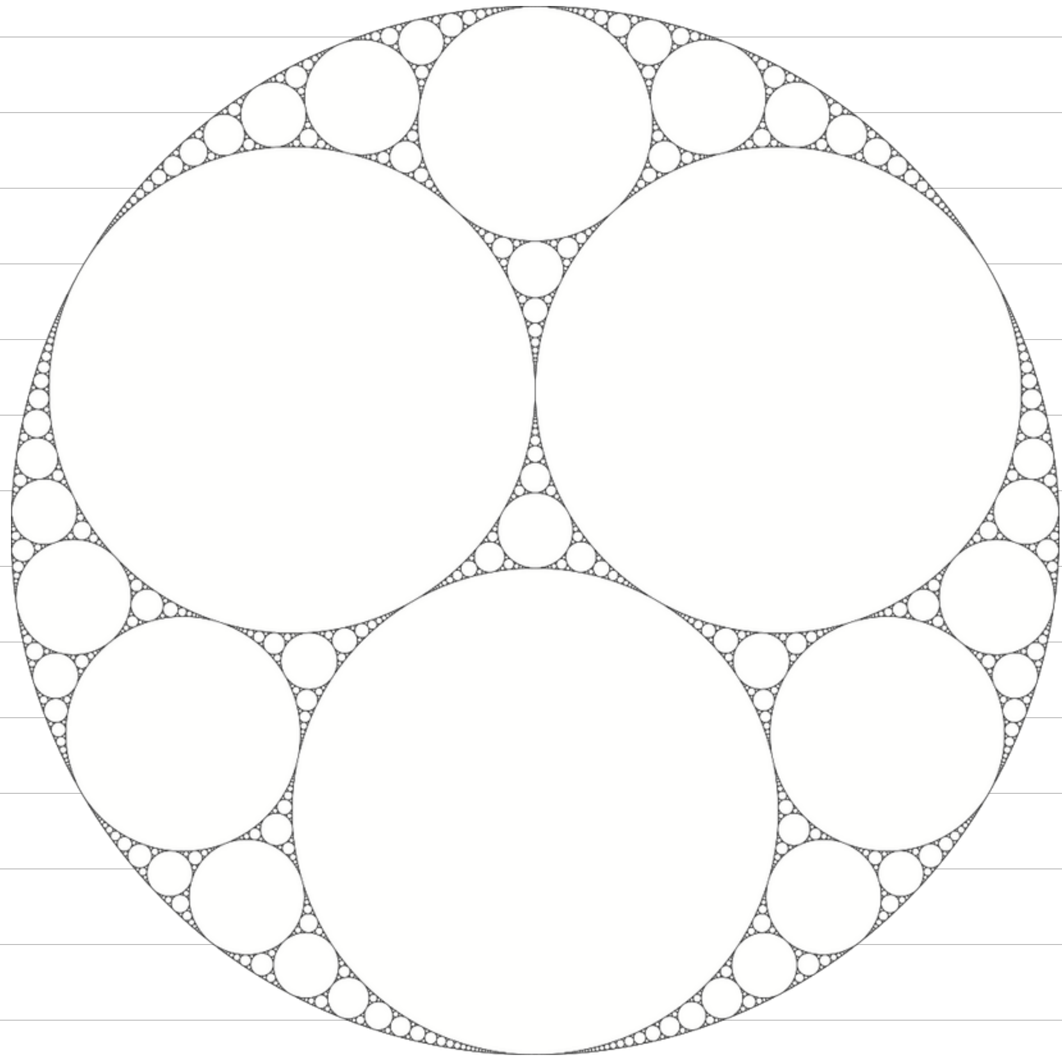
Circle Packings, Lagrangian Grassmannians, & Scattering Diagrams

Daniel Mathews

Monash topology seminar, 1/4/20



Overview

A few ideas from the theory of circle packings & recent work in progress
→ relations to spinors, symplectic geometry, Grassmannians & scattering theory.

1. (Koebe, Andreev, Thurston, Stephenson, ...) There is a wonderful theory of packing circles in the plane

→ like "discrete complex analysis"
→ related to geometry of hyperbolic polyhedra (Purcell, Hodgson, ...)

