Plane graphs, special alternating links, and contact geometry

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Recent results:

- Kálmán–M, *Tight contact structures on Seifert surface complements*, arXiv:1709.10304

Expository paper:

- M, *Polytopes, dualities, and Floer homology*, arXiv:1702.03630

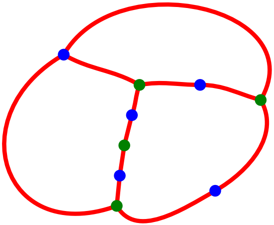
Outline

1. Graphs, trinities and trees
2. Knots and links and 3-manifolds
3. Contact geometry
4. Floer homology
5. Polytopes
6. Further details

Outline

Bipartite plane graphs

- Let G be a bipartite plane graph.
- Colour vertices blue/violet and green/emerald
- Colour edges red.
- Embedded in $\mathbb{R}^2 \subset S^2$.



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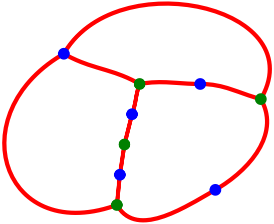

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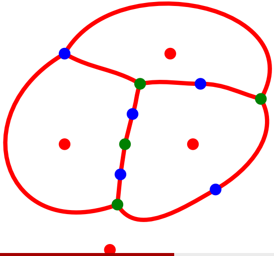

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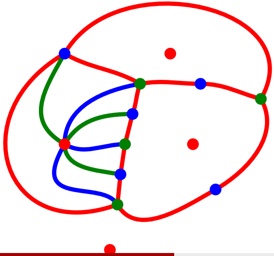
- We can colour each edge by the unique colour distinct from endpoints.



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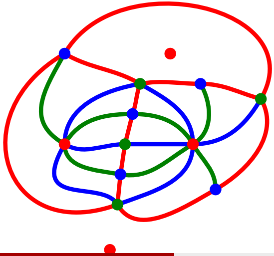

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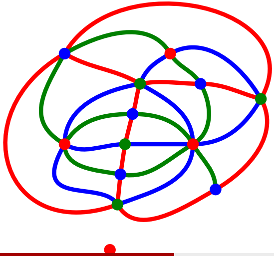

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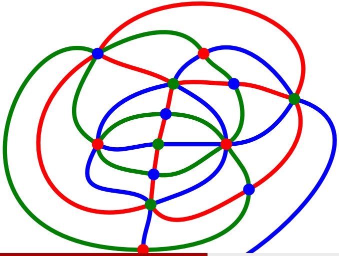

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- Add red vertices in complementary regions
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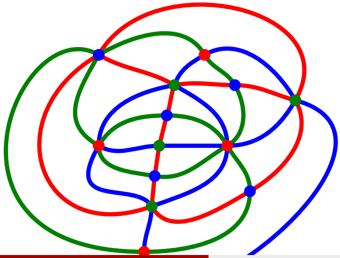
- This yields a *3-coloured* graph triangulating S^2.

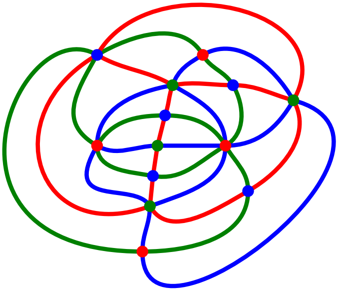
Definition

A 3-coloured *triangulation* of S^2 is called a *trinity*.



Bipartite graphs and trinities

A trinity naturally contains *three* bipartite planar graphs: take all edges of a single colour.



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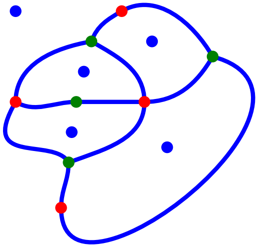

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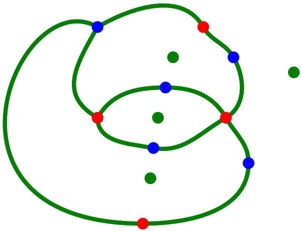

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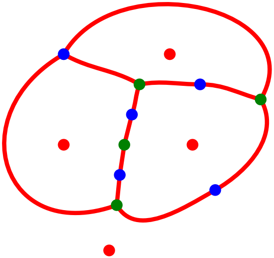

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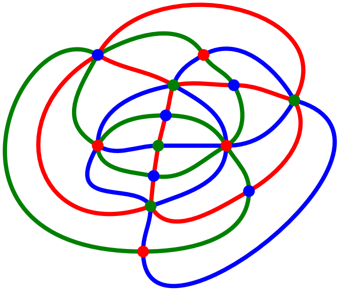
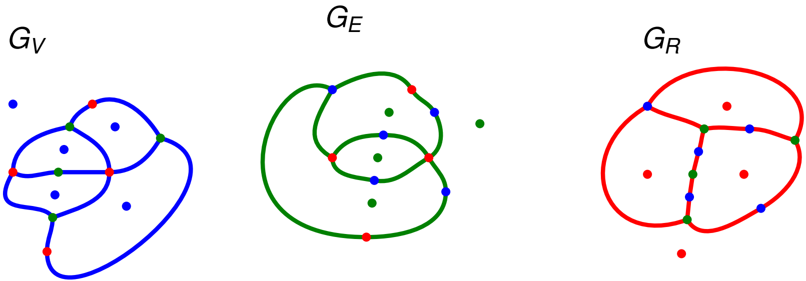


Bipartite graphs and trinities

The *violet graph* G_V, *emerald graph* G_E, *red graph* G_R are all bipartite plane graphs which yield (and are subsets of) the same trinity.



