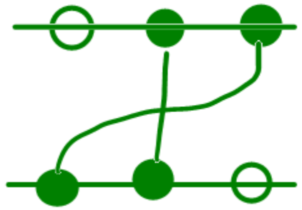
Strand Algebras & Contact categories



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Aust MS meeting, ANU 6/12/16

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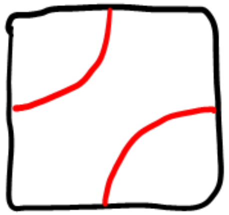
(Paper on arXiv with full details!)

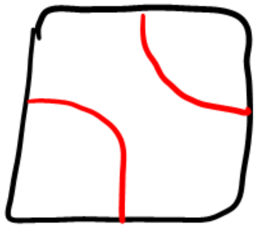
A game: Cube additions

Consider a (rounded) cube with some curves drawn on it.

→ On each (rounded) face, curves must appear as



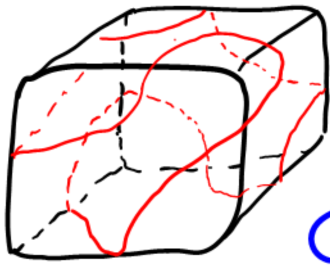
or



Eg.

Disconnected!

or



Connected!

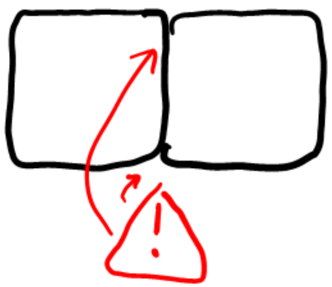
Aim: Keep curves "as connected as possible" under certain rules!

The rules of the game: Gluing, corners, rounding

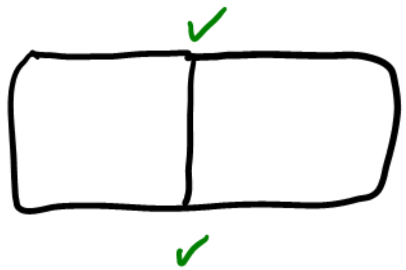
Try to glue together cubes to preserve connectedness,

BUT...

* Cubes must glue together smoothly...



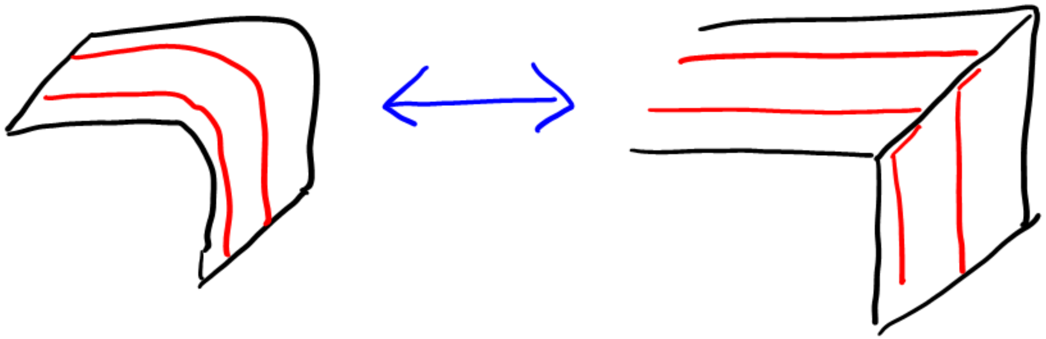
is not smooth!



is...

* We can sharpen a cube along edges (... any closed embedded 1-submanifold)

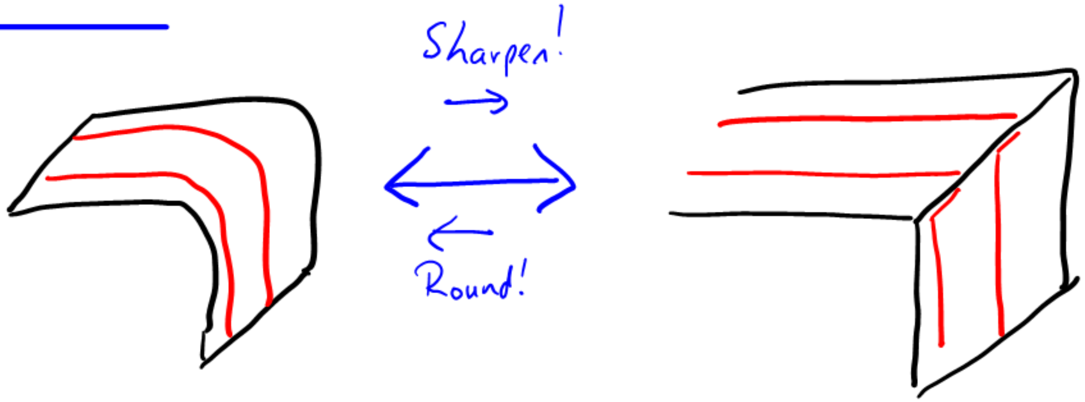
Rule:

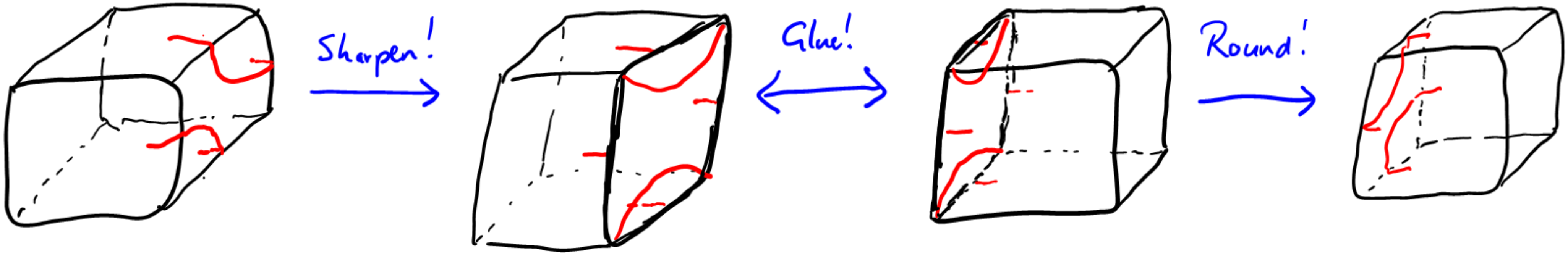


* We can glue faces together <u>if</u> curves match & result is smooth (up to isotopy in face)