2. (M.) One can understand & compute circle packings (& more general arrangements) as Lagrangian plane arrangements in a symplectic and complex vector space of spinors.

3. (M.) The resulting mathematics is closely related to recent developments in $\mathcal{N}=4$ SUSY Yang-Mills theory (on-shell diagrams, amplituhedron, ...).

4 (M) One can also understand & compute circle packings via De Sitter geometry.

Highlights from the Theory of Circle Packings

Fix a 2-complex K, a triangulation of a surface (w/ or w/o ∂).

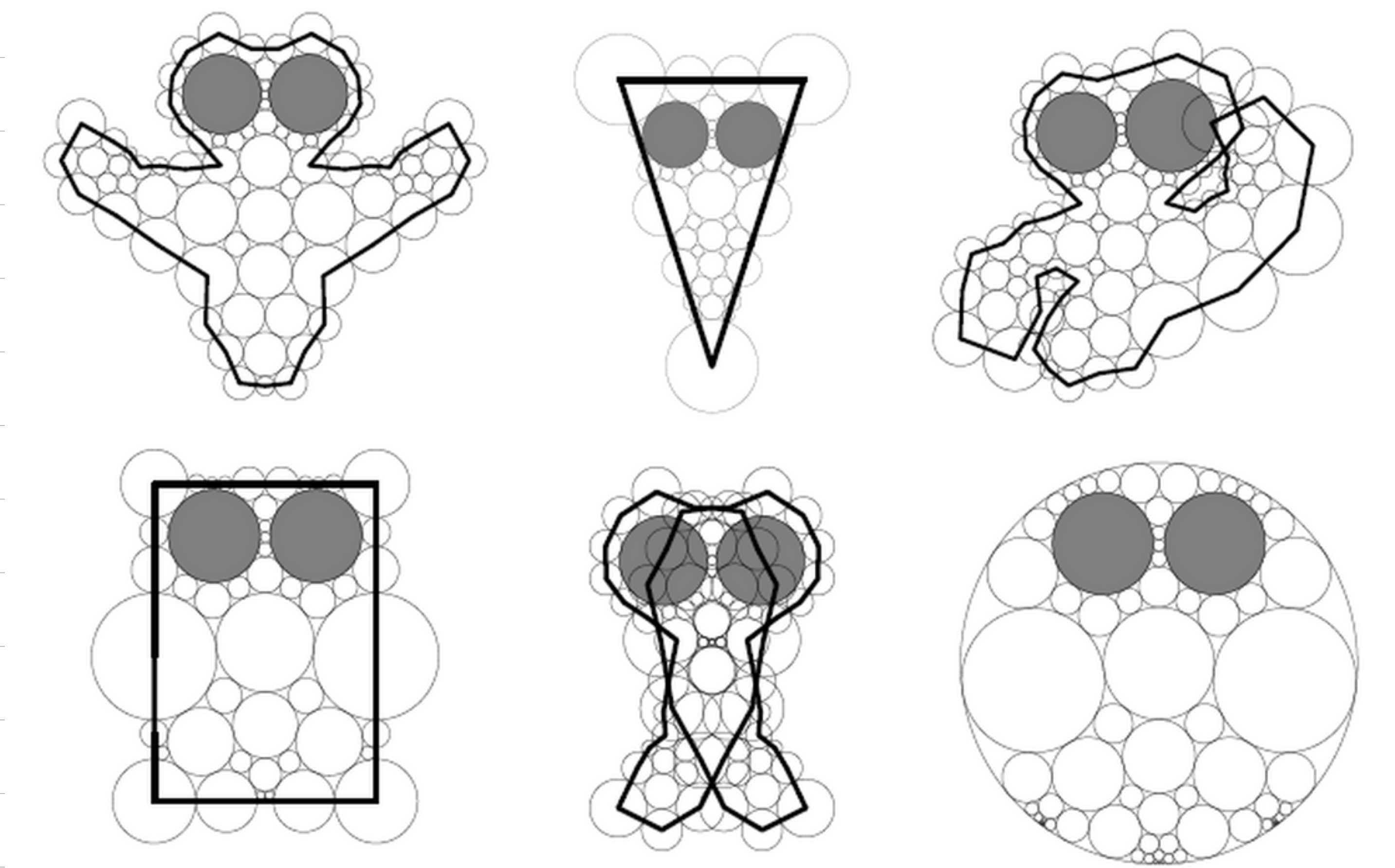
A collection of circles $\{C_v\}$ in a metric space is a circle packing for K if

(1) There is a circle C_v for each vertex v of K
(2) C_u, C_v are externally tangent when u,v are joined by an edge
(3) C_u, C_v, C_w form a positively oriented triple when u, v, w do.