So:

- Bipartite plane graphs naturally come in *threes* (*triality*).

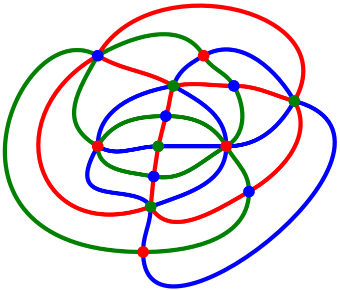
Compare to:

- Plane graphs have duals, so naturally come in *twos* (*duality*).

Trinities and triangulations

In the triangulation of S^2 from a trinity:

- Each triangle contains a vertex (edge) of each colour.



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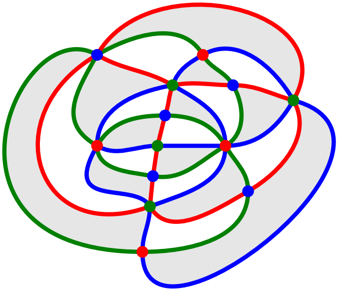

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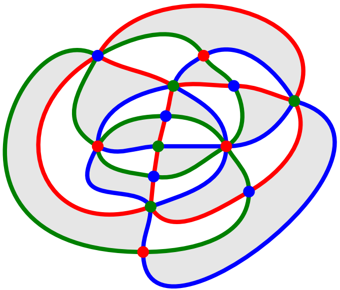
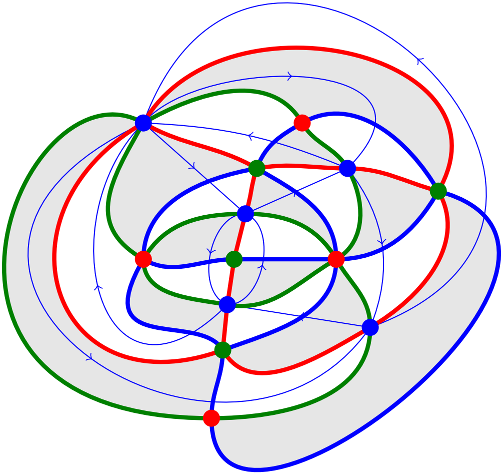
- Triangles sharing an edge must be *opposite* colours.
  (In other words, the planar dual of the trinity is bipartite.)



Planar duals of trinities

Consider the planar dual G_V^* of G_V in a trinity.



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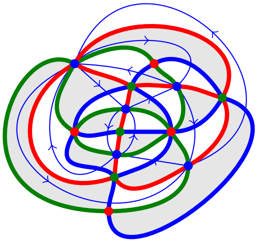

Observations on G_V^*:

- G_V^* has edges bijective with violet edges.
- Each edge of G_V^* crosses precisely *two* triangles of the trinity and hence is naturally *oriented*, say black to white.

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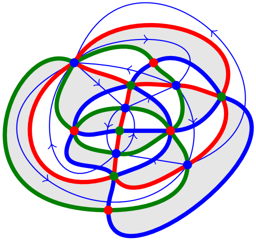

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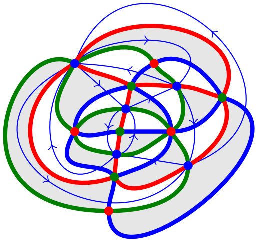

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- Around each vertex of G_V^*, edges alternate in and out.
- G_V^* is a *balanced directed planar graph*.
  (Balanced: in-deg = out-deg at each vertex.)

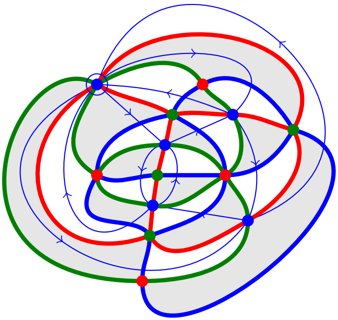
Arborescences

Let D be a directed graph D. Choose a *root* vertex r.

Definition

A (spanning) arborescence of D is a spanning tree T of D all of whose edges point away from r.

- I.e. for each vertex v of D there is a unique directed path in T from r to v.



