The algebra and topology of contact categories

Daniel V. Mathews

Monash University
Daniel.Mathews@monash.edu

University of Melbourne
23 July 2018

The point of this talk is to:

The point of this talk is to:

- Tell you what a contact category is.

The point of this talk is to:

- Tell you what a contact category is.
- Tell you some of the properties of contact categories.
- Argue that they are elementary but interesting things.

The point of this talk is to:

- Tell you what a contact category is.
- Tell you some of the properties of contact categories.
- Argue that they are elementary but interesting things.
- Mention some interesting connections:
  - A_∞ algebras
  - strand algebras
  - Floer homology
  - ...

The point of this talk is to:

- Tell you what a contact category is.
- Tell you some of the properties of contact categories.
- Argue that they are elementary but interesting things.
- Mention some interesting connections:
  - A_∞ algebras
  - strand algebras
  - Floer homology
  - ...

Standing assumptions:
manifolds are C^∞ smooth, compact, connected, oriented

Section 1:
What is contact geometry?

The *contact* in “contact category" refers to 3-dimensional contact geometry.

What is a contact category?

The *contact* in “contact category" refers to 3-dimensional contact geometry.

Contact geometry is:

- A type of geometric structure (contact structure) on a manifold
- Pure (obscure?) differential geometry

What is a contact category?

The *contact* in “contact category" refers to 3-dimensional contact geometry.

Contact geometry is:

- A type of geometric structure (contact structure) on a manifold
- Pure (obscure?) differential geometry
- Symplectic geometry’s sibling
- Connected to lots of fields of mathematics and physics
- In 3 dimensions, actually very combinatorial

What is a contact category?

The *contact* in “contact category" refers to 3-dimensional contact geometry.

Contact geometry is:

- A type of geometric structure (contact structure) on a manifold
- Pure (obscure?) differential geometry
- Symplectic geometry’s sibling
- Connected to lots of fields of mathematics and physics
- In 3 dimensions, actually very combinatorial

Roughly a contact category consists of:

- Objects = contact structures on/near a surface S
- Morphisms = contact structures on $S \times [0, 1]$

Definition

A contact form on a manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form.

Definition

A contact form on a manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form.

- Cf symplectic forms (closed 2-form ω s.t. ω^n volume form)
- Contact forms only exist in odd dimension
- $\xi := \ker \alpha$ is a codimension-1 plane field

Frobenius theorem:

- $\alpha \wedge (d\alpha)^n \neq 0$ means

Frobenius theorem:

- $\alpha \wedge (d\alpha)^n \neq 0$ means ξ is *maximally non-integrable*

Frobenius theorem:

- $\alpha \wedge (d\alpha)^n \neq 0$ means ξ is *maximally non-integrable*
  - any submanifold $S \subset M^{2n+1}$ tangent to ξ must have dimension $\leq n$

Frobenius theorem:

- $\alpha \wedge (d\alpha)^n \neq 0$ means ξ is *maximally non-integrable*
  - any submanifold $S \subset M^{2n+1}$ tangent to ξ must have dimension $\leq n$

Definition

A contact structure ξ on M^{2n+1} is a maximally non-integrable codimension-1 plane field.

Frobenius theorem:

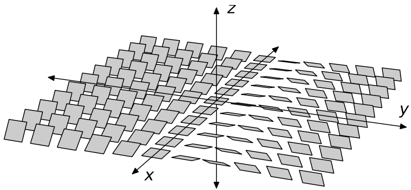
- $\alpha \wedge (d\alpha)^n \neq 0$ means ξ is *maximally non-integrable*
  - any submanifold $S \subset M^{2n+1}$ tangent to ξ must have dimension $\leq n$

Definition

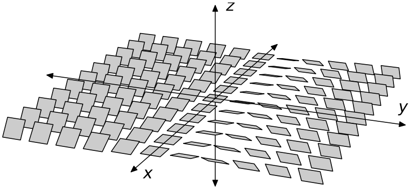
A contact structure ξ on M^{2n+1} is a maximally non-integrable codimension-1 plane field.

- In 3 dimensions, there is no surface tangent to ξ.

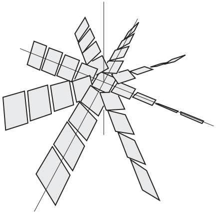
In $\mathbb{R}^3$, take $\alpha = dz - y\,dx$, $\xi = \text{span}\,\{\partial_y,\ \partial_x + y\partial_z\}$



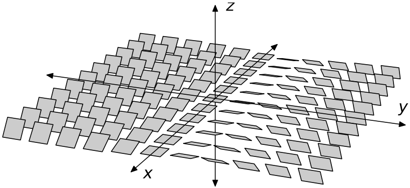
In $\mathbb{R}^3$, take $\alpha = dz - y\,dx$, $\xi = \text{span}\,\{\partial_y,\ \partial_x + y\partial_z\}$



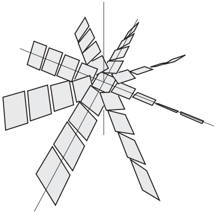
Also in $\mathbb{R}^3$, take $\alpha = dz + r^2\,d\theta$, $\xi = \text{span}\,\{\partial_r,\ r^2\partial_z - \partial_\theta\}$



In $\mathbb{R}^3$, take $\alpha = dz - y\,dx$, $\xi = \text{span}\,\{\partial_y,\ \partial_x + y\partial_z\}$



Also in $\mathbb{R}^3$, take $\alpha = dz + r^2\,d\theta$, $\xi = \text{span}\,\{\partial_r,\ r^2\partial_z - \partial_\theta\}$



In S^3 = unit quaternions, take $\xi_x = \text{span}\,\{ix, jx\}$.

The two examples on $\mathbb{R}^3$ are in fact equivalent/contactomorphic.

The two examples on $\mathbb{R}^3$ are in fact equivalent/contactomorphic.

Definition

A contactomorphism $(M_1, \xi_1) \to (M_2, \xi_2)$ *is a diffeomorphism* $f\colon M_1 \longrightarrow M_2$ *such that* $f_*\xi_1 = \xi_2$*.*

Equivalence of contact structures

The two examples on $\mathbb{R}^3$ are in fact equivalent/contactomorphic.

Definition

A contactomorphism $(M_1, \xi_1) \to (M_2, \xi_2)$ *is a diffeomorphism* $f\colon M_1 \longrightarrow M_2$ *such that* $f_*\xi_1 = \xi_2$.

In fact all contact manifolds are locally contactomorphic — no "contact curvature".

Theorem (Darboux 1882; 3D version)

Given any M^3, α *and* $p \in M$*, there exist coordinates* x, y, z *near* p *so that* $\alpha = dz - y\, dx$.

Contact structures have more symmetry than you might think!

Contact structures have more symmetry than you might think!

Definition

A contact vector field is a vector field whose flow preserves ξ.

Contact structures have more symmetry than you might think!

Definition

A contact vector field is a vector field whose flow preserves ξ.

