our purposes.

We conclude that each meromorphic form $\omega_{g,n}$ naturally decomposes into a finite sum of meromorphic forms $\omega_{g,n}^t$.

10.4. *Refined free energies*

From the refined meromorphic $\omega^t_{g,n}$, we can also refined our notion of free energy. So a *refined free energy* is a function $F^t_{g,n} : (\mathbb{CP}^1)^n \to (\mathbb{CP}^1)$ such that

$$d_{z_1} \cdots d_{z_n} F^t_{g,n}(z_1, \ldots, z_n) = \omega^t_{g,n}(z_1, \ldots, z_n).$$

In the case $(g,n) = (0,1)$, we have distinct $F^{G,t}_{0,1}$ and $F^{N,t}_{0,1}$.

In Proposition 1.16, we give expressions for free energy energy functions for $(g,n) = (0,1), (0,2), (0,3)$, which we now prove.

Proof of Proposition 1.16. We saw in Proposition 10.2 that $f^{G,0}_{0,1} = f^G_{0,1}$ and $f^{N,0}_{0,1} = f^N_{0,1}$, so $F^{G,0}_{0,1} = F^G_{0,1}$ and $F^{N,0}_{0,1} = F^N_{0,1}$, which were given in Proposition 1.15. Differentiating the claimed $F^0_{0,2}$ and $F^1_{0,2}$ with respect to z_1, z_2 gives the $f^{N,0}_{0,2}$ and $f^{N,1}_{0,2}$ from Proposition 10.3. Similarly, differentiating the $F^t_{0,3}$ with respect to z_1, z_2, z_3 gives the $f^{N,t}_{0,3}$ from Proposition 10.4. □

From Propositions 10.2, 10.3 and 10.4, we observe that the free energy functions of Proposition 1.16 are the only non-zero functions with $(g,n) = (0,1), (0,2), (0,3)$.

We can observe directly that these $F^t_{g,n}$ sum to the $F_{g,n}$ calculated previously. In the $(0,2)$ and $(0,3)$ cases especially, the terms of the rather complicated functions $F_{g,n}$ split up in a natural way.

10.5. *Putting the generating functions and differential forms together*

We can now combine refined generating functions, differential forms and free energies together over all values of r or t, to obtain more general generating functions and forms. These will eventually be put together into partition functions. We introduce variables α and β to keep track of r and t, respectively.

Definition 10.7. For integers $g \geq 0$ and $n \geq 1$, define the functions $\mathfrak{f}^G_{g,n}, \mathfrak{f}^N_{g,n}$, $\mathbf{f}^G_{g,n}, \mathbf{f}^N_{g,n}$ and differential forms $\Omega^G_{g,n}, \Omega^N_{g,n}$ by

$$\mathfrak{f}^G_{g,n}(x_1, \ldots, x_n; \alpha) = \sum_{r \geq 1} f^G_{g,n,r}(x_1, \ldots, x_n)\, \alpha^r,$$

$$\mathfrak{f}^N_{g,n}(z_1, \ldots, z_n; \alpha) = \sum_{r \geq 1} f^N_{g,n,r}(z_1, \ldots, z_n)\, \alpha^r$$

$$\Omega^G_{g,n}(x_1, \ldots, x_n; \alpha) = \mathfrak{f}^G_{g,n}(x_1, \ldots, x_n; \alpha)\, dx_1 \cdots dx_n = \sum_{r \geq 1} \omega^G_{g,n,r}(x_1, \ldots, x_n) \alpha^r$$

$$\Omega^N_{g,n}(z_1,\ldots,z_n;\alpha) = \mathfrak{f}^N_{g,n}(z_1,\ldots,z_n;\alpha)\, dz_1\cdots dz_n = \sum_{r\geq 1}\omega^N_{g,n,r}(z_1,\ldots,z_n)\alpha^r$$

$$\mathbf{f}^G_{g,n}(x_1,\ldots,x_n;\beta) = \sum_t f^{G,t}_{g,n}(x_1,\ldots,x_n)\,\beta^t,$$

$$\mathbf{f}^N_{g,n}(z_1,\ldots,z_n;\beta) = \sum_t f^{N,t}_{g,n}(z_1,\ldots,z_n)\,\beta^t.$$

Finally, for $(g,n)\neq(0,1)$, define the differential form

$$\Omega_{g,n}(\beta) = \sum_t \omega^t_{g,n}\,\beta^t.$$

For $(g,n)\neq(0,1)$, we have taken advantage of Proposition 10.6 to simply write $\omega^t_{g,n}$, which can be written in terms of the z_i or x_i; $\Omega_{g,n}$ then behaves similarly.