How gluing works

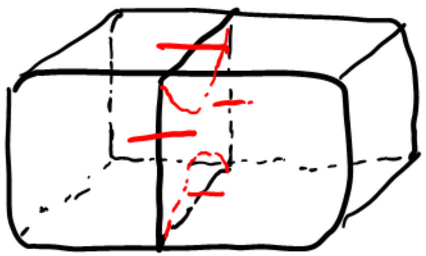
Rule :





Rounded cubes glue along a face (after sharpening) iff their curves appear to disagree!

Result :



Contact Cubes

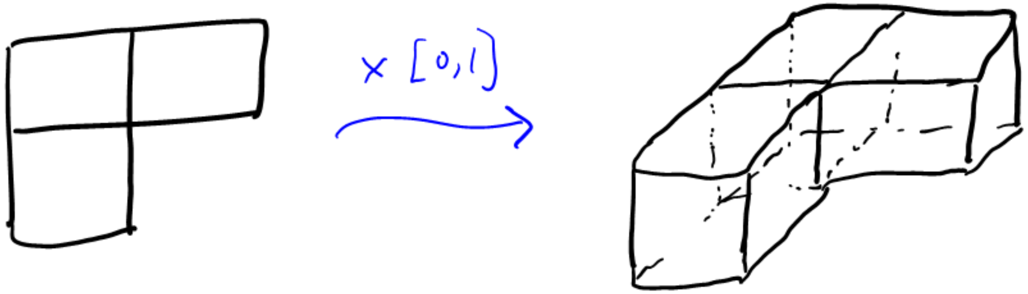
Defn: A <u>cubulated contact structure</u> consists of a collection of cubes with curves, as described above, glued together along faces according to rules described above.

E.g.

We will restrict to cubulated contact structures where cubes are glued along <u>side</u> faces only, i.e. as a

(gluing of squares)* $\times [0,1]$

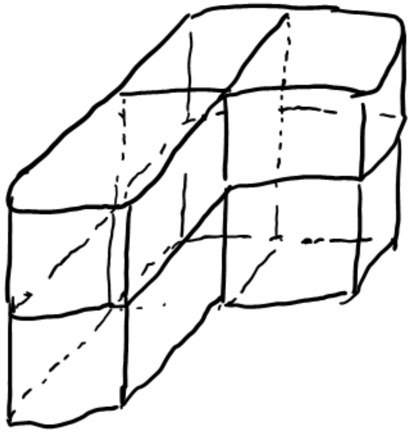
E.g.



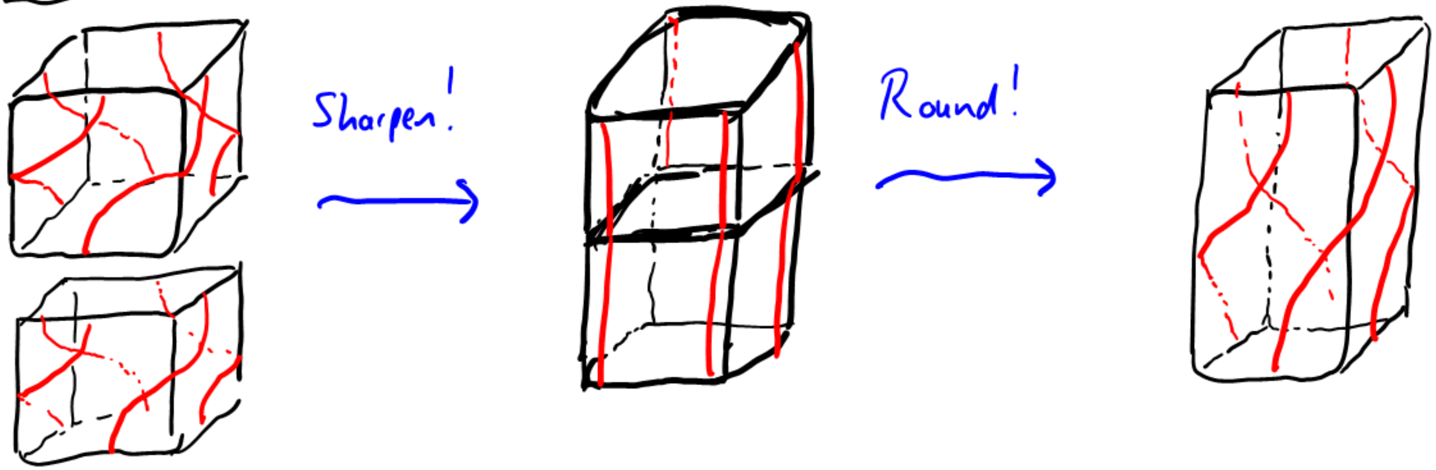
(* Actually to get this all to work certain conditions must be imposed on how squares glue)

Stacking

Our calculated contact structures can be stacked on top of each other!



Cubes with curves stack nicely if curves around outside are vertical...

