Trinities, sutured Floer homology, and contact structures

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joint with Tamás Kálmán [2]

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Outline

Overview

This talk is about

- trinities — a type of *triality* arising in graph theory
- sutured Floer homology — an extension of Heegaard Floer homology to 3-manifolds with (sutured) boundary
- contact structures on 3-manifolds.

In progress / joint with T. Kálmán.
Building on work of Friedl, Juhasz, Kálmán, Rasmussen, Etnyre, Honda, Kazez, Ozsváth, Szabó, Witten, Floer ...

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This talk will:

- Briefly introduce background ideas
  - sutured 3-manifolds
  - Heegaard Floer homology, sutured Floer homology
  - contact structures & classifying them
  - trinities in graph theory

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  - contact structures & classifying them
  - trinities in graph theory
- Introduce certain triples of sutured 3-manifolds associated to trinities
- Discuss how they are in *triality*
  - isomorphic sutured Floer homology
  - bijections between contact structures
  - isomorphisms of related *polytopes*

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A sutured 3-manifold is balanced if $\chi(R_+) = \chi(R_-)$.

We're interested in some specific sutured 3-manifolds...

Heegaard Floer homology

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Given a closed 3-manifold *M*, essentially:

- Heegaard Floer homology of *M* is Lagrangian intersection Floer homology of a manifold constructed from a Heegaard decomposition (Σ, α, β) of *M*.
  - Σ a closed surface of genus g
  - $\alpha = \{\alpha_1, \ldots, \alpha_g\}$, $\beta = \{\beta_1, \ldots, \beta_g\}$ Heegaard curves.

Much of this extends to the case of a sutured 3-manifold.

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Powerful invariant, categorifies Alexander polynomial, computes genus of knots, etc...

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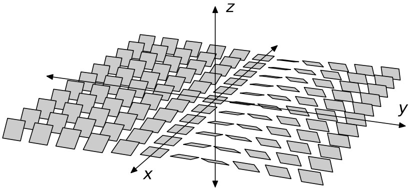
Generalises $\widehat{HF}$ in several cases:

- closed manifold
- knot Floer homology
- knots with Seifert surfaces

Contact structures

A contact structure on a 3-manifold *M* is a non-integrable 2-plane field $\xi = \ker \alpha$.



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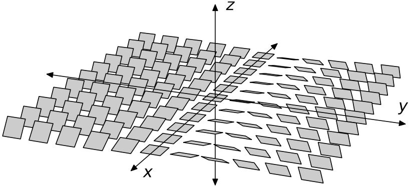

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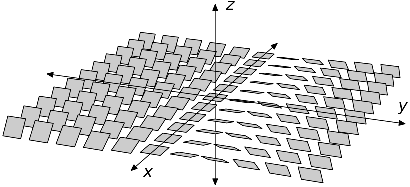

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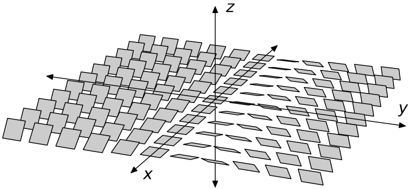

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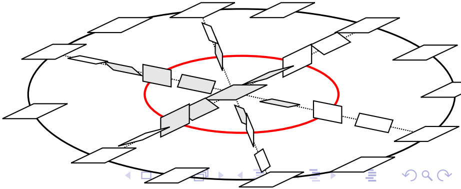
Two types of contact structures: tight and overtwisted.

An overtwisted contact structure is one containing an overtwisted disc:



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For 3-manifolds with boundary:

- natural boundary conditions for contact structures are given by sutures.
- Roughly, sutures Γ on ∂M prescribe a contact structure up to isotopy near ∂M.
- This is via Giroux's theory of convex surfaces (1991).

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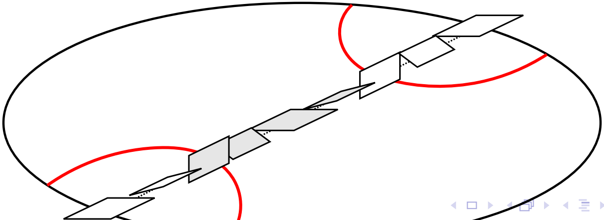

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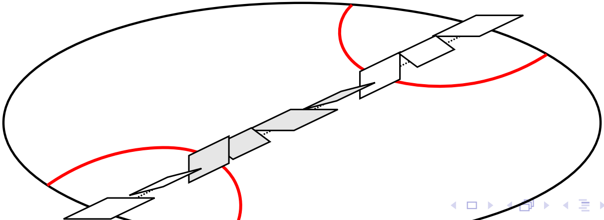
(Roughly: think of S as “horizontal", X as “vertical", Γ is “where ξ is vertical".)



