Plane graphs, special alternating links, and contact geometry

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Overview

A mathematical story involving many areas of mathematics:

- Graphs — especially plane & bipartite graphs
- Polytopes (in high-dimensional $\mathbb{R}^n$) — and their lattice points in $\mathbb{Z}^n$
- Knots and links (esp. special alternating links)
- Dualities and trialities (of links, of polytopes, of graphs)
- 3-manifold topology — especially sutured 3-manifolds
- Contact geometry — the odd-dim version of symplectic geometry
- Floer homology — invariants based on holomorphic curves

Work of many people:
- Friedl, Kálmán, Murakami, Postnikov, Rasmussen, Tutte

Recent results:
- Kálmán–M, Tight contact structures on Seifert surface complements, arXiv:1709.10304

Expository paper:
- M, Polytopes, dualities, and Floer homology, arXiv:1702.03630
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Outline

1. Graphs, trinities and trees
   - Bipartite plane graphs

2. Knots and links and 3-manifolds

3. Contact geometry

4. Floer homology

5. Polytopes

6. Further details
Let $G$ be a bipartite plane graph.

- Colour vertices blue/violet and green/emerald.
- Colour edges red.
- Embedded in $\mathbb{R}^2 \subset S^2$. 

\[ 
\text{Bipartite plane graphs} 
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Definition: A 3-coloured triangulation of $S^2$ is called a trinity.
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- Add red vertices in complementary regions
- Connect each red vertex to blue and green vertices around the boundary of its region.
- We can colour each edge by the unique colour distinct from endpoints.
- This yields a 3-coloured graph triangulating $S^2$.

**Definition**

A 3-coloured *triangulation* of $S^2$ is called a *trinity*. 
A trinity naturally contains *three* bipartite planar graphs: take all edges of a single colour.
Bipartite graphs and trinities

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The *violet graph* $G_V$, *emerald graph* $G_E$, *red graph* $G_R$ are all bipartite plane graphs which yield (and are subsets of) the same trinity.

So:
- Bipartite plane graphs naturally come in *threes* (*triality*).

Compare to:
- Plane graphs have duals, so naturally come in *twos* (*duality*).
Trinities and triangulations

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- Triangles sharing an edge must be *opposite* colours.
  (In other words, the planar dual of the trinity is bipartite.)
Planar duals of trinities

Consider the planar dual $G_v^*$ of $G_v$ in a trinity.
Observations on $G^*_V$:

- $G^*_V$ has edges bijective with violet edges.
- Each edge of $G^*_V$ crosses precisely *two* triangles of the trinity and hence is naturally *oriented*, say black to white.
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- Around each vertex of $G_V^*$, edges alternate in and out.
- $G_V^*$ is a *balanced directed planar graph*.
  (Balanced: in-deg = out-deg at each vertex.)
Let $D$ be a directed graph $D$. Choose a root vertex $r$.

**Definition**

A (spanning) arborescence of $D$ is a spanning tree $T$ of $D$ all of whose edges point away from $r$.

- I.e. for each vertex $v$ of $D$ there is a unique directed path in $T$ from $r$ to $v$. 

![Diagram of a directed graph with labeled vertices and directed edges pointing away from the root vertex r.]
Arborescences

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Tutte’s tree trinity theorem

Theorem (Tutte, 1948)

Let $D$ is a balanced finite directed graph. Then the number of spanning arborescences of $D$ does not depend on the choice of root point.

Hence we may define $\rho(D)$, the arborescence number of $D$, to be the number of spanning arborescences.

Theorem (Tutte’s tree trinity theorem, 1975)

Let $G^*_{V}$, $G^*_{E}$, $G^*_{R}$ be the planar duals of the coloured graphs of a trinity. Then $\rho(G^*_{V}) = \rho(G^*_{E}) = \rho(G^*_{R})$.

Many combinatorial questions about trinities have the same answer.

Definition

The magic number $\text{Magic}(G_V)$ of the plane bipartite graph $G$ is $\rho(G^*_V)$. 


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2. Knots and links and 3-manifolds
   - Special alternating links
   - Sutured 3-manifolds

3. Contact geometry

4. Floer homology

5. Polytopes

6. Further details
From plane graphs to knots and links

Given a plane graph $G$, there is a natural way to construct a knot or link $L_G$: the median construction.
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- Take a regular neighbourhood of $G$ in the plane (ribbon).
- Insert a negative half twist over each edge of $G$ to obtain a surface $F_G$. Then $L_G = \partial F_G$. 