Basically, you can always find a circle packing with spherical / Euclidean / hyp geometry.

<u>Theorem</u>: (Beardon-Stephenson 1990) If K is a disc then $\exists$ a (locally univalent) Euclidean circle packing for K. Boundary radii can be prescribed arbitrarily! It's unique up to Euclidean isometries.

Circle packings of a fixed triangulation K of a disc with varying boundary radii.



Source: K Stephenson, Circle packing & discrete analytic function theory

Spinors & Spacetime

Penrose, Rindler 1984: Think of spinors rather than tangent vectors on $S^2 \cong \mathbb{C}P^1$.

For present purposes, a spinor is just a vector in $\mathbb{C}^2$.

A spinor $\xi = (w, z)$ has a spin vector, a tangent vector to $\mathbb{C}P^1$ given by the vector $1/w^2$ at the point z/w.

<u>Note</u>: $re^{i\theta}\xi$ has spin vector at the same point as ξ but its length is multiplied by $1/r^2$ and it's rotated by -2θ.

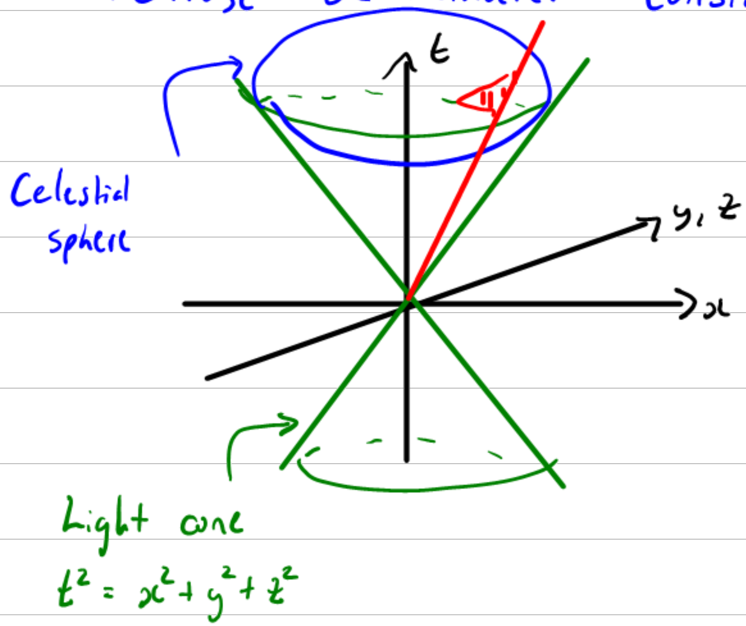
There's an antisymmetric bilinear form $\{\cdot,\cdot\} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ on spinors given by $\{\xi_0, \xi_1\} = w_0 z_1 - z_0 w_1$.

Its imaginary part is a symplectic form ω on $\mathbb{C}^2 \cong \mathbb{R}^4$
$\omega : \mathbb{R}^4 \otimes \mathbb{R}^4 \longrightarrow \mathbb{R}$, $\quad \omega(\xi_0, \xi_1) = \mathrm{Im}\{\xi_0, \xi_1\}$.

Are these really spinors?

Semi-Riemannian: for $v=(t,x,y,z)$, $\langle v,v\rangle = t^2-x^2-y^2-z^2$

Penrose & Rindler considered null vectors in Minkowski space. $\mathbb{R}^{3,1}$



Null lines through the origin, i.e. along the light cone L
i.e. for which $dt^2 - dx^2 - dy^2 - dz^2 = 0$

are determined by their intersection with the celestial sphere $L \cap \{t=1\}$
$= \{1, x, y, z : x^2+y^2+z^2=1\}$
which is naturally regarded as $\mathbb{C}P^1$.

