Some pure mathematics and Consciousness

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Overview

Two (mostly unrelated) topics:

① "Towards a neuronal gauge theory"
Biswa Sengupta, Arturo Tozzi, Gerald K Cooray,
Pamela K Douglas, Karl J Friston
PLOS Biology March 2016

→ Try to explain what this paper is on about...

② Categories, Topology & the Brain

→ Some prominent ideas in contemporary category theory & topology possibly of interest...

Disclaimer: I am a pure mathematician! Pls: Interrupt! Correct! Ask qns! Explain!

Towards "Towards a neuronal gauge theory"

Neuronal Gauge Theory...

Gauge Theory

Using gauge symmetries to simplify calculations

Information Geometry

Using coordinate changes to simplify calculations

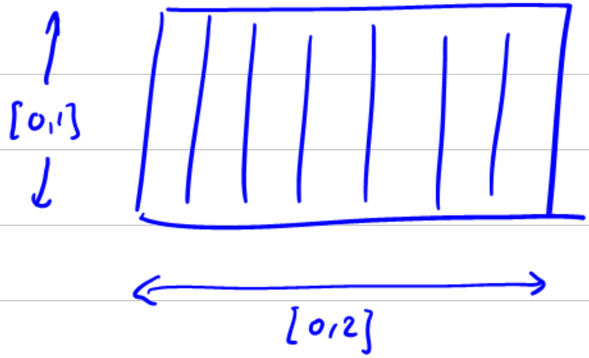
Connections & gauge transformations

Bayesian Brain / Free Energy Max

(Fibre / vector) Bundles

Neurons

Bundles — Simplest possible example!



E = Rectangle = $[0,2] \times [0,1]$ } Total space

constructed from

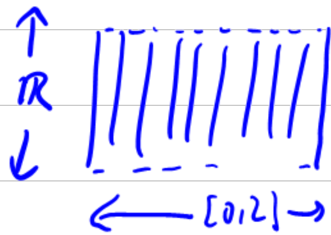
"vertical line" fibres $F_b = \{b\} \times [0,1]$

Each fibre is a copy of the same space $F = [0,1]$ } Fibre

Fibres parametrised by points $b \in [0,2] = B$ } Base space

— Same total space, fibred another way!

Bigger version!



"Infinitely high" rectangle $E = [0,2] \times \mathbb{R}$ } A vector bundle!

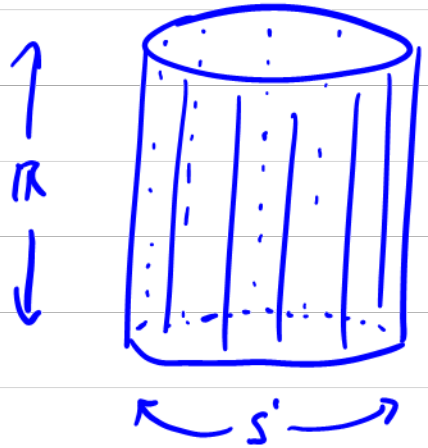
Fibre space is $\mathbb{R}$ ← A vector space!

Bundles - Slightly more complicated ones!

Take fibres $F = \mathbb{R}$ over base B = (unit) circle $= S^1$

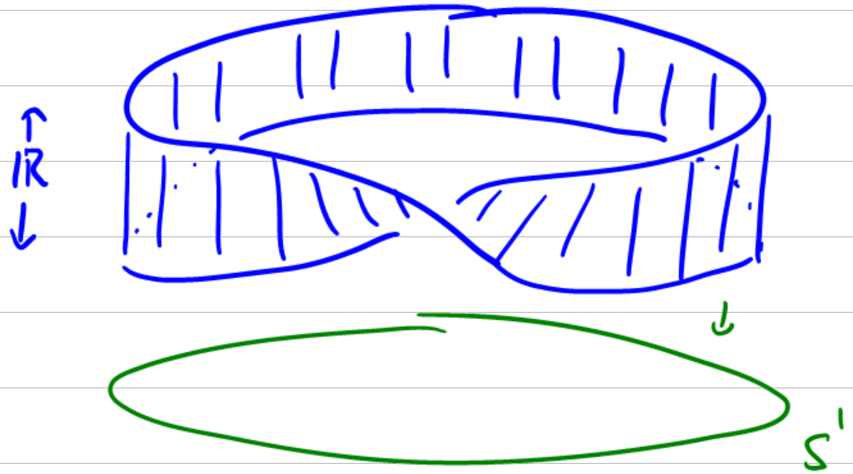
$= \bigcirc$

"a circle's worth of $\mathbb{R}$ lines"