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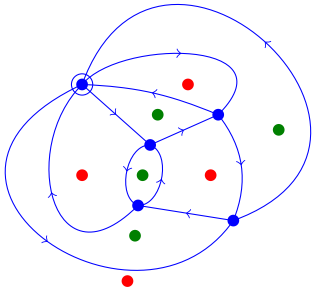

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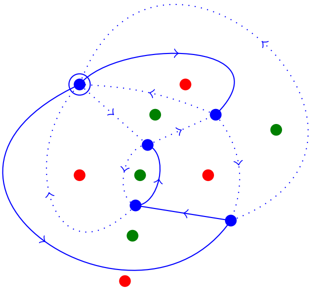

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Let G^*_V, G^*_E, G^*_R be the planar duals of the coloured graphs of a trinity. Then

$$\rho(G^*_V) = \rho(G^*_E) = \rho(G^*_R).$$

Many combinatorial questions about trinities have the same answer.

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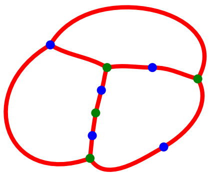
Definition

The *magic number* $\mathrm{Magic}(G_V)$ of the plane bipartite graph G is $\rho(G_V^*)$.

Outline

From plane graphs to knots and links

Given a plane graph G, there is a natural way to construct a knot or link L_G: the median construction.



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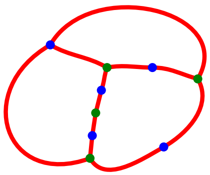

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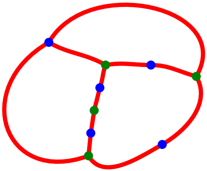

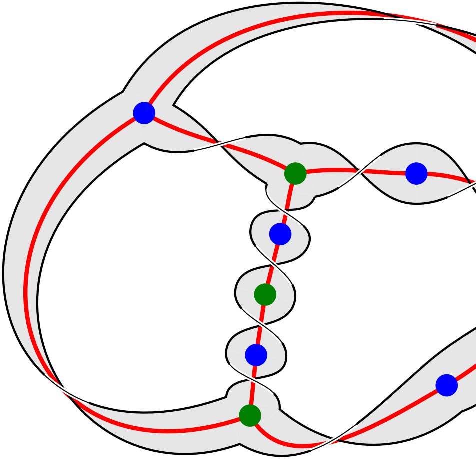

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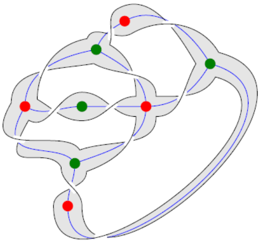

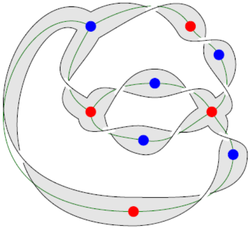

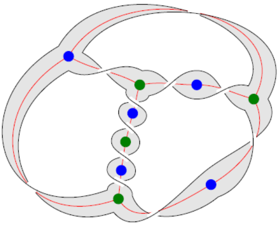

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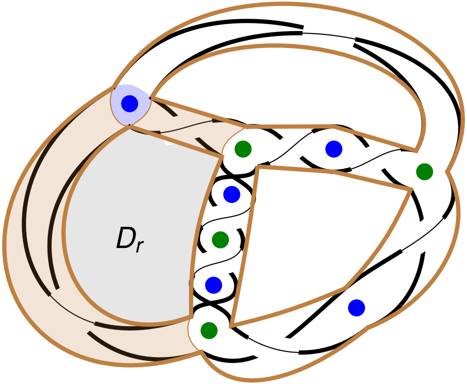

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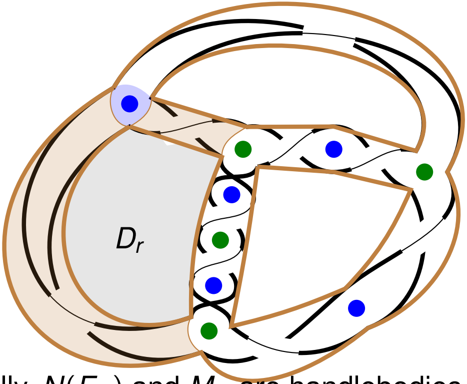

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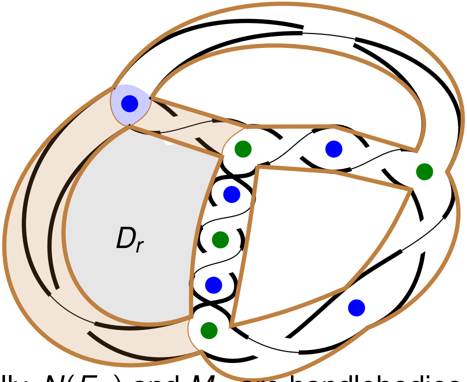

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The curves Γ split ∂M into positive and negative regions,

$$\partial M \backslash \Gamma = R_+ \sqcup R_-, \quad \partial R_+ = -\partial R_- = \Gamma$$

(I.e. when you cross Γ, you go from R_+ to R_-.)

