An introduction to contact geometry and topology

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Outline

Overview

An introduction to contact geometry and topology:

- What it is
- Background, fundamental results
- Some applications / “practical" examples
- Some areas of interest / research

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Standing assumptions/warnings:

- All manifolds are smooth, oriented, compact unless otherwise specified.
- All functions smooth unless otherwise specified
- Smooth $= C^\infty$
- Beware sign differences
- Biased!

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- A major area of research in contemporary low-dimensional geometry and topology
- Connected to many fields of mathematics:
  - symplectic geometry, Gromov-Witten theory, moduli spaces, quantum algebra, foliations, differential equations, mapping class groups, 3-and 4-manifolds, homotopy theory, homological algebra, category theory, knot theory...

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- Connected to many fields of physics:
    - classical mechanics, thermodynamics, optics, string theory, ice skating...

Contact geometry has a sibling: symplectic geometry.

Outline

Some differential geometry

Let M be a manifold.

Definition

A symplectic form on M is a closed 2-form ω such that ω^n is a volume form.
A contact form on M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form.

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- Insertion of vectors into ω yields an isomorphism between vectors and 1-forms, $v \leftrightarrow \iota_v \omega = \omega(v, \cdot)$.
- The kernel of a contact form $\xi = \ker \alpha$ is a codimension-1 plane field on M.

Integrability

Frobenius' theorem:

- $\alpha \wedge (d\alpha)^n \neq 0$ implies ξ is maximally non-integrable
  — any submanifold $S \subset M$ tangent to ξ must have dimension $\leq n$.

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A contact form co-orients ξ.

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Examples

Standard / “only" examples of symplectic & contact structures:

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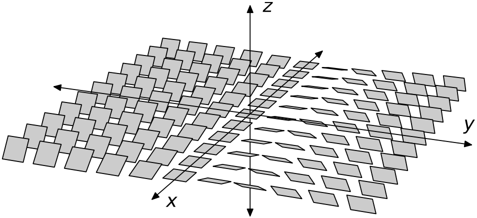

No tangent surfaces!
Only tangent/Legendrian curves! Legendrian knots and links.

Another example:

- Cylindrically symmetric standard contact structure on $\mathbb{R}^3$

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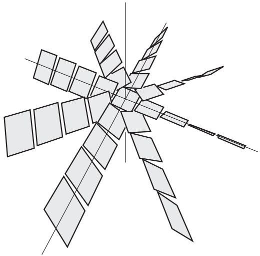

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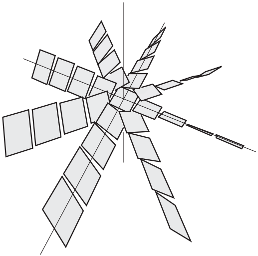


This is equivalent to the standard contact structure.

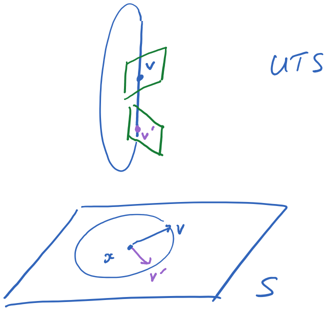
Definition

A contactomorphism is a diffeomorphism f between contact manifolds $(M_1, \xi_1) \longrightarrow (M_2, \xi_2)$ such that $f_* \xi_1 = \xi_2$.

Ice skating

The unit tangent bundle UTS of a smooth surface S has a canonical contact structure ξ:

- Take $(x, v) \in UTS$, with $x \in S$, $v \in T_x S$
- The contact plane there is spanned by v (“tautological direction") and the fibre direction.



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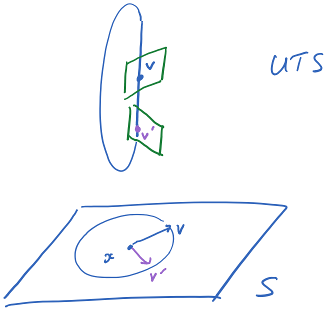

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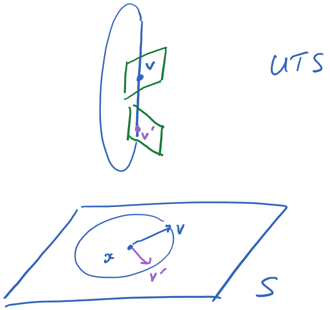

A smooth curve on *S* lifts uniquely to a Legendrian curve in *UTS*.
Ice skating on an ice rink *S*:

- Status of skater = (position of skater, direction of skates) $\in UTS$
- Ice skater's path is Legendrian iff she does not skid.