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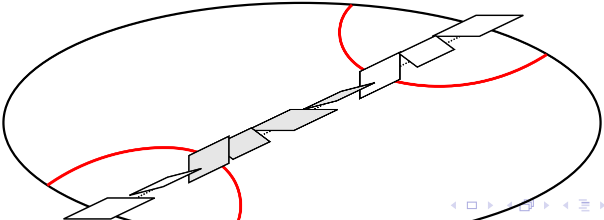

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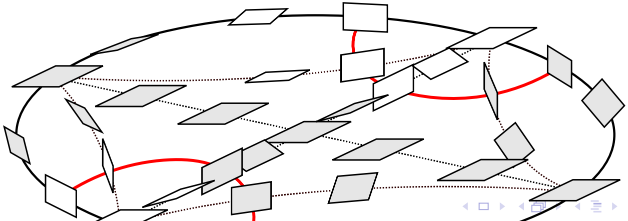
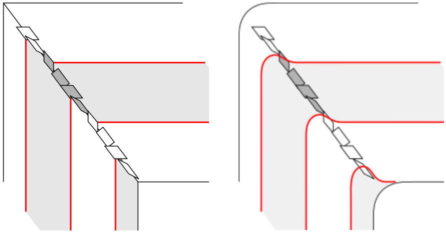


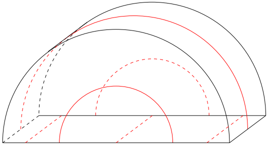
Contact corners

When two convex surfaces meet along a boundary, contact planes are arranged as shown.



Bypasses

Honda (2000): contact structures can be built up using a small contact 3-manifold called a *bypass* (=half an overtwisted disc).



"All topologically trivial contact topology is constructed from bypasses."

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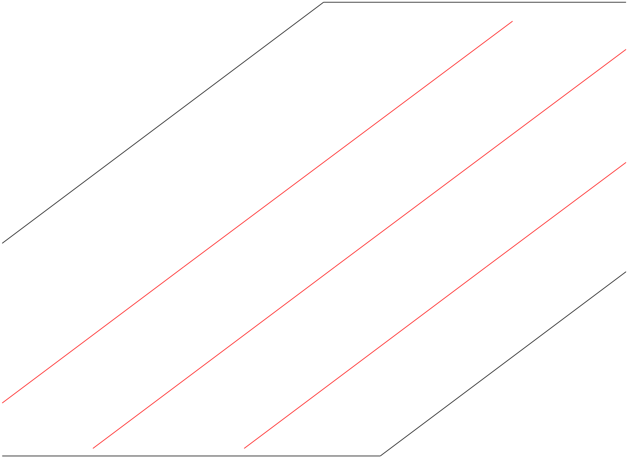

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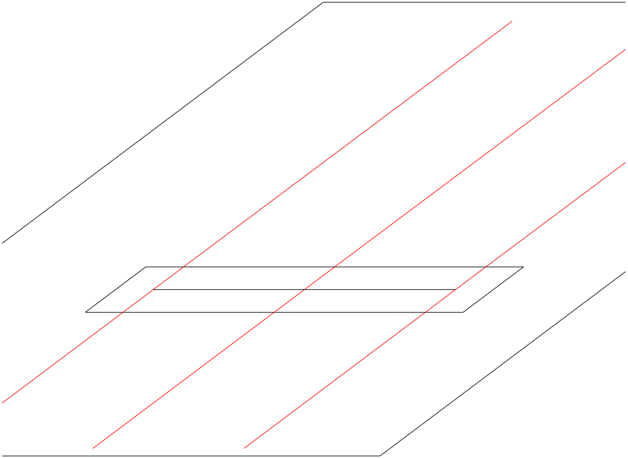

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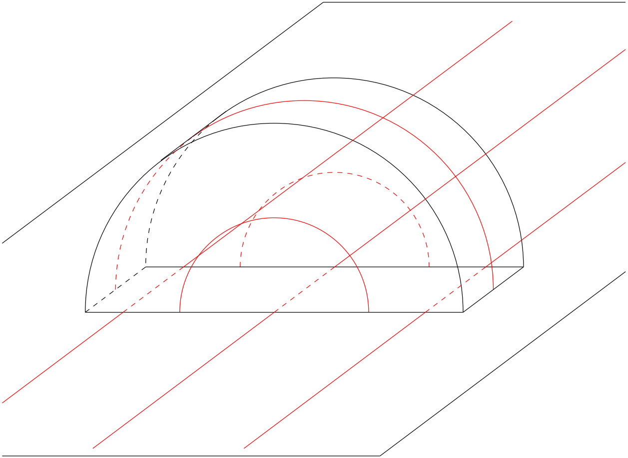

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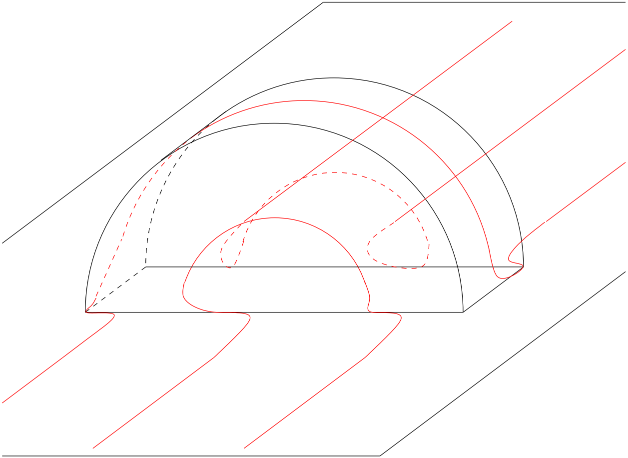