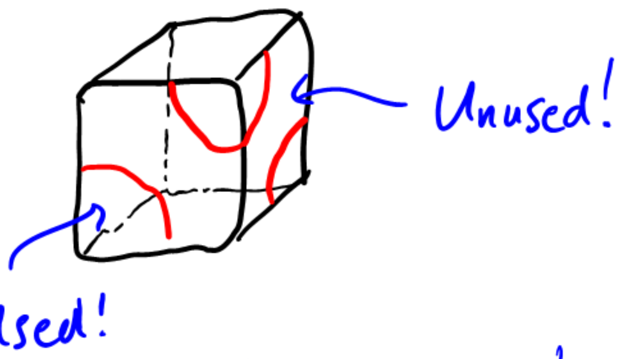


Defn: The curves on a side face are called <u>vertical</u> or <u>unused</u> if they appear as in the above pictures.
(Otherwise, <u>horizontal</u> or <u>used</u>.)

Stackable cubulated contact structures

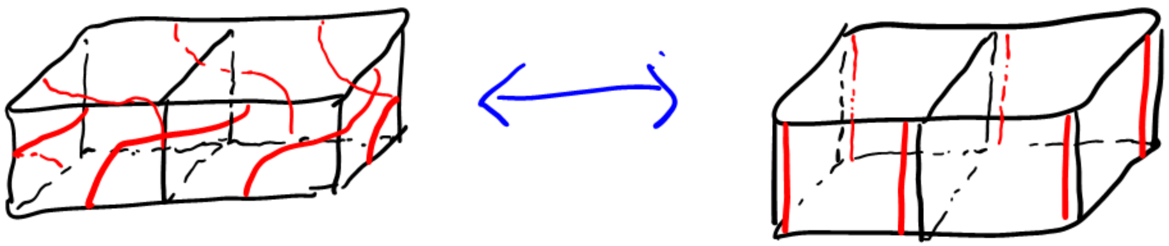
Note that on a cube, some sides might be used, others unused.

E.g.



Defn: A cubulated contact structure is **stackable** if all its exterior side faces are vertical / unused.

E.g.



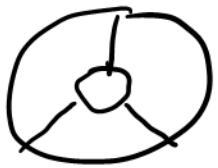
Game: * For a given cube arrangement, how can we arrange curves to be "always connected"?

* When we stack cubes, when do the curves stay connected?

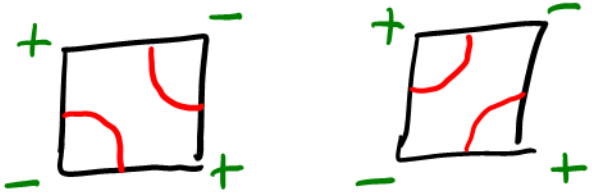
Some Notation

To keep track of curves on top & bottom, introduce **signs** on vertices of **squares**.

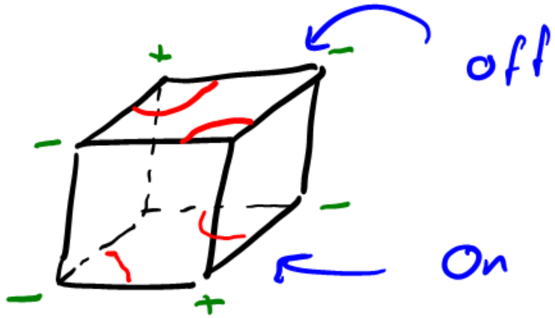
(Note this restricts arrangements of squares

Eg  impossible.)

Curves on top/bottom are called **on** or **off** as shown:



On **Off**

E.g. 

So: A stackable cubulated contact structure is completely determined by choices of

* each top & bottom face **on** or **off**
* each interior side face **used** or **unused**.

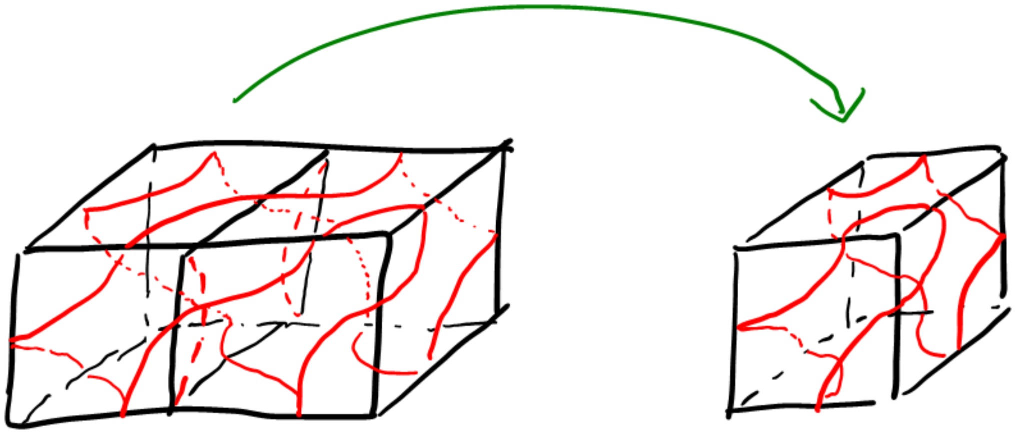
How to win: Tight contact structures

It's very easy to put curves on cubes which become disconnected.
The challenge is to place curves which always remain connected.

Defn: A stackable calculated contact structure (s.c.c.s.) is **tight** if for all subsets of cubes which glue up to give a ball topologically, the curves on the boundary of the ball are connected.

Examples:

Tight!

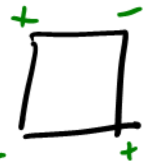


Not tight!