Similar: parking a car, rolling a suitcase.

Symplectic and contact examples

More examples:

Symplectic	Contact
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Not every 4-manifold has a symplectic structure	Not every 3-manifold has a tight contact structure

Hamiltonian mechanics

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Physics textbook version:

- State of a classical system given by coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$
- Set of states forms the phase space $\mathbb{R}^{2n}$
- Energy of states given by Hamiltonian function $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$.
- Hamilton's equations:

$$\dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}$$

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- The flow of X_H preserves ω and H:

$$L_{X_H}\omega = \iota_{X_H} d\omega + d\iota_{X_H}\omega = 0 + d(dH) = 0.$$

Dynamics on a contact manifold: Reeb vector field

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Proof.

$$L_{R_\alpha}\alpha = \iota_{R_\alpha} d\alpha + d\iota_{R_\alpha}\alpha = 0 + d1 = 0.$$

□

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Theorem

Let (M, ξ) be a contact 3-manifold with a contact form α. There is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Contact vector fields} \\ \text{on } (M,\xi) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Smooth functions} \\ H : M \longrightarrow \mathbb{R} \end{array} \right\}$$
$$X \mapsto \alpha(X)$$

Contact geometry as solving ODEs by geometry

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A first-order differential equation can be expressed as

$$F\left(x, z, \frac{dz}{dx}\right) = 0$$

for some smooth function $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and hence (generically) determines a *surface* S in $\mathbb{R}^3$, with coordinates $x, z, \frac{dz}{dx} = y$.

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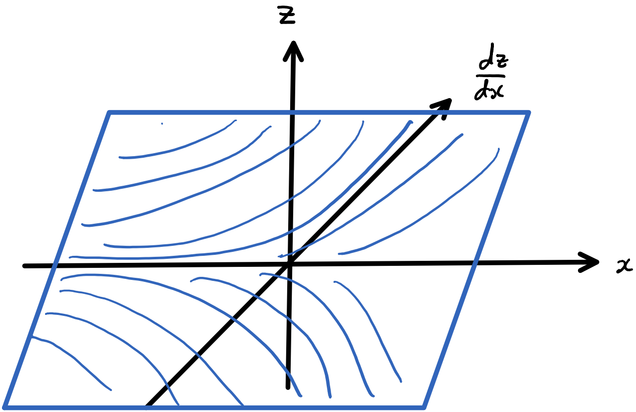
- The intersections of the plane field ξ with the surface $S = \{F = 0\}$ trace out curves on S which are solutions to the ODE.

E.g. $\frac{dz}{dx} = z$, solutions $z = Ae^x$.



Definition

The intersection of a contact structure ξ with a surface S gives a singular 1-dimensional foliation called the characteristic foliation of S.

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Theorem (Darboux theorem (Darboux 1882))

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Theorem (Gray stability theorem (Gray 1959))

Let ξ_t be a smooth family (an isotopy) of contact structures on M. Then there is an isotopy $\psi_t : M \longrightarrow M$ such that $\psi_{t*}\xi_0 = \xi_t$ for all t.

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- Similar statements hold for symplectic geometry.

The Weinstein conjecture

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(No: Schweizer, Harrison, Kuperberg, Kuperberg)

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Theorem (Taubes '07)

The Weinstein conjecture holds for contact 3-manifolds.

Proof uses Seiberg–Witten Floer homology.

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Generally:

- What is a contact structure like near a surface, and how does it change as a surface moves?
- What invariants are there of contact manifolds, what are their structures and relationships?
- What is the relationship between contact 3-manifolds and symplectic 4-manifolds?