![Diagram of a plane graph with a knot or link $L_G$ constructed using the median construction.]
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So if $G$ is a plane bipartite graph:
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- $L_G$ is in fact \textit{special alternating} (no nesting of Seifert circles)
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From plane bipartite graphs to 3-manifolds

Given $L_G$ and $F_G$, remove a neighbourhood $N(F_G)$ of $F_G$ to obtain an interesting 3-manifold $M_G = S^3 \setminus N(F_G)$. 

Topologically, $N(F_G)$ and $M_G$ are handlebodies (solid pretzels). The boundary $\partial M_G$ naturally has a copy of $L_G$ on it. $L_G$ splits $\partial M_G$ into two surfaces, both isotopic to $F_G$. 

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The curves \(\Gamma\) split \(\partial M\) into positive and negative regions,

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\partial M \setminus \Gamma = R_+ \sqcup R_-,
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(I.e. when you cross \(\Gamma\), you go from \(R_+\) to \(R_-\).)
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From a trinity we obtain a triple of sutured manifolds

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(M_{GV}, L_{GV}), \quad (M_{GE}, L_{GE}), \quad (M_{GR}, L_{GR}).
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Outline

1. Graphs, trinities and trees

2. Knots and links and 3-manifolds

3. Contact geometry
   - Contact structures on 3-manifolds
   - Convex surfaces and sutures
   - Classifications of contact structures
   - Our results

4. Floer homology

5. Polytopes

6. Further details
A contact structure on a 3-manifold $M$ is a non-integrable 2-plane field $\xi = \ker \alpha$.

E.g. $\alpha = dz - y \, dx$.

Non-integrability of $\xi \iff \alpha \wedge d\alpha \neq 0$. 
Contact structures

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Basic question of contact topology:

- Given a 3-manifold $M$, understand the (isotopy classes of) contact structures on $M$. 

![Diagram of a 3-manifold with arrows indicating the contact structure and coordinates x, y, z.](image-url)
Two types of contact structures: **tight** and **overtwisted**.
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Classification of overtwisted contact structures:

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A tight contact structure is a non-overtwisted one:

- For a closed oriented atoroidal 3-manifold \( M \), there are finitely many isotopy classes of tight contact structures (Colin–Giroux–Honda 2002).
- But understanding these is more subtle.
Convex surfaces and sutures

For 3-manifolds with boundary:

- *sutures* provide natural boundary conditions for contact structures.
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- This is via Giroux’s theory of **convex surfaces** (1991):
  \[ \Gamma = \text{dividing set}. \]

Roughly: think of $\partial M$ as horizontal, then

- $\Gamma$ is “where $\xi$ is vertical"
- $R_+, R_-$ say “which side of $\xi$ is facing up"
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Existing classification results

A *classification of contact structures* on \((M, \Gamma)\) is an explicit description of all contact structures on \((M, \Gamma)\).

Some known classification results:
- **Ball** \(B^3\) with one suture \(\Gamma\): 1 tight contact structure, \(cs(B^3, \Gamma) = 1\) (Eliashberg \(\approx 1992\))
- **Ball** \(B^3\) with more than one suture: no tight contact structures (already overtwisted!)
- **Sphere** \(S^3\): \(cs = 1\) (Eliashberg 1992)
- **Lens spaces** \(L(p, q)\): \(cs\) depends intricately on \(p, q\) (Giroux, Honda \(\sim 2000\))
- **Solid tori** with two sutures: \(cs\) depends intricately on slope of sutures (Giroux, Honda \(\sim 2000\))
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Some known classification results:

- **Ball** \(B^3\) with one suture \(\Gamma\): 1 tight contact structure, \(cs(B^3, \Gamma) = 1\) (Eliashberg \(\approx 1992\))
- **Ball** \(B^3\) with more than one suture: no tight contact structures (already overtwisted!)
- **Sphere** \(S^3\): \(cs = 1\) (Eliashberg 1992)
- **Lens spaces** \(L(p, q)\): \(cs\) depends intricately on \(p, q\) (Giroux, Honda \(\sim 2000\))
- **Solid tori with two sutures**: \(cs\) depends intricately on slope of sutures (Giroux, Honda \(\sim 2000\))
Contact structures on Seifert surface complements

With Kálmán, we investigated this question for sutured manifolds $(M_G, L_G)$:
- bipartite plane graph $G \rightsquigarrow$ sutured manifold $(M_G, L_G)$. 
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**Theorem (Kálmán-M.)**

$\text{cs}(M_G, L_G)$ is equal to the magic number of $G$. 