Algebra of cube additions / contact structures

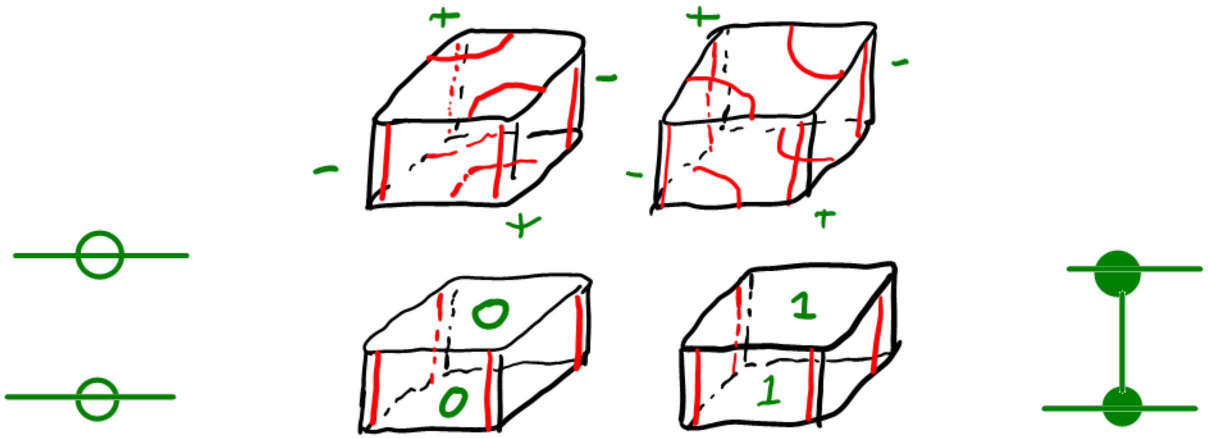
Defn: For a cube arrangement (square arrangement) Q, the **Contact Category Algebra** $CA(Q)$ is the $\mathbb{Z}_2$-algebra:

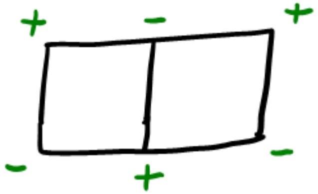
* freely generated as a $\mathbb{Z}_2$-module by tight s.c.c.s. on Q
* with multiplication defined by stacking s.c.c.s. (non-tight $= 0$).

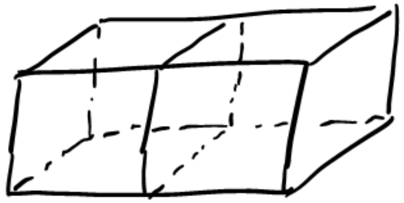
E.g. 1 cube $Q =$ 

$CA(Q) = \mathbb{Z}_2 \langle$  ,  $\rangle \big/ (a^2 = a,\ b^2 = b,\ ab = ba = 0)$