Outline

Contact structures on 3-manifolds

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Answer: ∞. (Lutz 1977) — We can perform a *Lutz* twist on an embedded solid torus arbitrarily many times.

The contact structures obtained by Lutz twists are usually overtwisted.

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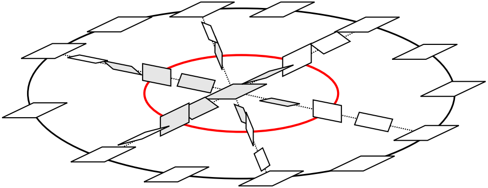
Definition

An overtwisted contact structure is one that contains a specific contact disc called an overtwisted disc.
A non-overtwisted contact structure is called tight.



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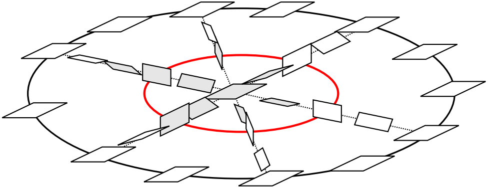

Classifying overtwisted contact structures reduces to homotopy theory!

Theorem (Eliashberg 1989)

$$\left\{ \begin{array}{c} \textit{Overtwisted contact structures} \\ \textit{on } M \end{array} \right\} \simeq \left\{ \textit{ 2-plane fields on } M \right\}$$

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Answer: Finitely many (Colin–Giroux–Honda 2003)

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- $M\#\overline{M}$ where M is Poincaré homology sphere:
  NO tight contact structure (Etnyre–Honda 1999).

Open book decompositions

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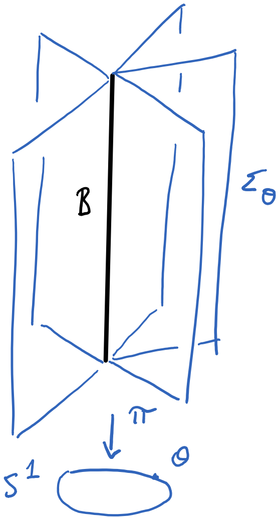
Definition

An open book decomposition of M consists of

- *an oriented link $B \subset M$ (the binding), and*
- *a map $\pi : M \setminus B \longrightarrow S^1$ where the preimage $\pi^{-1}(\theta)$ of every $\theta \in S^1$ is a surface Σ_θ with $\partial \Sigma_\theta = B$.*

The surfaces Σ_θ are the pages of the open book decomposition.

Open book decompositions



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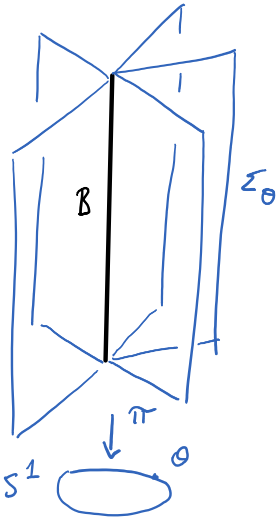

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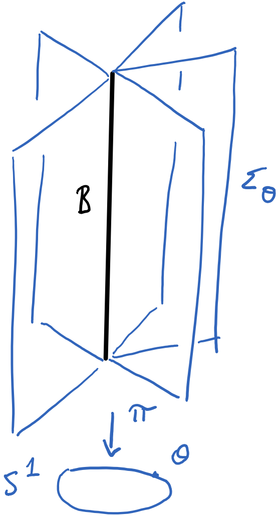

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From this data we can reconstruct M:

$$M = \frac{\Sigma \times [0,1]}{\begin{array}{r}(x,1) \sim (\phi(x),0)\ \forall x \in \Sigma, \\ (x,t) \sim (x,t')\ \forall x \in \partial\Sigma\end{array}}$$

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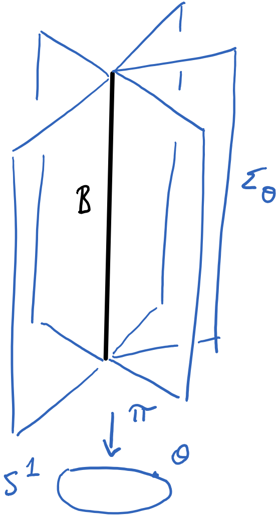

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In fact, from (Σ, ϕ) can reconstruct M and ξ.