We can regard $\mathfrak{f}^G_{g,n}$ and $\mathfrak{f}^N_{g,n}$ as families of functions $(\mathbb{CP}^1)^n\to\mathbb{CP}^1$, parametrized by $\alpha\in\mathbb{CP}^1$. Similarly, we can regard $\Omega^G_{g,n}$ and $\Omega^N_{g,n}$ as families of sections of $(T^*\mathbb{CP}^1)^{\boxtimes n}$, parametrized by α, and $\Omega_{g,n}$ as a family of sections parametrized by β. Setting $\alpha=1$ or $\beta=1$ recovers the unrefined generating functions $f^G_{g,n}$, $f^N_{g,n}$, and differential forms $\omega^G_{g,n}$, $\omega^N_{g,n}$.

While we know that each $f^G_{g,n,r}$, $f^N_{g,n,r}$, $\omega^G_{g,n,r}$ and $\omega^N_{g,n,r}$ is meromorphic, we do not yet know that $\mathfrak{f}^G_{g,n}$, $\mathfrak{f}^N_{g,n}$, $\Omega^G_{g,n}$ or $\Omega^N_{g,n}$ are meromorphic, as they are defined by infinite sums. (We will see this later in Proposition 10.12.) On the other hand, because of the bounds on t, namely $0\leq t\leq 2g+n-1$, each sum over t is a finite sum, immediately giving us the following.

Proposition 10.8. *Let $g\geq 0$ and $n\geq 1$. The functions $\mathbf{f}^G_{g,n}(x_1,\ldots,x_n;\beta)$ and $\mathbf{f}^N_{g,n}(z_1,\ldots,z_n;\beta)$ are meromorphic, and for each $\beta\in\mathbb{C}$, $\Omega_{g,n}(\beta)$ is a meromorphic form.* □

Using our calculations from Sec. 10.2, we can calculate small cases of these functions and forms.

Proposition 10.9. *The functions $\mathfrak{f}^G_{0,1}$, $\mathfrak{f}^N_{0,1}$, $\mathbf{f}^G_{0,1}$, $\mathbf{f}^N_{0,1}$ and differential forms $\Omega^G_{0,1}$, $\Omega^N_{0,1}$ are given as follows:*

$$\mathfrak{f}^G_{0,1}(x_1;\alpha) = \frac{x_1-\sqrt{x_1^2-4\alpha}}{2} \quad \textit{so} \quad \Omega^G_{0,1}(x_1;\alpha) = \frac{x_1-\sqrt{x_1^2-4\alpha}}{2}\,dx_1$$

$$\mathfrak{f}^N_{0,1}(z_1;\alpha) = z_1^{-1}\alpha \quad \textit{so} \quad \Omega^N_{0,1}(z_1;\alpha) = z_1^{-1}\alpha\,dz_1$$

$$\mathbf{f}^G_{0,1}(x_1;\beta) = z_1$$

$$\mathbf{f}^N_{0,1}(z_1;\beta) = z_1^{-1}.$$

Proof. All the claimed expressions except $\mathfrak{f}^G_{0,1}(x_1;\alpha)$ consist of sums with a single term, obtained immediately from Proposition 10.2. We compute $\mathfrak{f}_{0,1}(x_1;\alpha)$:

$$\begin{aligned}\mathfrak{f}^G_{0,1}(x_1;\alpha) &= \sum_{m=0}^{\infty} G_{0,1,m+1}(2m)x_1^{-2m-1}\alpha^{m+1}\\ &= \alpha^{1/2}\sum_{m=0}^{\infty} G_{0,1}(2m)(x_1\alpha^{-1/2})^{-2m-1} = \alpha^{1/2} f^G_{0,1}(x_1\alpha^{-1/2}).\end{aligned}$$

Since $f^G_{0,1}(x) = \frac{x-\sqrt{x^2-4}}{2}$ (Lemma 8.2), we obtain the desired result. □

We obtain $\mathbf{f}^N_{0,2}$, $\mathbf{f}^N_{0,3}$ immediately from Propositions 10.3 and 10.4; multiplying by dz_i gives the corresponding $\Omega_{0,2}, \Omega_{0,3}$.

Proposition 10.10. *The generating functions* $\mathbf{f}^N_{0,2}$, $\mathbf{f}^N_{0,3}$ *are given as follows*:

$$\begin{aligned}\mathbf{f}^N_{0,2}(z_1,z_2;\beta) &= \frac{1}{(1-z_1z_2)^2} + \frac{\beta}{z_1z_2}\\ \mathbf{f}^N_{0,3}(z_1,z_2,z_3;\beta) &= \frac{2(z_1+z_2+z_3+z_1z_2z_3)(1+z_1z_2+z_2z_3+z_3z_1)}{(1-z_1^2)^2(1-z_2^2)^2(1-z_3^2)^2}\\ &\quad + \beta\left(\sum_{\mathrm{cyc}} \frac{1+4z_1z_2+z_1^2+z_2^2+z_1^2z_2^2}{(1-z_1^2)^2(1-z_2^2)^2 z_3}\right)\\ &\quad + \beta^2\left(\frac{\begin{array}{c}1+16z_1^2z_2^2z_3^2+z_1^4z_2^4z_3^4+\sum_{\mathrm{cyc}} z_1^4\\ -4z_1^2z_2^2+z_1^4z_2^4-4z_1^4z_2^2z_3^2\end{array}}{z_1z_2z_3(1-z_1^2)^2(1-z_2^2)^2(1-z_3^2)^2}\right).\end{aligned}$$

□

In the proof of Proposition 10.9, we found an expression for $\mathfrak{f}^G_{0,1}(x_1;\alpha)$ by rewriting the sum as one involving $f^G_{0,1}(x_1\alpha^{-1/2})$. We can use a similar trick in general to write each $\mathfrak{f}$ in terms of an $\mathbf{f}$.