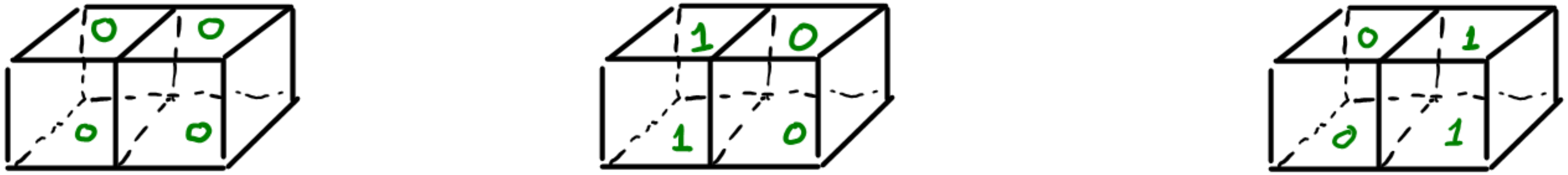
a   b

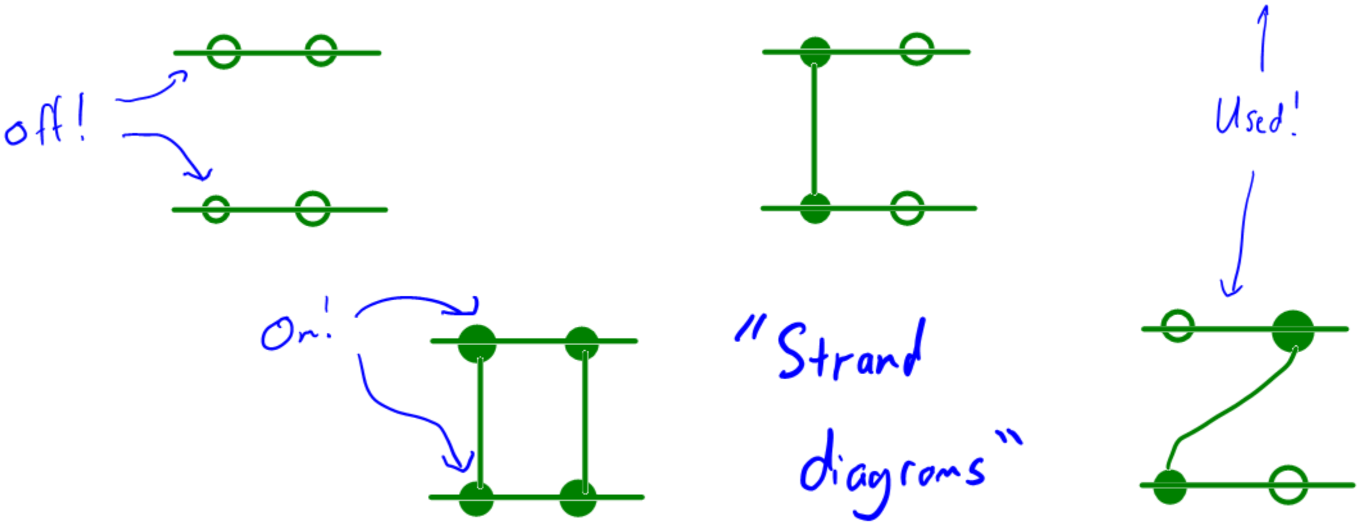


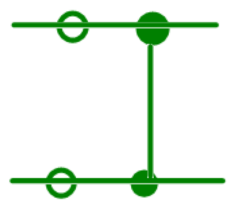
Eg 2 cubes: $Q =$ 

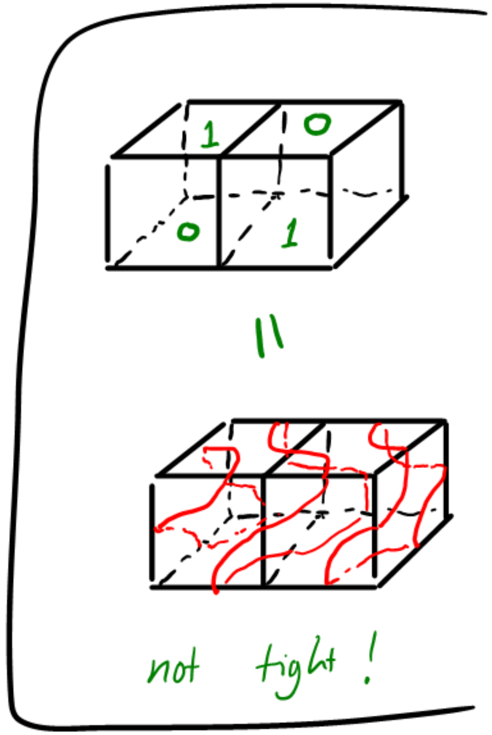


Turns out, there are precisely 5 s.c.c.s.



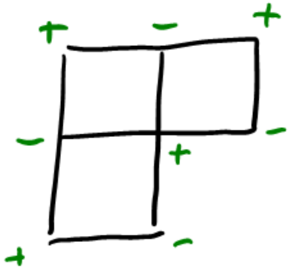




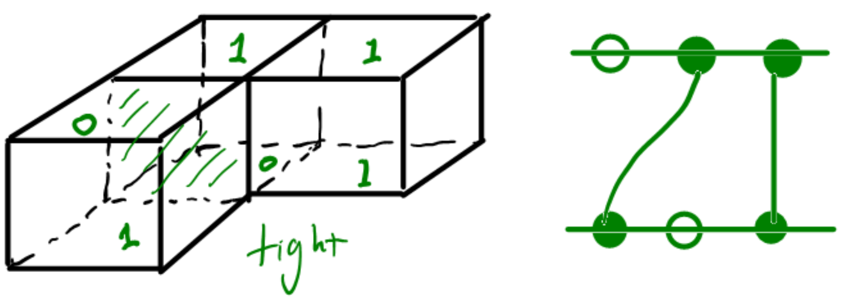


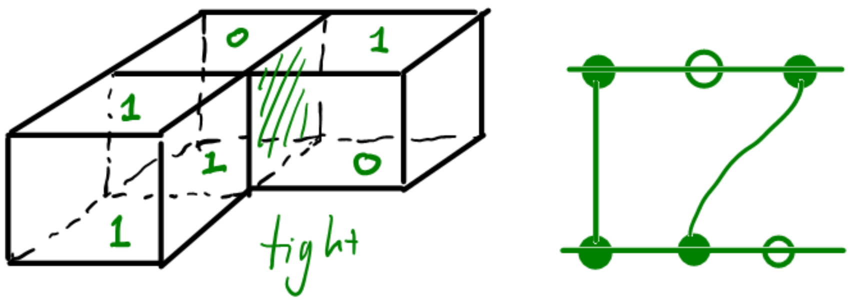
Stacking s.c.c.s = stacking strands!

Eg 3 cubes, Q =

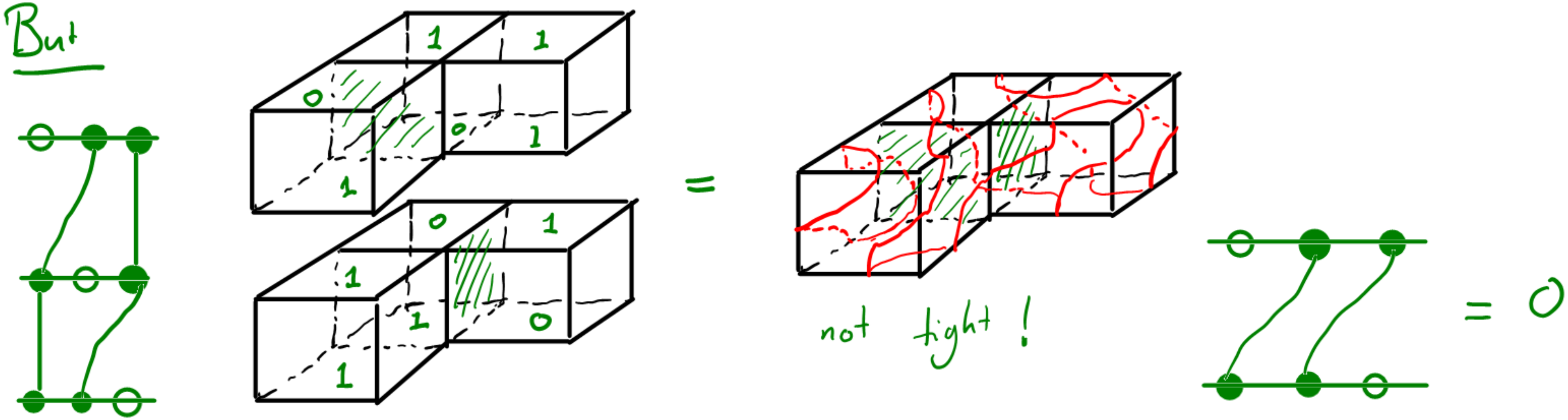


It's possible to stack tight s.c.c.s to obtain something not tight!