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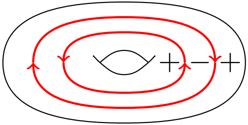

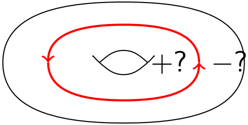

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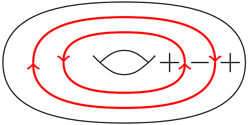
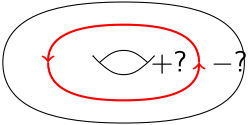
$$\partial M \backslash \Gamma = R_+ \sqcup R_-, \quad \partial R_+ = -\partial R_- = \Gamma$$

(I.e. when you cross Γ, you go from R_+ to R_-.)





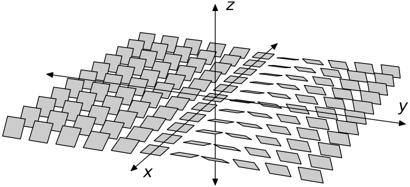
From a trinity we obtain a triple of sutured manifolds

$$(M_{G_V}, L_{G_V}), \quad (M_{G_E}, L_{G_E}), \quad (M_{G_R}, L_{G_R}).$$

Outline

Contact structures

A contact structure on a 3-manifold M is a non-integrable 2-plane field $\xi = \ker \alpha$.



E.g. $\alpha = dz - y\, dx$.
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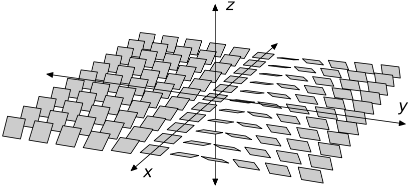

E.g. $\alpha = dz - y\, dx$.
Non-integrability of $\xi \Leftrightarrow \alpha \wedge d\alpha \neq 0$.
Basic question of contact topology:

- Given a 3-manifold M, understand the (isotopy classes of) contact structures on M.

Contact 3-manifolds

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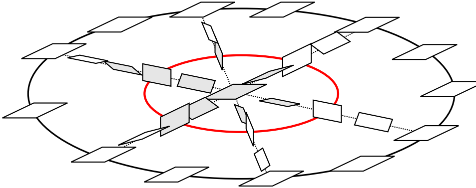

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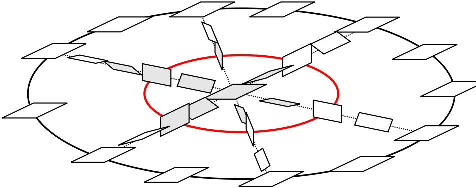

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- $\{$ Overtwisted contact structures on M $\}$ is weakly homotopy equivalent to $\{$ Homotopy classes of 2-plane fields on M $\}$
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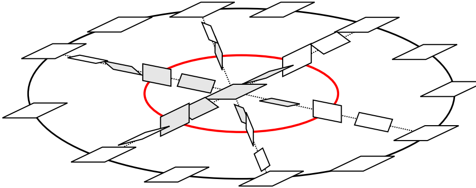

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A tight contact structure is a non-overtwisted one:

- For a closed oriented atoroidal 3-manifold M, there are finitely many isotopy classes of tight contact structures (Colin–Giroux–Honda 2002).
- But understanding these is more subtle.

Convex surfaces and sutures

For 3-manifolds with boundary:

- *sutures* provide natural boundary conditions for contact structures.
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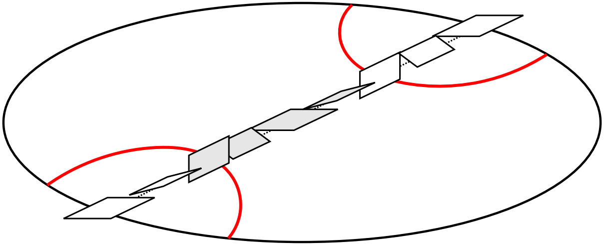

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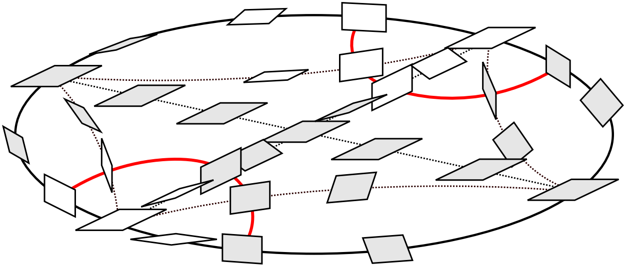

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Hence for a trinity with a triple of bipartite graphs G_V, G_E, G_R,

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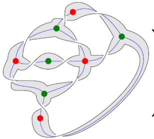
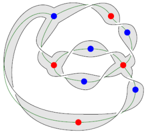
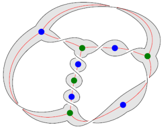
$cs\Big($  $\Big) = cs\Big($  $\Big) = cs\Big($  $\Big).$

Outline

Heegaard Floer homology

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All versions *define chain complexes* in which the *differential counts pseudoholomorphic curves* in some related manifold.
The *homology* then turns out to be invariant.