Cylinder $E = F \times B = \mathbb{R} \times S^1$

A product/trivial $\mathbb{R}$-bundle over S^1



The Möbius strip is a nontrivial $\mathbb{R}$-bundle over S^1.

(These are all vector bundles since the fibre is a 1-D vector space $\mathbb{R}$.)

Bundles in General

$$F \hookrightarrow E \xrightarrow{p} B$$

In general bundles can be constructed with any topological spaces B, F as base and fibre.

The total space is a union of fibres $E = \bigcup_{b \in B} F_b$

which "locally looks like a product.

Important example: Consider a smooth surface (manifold) S, and tangent vectors v to S at a point $p \in S$.

The set of tangent vectors to S at p

$T_pS \cong \mathbb{R}^2$ is a vector space.

The set of all tangent vectors at points of S

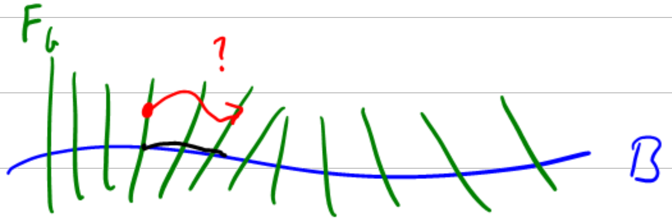
$$\{(p, v) : p \in S,\ v \in T_pS\} = \bigcup_{p \in S} T_pS$$ is a bundle over S with fibre $\mathbb{R}^2$.

$\longrightarrow$ the tangent bundle TS of S.

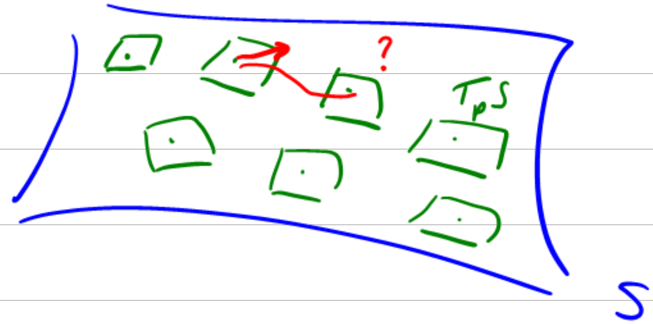
(More complicated examples:
* bundles of tensors over a manifold
* bundles of Lie groups/algebras)

Connections & Gauge

In a fibre bundle, we often draw fibres "vertically" & base "horizontally".



(but not always, e.g.



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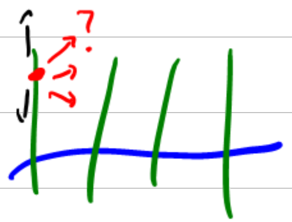
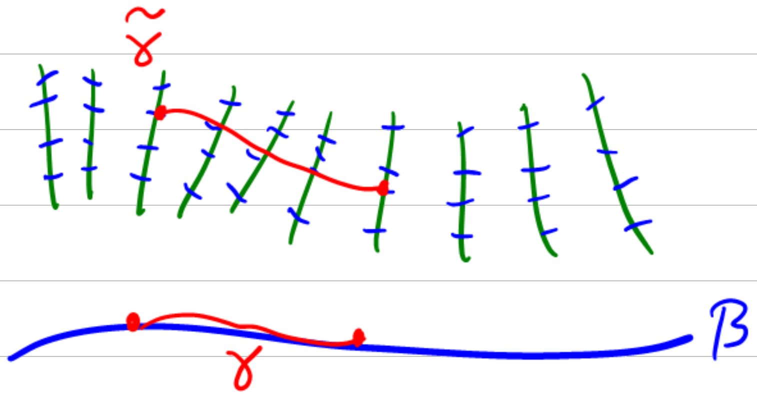
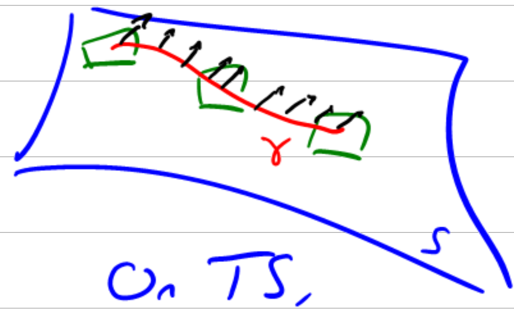