- (Roughly, ξ transverse to the binding, close to parallel to pages.)

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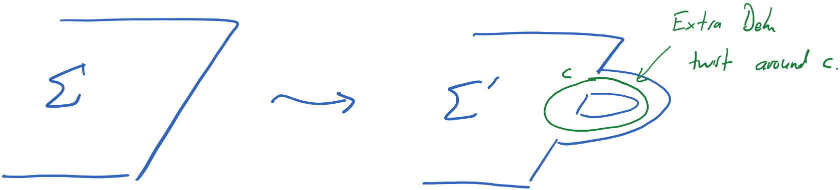
Definition

A stabilisation of an open book (Σ, ϕ) is an open book (Σ', ϕ') where

- *Σ' is Σ with a 1-handle attached along its boundary*
- *$\phi' = \phi \circ$ a Dehn twist around a simple closed curve intersecting the co-core of the 1-handle exactly once.*



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Which (Σ, ϕ) are tight and which are overtwisted?

- Wand (2014): “consistency".

Knots in contact 3-manifolds

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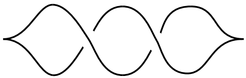

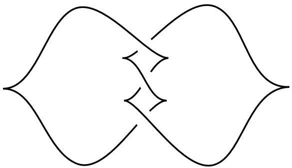

Thurston-Bennequin number

A framing of a knot K in a manifold M is any/all of:

- a trivialisation of the normal bundle of K;
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Definition

Let K be a nullhomologous Legendrian knot in (M, ξ). The Thurston-Bennequin number $tb(K)$ is the difference between the contact framing and the surface framing on K.

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Let K be a Legendrian knot in a contact 3-manifold (M, ξ). The manifold M' obtained by (-1)-Dehn surgery on K carries a natural contact structure ξ' that coincides with ξ away from K.

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Theorem (Wand 2014)

If (M, ξ) is tight then (M', ξ') is tight.

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Recall a surface S in (M, ξ) has a singular 1-dimensional foliation $\mathcal{F}$, the characteristic foliation, given by $\mathcal{F} = TS \cap \xi$.

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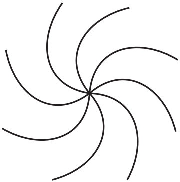
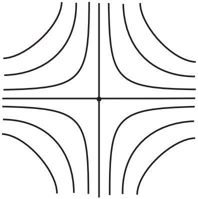
- Generically, singularities are only of two types: elliptic and hyperbolic.



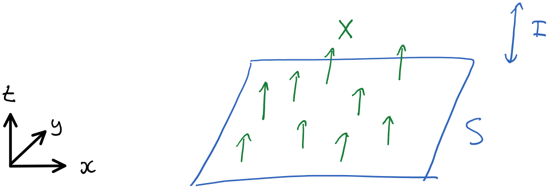


Roughly, $\mathcal{F}$ determines the “germ" of a contact structure near S.

Convex surfaces

Definition (Giroux 1991)

An embedded surface S in a contact 3-manifold is convex if there is a contact vector field X transverse to S.



We can take a neighbourhood $S \times I$ ($I = [0, 1]$), with X pointing along I.

- Local coordinates x, y on S, and t on I, so $X = \frac{\partial}{\partial t}$

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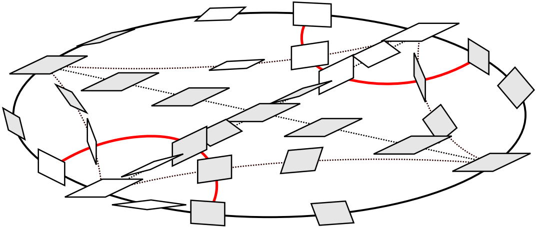

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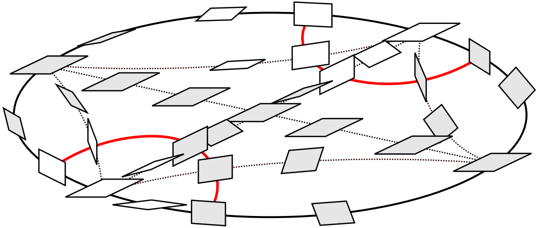

Definition

The ***dividing set*** *of S is $\Gamma = u^{-1}(0) = \{p \in S : X(p) \in \xi\}$.*

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The combinatorial structure of dividing sets and contact structures between them is captured by an algebraic object called the contact category.