Proposition 10.11. *For any* $g \geq 0$ *and* $n \geq 1$,

$$\begin{aligned}\mathfrak{f}^N_{g,n}(z_1,\ldots,z_n;\alpha) &= \alpha^{2-2g-\frac{n}{2}}\mathbf{f}^N_{g,n}(z_1\alpha^{1/2},\ldots,z_n\alpha^{1/2};\alpha)\\ \mathfrak{f}^G_{g,n}(x_1,\ldots,x_n;\alpha) &= \alpha^{2-2g-\frac{3n}{2}}\mathbf{f}^G_{g,n}(x_1\alpha^{-1/2},\ldots,x_n\alpha^{-1/2};\alpha)\end{aligned}$$

Note that the "usual" inputs to $\mathbf{f}^N_{g,n}$ are $(z_1,\ldots,z_n;\beta)$; we are saying that if we replace each z_i with $z_i\alpha^{1/2}$, and β with α, then up to a factor of $\alpha^{2-2g-\frac{n}{2}}$, we recover $\mathfrak{f}^N_{g,n}(z_1,\ldots,z_n;\alpha)$. Similarly, if we substitute z_i with $z_i\alpha^{-1/2}$ and β with α in $\mathbf{f}^G_{g,n}$, then we can recover $\mathfrak{f}^G_{g,n}$.

Thus, generating functions with respect to the number of regions r, and the variable α, can be recovered from generating functions with respect to the parameter t, and the variable β.

Proof. We compute

$$\begin{aligned}\alpha^{2-2g-\frac{n}{2}}\mathbf{f}^N_{g,n}(z_1\alpha^{1/2},\ldots,z_n\alpha^{1/2};\alpha) &= \alpha^{2-2g-\frac{n}{2}} \\ &\quad\times\sum_{t,\nu_1,\ldots,\nu_n} N^t_{g,n}(\nu_1,\ldots,\nu_n)(z_1\alpha^{1/2})^{\nu_1-1}\cdots(z_n\alpha^{1/2})^{\nu_n-1}\,\alpha^t \\ &= \sum_{t,\nu_1,\ldots,\nu_n} N^t_{g,n}(\nu_1,\ldots,\nu_n)z_1^{\nu_1-1}\cdots z_n^{\nu_n-1}\,\alpha^{2-2g-\frac{n}{2}+\frac{1}{2}\sum_{i=1}^n(\nu_i-1)} \\ &= \mathfrak{f}^N_{g,n}(z_1,\ldots,z_n;\alpha).\end{aligned}$$

Here, we used $r = t-(2-2g-n)-\frac{1}{2}\sum_{i=1}^n \nu_i$. The computation for the second equality is similar. □

Since $\mathbf{f}^N_{g,n}(z_1,\ldots,z_n;\beta)$ and $\mathbf{f}^G_{g,n}(x_1,\ldots,x_n;\beta)$ is meromorphic, the above proposition immediately yields the following.

Proposition 10.12. *The functions $\mathfrak{f}^G_{g,n}(x_1,\ldots,x_n;\alpha)$ and $\mathfrak{f}^N_{g,n}(z_1,\ldots,z_n;\alpha)$ are locally meromorphic, and for each $\alpha\in\mathbb{C}$, $\Omega^G_{g,n}(x_1,\ldots,x_n;\alpha)$ and $\Omega^N_{g,n}(z_1,\ldots,z_n;\alpha)$ are locally meromorphic differential forms.* □

10.6. *Refined differential equations*

We now return to the attempt to find differential equations satisfied by the generating functions $f_{g,n}(x_1,\ldots,x_n)$, which we left off in Sec. 8.6, and prove Proposition 1.17.

Recall in Sec. 8.6 that we took the recursion on $G_{g,n}(b_1,\ldots,b_n)$, multiplied by $x_1^{-b_1-1}\cdots x_n^{-b_n-1}$, and summed over all $b_1\geq 1$ and $b_2,\ldots,b_n\geq 0$. After suitable manipulation of the three terms I, II, III on the right-hand side, we arrived at Lemma 8.6. The problem was dealing with the terms on the left-hand side with $b_1=0$.