Theorem

There is a bijective correspondence

$$\begin{array}{ccc} \{\textit{Contact v. fields on } (M, \xi)\} & \longleftrightarrow & \{\textit{Smooth fns } M \to \mathbb{R}\} \\ X & \mapsto & \alpha(X) \end{array}$$

Contact structures have more symmetry than you might think!

Definition

A contact vector field is a vector field whose flow preserves ξ.

Theorem

There is a bijective correspondence

$$\begin{array}{ccc} \{\textit{Contact v. fields on } (M,\xi)\} & \longleftrightarrow & \{\textit{Smooth fns } M \to \mathbb{R}\} \\ X & \mapsto & \alpha(X) \end{array}$$

Moreover, “a deformation of ξ is equivalent to ξ":

Theorem (Gray stability theorem 1959)

Let ξ_t be a smooth family (an isotopy) of contact structures on M. Then there is an isotopy of diffeomorphisms $\psi_t \colon M \to M$ such that $\psi_{t*}\xi_0 = \xi_t$ for all t.

Q: Does any given (closed) 3-manifold M have a contact structure?

Q: Does any given (closed) 3-manifold *M* have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: Does any given (closed) 3-manifold *M* have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: How many contact structures does *M* have?

Q: Does any given (closed) 3-manifold *M* have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: How many contact structures does *M* have?

A: ∞, since non-integrability is an open condition.

Q: Does any given (closed) 3-manifold M have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: How many contact structures does M have?

A: ∞, since non-integrability is an open condition.

Q: How many *isotopy* classes of contact structures are there?

Q: Does any given (closed) 3-manifold *M* have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: How many contact structures does *M* have?

A: ∞, since non-integrability is an open condition.

Q: How many *isotopy* classes of contact structures are there?

A: ∞: a *Lutz twist* (Lutz 1970s) modifies ξ near an embedded simple closed curve and changes the homotopy class of ξ.

Contact structures on 3-manifolds

Q: Does any given (closed) 3-manifold *M* have a contact structure?

A: Yes. (Martinet 1971) — from Heegaard decomposition.

Q: How many contact structures does *M* have?

A: ∞, since non-integrability is an open condition.

Q: How many *isotopy* classes of contact structures are there?

A: ∞: a *Lutz twist* (Lutz 1970s) modifies ξ near an embedded simple closed curve and changes the homotopy class of ξ.

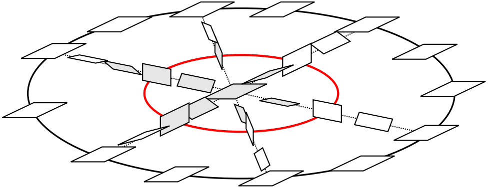
Theorem (Lutz-Martinet)

Every (co-oriented) 2-plane field on M is homotopic to a contact structure.

However, such contact structures are usually *overtwisted*.

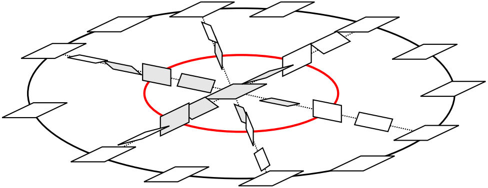
Definition

An overtwisted contact structure is one that contains a specific contact disc called an overtwisted disc.
A non-overtwisted contact structure is called tight.



Definition

An overtwisted contact structure is one that contains a specific contact disc called an overtwisted disc.
A non-overtwisted contact structure is called tight.



Theorem (Eliashberg 1989)

$\{$ *Overtwisted cont. str's on M* $\} \simeq \{$ *2-plane fields on M* $\}$

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers:

Tight contact structures on 3-manifolds

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)

Tight contact structures on 3-manifolds

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on M_1 $\times$ number on M_2. (Colin 2001)

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on M_1 $\times$ number on M_2. (Colin 2001)
- Toroidal: ∞. (Honda–Kazez–Matić 2002)

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on M_1 $\times$ number on M_2. (Colin 2001)
- Toroidal: ∞. (Honda–Kazez–Matić 2002)
- Irreducible and atoroidal: finitely many. (Colin–Giroux–Honda 2003)

Tight contact structures on 3-manifolds

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on M_1 $\times$ number on M_2. (Colin 2001)
- Toroidal: ∞. (Honda–Kazez–Matić 2002)
- Irreducible and atoroidal: finitely many. (Colin–Giroux–Honda 2003)
- $P \# \overline{P}$, $P =$ Poincaré homology sphere: 0. (Etnyre–Honda 1999)

Tight contact structures on 3-manifolds

Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on M_1 $\times$ number on M_2. (Colin 2001)
- Toroidal: ∞. (Honda–Kazez–Matić 2002)
- Irreducible and atoroidal: finitely many. (Colin–Giroux–Honda 2003)
- $P \# \overline{P}$, $P =$ Poincaré homology sphere: 0. (Etnyre–Honda 1999)
- $L(p, q)$: complicated & known! (Honda 2000, Giroux 2001)

Tight contact structures on 3-manifolds

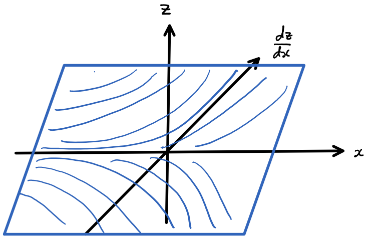
Q: How many *isotopy classes* of *tight* contact structures does M have?

Some known answers: If M is...

- S^3: 1. (Eliashberg 1991)
- B^3 with tight boundary: 1.
- Irreducible with $H_2(M) \neq 0$: ≥ 1 (Gabai–Eliashberg–W. Thurston 1990s)
- $M_1 \# M_2$: number on $M_1 \times$ number on M_2. (Colin 2001)
- Toroidal: ∞. (Honda–Kazez–Matić 2002)
- Irreducible and atoroidal: finitely many. (Colin–Giroux–Honda 2003)
- $P \# \overline{P}$, $P =$ Poincaré homology sphere: 0. (Etnyre–Honda 1999)
- $L(p, q)$: complicated & known! (Honda 2000, Giroux 2001)
- Complement of Seifert surface of special alternating link: complicated & known! (Kálmán–M. 2017)

Definition

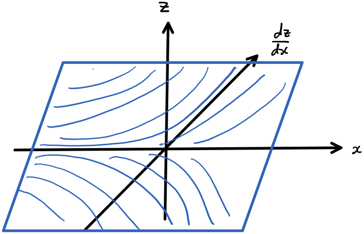
The characteristic foliation on an embedded surface S is the 1-dimensional singular foliation given by $TS \cap \xi$.



Roughly, $\mathcal{F}$ determines the germ of ξ near S.

Definition

The characteristic foliation on an embedded surface S is the 1-dimensional singular foliation given by $TS \cap \xi$.