Moving within a bundle, there is a natural notion of "vertical" but no natural "horizontal" direction.

A choice of horizontal direction at each point is called a <u>connection</u> or choice of <u>gauge</u>



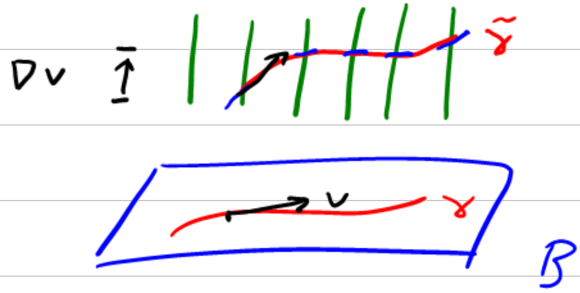
Given a path γ on B, it can be <u>lifted</u> to a horizontal path in E, thus <u>connecting</u> points in distinct fibres.



On TS, connection $\Leftrightarrow$ parallel transport $\Leftrightarrow$ covariant derivative ∇

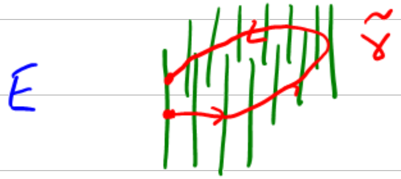
Calculus on Bundles – Covariant Derivative & Curvature

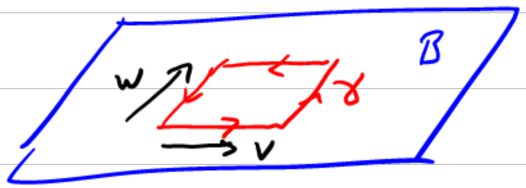
A connection tells you how much you move "vertically" (along fibre) when you move in a certain direction along the base



Gives a covariant derivative (often written ∇ or A) which is a differential 1-form on the base B

For a tangent vector v to B, $\nabla v = A(v) =$ infinitesimal movement along fibre as you go in v direction.

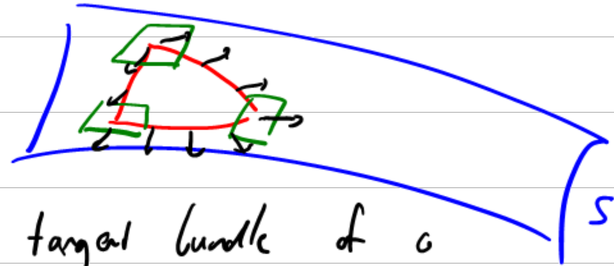




Now if you go around a loop γ on B, it lifts to a path in E which may or may not be a loop!

If γ̃ always a loop, the bundle is flat

If not, the bundle is curved.



Curvature is measured by the curvature 2-form $F = dA$

$$F(v,w) = \begin{pmatrix} \text{infinitesimal displacement along} \\ \text{fibre as you go around a} \\ \text{parallelogram spanned by } v, w \end{pmatrix}$$

On tangent bundle of a Riemannian manifold,

$A = \nabla =$ Levi-Civita connection

$F =$ Riemann curvature tensor

Gauge Theories in Physics - Electromagnetism

Many physical theories involve bundles & connections (Yang-Mills, Standard Model..)