To this end, we start again, refining the process by regions. So, we start from the recursion, on $G_{g,n,r}$ rather than $G_{g,n}$, applying Theorem 1.11. Multiplying that recursion by $x_1^{-b_1-1}\cdots x_n^{-b_n-1}\alpha^r$ and summing over $r>1$, $b_1>0$ and $b_2,\ldots,b_n\geq 0$, we obtain

$$\sum_{\substack{b_1\geq 1\\ b_2,\ldots,b_n\geq 0\\ r\geq 1}} G_{g,n}(b_1,\ldots,b_n)\,x_1^{-b_1-1}\cdots x_n^{-b_n-1}\alpha^r = I_\alpha+II_\alpha+III_\alpha,$$

where the left-hand side is "almost" $\mathfrak{f}^G_{g,n}(x_1,\ldots,x_n;\alpha)$ (except for terms with $b_1=0$), and

$$I_\alpha = \sum_{\substack{b_1\geq 1\\ b_2,\ldots,b_n\geq 0\\ r\geq 1}}\ \sum_{\substack{i,j\geq 0\\ i+j=b_1-2}} G_{g-1,n+1,r}(i,j,b_2,\ldots,b_n)x_1^{-b_1-1}\cdots x_n^{-b_n-1}\alpha^r,$$

$$II_\alpha = \sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0 \\ r \geq 1}} \sum_{k=2}^{n} b_k G_{g,n-1}(b_1 + b_k - 2, b_2, \ldots, \widehat{b}_k, \ldots, b_n) x_1^{-b_1-1} \cdots x_n^{-b_n-1} \alpha^r,$$

$$III_\alpha = \sum_{\substack{b_1 \geq 1 \\ b_2,\ldots,b_n \geq 0 \\ r \geq 1}} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \sum_{\substack{i,j \geq 0 \\ i+j=b_1-2}} \times \sum_{\substack{r_1,r_2 \geq 1 \\ r_1+r_2=r}} G_{g_1,|I_1|+1,r_1}(i, b_{I_1}) G_{g_2,|I_2|+1,r_2}(j, b_{I_2}) x_1^{-b_1-1} \cdots x_n^{-b_n-1} \alpha^r.$$

Proof of Proposition 1.17. The computations of Sec. 8.6 work equally here to simplify terms $I_\alpha, II_\alpha III_\alpha$. The only difference is that a factor of α^r is carried throughout; and in III_α, we have $\alpha^r = \alpha^{r_1}\alpha^{r_2}$, so we obtain a similar factorisation. We obtain

$$I_\alpha = x_1^{-1} \mathfrak{f}^G_{g-1,n+1}(x_1, x_1, x_2, \ldots, x_n; \alpha)$$

$$II_\alpha = x_1^{-1} \sum_{k=2}^{n} \frac{\partial}{\partial x_k} \frac{1}{x_k - x_1} (\mathfrak{f}^G_{g,n-1}(x_2, \ldots, x_n; \alpha) - \mathfrak{f}^G_{g,n-1}(x_1, x_2, \ldots, \widehat{x_k}, \ldots, x_n; \alpha))$$

$$III_\alpha = x_1^{-1} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \mathfrak{f}^G_{g_1,|I_1|+1}(x_1, x_{I_1}; \alpha)\, \mathfrak{f}^G_{g_2,|I_2|+1}(x_1, x_{I_2}; \alpha).$$

For the rest of this section, we simply write $\mathfrak{f}_{g,n}$ rather than $\mathfrak{f}^G_{g,n}$ to avoid clutter; we will not be writing $\mathfrak{f}^N_{g,n}$, so there will be no ambiguity. If we examine $\mathfrak{f}_{g,n}$, we find

$$\begin{aligned} \mathfrak{f}_{g,n}(x_1, \ldots, x_n; \alpha) &= \sum_{\substack{b_1,\ldots,b_n \geq 0 \\ r \geq 1}} G_{g,n,r}(b_1, \ldots, b_n)\, x_1^{-b_1-1} \cdots x_n^{-b_n-1}\, \alpha^r \\ &= I_\alpha + II_\alpha + III_\alpha + IV_\alpha, \end{aligned}$$

where IV_α is the sum arising from terms with $b_1 = 0$:

$$IV_\alpha = \sum_{\substack{b_2,\ldots,b_n \geq 0 \\ r \geq 1}} G_{g,n,r}(0, b_2, \ldots, b_n) x_1^{-1} x_2^{-b_1-1} \cdots x_n^{-b_n-1}\, \alpha^r.$$