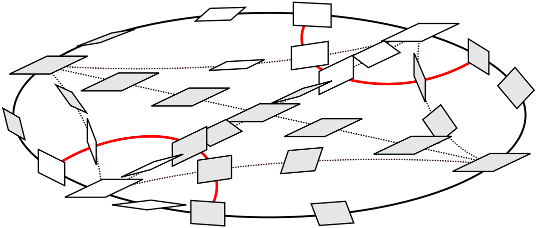


Roughly, $\mathcal{F}$ determines the germ of ξ near S.

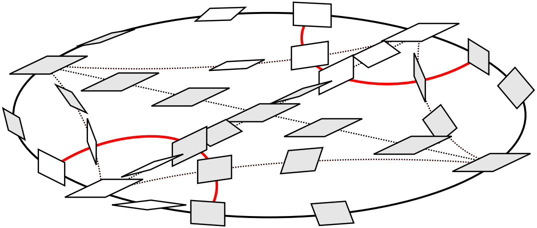
Definition (Giroux 1991)

An embedded surface S in a contact 3-manifold is convex if there is a contact vector field X transverse to S.

Draw S horizontal, X vertical. Colour ξ's sides black and white.



Draw S horizontal, X vertical. Colour ξ's sides black and white.



R_+ : where black side faces up. R_- : where white side faces up.

Definition

The dividing set Γ *is where* ξ *is vertical:* $\Gamma = \{p \in S : X(p) \in \xi\}$.

Γ is a smooth embedded 1-manifold transverse to $\mathcal{F}$

Giroux (1991) showed that, amazingly:

- Any embedded surface in M is C^∞ close to a convex surface.

Giroux (1991) showed that, amazingly:

- Any embedded surface in M is C^∞ close to a convex surface.
- $\mathcal{F}$ is adapted to Γ: has an area-expanding flow on $S \setminus \Gamma$
- If $\mathcal{F}'$ is another foliation adapted to Γ then there is a small isotopy of S in M making the characteristic foliation $\mathcal{F}'$.

Giroux (1991) showed that, amazingly:

- Any embedded surface in M is C^∞ close to a convex surface.
- $\mathcal{F}$ is adapted to Γ: has an area-expanding flow on $S \backslash \Gamma$
- If $\mathcal{F}'$ is another foliation adapted to Γ then there is a small isotopy of S in M making the characteristic foliation $\mathcal{F}'$.

Moral

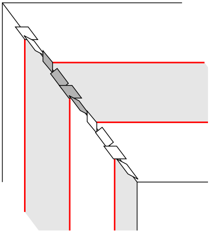
A contact structure near an embedded surface in a contact 3-manifold is described by the isotopy class of the dividing set.

3D contact topology is very discrete / combinatorial!

If S has boundary, we require ∂S be tangent to ξ (*Legendrian*).

If S has boundary, we require ∂S be tangent to ξ (*Legendrian*).

Two convex surfaces may meet along a Legendrian corner.

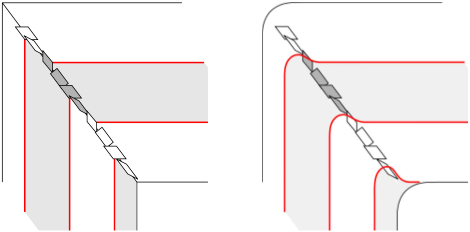


Dividing sets interleave along a corner.

Convex surfaces with boundary

If S has boundary, we require ∂S be tangent to ξ (*Legendrian*).

Two convex surfaces may meet along a Legendrian corner.



Dividing sets interleave along a corner.

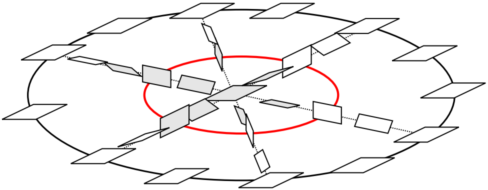
The corner can be rounded; the dividing set behaves as shown.

Dividing sets detect overtwistedness near a surface!

Dividing sets detect overtwistedness near a surface!

Theorem (Giroux's criterion, I)

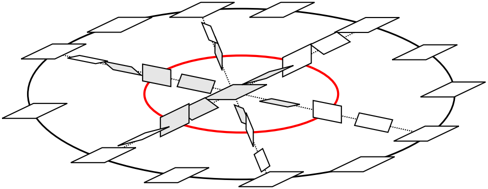
The contact structure near $S(\neq S^2)$ is overtwisted iff Γ contains a contractible closed curve.



Dividing sets detect overtwistedness near a surface!

Theorem (Giroux's criterion, I)

The contact structure near $S(\neq S^2)$ is overtwisted iff Γ contains a contractible closed curve.



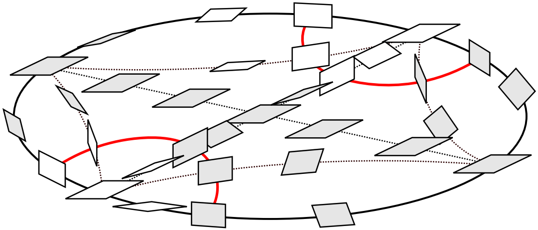
Theorem (Giroux's criterion, II)

The contact structure near $S \approx S^2$ is overtwisted iff Γ consists of more than one curve.

Section 2:
What is a contact category?

Let S be a surface, $F \subset \partial S$ a set of alternating signed points.

Let S be a surface, $F \subset \partial S$ a set of alternating signed points.



Let S be a surface, $F \subset \partial S$ a set of alternating signed points.

Let S be a surface, $F \subset \partial S$ a set of alternating signed points.

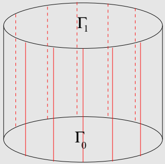
Definition (The contact category $\mathcal{C}(S, F)$)

Objects are:

- *Isotopy classes of tight dividing sets* Γ *on* S *with* $\partial\Gamma = F$

Morphisms $[\Gamma_0] \longrightarrow [\Gamma_1]$ *are:*

- *isotopy classes of tight contact structures on* $S \times [0, 1]$ *with boundary conditions shown*



Let S be a surface, $F \subset \partial S$ a set of alternating signed points.

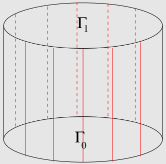
Definition (The contact category $\mathcal{C}(S, F)$)

Objects are:

- *Isotopy classes of tight dividing sets* Γ *on* S *with* $\partial\Gamma = F$
- *A single* object* $*$ *for overtwisted dividing sets.*

Morphisms $[\Gamma_0] \longrightarrow [\Gamma_1]$ *are:*

- *isotopy classes of tight contact structures on* $S \times [0, 1]$ *with boundary conditions shown*



Let S be a surface, $F \subset \partial S$ a set of alternating signed points.