Sutured Floer homology and contact structures

Sutured Floer homology yields, very roughly,

(M, Γ)	$\rightsquigarrow$	$SFH(M, \Gamma)$
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- Heegaard Floer homology gives invariants of contact structures ξ (Ozsváth–Szazó , Honda–Kazez–Matić $\approx$ 2005):
  - contact structure ξ on $(M,\Gamma) \rightsquigarrow c(\xi) \in SFH(-M,-\Gamma)$.

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Using grading of SFH, the Magic(G) $\mathbb{Z}$ *summands are naturally arranged at the lattice points of a polytope $\mathcal{P}_G$ in $\mathbb{Z}^{R-1}$.*

Here $R = \#$ complementary regions of G.

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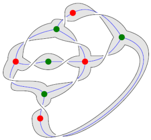
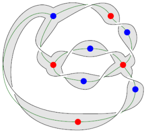
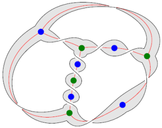
SFH of a trinity

It follows that

$$SFH(S^3 - F_{G_V}, L_{G_V}), \quad SFH(S^3 - F_{G_E}, L_{G_E}), \quad SFH(S^3 - F_{G_R}, L_{G_R})$$

all have dimension given by magic number.
Same # $\mathbb{Z}$ summands, but arranged in different polytopes!

$SFH\Bigg($  $\Bigg) \cong SFH\Bigg($  $\Bigg) \cong SFH\Bigg($  $\Bigg).$

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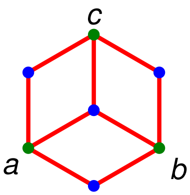
Hypergraphs $\sim$ bipartite graphs:

- A hypergraph (V, E) can be drawn as a bipartite graph $\mathrm{Bip}(V, E)$.
- A bipartite graph can be considered as a hypergraph (in 2 ways).

Hypertrees in hypergraphs



Bip(V, E)
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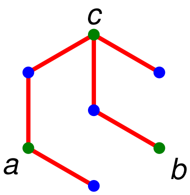

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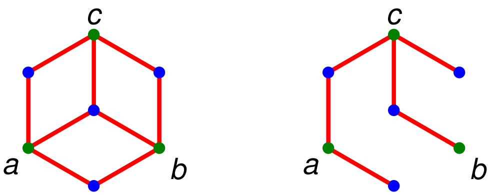

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A hypertree in (V, E) is a function $f\colon E \to \mathbb{N}_0$ such that there exists a spanning tree in Bip(V, E) *with degree $f(e) + 1$ at each $e \in E$.*

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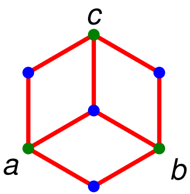

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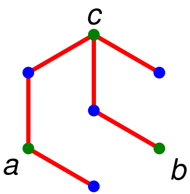

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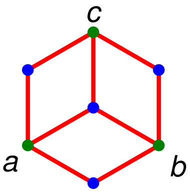
When (V, E) a graph, hypertrees = trees: choose edges with $f(e) = 1$.

The hypertree polytope

A hypertree $f\colon E \longrightarrow \mathbb{N}_0$ can be regarded as a point of $\mathbb{Z}^E$.

$$\text{Let} \quad Q_{(V,E)} = \{ f \in \mathbb{Z}^E \mid f \text{ is a hypertree of } (V,E) \}.$$



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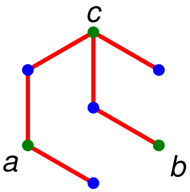

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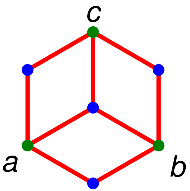

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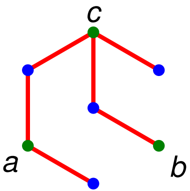

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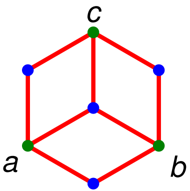

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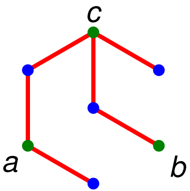

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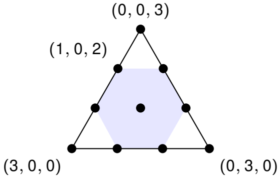

7 hypertrees

Hypertree polytopes and contact structures

This polytope reappears as the Euler classes of tight contact structures on (M_G, L_G) and the $\mathbb{Z}$ summands of $SFH(M_G, L_G)$.

Proofs are constructive: we obtain explicit bijections

$$\{\text{contact structures on } (M_{G_R}, L_{G_R})\}$$
$$\cong$$
$$\{\text{hypertrees on } (E, R)\}$$
$$\cong$$
$$\{\mathbb{Z} \text{ summands of } SFH(M_{G_R}, L_{G_R})\}.$$

Outline

Proof of classification of contact structures

Theorem (Kálmán–M.)