Simplest example: Maxwell's equations / electromagnetism
describing electric & magnetic fields $\vec{E}, \vec{B}$, charge density ρ, current density $\vec{j}$.

$$\left.\begin{aligned} \nabla\cdot\vec{E} &= 4\pi\rho \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{B} &= 4\pi\vec{J} + \frac{\partial\vec{E}}{\partial t} \end{aligned}\right\} \iff \begin{aligned} dF &= 0 \\ d^*F &= 4\pi {}^*J \end{aligned}$$

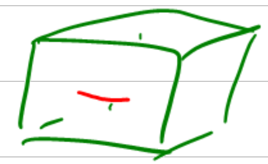
where $F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$, $J = \begin{pmatrix} \rho \\ j_x \\ j_y \\ j_z \end{pmatrix}$

Now Poincaré lemma says $dF = 0 \Rightarrow F = dA$ for some 1-form A: <u>electromagnetic potential</u>
There are many such A's. Replace $A \to A + d\theta$ for any real-valued function θ

$$d(A + d\theta) = dA + dd\theta = dA.$$

A can be regarded as a <u>connection</u> on a <u>bundle</u> over spacetime!

Many choices for A = choices of gauge.
Good choice of A simplifies analysis!



A is not fictitious e.g. Aharonov-Bohm effect.

Fundamentals of a Gauge Theory - Lagrangian formulation

Electromagnetism, like many other physical theories, has a Lagrangian formulation:

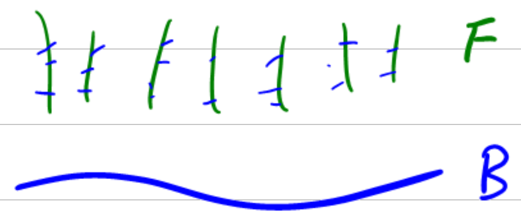
→ Specify a Lagrangian function L (= "kinetic energy - potential energy")

→ Dynamics can be recovered by finding "critical points" of L

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu$$

→ Different choices of gauge (gauge transformations) $(A \mapsto A + d\theta)$ leave L invariant.

Thus, a gauge theory (in Lagrangian form) has:



0. A setup involving a bundle over a manifold (hence connections/gauges A, with curvature F)

1. A Lagrangian function of A, F, (other parameters) which is invariant under gauge transformations and whose critical points describe dynamics.

2. Hence, a dynamical system with a vast family of symmetries
   - varying connection gives a continuous/smooth family of symmetries.

Using Symmetries to Simplify Problems

When a system is formulated in equations with many symmetries, we can change variables / coordinates to simplify the problem!

E.g. from Sengupta et al: Rotating frames & Coriolis effect!
(NB: Not gauge theory!)

For an object moving relative to a rotating object (like Earth),

$$\vec{F} = m\vec{a} + m\left(\frac{d\vec{\omega}}{dt} \times \vec{r}\right) + 2m\left(\vec{\omega} \times \vec{v}\right) + m\left(\vec{\omega} \times (\vec{\omega} \times \vec{r})\right)$$

($\vec{F}$ = force, $\vec{r}$ = position, $\vec{v}$ = velocity, $\vec{a}$ = acceleration, $\vec{\omega}$ = angular velocity of rotating frame)

But if we change coordinates to an inertial reference frame,

$\vec{F} = m\vec{a}$ — much simpler!

Similar / greater simplifications can occur in gauge theories by a clever choice of gauge!

Gauge Theory for the Brain

Sengupta et al suggest applying gauge theory to the brain, requiring:

0. A bundle over a manifold
1. A Lagrangian function invariant under gauge transformations
2. Hence, a dynamical system with large continuous families of symmetries

They propose using
* Bayesian brain hypothesis
* Statistical manifold: Manifold of probability distributions in Bayesian brain
* Variational free energy as Lagrangian.

But they do not give
* Any idea what bundle to use
* Any description of a gauge / connection on a statistical manifold

Indeed the details of fibre bundles on statistical manifolds barely exist in the mathematical literature!

The Bayesian Brain and Statistical Manifolds

Idea: Brain function as Bayesian inference.