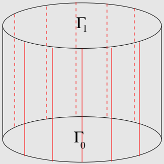
Definition (The contact category $\mathcal{C}(S, F)$)

Objects are:

- *Isotopy classes of tight dividing sets* Γ *on* S *with* $\partial\Gamma = F$
- *A single* object* $*$ *for overtwisted dividing sets.*

Morphisms $[\Gamma_0] \longrightarrow [\Gamma_1]$ *are:*

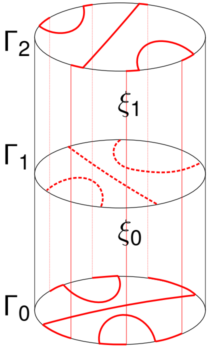
- *isotopy classes of tight contact structures on* $S \times [0, 1]$ *with boundary conditions shown*



- *A single morphism* $\mathbf{0}_{[\Gamma_0],[\Gamma_1]}$ *for OT contact structures.*

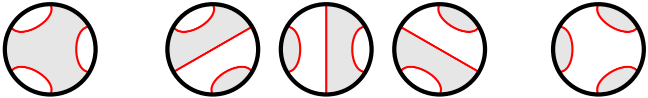
Definition (The contact category $\mathcal{C}(S, F)$ (cont.))

- *Composition* $[\Gamma_0] \stackrel{[\xi_0]}{\to} [\Gamma_1] \stackrel{[\xi_1]}{\to} [\Gamma_2]$*: stack* ξ_0 *and* ξ_1*.*
- *Identity* $[\Gamma] \to [\Gamma]$*: the I-invariant structure on* $S \times I$*.*

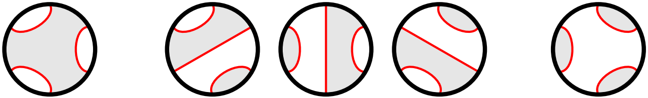


Take $(S, F) = (D^2, 6 \text{ points})$.

Take $(S, F) = (D^2, 6 \text{ points})$.
5 tight objects:



Take $(S, F) = (D^2, 6 \text{ points})$.
5 tight objects:

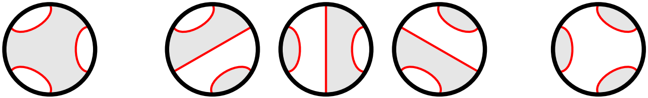


Morphisms?

Proposition

There exists a morphism $[\Gamma_0] \to [\Gamma_1]$ *iff they have the same Euler class* $e(\Gamma_0) = e(\Gamma_1)$.

Take $(S, F) = (D^2, 6 \text{ points})$.
5 tight objects:



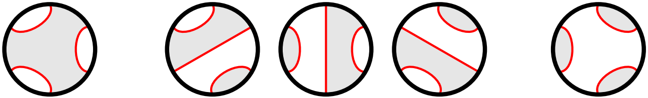
$e = -2$ $e = 0$ $e = 2$

Morphisms?

Proposition

There exists a morphism $[\Gamma_0] \to [\Gamma_1]$ iff they have the same Euler class $e(\Gamma_0) = e(\Gamma_1)$.

Take $(S, F) = (D^2, 6 \text{ points})$.
5 tight objects:



$e = -2$ $e = 0$ $e = 2$

Morphisms?

Proposition

There exists a morphism $[\Gamma_0] \to [\Gamma_1]$ iff they have the same Euler class $e(\Gamma_0) = e(\Gamma_1)$.

Definition (Euler class)

$$e(\Gamma) = e(\xi_\Gamma)[S] = \chi(R_+) - \chi(R_-)$$

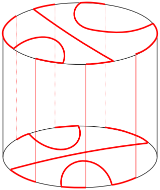
So $\mathcal{C}(S, F) = \bigsqcup_e \mathcal{C}(S, F, e)$.

An example (cont.)

Morphisms  $\rightarrow$  ?

An example (cont.)

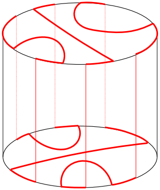
Morphisms  $\rightarrow$  ?



An example (cont.)

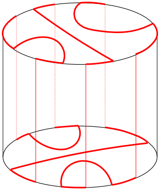
Morphisms  $\to$  ?

A B^3 with tight boundary: so 1 tight morphism.



An example (cont.)

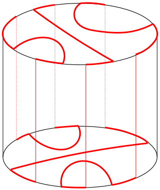
Morphisms  $\rightarrow$  ?



A B^3 with tight boundary: so 1 tight morphism.

- Bypass

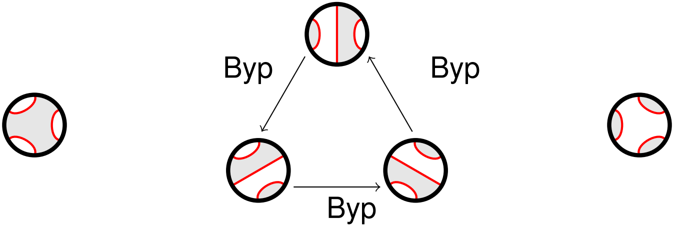
Morphisms  $\to$  ?

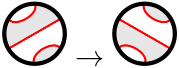


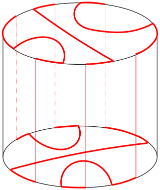
A B^3 with tight boundary: so 1 tight morphism.

- Bypass

Complete set of morphisms (between tight objects):



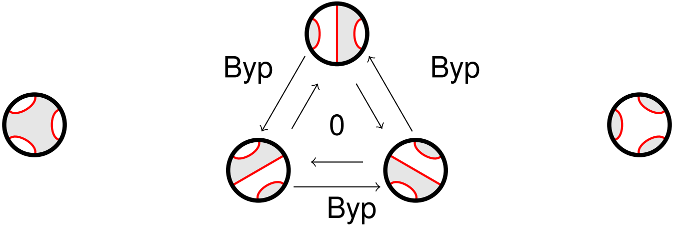
Morphisms  ?



A B^3 with tight boundary: so 1 tight morphism.

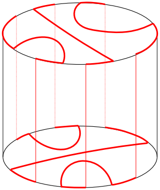
- Bypass

Complete set of morphisms (between tight objects):



An example (cont.)

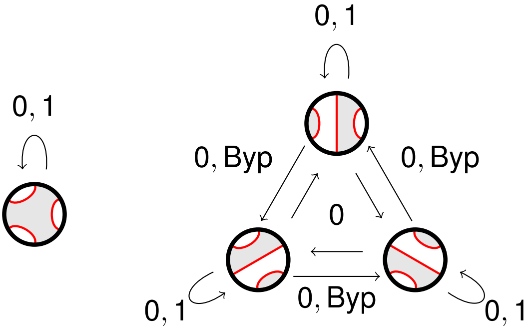
Morphisms  $\rightarrow$  ?



A B^3 with tight boundary: so 1 tight morphism.

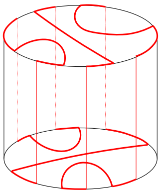
- Bypass

Complete set of morphisms (between tight objects):

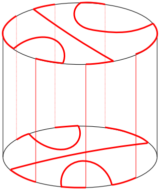


Section 3:
Properties of contact categories

A *bypass morphism* is a morphism made by adding a bypass to a convex surface.