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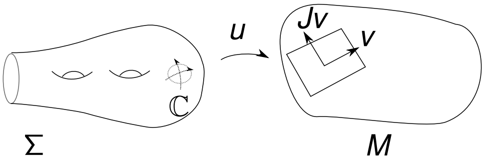
- Define an *almost complex structure* on (M, ω) and consider pseudo-holomorphic maps $u : \Sigma \longrightarrow M$, where Σ is a Riemann surface (Gromov 1985).



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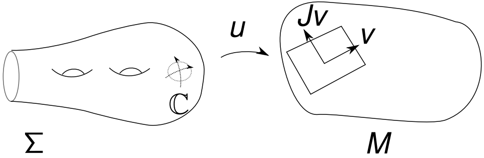
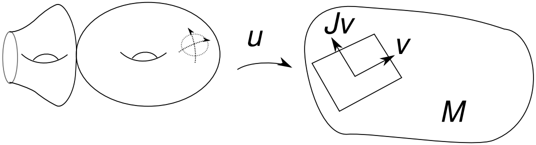


- Given appropriate constraints and assumptions, the space of holomorphic curves is a finite-dimensional *moduli space* $\mathcal{M}$.
- Can do analysis using Fredholm/index theory, compactness theorems, etc.

- Boundary of $\overline{\mathcal{M}}$ is stratified: boundary strata are moduli spaces for "degenerate" holomorphic curves (nodal surfaces, etc.)



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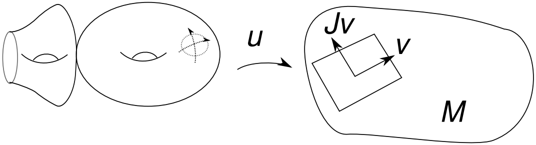

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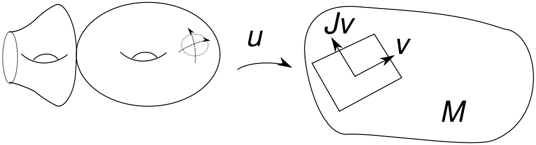

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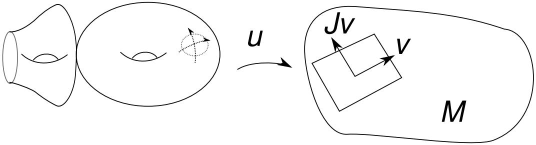

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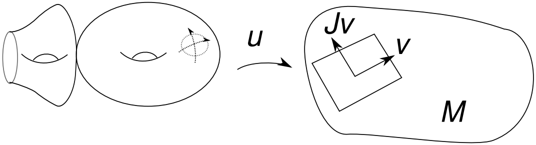

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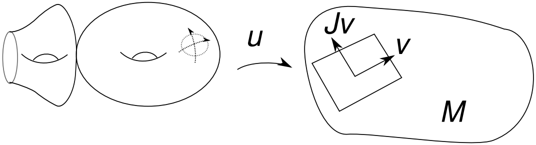

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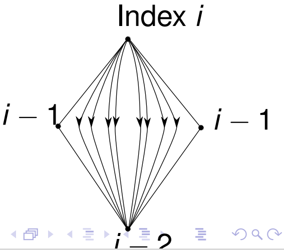
Analogy: singular homology via Morse complex.

- Complex generated by critical points of Morse function f.
- ∂ counts 0-dimensional families of trajectories of ∇f.



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  L, L' Legendrian isotopic $\Rightarrow (A, \partial), (A', \partial')$ have isomorphic homology.
  (Chekanov, Eliashberg, Ekholm–Etnyre–Sullivan, ...)

Thanks for listening!

References:

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