Brain maintains statistical models of outside world (perception, action, etc)
Models involve probability distributions with parameters $(\theta_1, \theta_2, \ldots, \theta_N) = \vec{\theta}$

$p(\vec{x}, \vec{\theta})$ — probability of data $\vec{x}$ using parameters $\vec{\theta}$

The full set of possible parameters $\{(\theta_1, \ldots, \theta_N)\}$ forms an (N-dimensional) statistical manifold with nice structure:

* Kullback-Leibler divergence $D_{KL}[p(\vec{x}, \vec{\alpha}) : p(\vec{x}, \vec{\beta})] = \int d\vec{x} \; p(\vec{x}, \vec{\alpha}) \log \frac{p(\vec{x}, \vec{\alpha})}{p(\vec{x}, \vec{\beta})}$

* Fisher-Rao information metric $g_{ij} = \int d\vec{x} \; \frac{\partial \log p(\vec{x}, \vec{\theta})}{\partial \theta_i} \frac{\partial \log p(\vec{x}, \vec{\theta})}{\partial \theta_j}$

* Distributions have a (variational) free energy and one can argue that maximising free energy ~ Bayesian inference...

Neuronal Gauge Theory

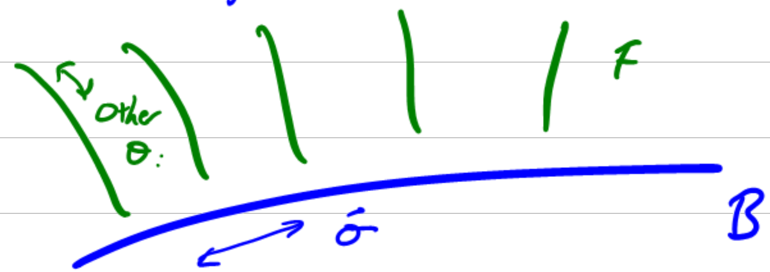
Manifold ⟶ Statistical manifold of neuronal probability distributions
Lagrangian ⟶ Variational free energy
Bundle ⟶ ???
Connector ⟶ ???

Some ideas:

* Taking a <u>subset</u> of (important?) parameters $\{\sigma_1, \sigma_2, .., \sigma_M\} \subset \{\theta_1, .., \theta_N\}$,

  consider the manifold $B = \{(\sigma_1 ... \sigma_M)\}$.
  Then the statistical manifold is a <u>product bundle</u>

  $$E = B \times F$$



Qn: What do connections mean in this context?

* Consciousness as "quantum field theory of statistical manifold"
  (Bayesian Penrose objective state reduction ??)

Category Theory & Consciousness

Idea: Use mathematical ideas from category theory to understand consciousness

→ More particularly, the relationship between consciousness & various theories

(E.g. Tsuchiya - Taguchi - Saigo 2015)

Question: If we accept this idea, then can we expand/refine our understanding of such relationships by using further ideas from category theory?

E.g.

→ Enriched categories

→ Higher categories

→ Topological invariants

(Reimann et al, "Cliques of neurons bound into cavities provide a missing link between structure & function", Frontiers in Computational Neuroscience 2015)

Category Theory ("Abstract Nonsense")

Definition: A category $\mathcal{C}$ consists of

* a collection of objects $ob(\mathcal{C}) = \{A, B, C, \dots\}$
* a collection of morphisms / arrows $= \{a, b, c, \dots\} = Mor(\mathcal{C})$

such that

(i) each morphism has a source and target object $\quad A \xrightarrow{f} B$

(ii) morphisms compose associatively $\quad A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{g \circ f} C$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

(iii) each object A has an identity morphism $A \xrightarrow{1_A} A$ and $1_A \circ f = f$, $f \circ 1_A = f$.

No notion of: functions, invertible, bijection, $+/-/\times/\div$, group operation, or anything really!
("Empty vessel") $\rightsquigarrow$ Examples appear everywhere!

Mathematical field	Algebra	Topology	Geometry	Linear algebra
Objects	Groups	Spaces	(Riem.) manifolds	Vector spaces
Morphisms	Homomorphisms	Continuous maps	Smooth maps (isometric)	Linear maps

Functors — "Functions on categories"

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ consists of

* an object function $F_0 : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
* a morphism function $F_1 : Mor(\mathcal{C}) \to Mor(\mathcal{D})$ which takes any morphism $A \xrightarrow{f} B$ of $\mathcal{C}$ to a morphism $F_0(A) \xrightarrow{F_1(f)} F_0(B)$ of $\mathcal{D}$.

such that identity and composition is preserved.