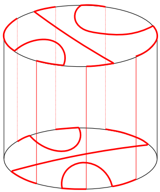
A *bypass morphism* is a morphism made by adding a bypass to a convex surface.



Theorem (Colin, Honda)

Every morphism in $\mathcal{C}(S, F)$ is a composition of bypass morphisms.

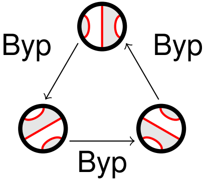
A *bypass morphism* is a morphism made by adding a bypass to a convex surface.



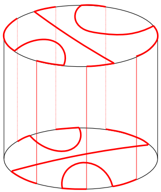
Theorem (Colin, Honda)

Every morphism in $\mathcal{C}(S, F)$ is a composition of bypass morphisms.

Bypass morphisms form *bypass triangles*.



A *bypass morphism* is a morphism made by adding a bypass to a convex surface.



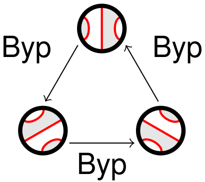
Theorem (Colin, Honda)

Every morphism in $\mathcal{C}(S, F)$ is a composition of bypass morphisms.

Bypass morphisms form *bypass triangles*.

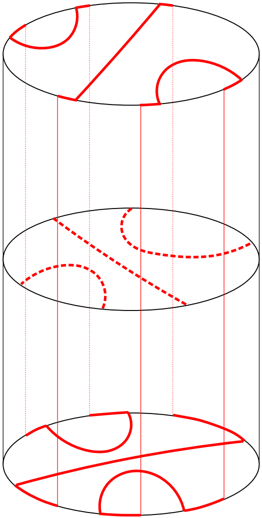
The composition of two is zero/overtwisted.

Composition of all three "has homotopy class -1".



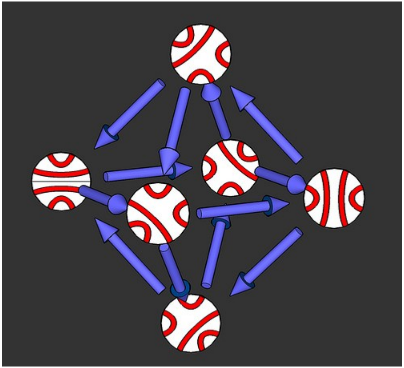
Huang 2014

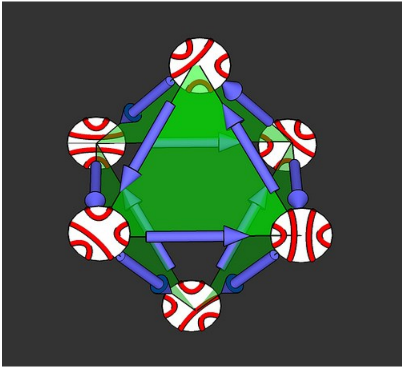
Two bypasses form an overtwisted disc



In $\mathcal{C}(D^2, 8\text{ points}, e = 1)$ the tight objects and bypass morphisms are:

In $\mathcal{C}(D^2, 8 \text{ points}, e = 1)$ the tight objects and bypass morphisms are:





Source: Ken Baker's Sketches of Topology

A contact category $\mathcal{C}(S, F, e)$ has many of the properties of a *triangulated category*.

A contact category $\mathcal{C}(S, F, e)$ has many of the properties of a *triangulated category*.

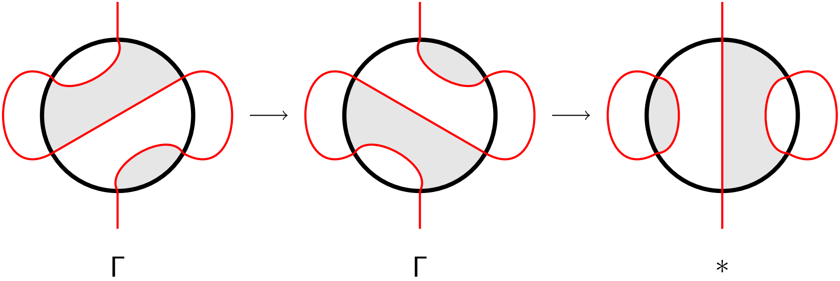
Theorem (Honda?, M.)

1. *Mor*$(*, \Gamma)$ *and Mor*$(\Gamma, *)$ *are trivial.*
2. *Any triple of morphisms* $\Gamma \stackrel{1}{\longrightarrow} \Gamma \longrightarrow * \longrightarrow \Gamma$ *is a bypass triangle.*
3. *If a triple of morphisms is isomorphic to a bypass triangle, then it is a bypass triangle.*
4. *Any upper cap diagram, with all morphisms arising from bypass triangles, can be completed to an octahedron diagram.*

$$\Gamma \xrightarrow{1} \Gamma \longrightarrow * \longrightarrow \Gamma$$

Distinguished triangles

$$\Gamma \xrightarrow{1} \Gamma \longrightarrow * \longrightarrow \Gamma$$



The $\mathbb{Z}_2$ *Grothendieck group* $K(\mathcal{C})$ of $\mathcal{C}$ is the free $\mathbb{Z}_2$-module on objects modulo the relation

$$X + Y + Z = 0 \text{ when } X \to Y \to Z \to X \text{ is bypass triangle.}$$

The $\mathbb{Z}_2$ *Grothendieck group* $K(\mathcal{C})$ of $\mathcal{C}$ is the free $\mathbb{Z}_2$-module on objects modulo the relation

$$X + Y + Z = 0 \text{ when } X \to Y \to Z \to X \text{ is bypass triangle.}$$

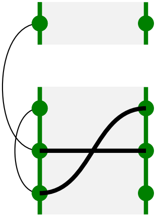
Theorem (Honda?, M.)

$$K(\mathcal{C}(S, F)) \cong SFH(S \times S^1, F \times S^1)$$

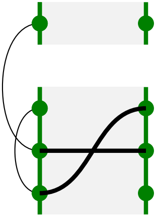
where $SFH(M, \Gamma)$ *is the sutured Floer homology of the sutured 3-manifold* (M, Γ)*.*

Section 4:
Let’s all go down the strand

A *strand algebra* is made up of diagrams like this:



A *strand algebra* is made up of diagrams like this:



They do not look like contact geometry!

And yet they are...

Z: an oriented interval.
$\mathbf{a} = (a_1 < \cdots < a_n)$: a set of points on Z.

Z: an oriented interval.

$\mathbf{a} = (a_1 < \cdots < a_n)$: a set of points on Z.

Definition

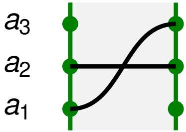
An *unconstrained strand diagram* is a triple (S, T, ϕ), where $S, T \subseteq \mathbf{a}$ and $\phi \colon S \to T$ is a non-decreasing bijection, $x \leq \phi(x)$.