But



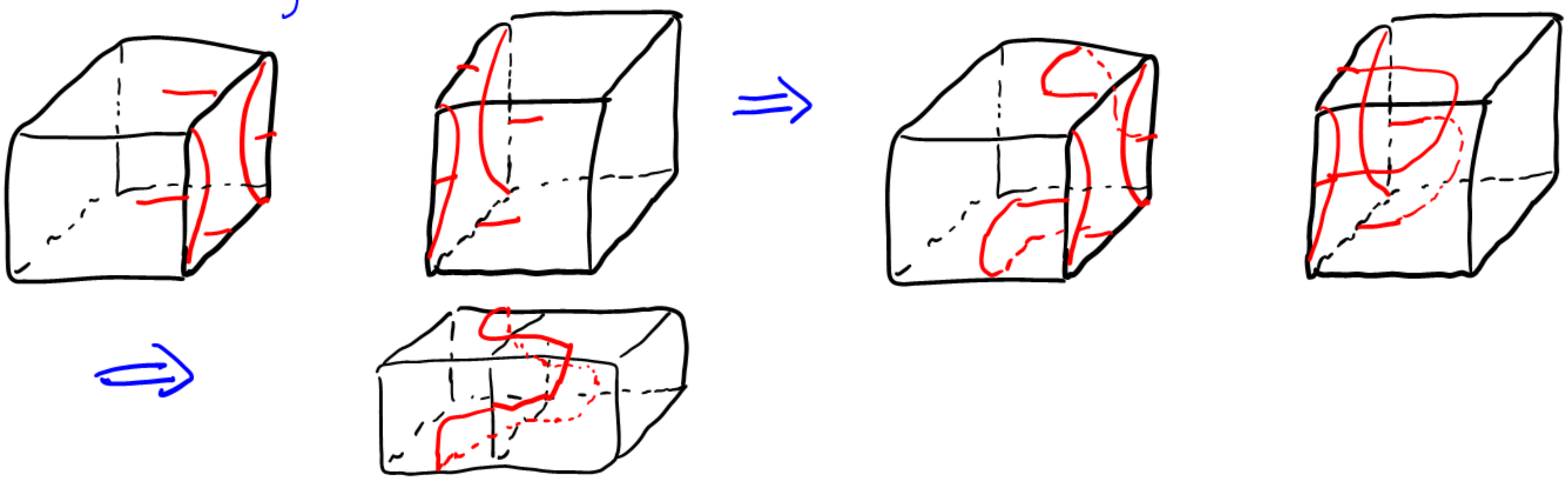
Some elementary statements

Lemma: If a s.c.c.s. is tight, then
"on squares" on top = # "on squares" on bottom

Pf: Signs on Q mean that the curve Γ divides the boundary of the s.c.c.s. into +, − regions R_+, R_-. If Γ is connected then $\chi(R_+) = \chi(R_-)$, which can be given in terms of # on/off squares. □

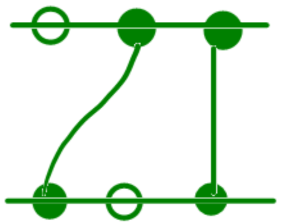
Lemma: If each cube in a s.c.c.s. is tight, then the s.c.c.s. is tight.

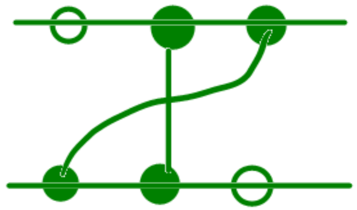
Pf:

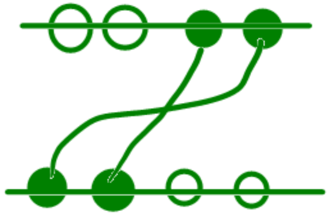


Strand Algebra

A strand algebra is generated over $\mathbb{Z}_2$ by diagrams like





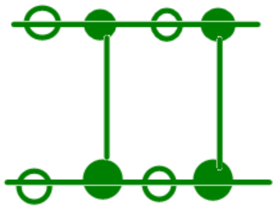


consisting of strands running between points on an interval

Points are: * <u>on</u> if a strand begins/ends there
* <u>off</u> otherwise.

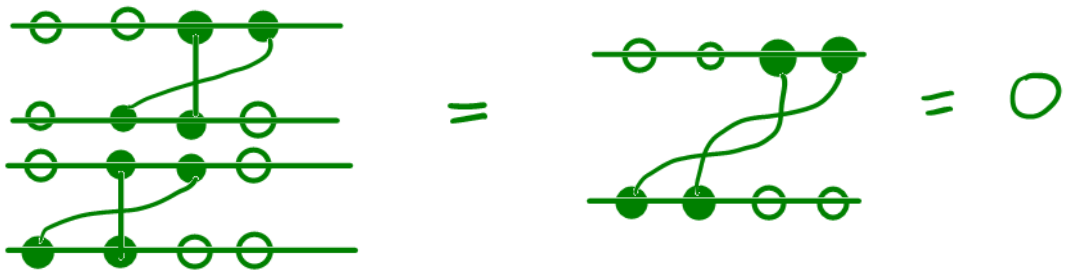
Strands never move left as they go up! (Directionality/partial order)

Multiplication: stack strand diagrams.
(If strands don't match, 0.)

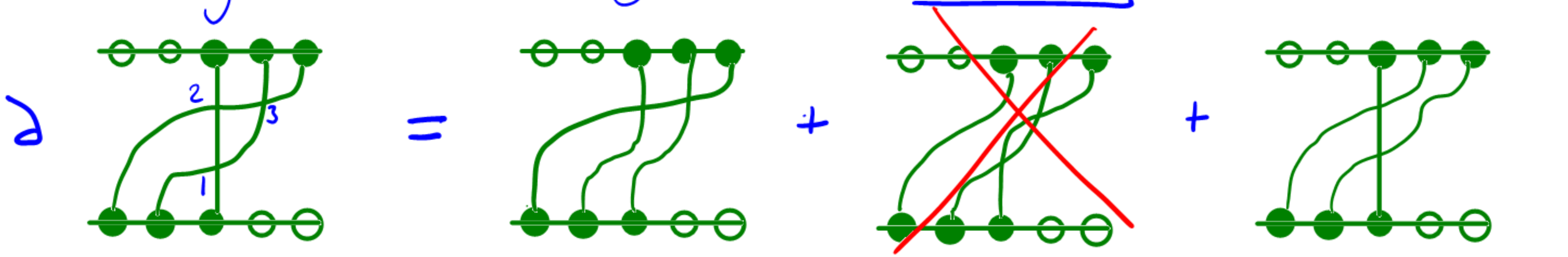
Idempotents:  strand diagrams with vertical strands only.