Applying Proposition 9.4 to IV_α, we obtain

$$IV_\alpha = \sum_{\substack{b_2,\ldots,b_n \geq 0 \\ r \geq 1}} r\, G_{g,n-1,r}(b_2, \ldots, b_n)\, x_1^{-1} x_2^{-b_2-1} \cdots x_n^{-b_n-1} \alpha^r$$

$$= x_1^{-1} \alpha \frac{d}{d\alpha} \sum_{\substack{b_2,\ldots,b_n \geq 0 \\ r \geq 1}} G_{g,n-1,r}(b_2,\ldots,b_n)\, x_2^{-b_2-1} \cdots x_n^{-b_n-1} \alpha^r$$

$$= x_1^{-1} \alpha \frac{d}{d\alpha} \mathfrak{f}_{g,n-1}(x_2,\ldots,x_n;\alpha).$$

Putting I_α through IV_α all together yields the desired differential equation. □

10.7. *Differential equation in free energies, and quantum curve?*

We now integrate the differential equation of Proposition 1.17 to obtain a differential equation on a free energy. We obtained free energies by integrating the form $\omega_{g,n}^t$; it is this form, rather than $\omega_{g,n,r}$, which was natural. Similarly, it is the $\mathbf{f}_{g,n}^G$ and $\mathbf{f}_{g,n}^N$, which produce the natural differential form $\Omega_{g,n}$. However, as we have seen, a nice recursion can be obtained on $G_{g,n,r}$, and from it we have derived a differential equation for $\mathfrak{f}_{g,n}^G$. If we integrate *this* function, we obtain another set of "free energies" which obey a differential equation. In this, we follow Mulase–Sułkowski [29].

We therefore define $\mathfrak{F}_{g,n}(x_1,\ldots,x_n;\alpha)$ to be a *free energy* if

$$\frac{\partial^n \mathfrak{F}}{\partial x_1 \cdots \partial x_n} = \mathfrak{f}_{g,n}^G(x_1,\ldots,x_n;\alpha).$$

Proof of Theorem 1.18. Integrate both sides of the equation in Proposition 1.17 with respect to $x_2,\ldots,x_n$. From $\mathfrak{f}_{g,n}(x_1,\ldots,x_n;\alpha)$, we obtain $\frac{\partial}{\partial x_1}\mathfrak{F}(x_1,\ldots,x_n;\alpha)$; and similarly, we obtain the desired terms on the right-hand side. □

We now assemble the ingredients for a *partition function.*

Definition 10.13. For integers $m \geq 0$, define

$$S_m(x) = \sum_{2g+n-1=m} \frac{1}{n!} \mathfrak{F}_{g,n}(x,\ldots,x), \quad \mathbf{F} = \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \quad \text{and} \quad \mathbf{Z} = e^{\mathbf{F}}.$$

Here, $\hbar$ is a formal parameter and we regard these as formal Laurent series. We refer to $\mathbf{F}$ as a logarithmic partition function, and $\mathbf{Z}$ as the partition function.

Lemma 10.14. *For each $m \geq 0$,*

$$x \frac{\partial}{\partial x} S_{m+1} = \frac{\partial^2 S_m}{\partial x^2} + \sum_{a+b=m+1} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x} + \alpha \frac{\partial S_m}{\partial \alpha}.$$

Proof. This proof follows the method of [29, Appendix A]. We drop α from $\mathfrak{F}_{g,n}(x_1,\ldots,x_n;\alpha)$ for convenience. From Theorem 1.18, set $x_1 = \cdots = x_n = x$, multiply by $\frac{1}{(n-1)!}$, and sum over all g, n such that $2g+n-2=m$. Taking the terms of the equation separately, we first have

$$\sum_{2g+n-2=m} \frac{1}{(n-1)!} x_1 \frac{\partial}{\partial x_1} \mathfrak{F}_{g,n}(x_1, \ldots, x_n)\Big|_{x_1, \ldots, x_n}$$
$$= x \frac{\partial}{\partial x} \sum_{2g+n-2=m} \frac{1}{n!} \mathfrak{F}_{g,n}(x, \ldots, x) = x \frac{\partial}{\partial x} S_{m+1}.$$

Here, we used the general fact that, for a symmetric function f of n variables,

$$\frac{d}{dt} f(t, \ldots, t) = n \frac{\partial}{\partial u} f(u, t, \ldots, t)\Big|_{u=t}.$$

Doing the same for the first term on the right-hand side, we obtain

$$\sum_{2g+n-2=m} \frac{1}{(n-1)!} \frac{\partial^2}{\partial u\, \partial v} \mathfrak{F}_{g-1,n+1}(u, v, x_2, \ldots, x_n)\Big|_{u=v=x_2=\cdots=x_n}$$
$$= \sum_{2g+n-2=m} \frac{1}{(n-1)!} \frac{\partial^2}{\partial u\, \partial v} \mathfrak{F}_{g-1,n+1}(u, v, x, \ldots, x)\Big|_{u=v=x}.$$