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Proof ideas:

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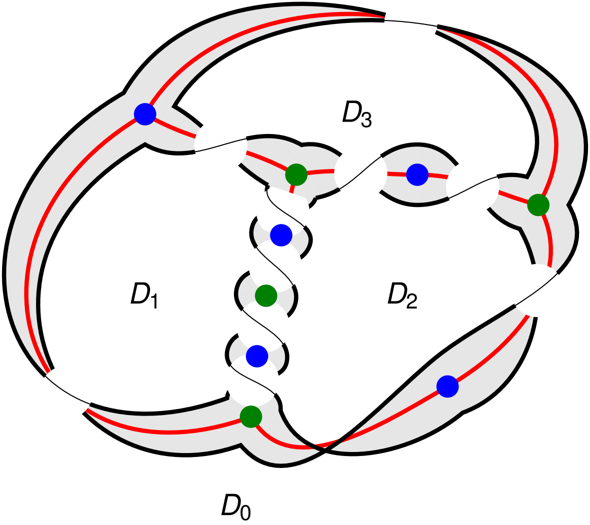
5. Kálmán's previously showed there are ρ hypertrees; show two spanning trees representing same hypertree produce isotopic contact structures.

Contact structures on trinities

Consider complements of tubular neighbourhood of G:



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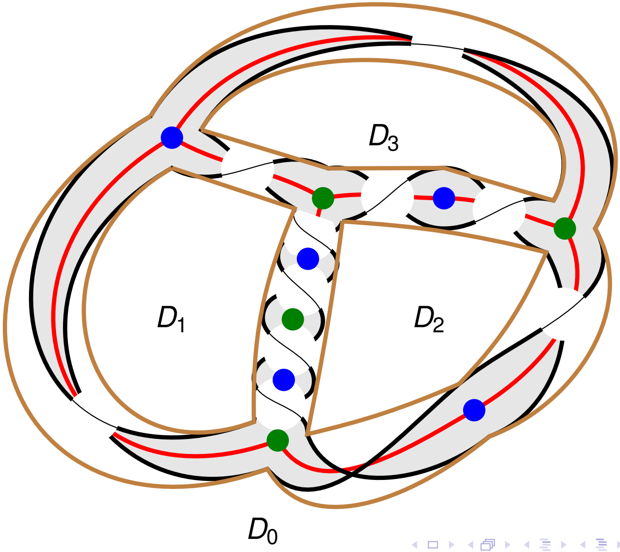

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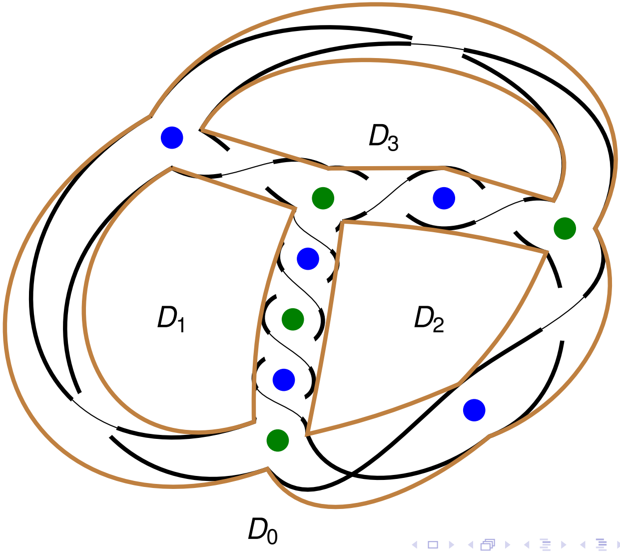

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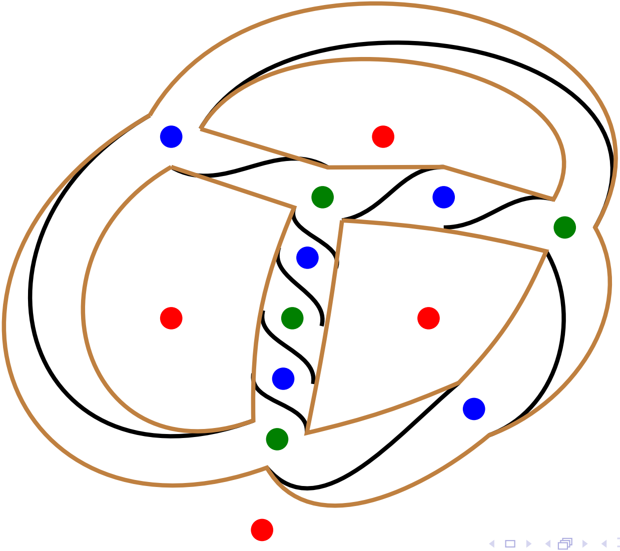