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With Kálmán, we investigated this question for sutured manifolds \((M_G, L_G)\):
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**Theorem (Kálmán-M.)**

\(cs(M_G, L_G)\) is equal to the magic number of \(G\).

Hence for a trinity with a triple of bipartite graphs \(G_V, G_E, G_R\),

\[
\begin{align*}
\text{cs}(M_{G_V}, L_{G_V}) &= \text{cs}(M_{G_E}, L_{G_E}) = \text{cs}(M_{G_R}, L_{G_R}) = \text{Magic}(G) \\
&= \rho(G_V^*) = \rho(G_E^*) = \rho(G_R^*)
\end{align*}
\]
Outline

1. Graphs, trinities and trees
2. Knots and links and 3-manifolds
3. Contact geometry
4. Floer homology
   - The idea of Heegaard Floer homology
   - Contact invariants of Seifert surface complements
5. Polytopes
6. Further details
Heegaard Floer homology

Heegaard Floer homology is a theory which gives invariants of various 3-dimensional objects:

- Introduced by Ozsváth–Szabó ≈ 2004
- Now many versions... for closed 3-manifolds, knots and links

Powerful invariant: categorifies Alexander polynomial, computes genus of knots, often hard to compute!

The version useful here: Sutured Floer homology (SFH) is an invariant of sutured 3-manifolds, introduced by Juhász ≈ 2006.

All versions define chain complexes in which the differential counts pseudoholomorphic curves in some related manifold. The homology then turns out to be invariant.
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Sutured Floer homology and contact structures

Sutured Floer homology yields, very roughly,

\[(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)\]

Sutured 3-mfld \quad finitely gen. abelian group

Properties:

- \(SFH(M, \Gamma)\) is graded (in several ways)
  
  - Think of \(SFH(M, \Gamma)\) as an array of groups
  
  - One group at each point of a lattice \(\mathbb{Z}^d\)

Heegaard Floer homology gives invariants of contact structures \(\xi\) (Ozsváth–Szabó, Honda–Kazez–Matić \(\approx 2005\)):

- Contact structure \(\xi\) on \((M, \Gamma)\) \(\mapsto c(\xi) \in SFH(-M, -\Gamma)\).
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Contact invariants of Seifert surface complements

Return to a plane bipartite graph $G$ and the sutured manifold $(M_G, L_G)$. 

Standard contact geometry/topology questions:

1. What is $SFH(M_G, L_G)$?
2. What are the (isotopy classes of) tight contact structures $\xi$ on $(M_G, L_G)$?
3. What are the contact invariants $c(\xi) \in SFH$?

Answer to Q1 is known:

Theorem (Juhász–Kálmán–Rasmussen (2012))

$SFH(M_G, L_G) \cong \mathbb{Z} \cdot \text{Magic}(G)$. 

Using grading of $SFH$, the $\text{Magic}(G) \cdot \mathbb{Z}$ summands are naturally arranged at the lattice points of a polytope $P_G$ in $\mathbb{Z}^{R-1}$.

Here $R = \# \text{complementary regions of } G$. 

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Here $R = \#$ complementary regions of $G$. 
SFH for bipartite graph complement

Answer to Q2 given earlier: \( cs(M_G, L_G) = \text{Magic}(G) \). But more:

Theorem (Kálmán–M)

The distinct tight contact structures \( \xi \) on \( (M_G, L_G) \) have distinct Euler classes \( e(\xi) \in H^2(M_G, \partial M_G) \sim = \mathbb{Z} \).

These Euler classes, points in \( \mathbb{Z} \), are naturally arranged at the lattice points of \( P_G \).

Answer to Q3 (what are contact invariants of contact structures in SFH?)