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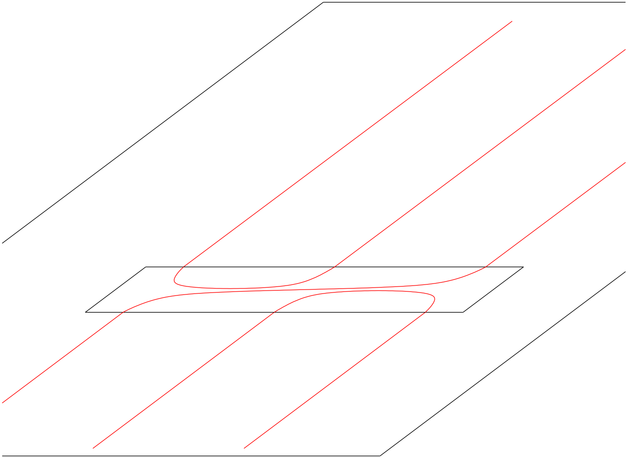

Contact invariants in Heegaard Floer homology

Heegaard Floer homology gives invariants of contact structures:

- ξ on closed $M \rightsquigarrow c(\xi) \in \widehat{HF}(-M)$.
- ξ on sutured $(M, \Gamma) \rightsquigarrow c(\xi) \in SFH(-M, -\Gamma)$.

(Ozsváth–Szazó, Honda–Kazez–Matić)
Can be defined via open book decompositions...

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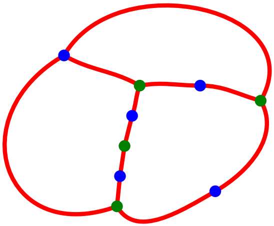

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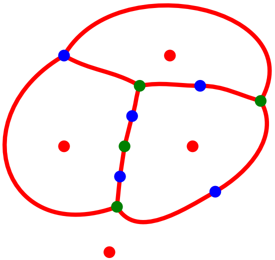

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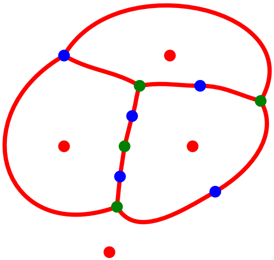

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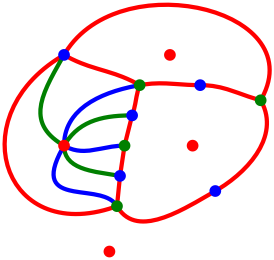

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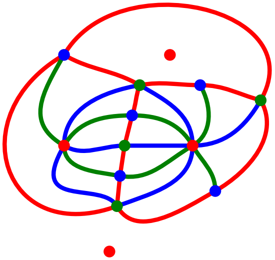

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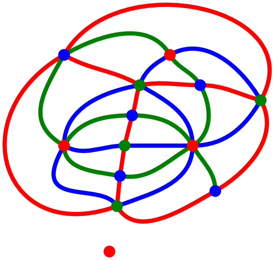

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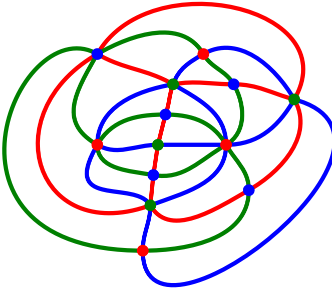

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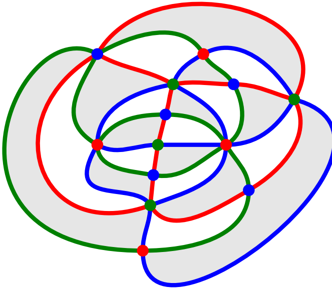

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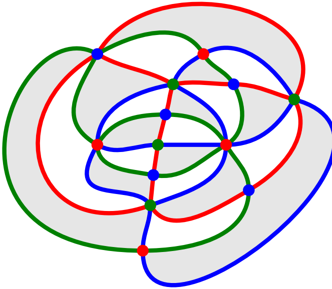

(Alternately can start from triangulation and colour in.)

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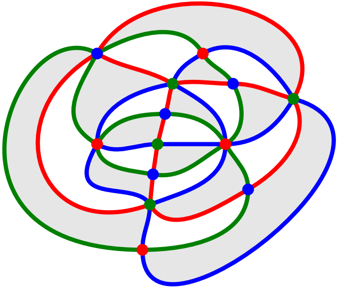

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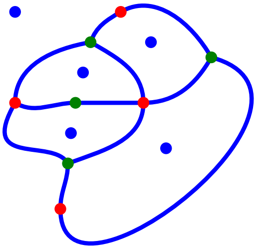

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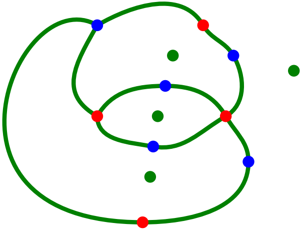

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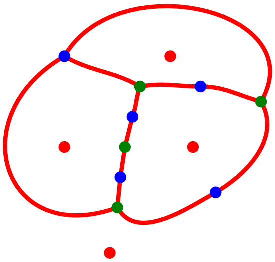

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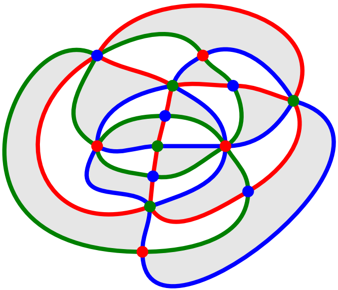