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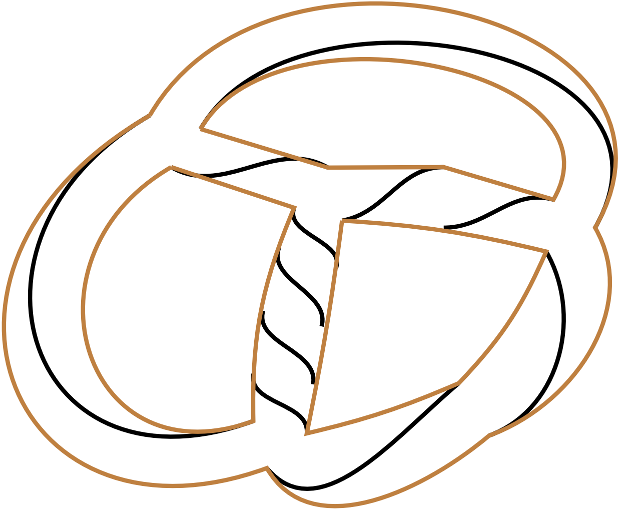

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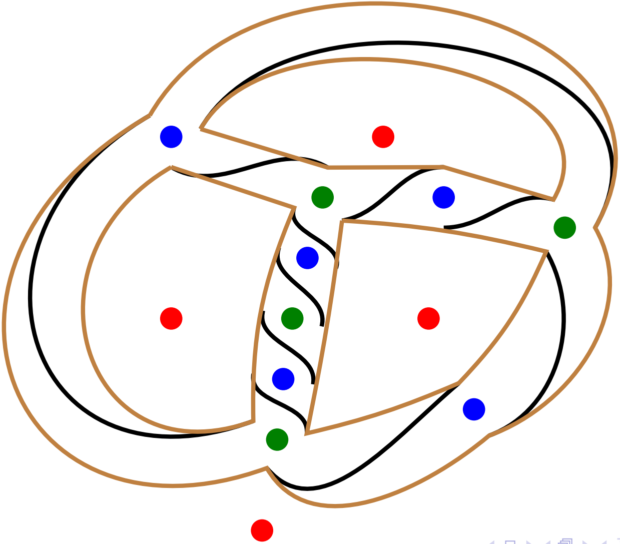

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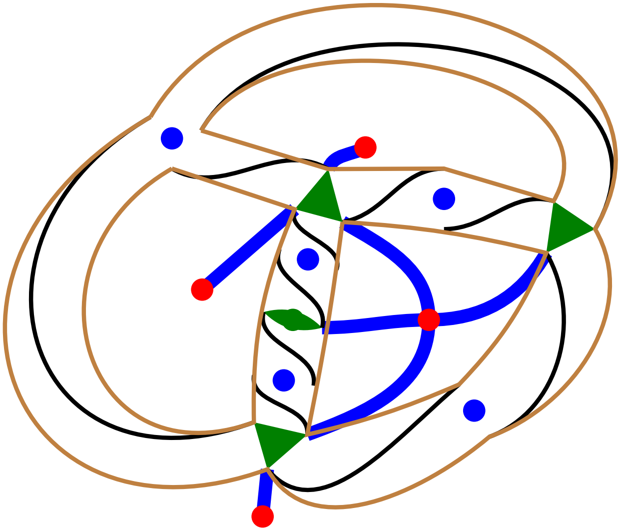

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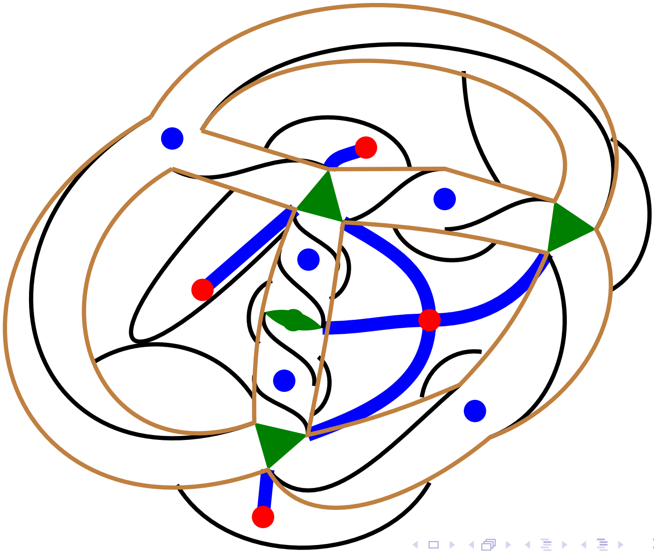