$$F_1(1_A) = 1_{F_0(A)} \qquad F_1(g \circ f) = F_1(g) \circ F_1(f)$$

$$\begin{tikzcd} A \arrow[loop right, "1_A"] \arrow[r, squiggly, "F"] & F_0(A) \arrow[loop right, "1_{F_0(A)}"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dd, bend right=60, "g \circ f"'] & & F(A) \arrow[d, "F(f)"] \arrow[dd, bend left=60, "F(g \circ f)"] \\ B \arrow[d, "g"] \arrow[rr, squiggly, "F"] & & F(B) \arrow[d, "F(g)"] \\ C & & F(C) \end{tikzcd}$$

Functors map objects from one category to another, which may be of a completely different nature!

But they preserve some "essential" "abstract" structure of morphisms.

More structure in category $\rightsquigarrow$ functors give more information.

Examples (using nothing but numbers and sets!)

① $\underline{3}$ $\quad Ob = \{0,1,2\}$

A morphism $a \to b$ whenever $a \leq b$

$$\begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] \arrow[loop below] & 1 \arrow[r] \arrow[loop below] & 2 \arrow[loop below] \end{tikzcd}$$

② More generally, for any set S with an ordering $\leq$ on elements.

Objects = Elements of S

A morphism $a \to b$ whenever $a \leq b$.

④ Any order-preserving map
$F: \{0,1,2\} \to \{0,1,2,3\}$
yields a functor
$F: \underline{3} \to \underline{4}$

⑤ More generally, any map between ordered sets $F: (S, \leq) \to (T, \leq)$ yields a functor
$F: S \to T$.

③ For any set S consider its subsets: these form $\mathbb{P}(S)$, the power set of S.

Objects: subsets of S

A morphism $a \to b$ whenever $a \subseteq b$.

$S = \{0,1\}$

$$\begin{tikzcd} & \{0\} \arrow[dr] \arrow[loop above] & \\ \phi \arrow[ur] \arrow[rr] \arrow[dr] \arrow[loop above] & & \{0,1\} \arrow[loop right] \\ & \{1\} \arrow[ur] \arrow[loop below] & \end{tikzcd}$$

Examples – Functors between completely different objects!

① Homology tells you about topological spaces via algebra!

$$\left\{ \underline{\text{Top}} \quad \begin{array}{l} \text{Objects} = \text{Topological spaces} \\ \text{Morphisms} = \text{Continuous maps} \end{array} \right\} \xrightarrow{H_n} \left\{ \underline{\text{Groups}} \quad \begin{array}{l} \text{Objects} = \text{groups} \\ \text{Morphisms} = \text{group homomorphisms} \end{array} \right\}$$

E.g. $X = S^1 =$ circle :  $\rightsquigarrow H_1(S^1) = \mathbb{Z}$

$X =$ donut / torus / coffee cup $\rightsquigarrow H_1(S^1) = \mathbb{Z}^2$

② Calculus / Derivative is 'really' a functor from manifolds to bundles!

$$\left\{ \underline{\text{Man}} \quad \begin{array}{l} \text{Objects} = \text{smooth manifolds} \\ \text{Morphisms} = \text{smooth maps} \end{array} \right\} \xrightarrow{D} \left\{ \underline{\text{Vec. Bdl.}} \quad \begin{array}{l} \text{Objects} = \text{Vector bundles} \\ \text{Morphisms} = \text{Bundle maps} \end{array} \right\}$$

 $\longrightarrow$ 

Functors, brain, consciousness

Tsuchiya et al 2014:
Consider categories / functors relating Integrated Information Theory (IIT) to qualia

Integrated Information Theory		Qualia
Systems / subsystems	?	Intrinsic existence
Maximally irreducible conceptual structure (MICS)	$\rightsquigarrow$	Composition
Cause/effect information		Information
Integrated information		Integration

IIT (Oizumi et al eg 2014) proposes that qualia be identified with MICS...
Tsuchiya et al idea of a functor is a less demanding (more plausible?) condition.

Category theory has several notions of categories being "the same" or "similar"
→ Identity, isomorphism, equivalence, adjunctions...

But categories have so little structure
→ it may be worth considering <u>enriched</u> categories

Enriched Categories

A very versatile way to "add a little more structure" to a category than merely objects & morphisms.
Best defined by examples.