Strand Algebras (cont...)

More relations: Double intersections = 0.



A strand algebra also naturally has a <u>differential</u>: resolve crossings.



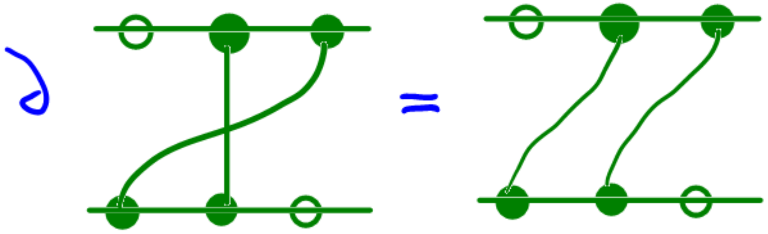
Obeys Leibniz rule!

<u>Graded</u> by # strands / # points "on", <u>or</u> by # intersections of strands

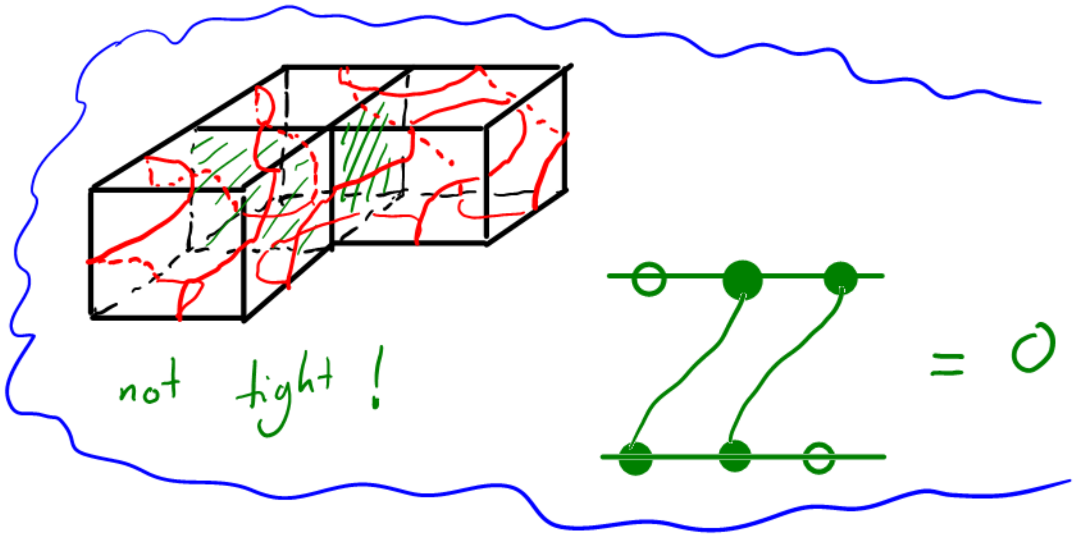
$\rightsquigarrow$ A Differential Graded Algebra!

Strand Algebra / Contact Category Correspondence

Now



and recall from previously



So look at <u>homology</u> !

Theorem (M): Let Q be a square/cube arrangement (more generally, quadrangulated surface). Then there exists a strand algebra $\mathcal{A}$ such that

$$CA(Q) \cong H(\mathcal{A}).$$

Note: In general the strand algebras required are slightly more complicated than the ones described here... multiple intervals, restrictions on starting/ending points, symmetrisation...