Lorentz transformations of $\mathbb{R}^{3,1}$ effect Möbius transformations on $\mathbb{C}P^1$
$SO(3,1)^+ \cong PSL(2,\mathbb{C})$

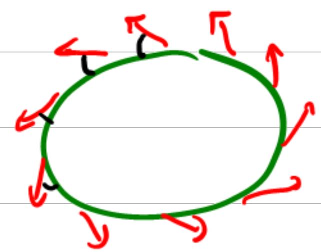
Represent points of $\mathbb{C}P^1$ as $[z:w] \cong z/w$.
The redundancy in (z,w) gives a null flag — for us, a vector will do.

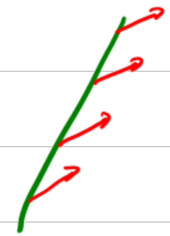
Spin vectors and Lagrangian subspaces

Let Π be a 2-(real)-dimensional subspace of $\mathbb{C}^2$.

Facts (Penrose, Rindler): Let X denote the set of spin vectors of Π.

* Π is complex $\Leftrightarrow$ All vectors in X lie at a point
* Otherwise, X consists of vectors along a circle in $\mathbb{CP}^1$, making a constant angle.





* The spin vectors are tangent to the circle $\Leftrightarrow$ Π is Lagrangian, i.e. ω vanishes on Π.

Indeed, there are natural bijections

Grassmannian $Gr(2,4) = \{$2-planes in $\mathbb{R}^4\} \longleftrightarrow \{$points and circles with directions in $\mathbb{CP}^1\}$

Lagrangian Grassmannian $LGr(2,4) = \{$Lagrangian 2-planes in $\mathbb{R}^4\}$
$\longleftrightarrow \{$points and oriented circles in $\mathbb{CP}^1\}$

Lagrangian Plane Arrangements

Thus "Lagrangian planes have centres & radii" (= "distance from being complex")

Note: Π Lagrangian $\Longrightarrow$ $i\Pi$ Lagrangian

$\rightarrow$ Corresponds to reversing orientations of circles!

Statements about circle packings now translate into statements about Lagrangian planes

Eg ..

Theorem (M): Suppose K triangulates a disc. Then $\exists$:

* for each vertex v of K, a Lagrangian plane Π_v
* for each u, v connected by an edge, $\Pi_u \cap i\Pi_v$ is 1-Dimensional.
* for each u, v, w around a positively oriented triangle,

  Π_u, Π_v, Π_w have centres forming a positively oriented triangle.

Moreover, every boundary Π_v can have any prescribed radius.

These planes are unique up to the action of $GL^+(2, \mathbb{C})$ on $\mathbb{C}^2$

(det real & positive)

Plücker Coordinates & Inversive Vectors

The Grassmannian $Gr(2,4)$ has projective Plücker coordinates

$$[\Delta_{12} : \Delta_{13} : \Delta_{14} : \Delta_{23} : \Delta_{24} : \Delta_{34}]$$

* Given a 2-plane $\Pi \subset \mathbb{R}^4$, form a 2×4 matrix whose rows form a basis
* Δ_{ij} = determinant of 2×2 submatrix formed by i'th & j'th columns

Lemma (M): For $\Pi \in LGr(2,4)$, its Plücker coordinates are

$$[\kappa : -\kappa q + 1 : \kappa p : -\kappa p : -\kappa q - 1 : \tilde{\kappa}]$$

where Π corresponds to a circle centred at $p + qi$ with curvature κ (and $\tilde{\kappa} = \kappa(p^2+q^2) - \frac{1}{\kappa}$)

Thus consider the vector $v_\Pi = (\kappa, -\kappa q, \kappa p, \tilde{\kappa})$ Inversive vector of Π.

Note

$$[\kappa \;\; -\kappa q \;\; \kappa p \;\; \tilde{\kappa}] \underbrace{\begin{bmatrix} & & & 1 \\ & -2 & & \\ & & -2 & \\ 1 & & & \end{bmatrix}}_{B} \begin{bmatrix} \kappa \\ -\kappa q \\ \kappa p \\ \tilde{\kappa} \end{bmatrix} = -2$$

so define a bilinear form:

$$J(v,w) = v^T B w.$$

Circle Packing Equations

For any Lagrangian plane Π then $J(v_\Pi, v_\Pi) = -2$

But more...