Theorem (Kálmán–M)

The contact invariants \( c(\xi) \) of the tight contact structures \( \xi \) on \( (M_G, L_G) \) are precisely generators of the corresponding \( \mathbb{Z} \) summands of SFH.
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The contact invariants $c(\xi)$ of the tight contact structures $\xi$ on $(M_G, L_G)$ are precisely generators of the corresponding $\mathbb{Z}$ summands of SFH.
SFH of a trinity

It follows that

\[ SFH(S^3 - F_{G_V}, L_{G_V}), \quad SFH(S^3 - F_{G_E}, L_{G_E}), \quad SFH(S^3 - F_{G_R}, L_{G_R}) \]

all have dimension given by magic number. Same \# \mathbb{Z} summands, but arranged in different polytopes!
Outline

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Polytopes from hypergraphs

What are these polytopes arising in contact topology & Floer homology of $(M_G, L_G)$?

Back to graph theory: Combinatorial theory of polytopes associated to graphs & hypergraphs (Postnikov, Kálmán).

Consider hypergraphs.

A graph has edges — each edge joins two vertices.

A hypergraph has hyperedges — each hyperedge joins many vertices.

Definition

A hypergraph is a pair $(V, E)$, where $V$ is a set of vertices and $E$ is a set of hyperedges. Each hyperedge is a nonempty subset of $V$.

Hypergraphs $\sim$ bipartite graphs: A hypergraph $(V, E)$ can be drawn as a bipartite graph $\text{Bip}(V, E)$.

A bipartite graph can be considered as a hypergraph (in 2 ways).
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Hypertrees in hypergraphs

Bip($V, E$)
$E = \{a, b, c\}$

Spanning tree, hypertree $f$
$(f(a), f(b), f(c)) = (1, 0, 2)$
Hypertrees in hypergraphs

Consider spanning trees in a hypergraph \((V, E)\).

**Definition**

A hypertree in \((V, E)\) is a function \(f : E \to \mathbb{N}_0\) such that there exists a spanning tree in \(\text{Bip}(V, E)\) with degree \(f(e) + 1\) at each \(e \in E\).
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When \((V, E)\) a graph, hypertrees = trees: choose edges with \(f(e) = 1\).
The hypertree polytope

A hypertree \( f : E \rightarrow \mathbb{N}_0 \) can be regarded as a point of \( \mathbb{Z}^E \).

Let \( Q_{(V,E)} = \{ f \in \mathbb{Z}^E \mid f \text{ is a hypertree of } (V,E) \} \).

![Diagram showing a bipartite graph and hypertree]

- Bip\((V,E)\)
- \( E = \{a, b, c\} \)
- Spanning tree, hypertree \( f \)
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**Theorem (Postnikov 2009, Kálmán 2013)**

$Q_{(V,E)}$ consists of a magic number of points, forming the lattice points of a convex polytope in $\mathbb{R}^E$.

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7 hypertrees
Hypertree polytopes and contact structures

This polytope reappears as the Euler classes of tight contact structures on \((M_G, L_G)\) and the \(\mathbb{Z}\) summands of \(SFH(M_G, L_G)\).

Proofs are constructive: we obtain explicit bijections

\[
\{ \text{contact structures on } (M_{GR}, L_{GR}) \} \overset{\sim}{\cong} \{ \text{hypertrees on } (E, R) \} \overset{\sim}{\cong} \{ \mathbb{Z} \text{ summands of } SFH(M_{GR}, L_{GR}) \}.
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Outline

1. Graphs, trinities and trees
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4. Floer homology
5. Polytopes
6. Further details
   - Classification of contact structures
   - Computing contact invariants
   - Details of Heegaard Floer homology
Theorem (Kálmán–M.)

The number of isotopy classes of tight contact structures on $(S^3 - F_G, L_G)$ is given by $\rho$, the magic number of the trinity.

Proof ideas:

1. $S^3 - F_G$ can be cut into two 3-balls by $|R|$ convex discs in the complementary regions of $G$.
   A choice of dividing set $\Gamma_i$ on each $D_i$ determines at most one tight contact structure on $(S^3 - F_G, L_G)$.

2. A spanning tree $T$ representing a hypertree yields a dividing set on each $D_i$ by taking the boundary of a ribbon.

3. Analyse bypasses (small contact isotopies) across the discs $D_i$ between the two 3-balls, use a gluing theorem of Honda to prove no overtwisted discs exist — contact structures are tight.