Trinities

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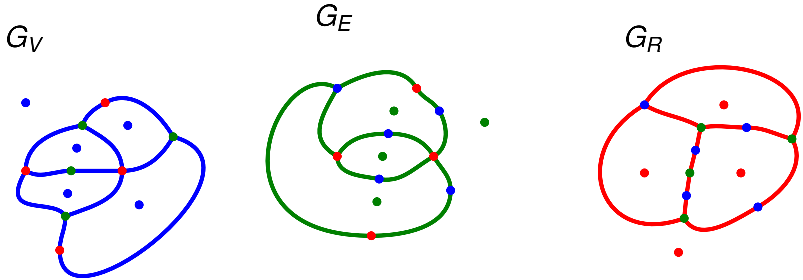

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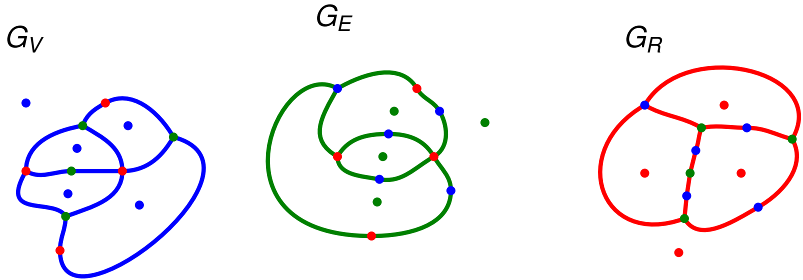

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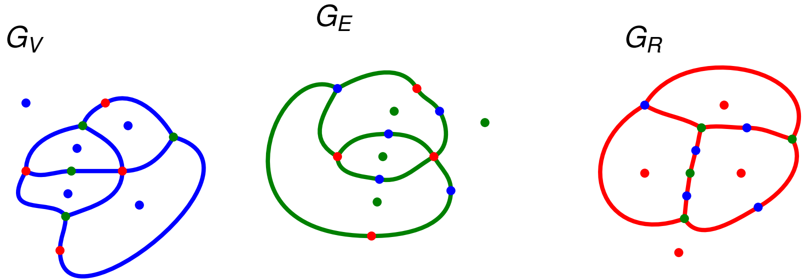

- Dual graphs are naturally oriented.
- Tutte's tree trinity theorem: G_V^*, G_E^*, G_R^* have same number ρ of spanning arborescences.
- Define this number as the arborescence number or magic number of the trinity.

Hypertrees

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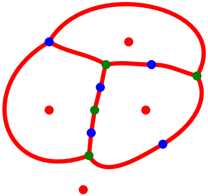

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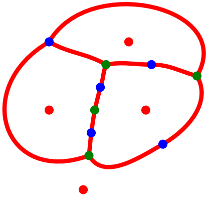

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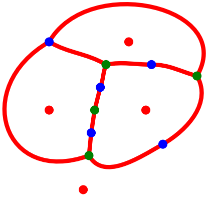

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A hypertree in $\mathcal{H} = (V, E)$ is a function $f : E \to \mathbb{N}_0$ such that $\exists$ a spanning tree in $\mathrm{Bip}\,\mathcal{H}$ with degree $f(e) + 1$ at each $e \in E$.

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In a trinity (V, E, R), there are naturally six hypergraphs:

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Theorem (Postnikov 2009, Kálmán 2013)

The number of hypertrees in any of these hypergraphs is equal to the magic number of the trinity:

$$\rho = |Q_{(V,E)}| = |Q_{(E,V)}| = |Q_{(E,R)}| = |Q_{(R,E)}| = |Q_{(R,V)}| = |Q_{(V,R)}|.$$

Outline

From bipartite planar graphs to sutured manifolds

Given a planar graph G, there is a natural way to construct a surface F_G bounding an alternating link L_G: the median construction.

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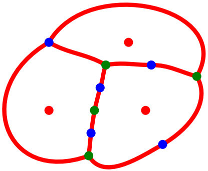

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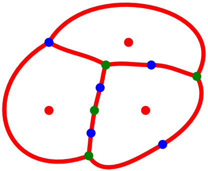

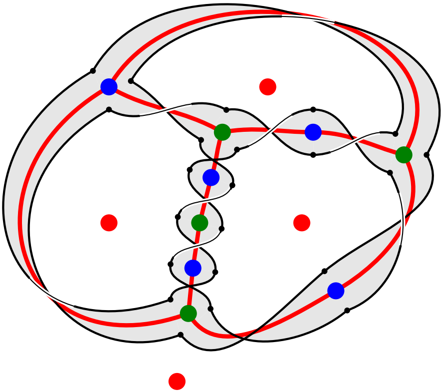

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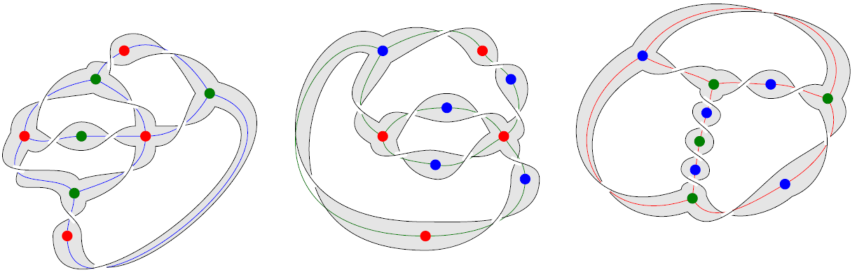
- In this case L_G is *special alternating* and F_G is a minimal genus Seifert surface (Murasugi 1958, Crowell 1959, Gabai 1986).
- In fact, any minimal genus Seifert surface of a non-split prime special alternating link arises as such an F_G (Hirasawa–Sakuma 1996, Banks 2011).