Proposition (M, Lagarias - Mallows - Wilks 2001) For Lagrangian Π, Π' corresponding to oriented circles C, C'

$$J(v_\Pi, v_{\Pi'}) = -2\cos(\text{angle b/w } C, C') = \text{Inversive distance}(C, C')$$

So we obtain equations for tangency of circles! (Or any prescribed angle!)

$J(v,v) = -2$ for each circle, $J(v,v') = 2$ for tangency!

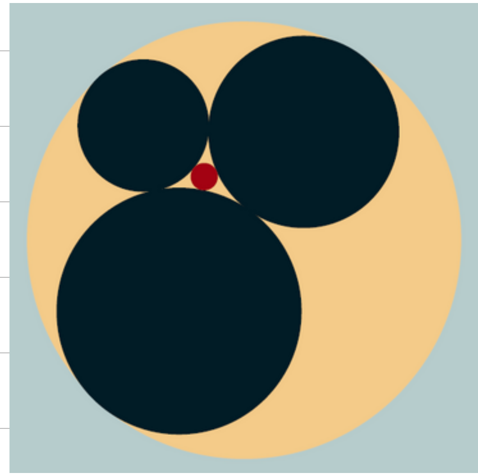
One can find a circle packing for a triangulation K by solving quadratic equations of the form $J(v,w) = \pm 2$

$\rightsquigarrow$ Exercise in elimination theory

Generalisations of Descartes' Circle Theorem

Consider K_n =  (n triangles around a central point)

A circle packing for K_n is a flower with n petals.
→ A central circle C_0, surrounded by tangent circles $C_1, \dots, C_n$.



Source: P Levrie

n=3: Flower = 4 mutually externally tangent circles.
Let k_i : curvature of C_i.

Descarte's Circle Theorem (1634):

$$(k_0 + k_1 + k_2 + k_3)^2 = 2(k_0^2 + k_1^2 + k_2^2 + k_3^2)$$

In general, circle packing theory says k_0 is a function of $k_1, \dots, k_n$.
Using our methods we can find generalisations for higher n.

n=4 Thm (M):

$$16k_0^4 - 8k_0^2\left(k_1k_2 + k_2k_3 + k_3k_4 + k_4k_1 + 2k_1k_3 + 2k_2k_4\right) + (k_1^2+k_3^2)(k_2^2+k_4^2) - 16k_0k_1k_2k_3k_4\left(\sum_{i=1}^{4} \frac{1}{k_i}\right) - 12k_1k_2k_3k_4 - 2(k_1k_2+k_3k_4)(k_2k_3+k_4k_1) = 0$$

De Sitter Geometry

The quadratic form J has signature $(3,1)$ & in fact $J(v,v) = -2$ can be rewritten as

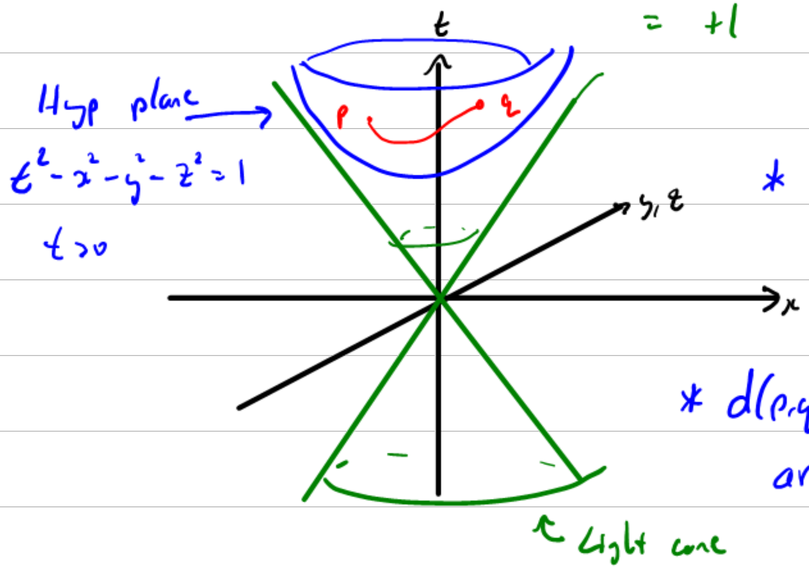
$$\begin{bmatrix} \frac{k+\tilde{k}}{2} & -kq & kp & \frac{\tilde{k}-k}{2} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} \frac{k+\tilde{k}}{2} \\ -kq \\ kp \\ \frac{\tilde{k}-k}{2} \end{bmatrix} = -1$$