Z: an oriented interval.
$\mathbf{a} = (a_1 < \cdots < a_n)$: a set of points on Z.

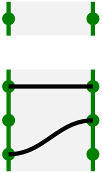
Definition

An *unconstrained strand diagram* is a triple (S, T, ϕ), where $S, T \subseteq \mathbf{a}$ and $\phi\colon S \to T$ is a non-decreasing bijection, $x \leq \phi(x)$.

But we draw it as a diagram.



$$\phi : \{a_1, a_2\} \to \{a_2, a_3\}$$
$$\phi(a_1) = a_3$$
$$\phi(a_2) = a_2$$



Strands never go down.

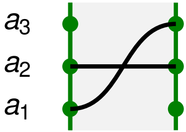
Let's all go *up* the strand

Z: an oriented interval.
$\mathbf{a} = (a_1 < \cdots < a_n)$: a set of points on Z.

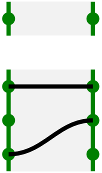
Definition

An *unconstrained strand diagram* is a triple (S, T, ϕ), where $S, T \subseteq \mathbf{a}$ and $\phi\colon S \to T$ is a non-decreasing bijection, $x \leq \phi(x)$.

But we draw it as a diagram.



$$\phi : \{a_1, a_2\} \to \{a_2, a_3\}$$
$$\phi(a_1) = a_3$$
$$\phi(a_2) = a_2$$

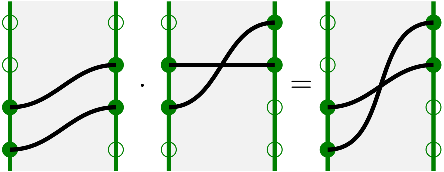


Strands never go down.

We can do the same with several intervals $\mathbf{Z} = (Z_1, \ldots, Z_l)$.

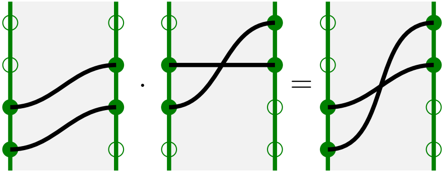
Multiplication: the name of the game

Multiply strand diagrams by composition, if you can (else 0).

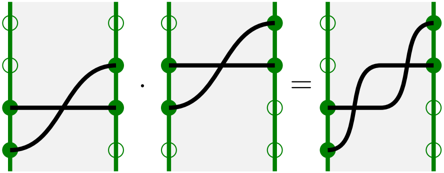


Multiplication: the name of the game

Multiply strand diagrams by composition, if you can (else 0).



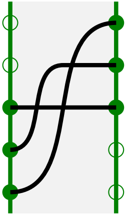
Except if a Reidemeister 2 move simplifies diagram. Then 0.



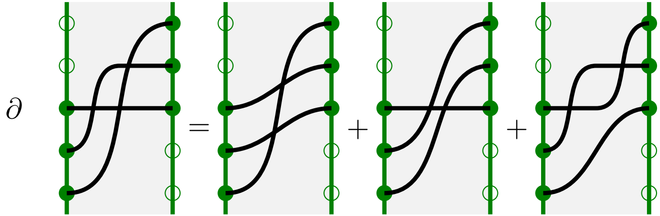
$= 0$

The differential operator ∂ resolves each crossing.

The differential operator ∂ resolves each crossing.

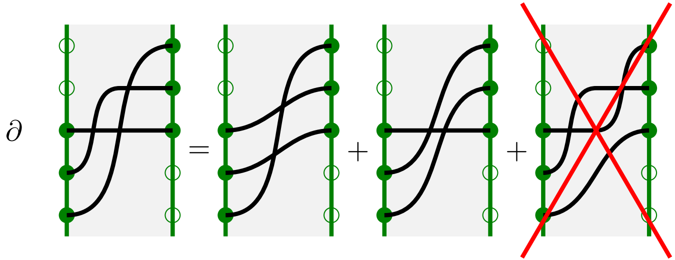
∂  $=$

The differential operator ∂ resolves each crossing.



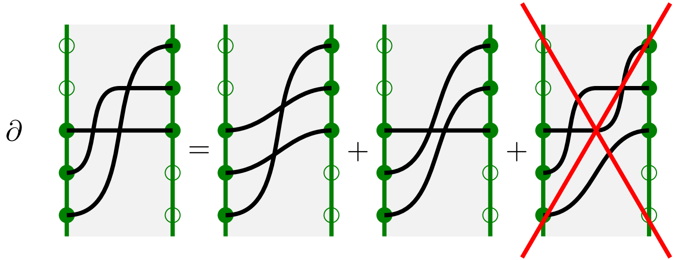
Differentiators gonna differentiate

The differential operator ∂ resolves each crossing.



Again, Reidemeister 2 means 0. “Crossings resolve efficiently."

The differential operator ∂ resolves each crossing.

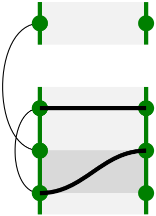


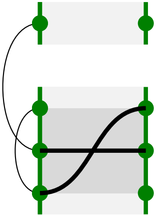
Again, Reidemeister 2 means 0. “Crossings resolve efficiently."

Definition/lemma

The *unconstrained strand algebra* $\mathcal{A}(\mathbf{Z}, \mathbf{a})$ is the $\mathbb{Z}_2$ differential graded algebra generated by strand diagrams.

Join the points **a** on the intervals **Z** in pairs by a matching *M*.
The data $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is called an *arc diagram*.*

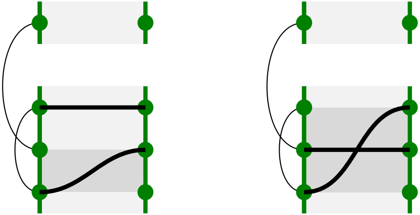




*Conditions apply.

Join the points **a** on the intervals **Z** in pairs by a matching *M*. The data $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is called an *arc diagram*.*

Prohibit strand diagrams from beginning at paired points! Or ending!

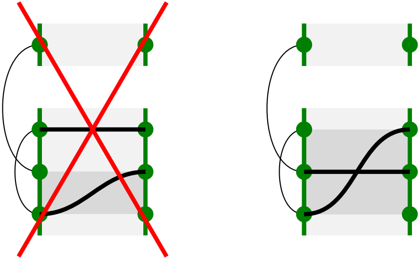


*Conditions apply.

Constraining strand diagrams

Join the points **a** on the intervals **Z** in pairs by a matching *M*. The data $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is called an *arc diagram*.*

Prohibit strand diagrams from beginning at paired points! Or ending!

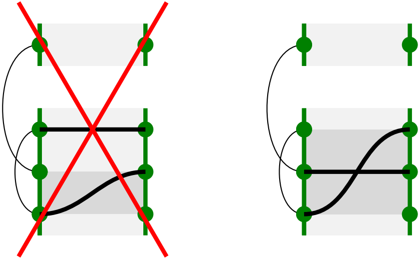


*Conditions apply.