Turning to the second term on the right-hand side yields

$$\sum_{2g+n-2=m} \frac{1}{(n-1)!} \sum_{k=2}^{n} \frac{1}{x_k - x_1} \left(\frac{\partial}{\partial x_k} \mathfrak{F}_{g,n-1}(x_2, \ldots, x_n) \right.$$
$$\left. - \frac{\partial}{\partial x_1} \mathfrak{F}_{g,n-1}(x_1, \ldots, \widehat{x_k}, \ldots, x_n) \right) \Big|_{x_2=\cdots=x_n=x}$$
$$= \sum_{2g+n-2=m} \frac{1}{(n-1)!} \sum_{k=2}^{n} \frac{\partial^2}{\partial x^2} \mathfrak{F}_{g,n-1}(x, x_2, \ldots, \widehat{x_k}, \ldots, x_n)\Big|_{x_2=\cdots=\widehat{x_k}=\cdots=x_n=x}$$
$$= \sum_{2g+n-2=m} \frac{1}{(n-1)!} \sum_{k=2}^{n} \frac{\partial^2}{\partial u^2} \mathfrak{F}_{g,n-1}(u, x, \ldots, x)\Big|_{u=x}$$
$$= \sum_{2g+n-2=m} \frac{1}{(n-2)!} \frac{\partial^2}{\partial u^2} \mathfrak{F}_{g,n-1}(u, x, \ldots, x)\Big|_{u=x}.$$

In the second line, we used the general fact that for functions f and g,

$$\frac{1}{x-y} \left(g(x) \frac{df(x)}{dx} - g(y) \frac{df(y)}{dy} \right) \Big|_{x=y} = g'(x) f'(x) + g(x) f''(x).$$

Now adding the first and second terms on the right-hand side gives

$$\sum_{2g+n-2=m} \frac{1}{(n-1)!} \frac{\partial^2}{\partial u\, \partial v} \mathfrak{F}_{g-1,n+1}(u,v,x,\ldots,x)\Big|_{u=v=x}$$

$$+ \frac{1}{(n-2)!} \frac{\partial^2}{\partial u^2} \mathfrak{F}_{g,n-1}(u,x,\ldots,x)\Big|_{u=x}$$

$$= \sum_{2g+n-2=m} \frac{1}{n!} \frac{\partial^2}{\partial x^2} \mathfrak{F}_{g,n-1}(x,\ldots,x) = \frac{\partial^2 S_m}{\partial x^2}.$$

Here, we have used the general fact that, for a symmetric function f of n variables,

$$\frac{d^2}{dt^2} f(t,\ldots,t)$$

$$= n \frac{\partial^2}{\partial u^2} f(u,t,\ldots,t)\Big|_{u=t} + n(n-1) \frac{\partial^2}{\partial u_1\, \partial u_2} f(u_1,u_2,t,\ldots,t)\Big|_{u_1=u_2=t}.$$

For the final term, we find

$$\sum_{2g+n-2=m} \frac{1}{(n-1)!}$$

$$\times \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} \frac{\partial}{\partial x_1} \mathfrak{F}_{g_1,|I_1|+1}(x_1, x_{I_1}) \frac{\partial}{\partial x_1} \mathfrak{F}_{g_2,|I_2|+1}(x_1, x_{I_2})\Big|_{x_1=\cdots=x_n=x}$$

$$= \sum_{2g+n-2=m} \frac{1}{(n-1)!} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n-1}} \binom{n-1}{n_1} \frac{\partial}{\partial x_1} \mathfrak{F}_{g_1,n_1+1}$$

$$\times (x_1,x,\ldots,x) \frac{\partial}{\partial x_1} \mathfrak{F}_{g_2,n_2+1}(x_1,x,\ldots,x)\Big|_{x_1=x}$$

$$= \sum_{2g+n-2=m} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n-1}} \frac{1}{n_1!} \frac{\partial}{\partial x_1} \mathfrak{F}_{g_1,n_1+1}$$

$$\times (x_1,x,\ldots,x) \frac{1}{n_2!} \frac{\partial}{\partial x_1} \mathfrak{F}_{g_2,n_2+1}(x_1,x,\ldots,x)\Big|_{x_1=x}$$

$$= \sum_{a+b=m+1} \left(\sum_{2g_1+n_1-2=a-2} \frac{1}{(n_1+1)!} \frac{\partial}{\partial x} \mathfrak{F}_{g_1,n_1+1}(x,\ldots,x) \right)$$

$$\times \left(\sum_{2g_2+m_2=2=b-2} \frac{1}{(n_2+1)!} \frac{\partial}{\partial x} \mathfrak{F}_{g_2,n_2+1}(x,\ldots,x) \right)$$

$$= \sum_{a+b=m+1} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x}.$$

Adding together all the terms then gives the desired result. $\square$

Next, we find a differential equation satisfied by the logarithmic partition function $\mathbf{F}$.