A <u>category enriched over vector spaces</u> is a category $\mathcal{C}$ such that for any two objects X, Y, the morphisms $X \to Y$ form a <u>vector space</u>

(I.e. you can add/subtract/scalar multiply morphisms!

$$X \xrightarrow{\substack{f \\ g \\ 2f \\ f+3g}} Y \xrightarrow{h} Z \qquad \text{Compose!} \qquad X \xrightarrow{\substack{h\circ f \\ h\circ g \\ 2\,h\circ f \\ h\circ f + 3\,h\circ g}} Z$$

In general, categories can be enriched over any (monoidal) category!

Enriched Categories — Speculations

① Perhaps apply ideas of Stanley of "Qualia space"?

Closed pointed cones in ∞-dim separable real topological vector space.

$X \longrightarrow Y$ Enriched category over

"state" "state" ∞-dim sep $\mathbb{R}$ top. vec. spaces?

$Mor(X,Y)$

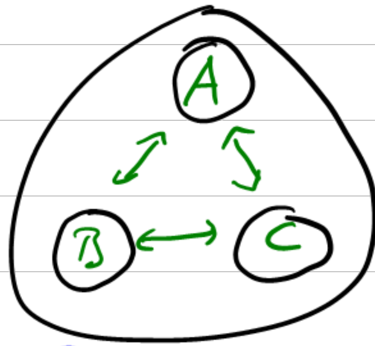
∞-dim sep $\mathbb{R}$ top vec space?

Qn: In what ways (i.e. over which categories) might it be relevant to enrich categories to describe qualia?

② IIT discusses many "mechanisms" & "composite mechanisms"

Objects = mechanisms (composite or not)? MICS?

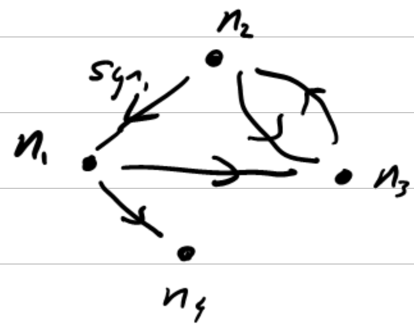
Morphisms = inclusions of mechanisms / MICS?



Inclusion of a sub-mechanism into a larger mechanism has associated information integration — enrich to incorporate this?

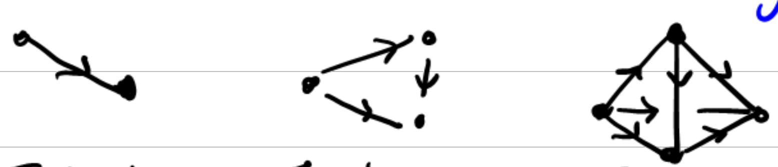
An interesting recent application of category theory & topology to the brain

M. W. Reimann, M. Nolte, M. Scolamiero, K. Turner, R. Perin, G. Chindemi, P. Dłotko, R. Levi, K. Hess, A. Markram,
"Cliques of neurons bound into cavities provide a missing link between structure and function", Frontiers in Computational Neuroscience, June 2017.



Rough idea: Start from the brain as a directed graph.
Vertices = neurons
Directed edge = synapses in direction of communication

This graph is huge! The authors consider directed cliques / simplices:
Subsets of vertices/neurons & edges/synapses which are all connected to each other. (without cycles)



...

Interval 1-simplex | Triangle 2-simplex | Tetrahedron 3-simplex

Algebra of Cliques

Authors consider cliques, as well as inclusions of cliques

They build a topological space from this information (directed flag complex). They then consider its homology groups.

This is very categorical! Several categories & functors involved.

$$\left\{\begin{array}{c}\text{Directed}\\ \text{graphs}\end{array}\right\} \xrightarrow{\text{Cliques}} \left\{\begin{array}{c}\text{Ordered}\\ \text{sets}\end{array}\right\} \xrightarrow{\text{Directed flag complex}} \underline{\mathbf{Top}} \xrightarrow{\text{Homology}} \underline{\mathbf{Groups}}$$

Qn: Can similar ideas be applied to IIT, qualia, etc?