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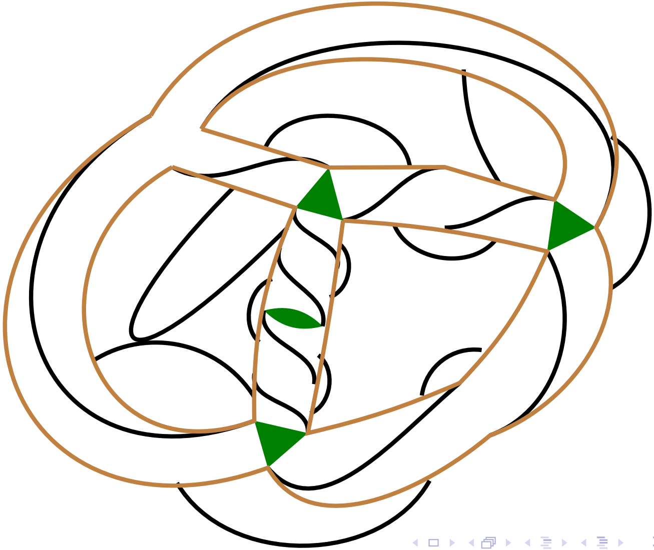

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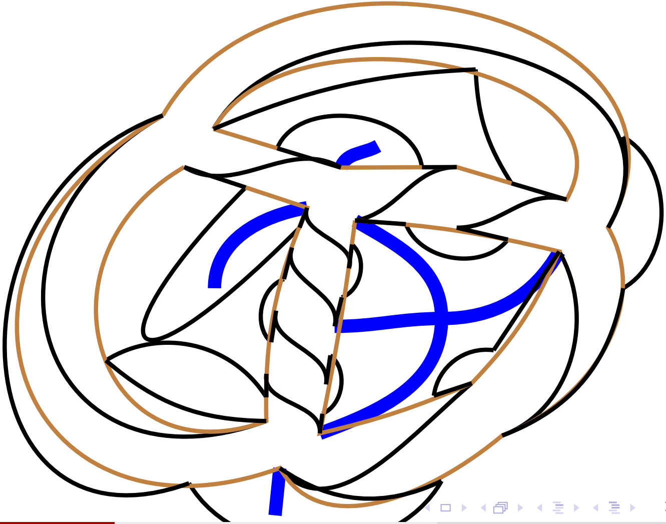

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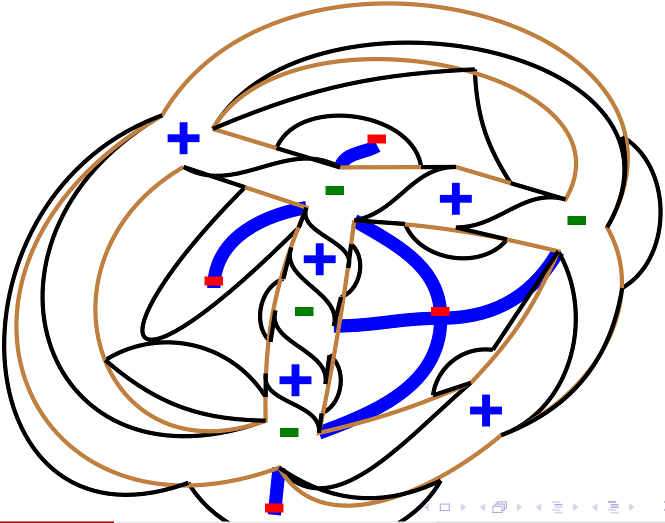

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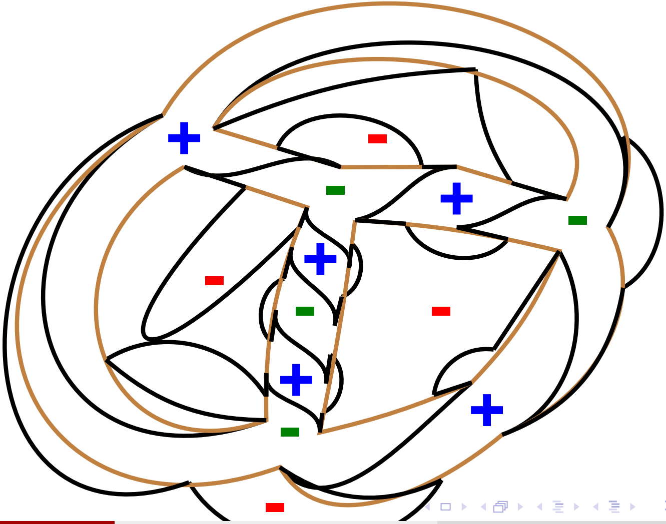

Contact invariants of sutured manifold trinities

What about contact invariants of these contact structures in *SFH*?
We showed there is one tight contact structure ξ for each $\mathbb{Z}$-summand of *SFH*, and $c(\xi)$ is known to lie in this summand.

Theorem (Kálmán–M.)

$c(\xi)$ generates the appropriate $\mathbb{Z}$ summand of $SFH(M_G, L_G)$.

Proof uses Honda–Kazez–Matić TQFT map on SFH:

- Each contact structure on (M_G, L_G) includes into the standard tight contact structure on S^3.

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- $\widehat{HF}(M)$ is the homology of this complex and is an invariant of M (independent of all other choices).

Thanks for listening!

References:

- Kálmán and Mathews, *Tight contact structures on Seifert surface complements*, arXiv:1709.10304
- Mathews, *Polytopes, dualities, and Floer homology*, arXiv:1702.03630