$\frac{k+\tilde{k}}{2} = t$, $-kq = x$, $kp = y$, $\frac{\tilde{k}-k}{2} = z$

$\langle v, v \rangle = -1$ in $R^{3,1}$!

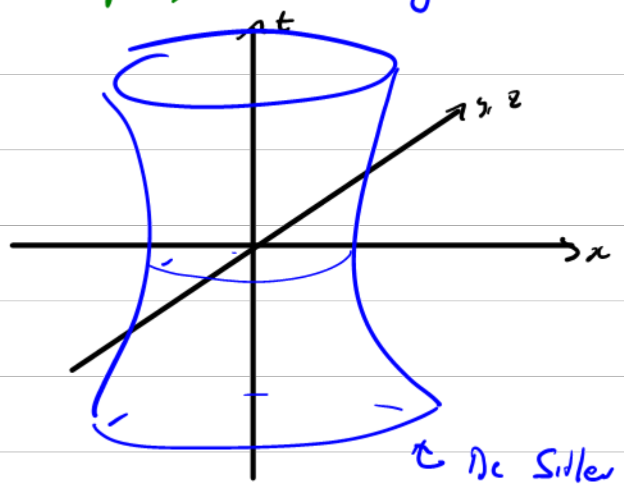
$\Leftrightarrow$ $t^2 - x^2 - y^2 - z^2 = -1$ ... So circles correspond to points in $\mathbb{R}^{3,1}$ with $\|p\| = -1$!

$= +1$ gives hyperbolic space, -1 gives De Sitter space!



* Riemannian.

* $d(p,q) =$ $\operatorname{arccosh} \langle p, q \rangle$



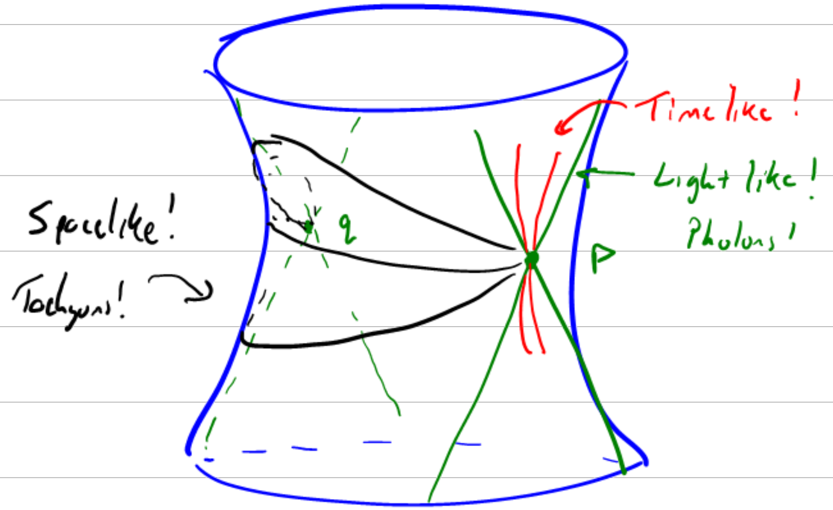
* Semi-Riemannian

* $d(p,q) = \begin{cases} \operatorname{arccosh}(p,q) \\ \arccos(p,q) \\ \infty \end{cases}$ in different cases

De Sitter space $t^2 - x^2 - y^2 - z^2 = -1$

Circle Packing as Extreme Physical Distancing

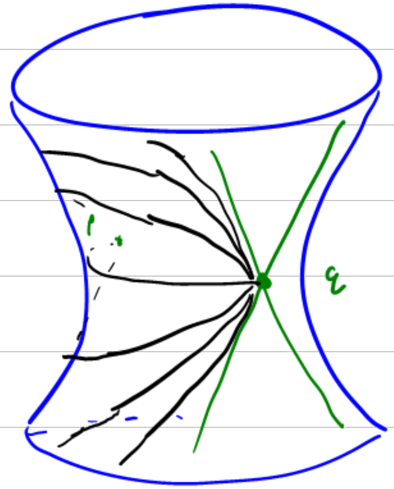
* From a point p in De Sitter space, there are geodesics to some but not all points.