Constraining strand diagrams

Join the points **a** on the intervals **Z** in pairs by a matching M.
The data $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ is called an *arc diagram*.*

Prohibit strand diagrams from beginning at paired points! Or ending!

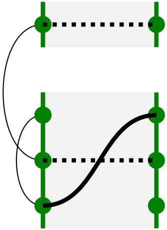


Such strand diagrams are *$\mathcal{Z}$-constrained*.

*Conditions apply.

Observation

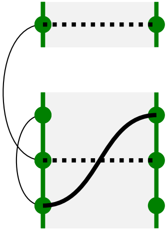
If D is $\mathcal{Z}$-constrained and a, a' are paired, then a horizontal strand can be added at a iff one can also be added at a.



Observation

If D is $\mathcal{Z}$-constrained and a, a' are paired, then a horizontal strand can be added at a iff one can also be added at a.

Dotted horizontal strands denote a sum over both choices.

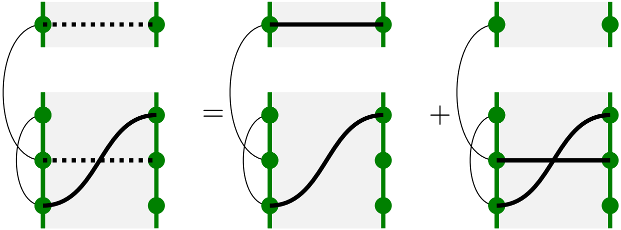


Symmetrising strands

Observation

If D is $\mathcal{Z}$-constrained and a, a' are paired, then a horizontal strand can be added at a iff one can also be added at a.

Dotted horizontal strands denote a sum over both choices.

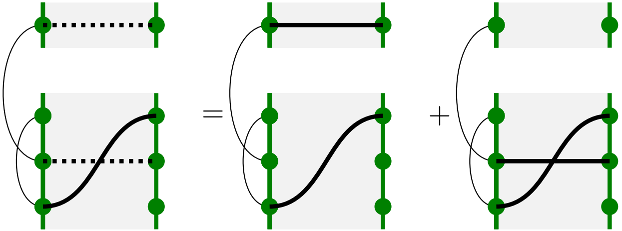


Symmetrising strands

Observation

If D is $\mathcal{Z}$-constrained and a, a' are paired, then a horizontal strand can be added at a iff one can also be added at a.

Dotted horizontal strands denote a sum over both choices.



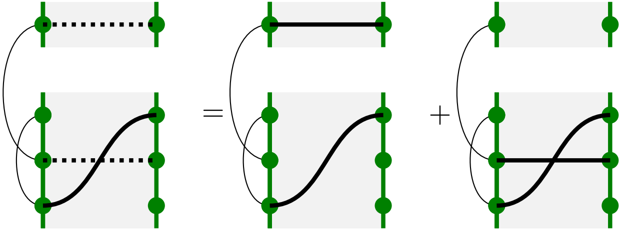
A diagram is *symmetrised* if all horizontal strands are dotted.

Symmetrising strands

Observation

If D is $\mathcal{Z}$-constrained and a, a' are paired, then a horizontal strand can be added at a iff one can also be added at a.

Dotted horizontal strands denote a sum over both choices.



A diagram is *symmetrised* if all horizontal strands are dotted.

Definition

The strand algebra $\mathcal{A}(\mathcal{Z})$ *is the sub-DGA of* $\mathcal{A}(\mathbf{Z}, \mathbf{a})$ *generated by symmetrised* $\mathcal{Z}$*-constrained diagrams.*

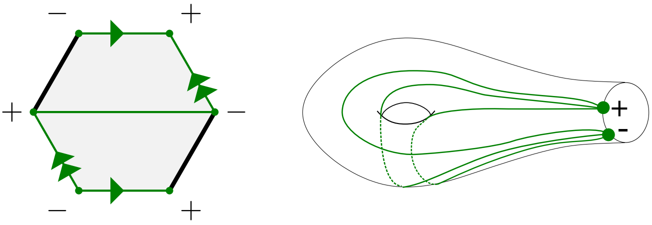
Arc diagrams correspond to *quadrangulated surfaces*:

Arc diagrams correspond to *quadrangulated surfaces*:

- a compact oriented surface with boundary Σ
- with alternating signed points F on $\partial\Sigma$, and
- properly embedded arcs in (Σ, F) cutting Σ into squares

Arc diagrams correspond to *quadrangulated surfaces*:

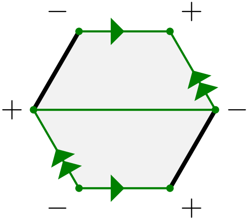
- a compact oriented surface with boundary Σ
- with alternating signed points F on $\partial\Sigma$, and
- properly embedded arcs in (Σ, F) cutting Σ into squares

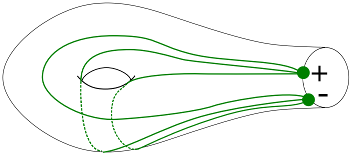


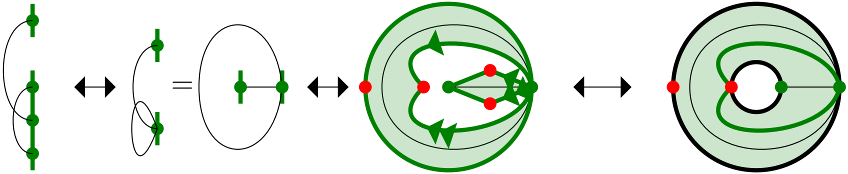
Quadrangulated surfaces

Arc diagrams correspond to *quadrangulated surfaces*:

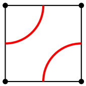
- a compact oriented surface with boundary Σ
- with alternating signed points F on $\partial\Sigma$, and
- properly embedded arcs in (Σ, F) cutting Σ into squares

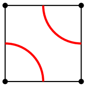






On quadrangulated surface (Σ, Q), a natural class of “basic" dividing sets generates a full subcategory $\mathcal{C}(\Sigma, Q)$ of $\mathcal{C}(\Sigma, F)$.





An isomorphism

On quadrangulated surface (Σ, Q), a natural class of “basic" dividing sets generates a full subcategory $\mathcal{C}(\Sigma, Q)$ of $\mathcal{C}(\Sigma, F)$.

Theorem (M.)

There is an isomorphism of $\mathbb{Z}_2$-algebras

$$H(\mathcal{A}(\mathcal{Z})) \cong CA(\Sigma, Q),$$

where

- *the arc diagram $\mathcal{Z}$ corresponds to quadrangulated (Σ, Q)*
- *$H(\mathcal{A}(\mathcal{Z}))$ is the homology of $\mathcal{A}(\mathcal{Z})$*
- *$CA(\Sigma, Q)$ is the category algebra of $\mathcal{C}(\Sigma, Q)$.*

Theorem (Kadeishvili 1980)

The homology of a DGA has a natural A_∞ algebra structure.

