The Tutte polynomial and knot theory

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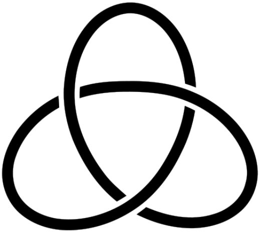
Key point:

The Tutte polynomial is relevant not just to graph theory, but also to *topology* — in particular, to *knot theory*.



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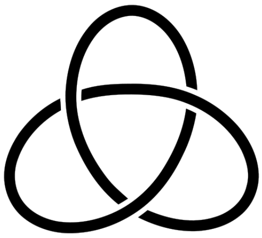


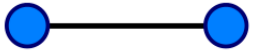
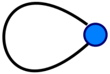
Knots and graphs are both:

- 1-dimensional objects
- hard to tell apart
- have many invariants, few of them complete

The Tutte polynomial

$$\text{Graph } G \quad \rightsquigarrow \quad \text{Tutte polynomial } T_G(x,y)$$

Can be defined recursively:

0 edges		$\rightsquigarrow$	1
1 edge		$\rightsquigarrow$	x
		$\rightsquigarrow$	y
Graph with a bridge e	$T_G(x,y)$	$=$	$xT_{G/e}(x,y)$
Graph with a loop e	$T_G(x,y)$	$=$	$yT_{G-e}(x,y)$
Any other edge e	$T_G(x,y)$	$=$	$T_{G-e}(x,y) + T_{G/e}(x,y)$

Specialisations of $T_G(x,y)$ give other polynomial invariants.

- E.g. chromatic polynomial

Knot polynomials

Knots, more generally *links* $L \subset \mathbb{R}^3$, have *polynomial invariants*.

- Alexander polynomial $\Delta_L(t)$ (1923)
- Jones polynomial $V_L(t)$ (1984)
- HOMFLY-PT polynomial (1985) (Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter; Przytycki, Traczyk) :
  - $P_L(a, z)$ (2-var) or $P_L(x, y, z)$ (homogeneous 3-var)

These can all be defined by *skein relations*.

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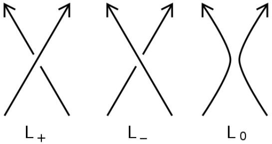
These can all be defined by *skein relations*. For HOMFLY:

$$\text{Unknot} \rightsquigarrow 1$$

$$xP_{L_+} + yP_{L_-} + zP_{L^0} = 0$$



Specialisations of $P_L(x, y, z)$ give Jones and Alexander polynomials.

From plane graph G, construct a link $D(G)$:

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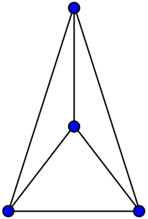

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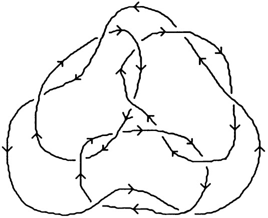

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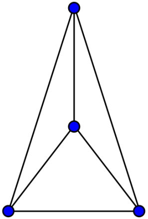

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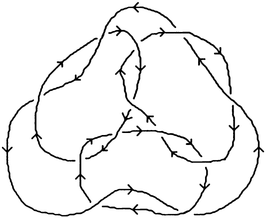

Theorem (Francois Jaeger 1988)

For a connected plane graph G,

$$P_{D(G)}(x,y,z) = \left(\frac{y}{z}\right)^{V(G)-1} \left(-\frac{z}{x}\right)^{E(G)} T_G\left(-\frac{x}{y},\ \frac{-(xy+y^2)}{z^2}\right)$$

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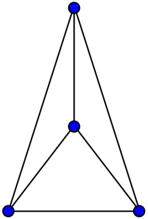

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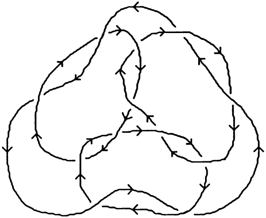

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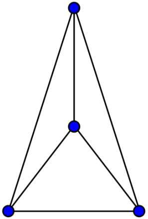
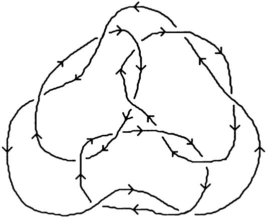
 $\rightsquigarrow$ 

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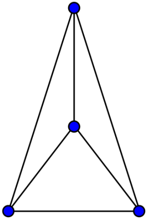

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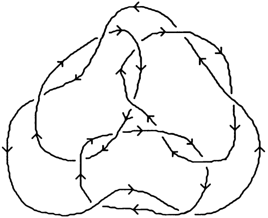

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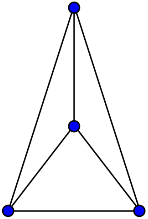
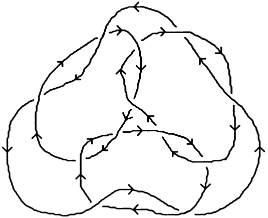
 $\rightsquigarrow$ 

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Corollaries:

- G has a four-colouring if and only if $P_{D(G)}(3,1,2) \neq 0$
- Calculating the HOMFLY polynomial is NP-hard.