* At p there is a light cone - lightlike geodesics from p.
* p has an antipodal point q.
  All spacelike geodesics from p go to q.

* Geodesics from p approach but cannot reach the light cone at q. → The "horizon" of p.
  Extremely separated from p!

Prop: $\langle p, p' \rangle = 1$ iff p' is on the horizon of p.



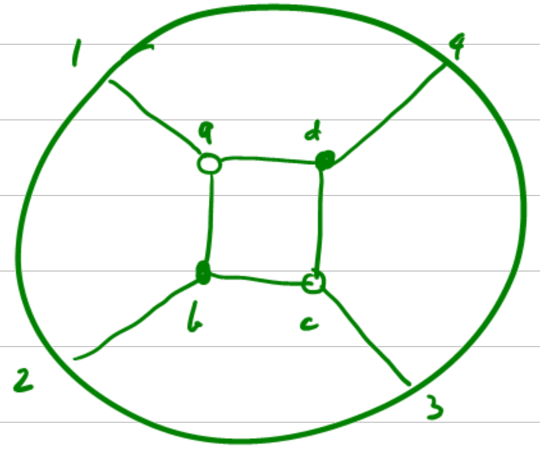
Circles with inversive vectors v, v' correspond to points $p, p' \in$ De Sitter.
They are externally tangent iff $J(v, v') = 2$ iff $\langle p, p' \rangle = 2$

So...

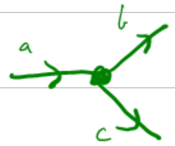
Thm: A circle packing corresponds to an arrangement of points in De Sitter space where tangent circles correspond to extremely separated points!

Circle Packings as Scattering Diagrams

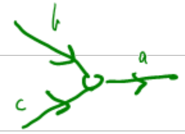
In the scattering theory of $\mathcal{N}=4$ supersymmetric Yang-Mills theory, one considers "on-shell diagrams" which are planar bicoloured graphs with "external" vertices arranged in a circle. (A Postnikov, N Arkani-Hamed, many others...)



Each oriented edge E is decorated with a spinor ρ_E



At a black vertex, spinors are proportional
$$\rho_c = \alpha_c \rho_a, \qquad \rho_b = \alpha_b \rho_a$$



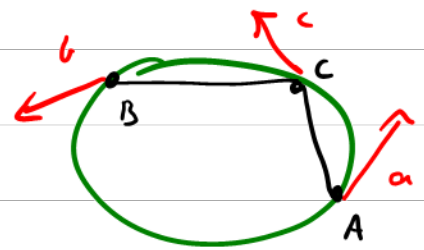
At a white vertex, two spinors sum to the third
$$\rho_a = \rho_b + \rho_c.$$

Find space of solutions to these equations given "external" spinors.
Given "external" spinors, solve for others in terms of parameters α_e.
The space of solutions, with positivity conditions, is a polytope from which one can find scattering amplitudes as a volume – "amplituhedron"

On-shell Diagrams are Circle Diagrams

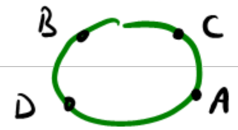
Certain quantities in scattering theory then have an interpretation in Euclidean geometry!

Edge parameters $\rightsquigarrow$ ratios
α_a, α_b

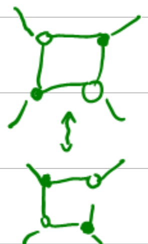


$$\frac{C-B}{C-A} = \frac{\alpha_b}{\alpha_a}$$

Face parameters $\longrightarrow$ Generalised cross-ratios
(Π edge parameters around a face)



$$\frac{C-B}{C-A} \cdot \frac{D-A}{D-B}$$



Cluster transformations $\rightarrow$ Cross-ratio transformations !

Ptolemy relation $\rightsquigarrow$ Ptolemy's theorem for cyclic quadrilaterals!