Trinity of sutured manifolds

From a trinity of bipartite graphs, we obtain a trinity of alternating links with Seifert surfaces



and sutured 3-manifolds

$$(S^3 - F_{G_V}, L_{G_V}), (S^3 - F_{G_E}, L_{G_E}), (S^3 - F_{G_R}, L_{G_R}).$$

Trinity of sutured manifolds

Question: For each of $(S^3 - F_{G_V}, L_{G_V})$, $(S^3 - F_{G_E}, L_{G_E})$, $(S^3 - F_{G_R}, L_{G_R})$:

1. What is *SFH*?
2. How many (isotopy classes of) tight contact structures ξ?
3. What are contact invariants $c(\xi) \in SFH$?

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- homological grading
- spin-c grading

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SFH decomposes over spin-c structures:

$$SFH(M, \Gamma) \cong \bigoplus_{\mathfrak{s} \in \text{Spin}^c(M, \Gamma)} SFH(M, \Gamma, \mathfrak{s}).$$

Spin-c decomposition of SFH of a trinity

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Thus to know SFH of a sutured L-space it is sufficient to know the support

$$S(M, \Gamma) = \{\mathfrak{s} \in \mathrm{Spin}^c(M, \Gamma) \ : \ SFH(M, \Gamma, \mathfrak{s}) \neq 0\}.$$

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Taking $G = G_R$ planar bipartite with vertex classes (V, E),

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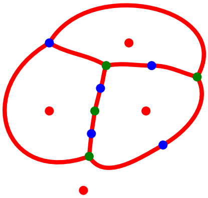

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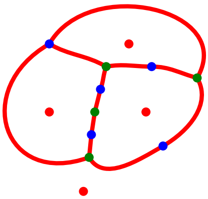

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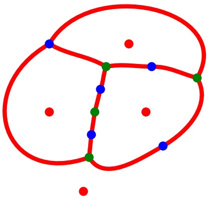

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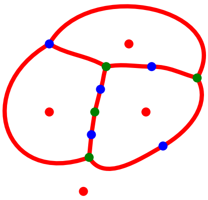

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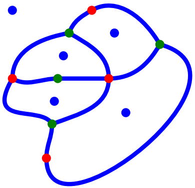

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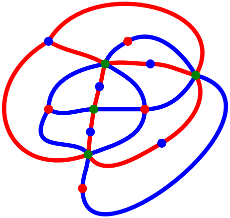

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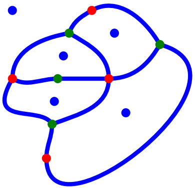

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Similarly, $S(S^3 - F_{G_R}, L_{G_R}) \cong -Q_{(V,R)}$.

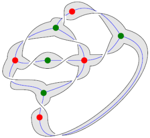
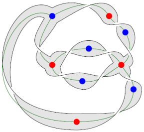
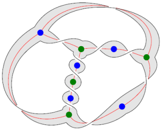
SFH of a trinity

It follows that

$$SFH(S^3 - F_{G_V}, L_{G_V}), \quad SFH(S^3 - F_{G_E}, L_{G_E}), \quad SFH(S^3 - F_{G_R}, L_{G_R})$$

all have dimension given by magic number, corresponding to

$$|Q_{(V,E)}| = |Q_{(E,V)}| = |Q_{(E,R)}| = |Q_{(R,E)}| = |Q_{(R,V)}| = |Q_{(V,R)}| = \rho.$$

SFH () ≅ SFH () ≅ SFH ().

Duality of polytopes

Postnikov related dual polytopes $Q_{(V,E)} \subset \mathbb{Z}^E$, $Q_{(E,V)} \subset \mathbb{Z}^V$.

$$Q_{(V,E)} = \left(\sum_{v \in V} \Delta_v \right) - \Delta_E = Q^+_{(V,E)} - \Delta_E,$$

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Essentially they are projections of Q, e.g.: $\pi_V : \mathbb{R}^V \oplus \mathbb{R}^E \to \mathbb{R}^V$

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Question

Does this root polytope have a symplectic or Floer-theoretic interpretation?

SFH of a trinity

Theorem (Juhász–Kálmán–Rasmussen)

$$S(S^3 - F_{G_R}, L_{G_R}) \cong Q_{(E,R)}$$

Ideas of proof:

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- This torsion can be calculated by using Fox calculus and the map $\pi_1(R_-) \to \pi_1(M)$.
- The Fox calculus yielding Turaev torsion of $(S^3 - F_{G_R}, L_{G_R})$ is equal to the determinant of a certain adjacency matrix for the trinity.
- Terms in this determinant are monomials corresponding to lattice points in the polytope $Q_{(E,R)}$.

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The number of isotopy classes of tight contact structures on $(S^3 - F_{G_R}, L_{G_R})$ is given by ρ, the magic number of the trinity. Moreover, there is precisely one isotopy class of tight contact structure in each Spin^c *class in the support $S(S^3 - F_G, L_G)$ of SFH.*

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Proof gives explicit bijections

$$\{\text{contact structures}\} \cong \{\text{hypertrees on } (E, R)\} \cong \{\text{Spin}^c \text{ structures}\},$$

and is constructive.