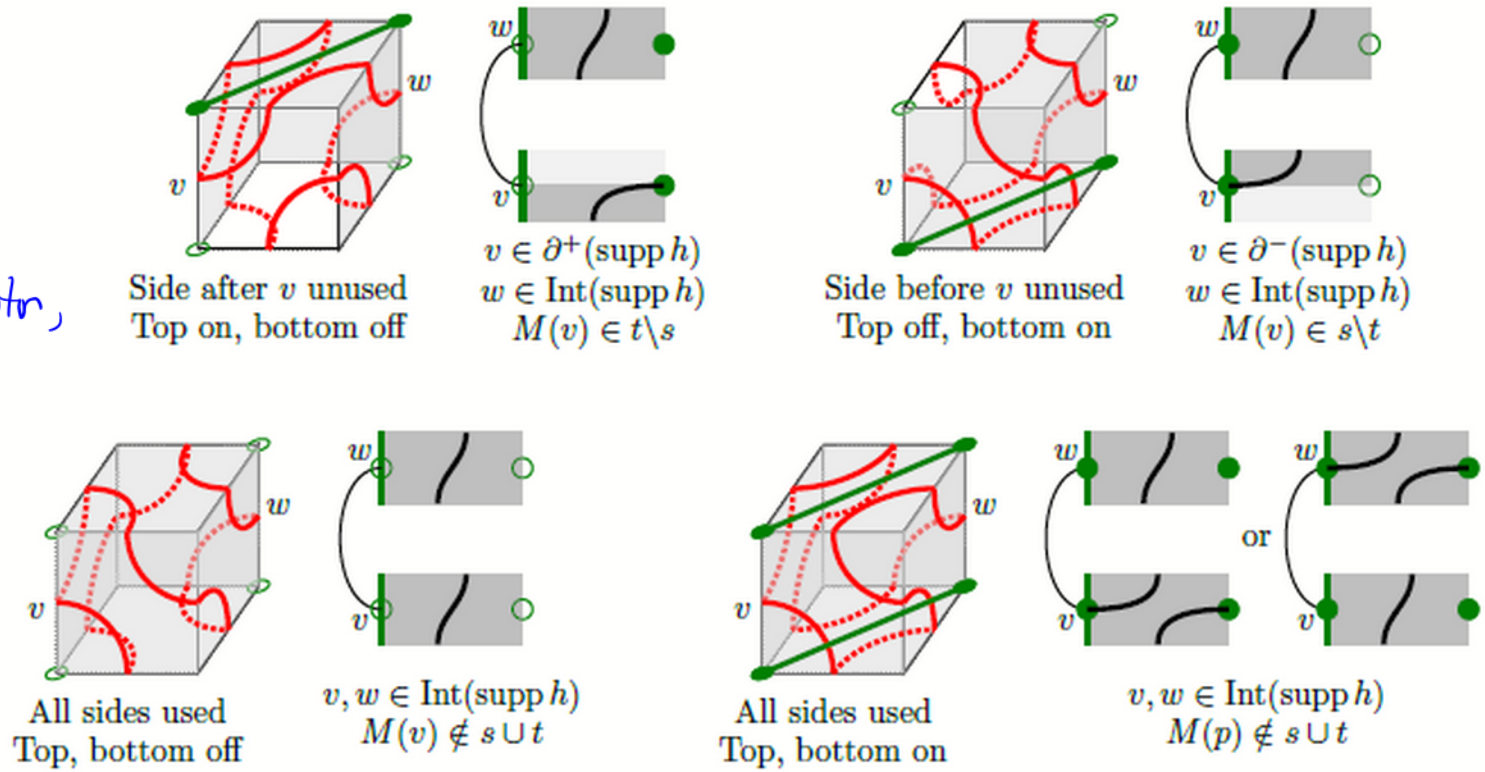
Idea of proof

Show that both $CA(Q)$, $H(A)$ are "determined locally"

and show that these local parts match up.

Cubes $\longleftrightarrow$ strands (with $\partial\cdot = 0$, up to boundaries)

(Use results of Lipshitz-Ozsváth-Thurston, ideas of Zarev.)

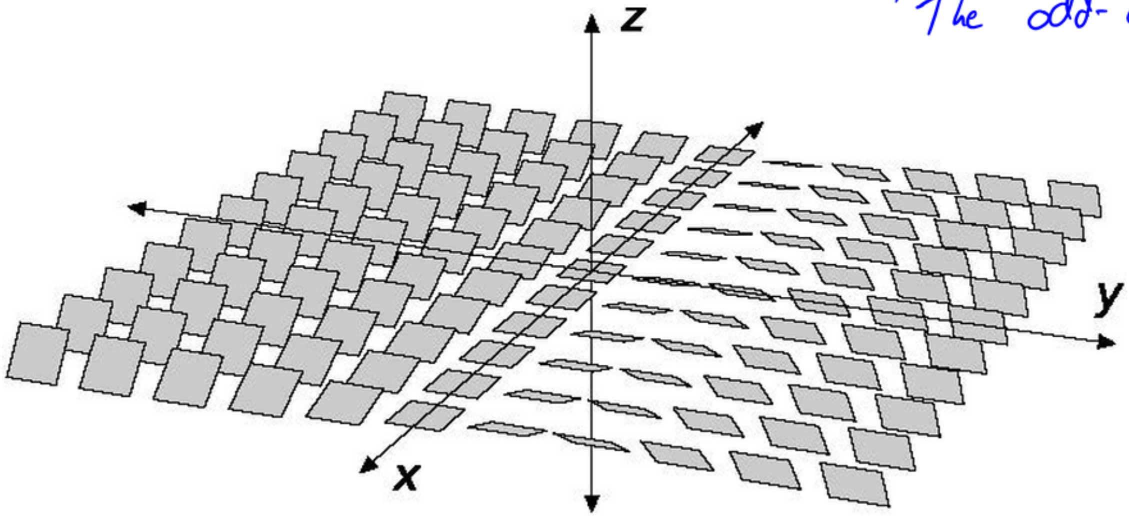


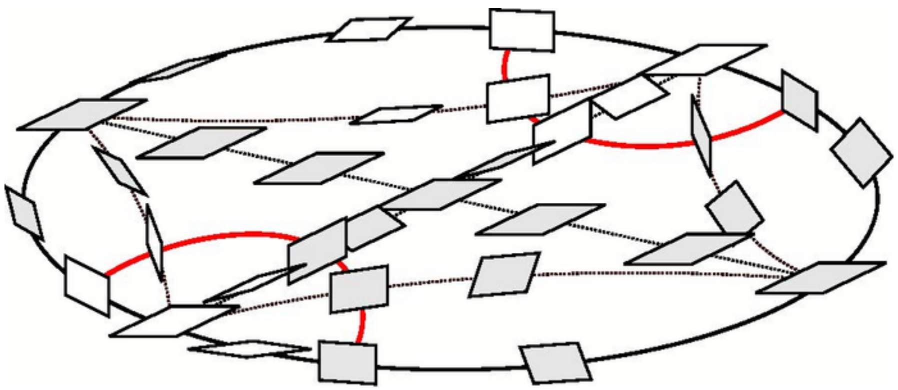
Context, motivation, remarks

* Cubulated contact structures are a combinatorial version of contact structures.

A contact structure on a 3-manifold is a non-integrable 2-plane field.

"The odd-dimensional version of symplectic geometry."





* So we have (a) reduced differential geometry to combinatorics,

(b) precisely described how they combine.

Context, motivation, remarks (cont.)

Strand algebras were actually introduced to describe something apparently completely different!

Lipshitz - Ozsváth - Thurston: Bordered Heegaard Floer homology

* A powerful invariant of 3-manifolds with boundary M defined by analysing holomorphic curves in an auxiliary 4-manifold $\Sigma \times I \times \mathbb{R}$
* Strand algebras encode asymptotic behaviour of holomorphic curves near ∂M.

Contact geometry is known to have many connections to Heegaard Floer theory ... this is a new one.

"Geometry of holomorphic boundary asymptotics = Contact geometry near boundary surface"