Proposition 10.15. *The function* $\mathbf{F}$ *satisfies*

$$\hbar^2\left(\frac{\partial^2\mathbf{F}}{\partial x^2}+\left(\frac{\partial\mathbf{F}}{\partial x}\right)^2+\alpha\frac{\partial\mathbf{F}}{\partial\alpha}\right)-\hbar x\frac{\partial\mathbf{F}}{\partial x}+\alpha=0.$$

Proof. Take Lemma 10.14, multiply by $\hbar^{m+1}$ and sum over $m\geq 0$. The left-hand side becomes

$$\sum_{m=0}^{\infty}x\frac{\partial S_{m+1}}{\partial x}\hbar^{m+1}=x\hbar\frac{\partial\mathbf{F}}{\partial x}-x\frac{\partial S_0}{\partial x}.$$

The first term on the right-hand side becomes

$$\sum_{m=0}^{\infty}\hbar^{m+1}\frac{\partial^2 S_m}{\partial x^2}=\hbar^2\frac{\partial^2\mathbf{F}}{\partial x^2}.$$

The second term yields

$$\sum_{m=0}^{\infty}\sum_{a+b=m+1}\hbar^{m+1}\frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x}=\sum_{a+b\geq 1}\hbar^{a}\frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x}$$

$$=\left(\sum_{a=0}^{\infty}\hbar^a\frac{\partial S_a}{\partial x}\right)\left(\sum_{b=0}^{\infty}\hbar^b\frac{\partial S_b}{\partial x}\right)-\left(\frac{\partial S_0}{\partial x}\right)^2=\hbar^2\left(\frac{\partial\mathbf{F}}{\partial x}\right)^2-\left(\frac{\partial S_0}{\partial x}\right)^2.$$

The final term gives

$$\sum_{m=0}^{\infty}\hbar^{m+1}\alpha\frac{\partial S_m}{\partial\alpha}=\hbar^2\alpha\frac{\partial\mathbf{F}}{\partial\alpha}.$$

Summing the terms and rearranging, then gives

$$\hbar^2\frac{\partial^2\mathbf{F}}{\partial x^2}+\hbar^2\left(\frac{\partial\mathbf{F}}{\partial x}\right)^2+\hbar^2\alpha\frac{\partial\mathbf{F}}{\partial\alpha}-x\hbar\frac{\partial\mathbf{F}}{\partial x}+x\frac{\partial S_0}{\partial x}-\left(\frac{\partial S_0}{\partial x}\right)^2=0.$$

It remains to compute the S_0 terms. Now $S_0(x)=\mathfrak{F}_{0,1}(x)$, which is an integral of $\mathfrak{f}^G_{0,1}(x_1,\ldots,x_n;\alpha)$. Using Proposition 10.9,

$$\frac{\partial S_0}{\partial x}=\mathfrak{f}^G_{0,1}(x;\alpha)=\frac{x-\sqrt{x^2-4\alpha}}{2},\quad\text{so that}\quad x\frac{\partial S_0}{\partial x}-\left(\frac{\partial S_0}{\partial x}\right)^2=\alpha$$

giving the desired result. □

Finally, we obtain a differential equation satisfied by the partition function $\mathbf{Z}$. This is reminiscent of the "quantum curve" that appears in the general theory of the topological recursion [29, 30].

Proof of Theorem 1.19. Since $\mathbf{Z}=e^{\mathbf{F}}$, we have $\frac{\partial\mathbf{Z}}{\partial x}=\frac{\partial\mathbf{F}}{\partial x}\mathbf{Z}$, so $\frac{\partial^2\mathbf{Z}}{\partial x^2}=(\frac{\partial^2\mathbf{F}}{\partial x^2}+(\frac{\partial\mathbf{F}}{\partial x})^2)\mathbf{Z}$. Also $\frac{\partial\mathbf{Z}}{\partial\alpha}=\frac{\partial\mathbf{F}}{\partial\alpha}\mathbf{Z}$. Substituting these into Proposition 10.15 gives the claimed result. □

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References

[1] G. Borot, Blobbed topological recursion, *Theoret. and Math. Phys.* **185**(3) (2015) 1729–1740.

[2] V. Bouchard, D. Hernández Serrano, X. Liu and M. Mulase, Mirror symmetry for orbifold Hurwitz numbers, *J. Differential Geom.* **98**(3) (2014) 375–423.

[3] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, Remodeling the B-model, *Comm. Math. Phys.* **287**(1) (2009) 117–178.

[4] V. Bouchard and M. Mariño, Hurwitz numbers, matrix models and enumerative geometry, in *From Hodge Theory to Integrability and TQFT tt*-Geometry*, *Proceedings of Symposia in Pure Mathematics*, Vol. 78 (American Mathematical Society, Providence, RI, 2008), pp. 263–283.

[5] M. Brion and M. Vergne, Lattice points in simple polytopes, *J. Amer. Math. Soc.* **10**(2) (1997) 371–392.

[6] L. Chekhov and B. Eynard, Hermitian matrix model free energy: Feynman graph technique for all genera, *J. High Energy Phys.* **14**(3) (2006), 18 pp. (electronic).