Theorem (Kadeishvili 1980)

The homology of a DGA has a natural A_∞ algebra structure.

Corollary

The contact category algebra $CA(\Sigma, Q)$ has an A_∞ structure.

Theorem (Kadeishvili 1980)

The homology of a DGA has a natural A_∞ algebra structure.

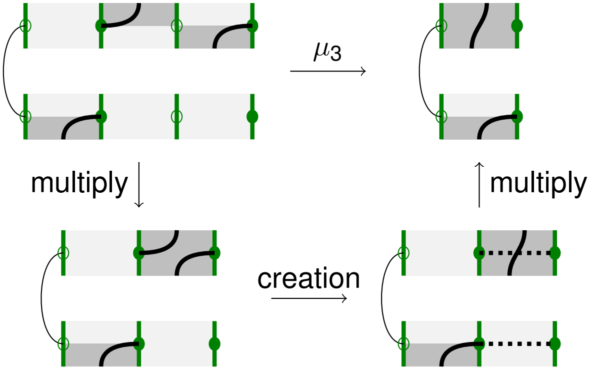
Corollary

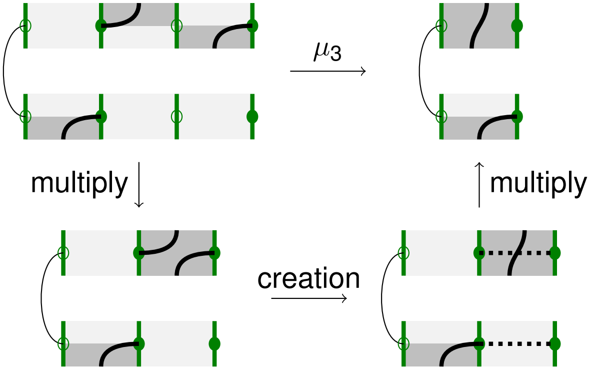
The contact category algebra $CA(\Sigma, Q)$ has an A_∞ structure.

Theorem (M.)

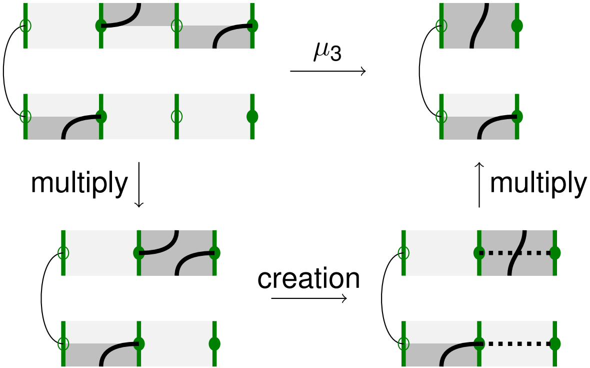
The A_∞ structure on $CA(\Sigma, Q)$ can be given explicitly in terms of bypasses or strands.

A_∞ mechanics





What about the full category $\mathcal{C}(\Sigma, F)$?



What about the full category $\mathcal{C}(\Sigma, F)$?

Conjecture (Proved for discs by Honda–Tian, 2016)

The universal cover of $\mathcal{C}(\Sigma, F)$ embeds into the homotopy category of bounded chain complexes of finitely generated projective left $H(\mathcal{A}(\mathcal{Z}))$-modules.

“Floer homology" describes a large family of powerful invariants of various types of manifolds

“Floer homology" describes a large family of powerful invariants of various types of manifolds, all involving words like:

“Floer homology" describes a large family of powerful invariants of various types of manifolds, all involving words like:

Holomorphic curves, PDEs, symplectic structures, almost complex structures, moduli spaces, Fredholm theory, chain complex generated by asymptotics, Reeb orbits, differential counting holomorphic curves, “infinite-dimensional Morse theory", “homology of the loop space", “quantum cup product", Gromov, Witten, Floer, Eliashberg, Givental, Hofer, Ozsváth, Szabó, Lipshitz, D. Thurston, ...

Floer homology

“Floer homology" describes a large family of powerful invariants of various types of manifolds, all involving words like:

Holomorphic curves, PDEs, symplectic structures, almost complex structures, moduli spaces, Fredholm theory, chain complex generated by asymptotics, Reeb orbits, differential counting holomorphic curves, “infinite-dimensional Morse theory", “homology of the loop space", “quantum cup product", Gromov, Witten, Floer, Eliashberg, Givental, Hofer, Ozsváth, Szabó, Lipshitz, D. Thurston, ...

Heegaard Floer homology is a powerful set of invariants of 3-manifolds, knots, links... (Oszváth–Szabó, 2004)

"Floer homology" describes a large family of powerful invariants of various types of manifolds, all involving words like:

Holomorphic curves, PDEs, symplectic structures, almost complex structures, moduli spaces, Fredholm theory, chain complex generated by asymptotics, Reeb orbits, differential counting holomorphic curves, "infinite-dimensional Morse theory", "homology of the loop space", "quantum cup product", Gromov, Witten, Floer, Eliashberg, Givental, Hofer, Ozsváth, Szabó, Lipshitz, D. Thurston, ...

Heegaard Floer homology is a powerful set of invariants of 3-manifolds, knots, links... (Oszváth–Szabó, 2004)

- The Grothendieck group $K(\mathcal{C}(S, F))$ is isomorphic to sutured Floer homology (Juhász 2006), the version for sutured manifolds.
- Bordered Floer homology (Lipshitz–Ozsváth–D. Thurston 2014) constructs various A_∞-*-modules over the strands algebra

Thanks for listening!

References:

- D. V. Mathews, *A-infinity algebras, strand algebras, and contact categories*, arXiv 1803.06455
- Tamás Kálmán and D. V. Mathews, *Tight contact structures on Seifert surface complements* arxiv 1709.10304
- D. V. Mathews, *Strand algebras and contact categories* accepted for publication in Geom. & Top., arXiv 1608.02710
- D. V. Mathews, *Strings, fermions and the topology of curves on annuli* (2014) arXiv 1410.2141
- D. V. Mathews, *Contact topology and holomorphic invariants via elementary combinatorics* arXiv 1212.1759.
- D. V. Mathews and E. Schoenfeld, *Dimensionally-reduced sutured Floer homology as a string homology* (2012) arXiv 1210.7394.
- D. V. Mathews, *Itsy bitsy topological field theory* (2012) arXiv 1201.4584.
- D. V. Mathews, *Sutured TQFT, torsion, and tori* (2011) arXiv 1102.3450.
- D. V. Mathews, *Sutured Floer homology, sutured TQFT and non-commutative QFT* Alg. & Geom. Top. 11 (2011) 2681–2739, arXiv 1006.5433
- D. V. Mathews, *Chord diagrams, contact-topological quantum field theory, and contact categories* Alg. & Geom. Top. 10 (2010) 2091–2189, arxiv 0903.1453