4. Kálmán's previously showed there are $\rho$ hypertrees; show two spanning trees representing the same hypertree produce isotopic contact structures.
Proof of classification of contact structures

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Contact structures on trinities

Consider complements of tubular neighbourhood of $G$: 

![Diagram of a trinity with contact structures]

- $D_0$
- $D_1$
- $D_2$
- $D_3$
Contact structures on trinities

Consider complements of tubular neighbourhood of $G$: 
Contact structures on trinities

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Contact structures on trinities

Now just take one side of the plane:
Contact structures on trinities

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A spanning tree in $(E, R)$
Contact structures on trinities

A spanning tree in \((E, R)\) yields a dividing set
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Now round corners and consider dividing set: it is a neighbourhood of tree, hence connected, so gives tight 3-balls.
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Now round corners and consider dividing set: it is a neighbourhood of tree, hence connected, so gives tight 3-balls.
What about contact invariants of these contact structures in $SFH$? We showed there is one tight contact structure $\xi$ for each $\mathbb{Z}$-summand of $SFH$, and $c(\xi)$ is known to lie in this summand.

**Theorem (Kálmán–M.)**

$c(\xi)$ generates the appropriate $\mathbb{Z}$ summand of $SFH(M_G, L_G)$.

Proof uses Honda–Kazez–Matić TQFT map on SFH:

- Each contact structure on $(M_G, L_G)$ includes into the standard tight contact structure on $S^3$. 
Very rough idea of Heegaard Floer homology (of closed 3-manifold $M$):

1. Take a Heegaard decomposition $(\Sigma, \alpha, \beta)$ of $M$.
   - $\Sigma$ is a closed surface of genus $g$.
   - $\alpha = \{\alpha_1, \ldots, \alpha_g\}$, $\beta = \{\beta_1, \ldots, \beta_g\}$ are Heegaard curves.

2. Consider the $2g$-dimensional symmetric product $\text{Sym}^g \Sigma$ and the $g$-dimensional tori given by $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $T_\beta = \beta_1 \times \cdots \times \beta_g$ in $\text{Sym}^g \Sigma$.

3. Form a chain complex generated by intersection points $T_\alpha \cap T_\beta$.

4. Consider holomorphic curves in $\text{Sym}^g \Sigma$, i.e., $u: S \to \text{Sym}^g \Sigma$ satisfying Cauchy–Riemann equations, where $S$ is a Riemann surface.

5. For $x, y \in T_\alpha \cap T_\beta$, consider holomorphic curves "from" $x$ "to" $y$ in $\text{Sym}^g \Sigma$.

6. Such curves which are suitably rigid give a boundary operator $\partial$ and $\partial^2 = 0$.

7. $\hat{\text{HF}}(M)$ is the homology of this complex and is an invariant of $M$ (independent of all other choices).
Rough idea of Heegaard Floer homology

Very rough idea of Heegaard Floer homology (of closed 3-manifold $M$):

- Take a Heegaard decomposition $(\Sigma, \alpha, \beta)$ of $M$.
  - $\Sigma$ a closed surface of genus $g$
  - $\alpha = \{\alpha_1, \ldots, \alpha_g\}$, $\beta = \{\beta_1, \ldots, \beta_g\}$ Heegaard curves.

Consider the $2g$-dimensional symmetric product $\text{Sym}_g \Sigma$ and the $g$-dimensional tori $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $T_\beta = \beta_1 \times \cdots \times \beta_g$ in $\text{Sym}_g \Sigma$.

Form a chain complex generated by intersection points $T_\alpha \cap T_\beta$.

Consider holomorphic curves $\text{holomorphic curves}$ in $\text{Sym}_g \Sigma$, i.e., $u : S \to \text{Sym}_g \Sigma$ satisfying Cauchy–Riemann equations, where $S$ is a Riemann surface.

For $x, y \in T_\alpha \cap T_\beta$, consider holomorphic curves "from" $x$ "to" $y$ in $\text{Sym}_g \Sigma$.

Such curves which are suitably rigid give a boundary operator $\partial$ and $\partial^2 = 0$.

$\hat{\text{HF}}(M)$ is the homology of this complex and is an invariant of $M$ (independent of all other choices).
Rough idea of Heegaard Floer homology

Very rough idea of Heegaard Floer homology (of closed 3-manifold $M$):

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- $\widehat{HF}(M)$ is the homology of this complex and is an invariant of $M$ (independent of all other choices).
Thanks for listening!

References:

- Kálmán and Mathews, *Tight contact structures on Seifert surface complements*, arXiv:1709.10304