[7] N. Do, O. Leigh and P. Norbury, Orbifold Hurwitz numbers and Eynard–Orantin invariants, to appear *Math. Res. Lett.* (2015).

[8] N. Do and D. Manescu, Quantum curves for the enumeration of ribbon graphs and hypermaps, *Commun. Number Theory Phys.* **8**(4) (2014) 677–701.

[9] N. Do and P. Norbury, Counting lattice points in compactified moduli spaces of curves, *Geom. Topol.* **15**(4) (2011) 2321–2350.

[10] N. Do and P. Norbury, Pruned Hurwitz numbers, http://arXiv.org/abs/1312.7516, 2013. To appear in the *Transactions of the American Mathematical Society*.

[11] P. Drube and P. Pongtanapaisan, Annular non-crossing matchings, *J. Integer Seq.* **19**(2) (2016), Articl 16.2.4, 17.

[12] O. Dumitrescu, M. Mulase, B. Safnuk and A. Sorkin, The spectral curve of the Eynard–Orantin recursion via the Laplace transform, in *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices*, Contemporary of Mathematics, Vol. 593 (American Mathematical Society, Providence, RI, 2013), pp. 263–315.

[13] P. Dunin-Barkowski, N. Orantin, S. Shadrin and L. Spitz, Identification of the Givental formula with the spectral curve topological recursion procedure, *Comm. Math. Phys.* **328**(2) (2014) 669–700.

[14] P. Dunin-Barkowski, N. Orantin, A. Popolitov and S. Shadrin, Combinatorics of loop equations for branched covers of sphere, http://arXiv.org/abs/1412.1698, 2014.

[15] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion, *Commun. Number Theory Phys.* **1**(2) (2007) 347–452.

[16] B. Eynard and N. Orantin, Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture, *Comm. Math. Phys.* **337**(2) (2015) 483–567.

[17] B. Eynard, M. Mulase and B. Safnuk, The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers, *Publ. Res. Inst. Math. Sci.* **47**(2) (2011) 629–670.

[18] B. Eynard and N. Orantin, Topological recursion in enumerative geometry and random matrices, *J. Phys. A* **42**(29) (2009) 293001, 117.

[19] B. Fang, C.-C. Melissa Liu and Z. Zong, All genus mirror symmetry for toric Calabi–Yau 3-orbifolds, String-Math 2014, *Proc. Sympos. Pure Math.*, Vol. 93 (Amer. Math. Soc., Providence, RI, 2016), pp. 1–19.

[20] E. Giroux, Convexité en topologie de contact, *Comment. Math. Helv.* **66**(4) (1991) 637–677.

[21] J. Harris and I. Morrison, *Moduli of Curves*, Graduate Texts in Mathematics, Vol. 187 (Springer-Verlag, New York, 1998).

[22] K. Honda, On the classification of tight contact structures. I, *Geom. Topol.* **4** (2000) 309–368 (electronic).

[23] M. Kazarian and P. Zograf, Virasoro constraints and topological recursion for Grothendieck's dessin counting, *Lett. Math. Phys.* **105**(8) (2015) 1057–1084.

[24] J. S. Kim, Cyclic sieving phenomena on annular noncrossing permutations, *Sém. Lothar. Combin.* **69** (2012) Art. B69b, 20.

[25] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, *Comm. Math. Phys.* **147**(1) (1992) 1–23.

[26] D. V. Mathews, Chord diagrams, contact-topological quantum field theory, and contact categories, *Algebraic Geom. Topol.* **10**(4) (2010) 2091–2189.

[27] M. Mirzakhani, Weil-Petersson volumes and intersection theory on the moduli space of curves, *J. Amer. Math. Soc.* **20**(1) (2007) 1–23 (electronic).

[28] M. Mulase and M. Penkava, Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves, *Adv. Math.* **230**(3) (2012) 1322–1339.

[29] M. Mulase and P. Sułkowski, Spectral curves and the Schrödinger equations for the Eynard–Orantin recursion, *Adv. Theor. Math. Phys.* **19**(5) (2015) 955–1015.

[30] P. Norbury, Quantum curves and topological recursion, String-Math 2014, *Proc. Sympos. Pure Math.*, Vol. 93 (Amer. Math. Soc., Providence, RI, 2016), pp. 41–65.

[31] P. Norbury, Counting lattice points in the moduli space of curves, *Math. Res. Lett.* **17**(3) (2010) 467–481.

[32] P. Norbury, String and dilaton equations for counting lattice points in the moduli space of curves, *Trans. Amer. Math. Soc.* **365**(4) (2013) 1687–1709.

[33] P. Norbury and N. Scott, Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard–Orantin invariants, *Geom. Topol.* **18**(4) (2014) 1865–1910.

[34] J. H. Przytycki, Fundamentals of Kauffman bracket skein modules, *Kobe J. Math.* **16**(1) (1999) 45–66.

[35] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus. I, *J. Combin. Theory Ser. B* **13** (1972) 192–218.