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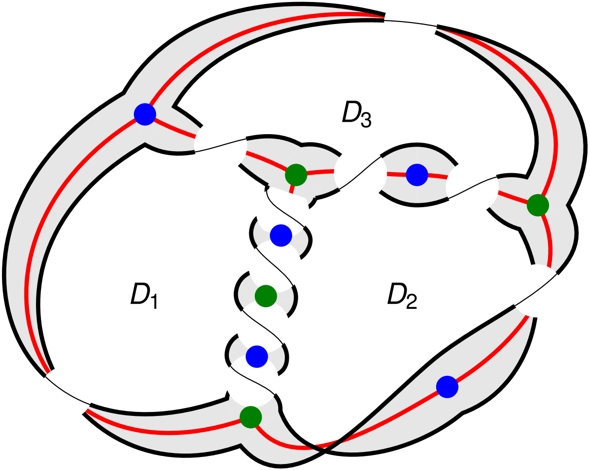
5. Use Kálmán's work: there are ρ hypertrees; two spanning trees representing same hypertree produce contact structures related by bypasses.

Contact structures on trinities

Consider complements of tubular neighbourhood of G:



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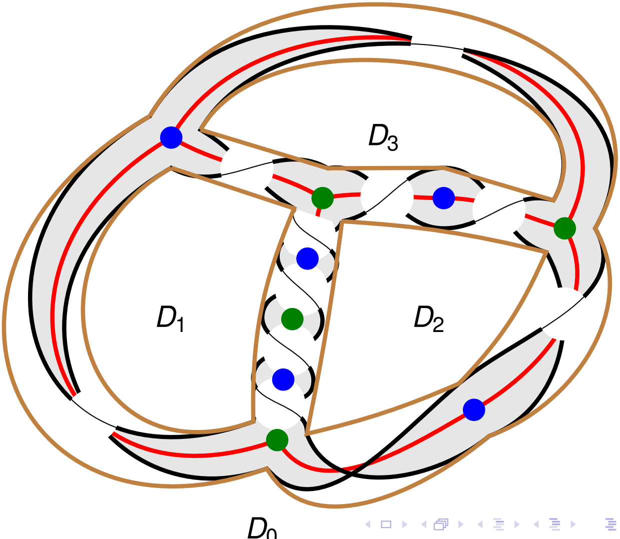

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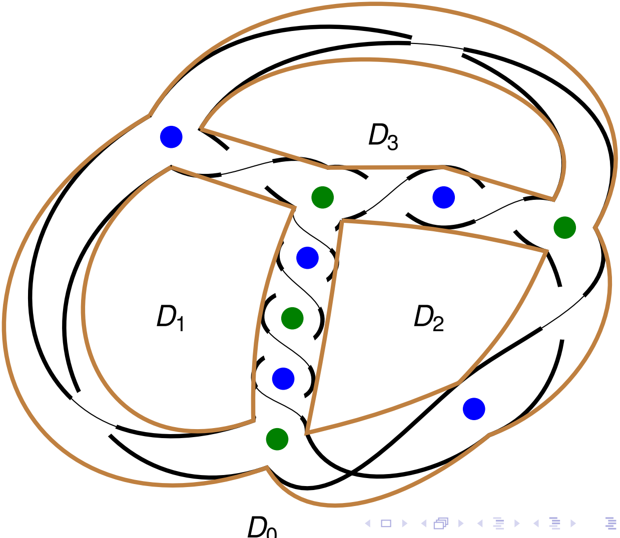

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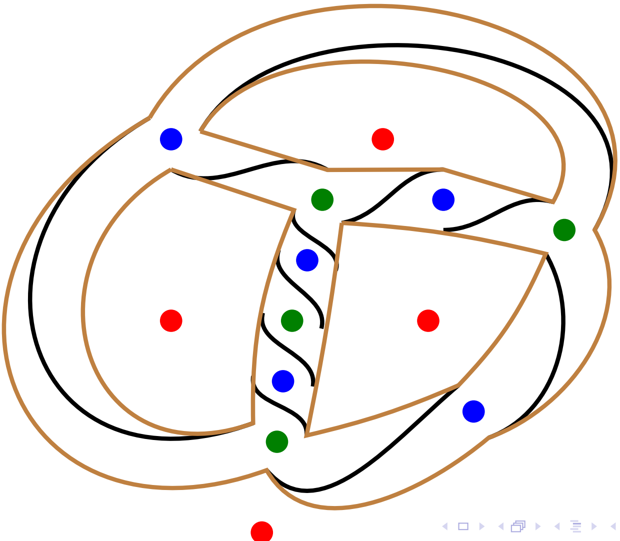

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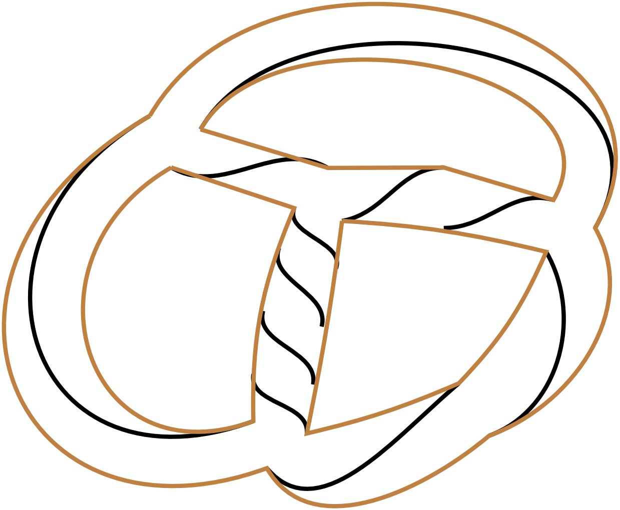

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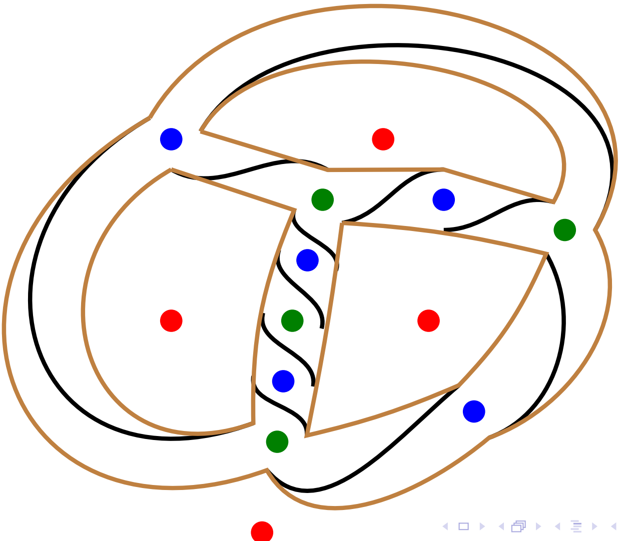

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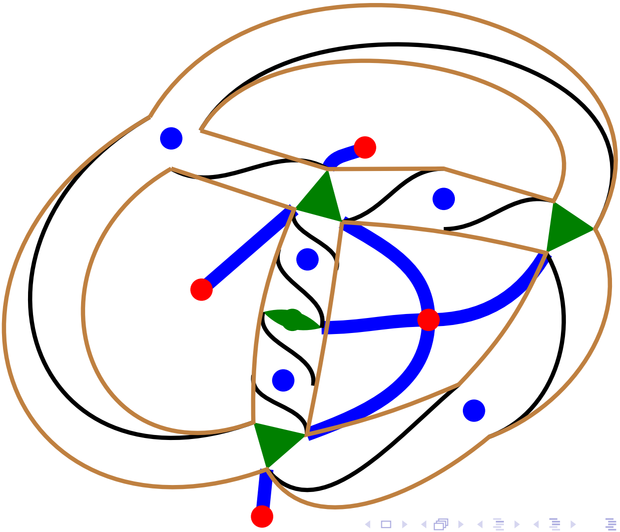

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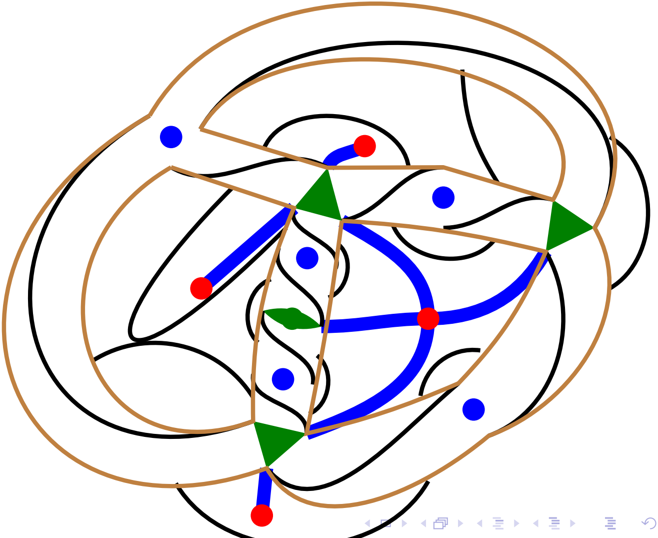

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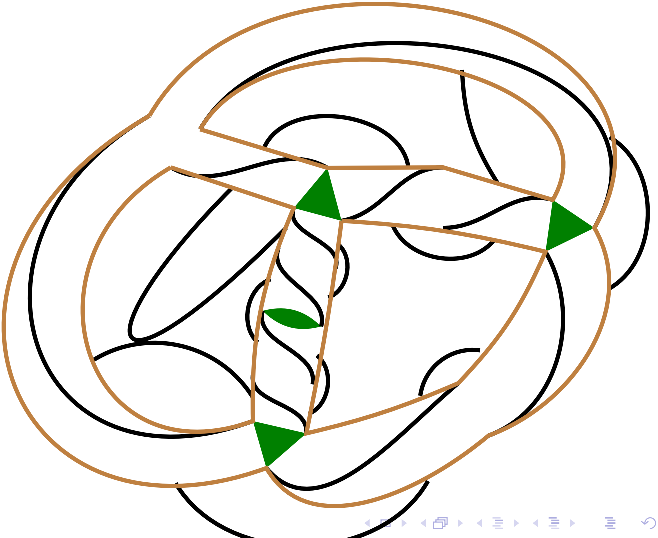

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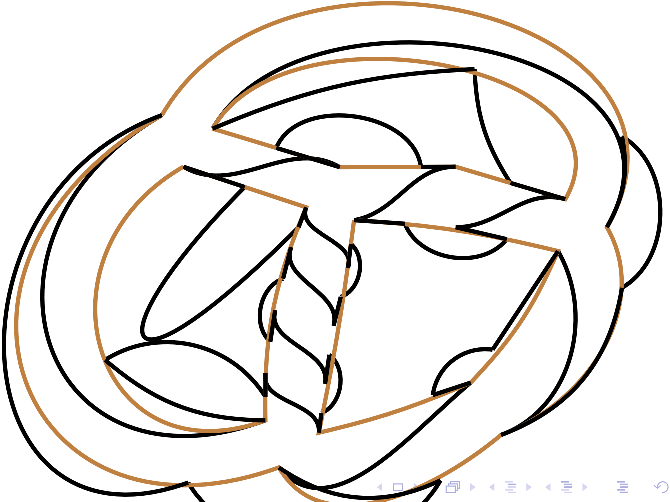

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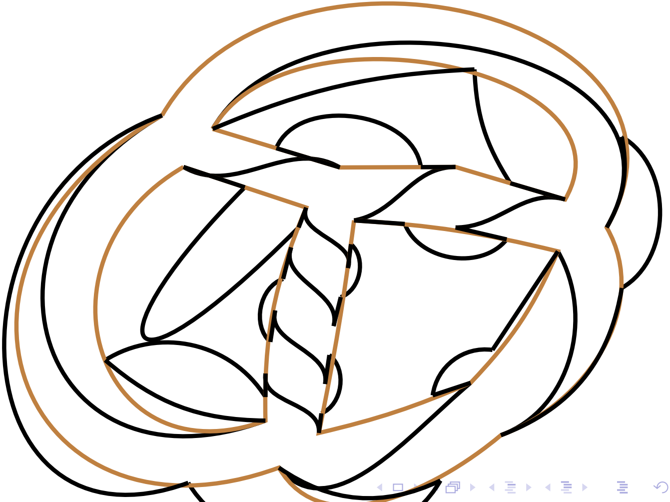

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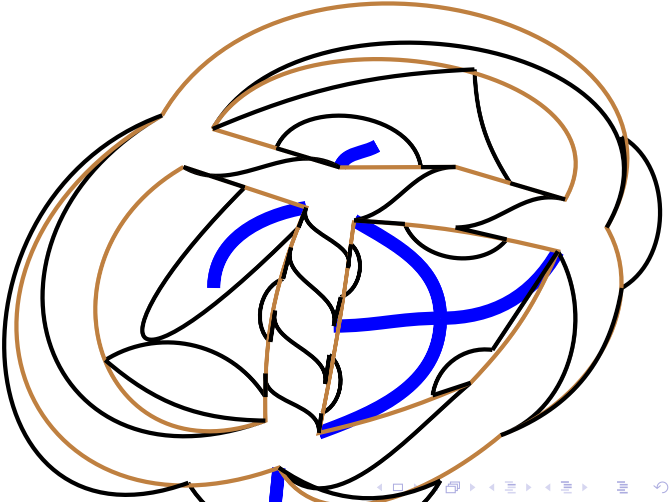

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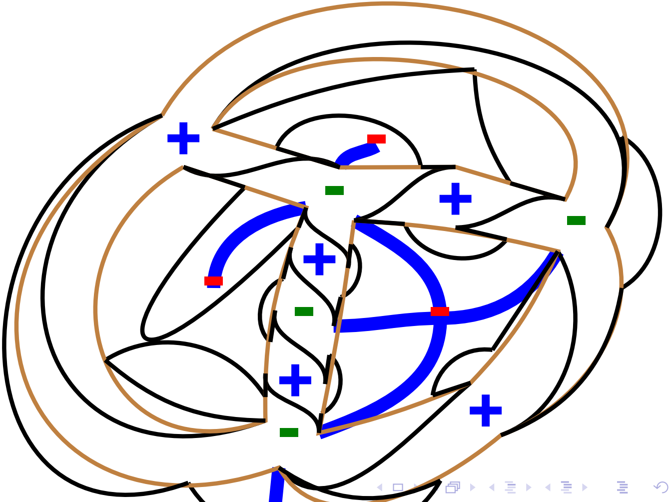

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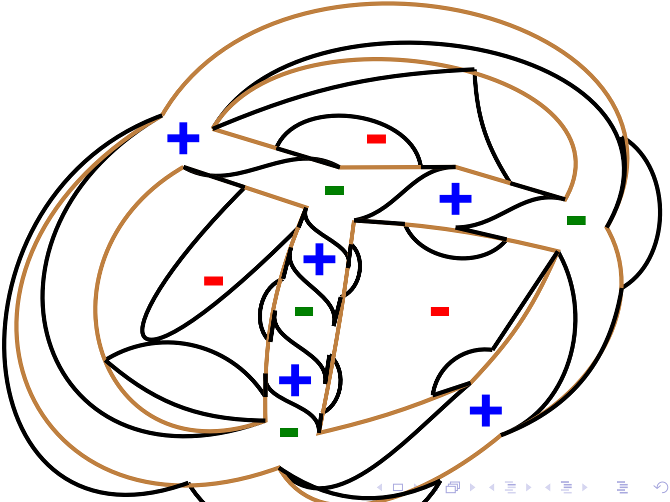

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- Each $\xi_\mathfrak{s}$ is the complement of a neighbourhood of a Legendrian graph in S^3.

Thanks for listening!