A-polynomials, Ptolemy varieties, and Dehn filling

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joint with Joshua A. Howie and Jessica S. Purcell
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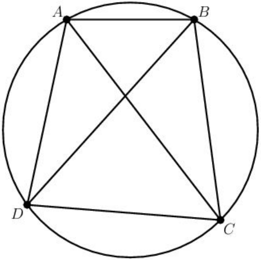
This talk is about connections between

- Ptolemy equations (arise in many places, esp. cluster algebras)
- the A-polynomial (a knot invariant)
- hyperbolic geometry (2D and 3D)
- triangulations of manifolds (2D and 3D)
- symplectic geometry, and
- Dehn filling (an operation on 3-manifolds).

Theorem (Claudius Ptolemaeus c. AD 160)

For any cyclic quadrilateral ABCD,

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$



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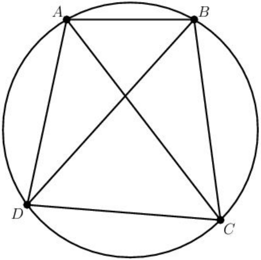

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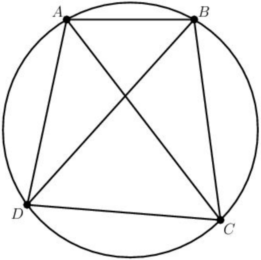

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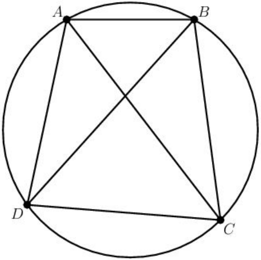

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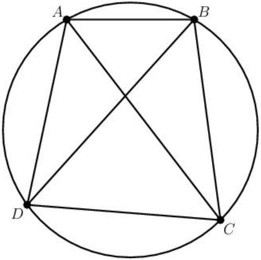

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Two (hence three) terms above have the same argument, so their lengths sum. □

Ptolemy equations in hyperbolic geometry

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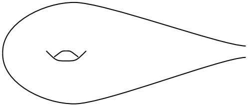

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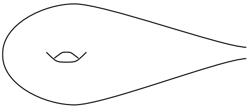

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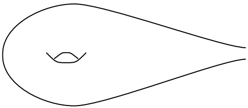

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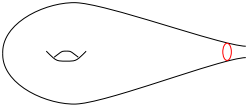

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Space of such surfaces forms decorated T-space $\widetilde{T_{g,n}} \cong \mathbb{R}^{6g-6+3n}$.

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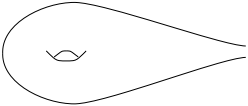

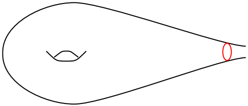

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Consider an ideal triangulation of S.
This requires $6g - 6 + 3n$ edges.

Each edge *e* of the triangulation has infinite length, but if we truncate at horocycles, the length $l_e \in \mathbb{R}$.

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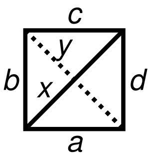

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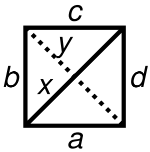
The algebra of the λ_a is full of amazing surprises.
A prototypical example of a cluster algebra (Fomin-Zelevinsky $\sim$ 2000).



The A-polynomial — intuitive idea

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Work in the upper half space model of hyperbolic space $\mathbb{H}^3$.

$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$

Metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$

Sphere at infinity S_∞
$= \{z = 0\} \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{C}P^1$.

$\text{Isom}^+\mathbb{H}^3 \cong PSL_2\mathbb{C} \cong SL_2\mathbb{C}/\{\pm I\}$.
Acts by Möbius transformations on S_∞

$$\pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow \left(z \mapsto \frac{az+b}{cz+d} \right).$$

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The peripheral subgroup $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$ has basis given by a longitude $\mathfrak{l}$ & meridian $\mathfrak{m}$.

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Now $\mathfrak{l}, \mathfrak{m} \in \pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$ commute.
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Answer (Cooper-Culler-Gilet-Long-Shalen 1994)

Those l, m satisfying the A-polynomial!

$$A_K(l, m) = 0.$$

Ways to calculate the A-polynomial

1. Original definition: representation theory / algebraic geometry
   (CCGLS 1994)
2. Hyperbolic geometry
   (Champanerkar 2003)
3. Sophisticated representation theory, Ptolemy varieties
   (Zickert 2016)
4. Hyperbolic geometry + symplectic geometry
   (Dimofte 2013, HMP 2020).

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Precisely (Cooper-Long 1996):
Let $R_U(M) = \{\rho \in R(M)\colon \rho(\lambda), \rho(\mu) \text{both upper triangular}\}$. Every $\rho \in R(M)$ is conjugate to one in $R_U(M)$.
Consider the map $\xi\colon R_U \longrightarrow \mathbb{C}^2$ which takes ρ to the top left entries of $\rho(\lambda)$ and $\rho(\mu)$.
After taking components with 1-dimensional Zariski closure, $\xi(R_U)$ defines the A-polynomial.

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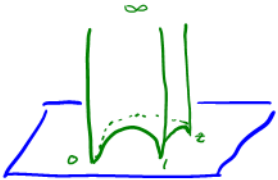
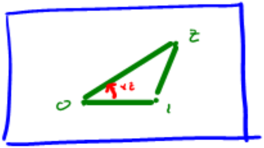


$\{\text{Hyp ideal tetrahedra}\} \cong \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$\operatorname{Isom}^+ \mathbb{H}^3$ acts triply transitively on S_∞.
$\exists!$ isometry taking 3 vertices to $0, 1, \infty$.
Fourth vertex then goes to z (cross ratio).

$\arg(z) =$ dihedral angle



Given edge/shape parameter z, other edges have parameters

$$z' = \frac{1}{1-z}, \quad z'' = \frac{z}{z-1}.$$

Opposite edges have same parameter.

In an ideal triangulation, tetrahedra fit together around an edge e.

$$\prod_{z \text{ parameter around } e} z = 1.$$

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I.e. $\rho(\mu) = (z \mapsto lz + \cdot)$ and $\rho(\lambda) = (z \mapsto mz + \cdot)$,
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Then l, m are given by a product of parameters obtained by walking through a cusp triangulation: cusp equations.

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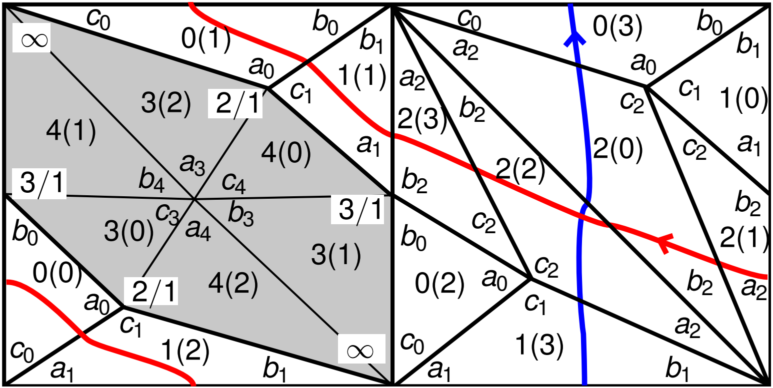

$$m = z_{b_1}^{-1} z_{a_2}^{-1} z_{a_2} z_{c_0}, \quad l = z_{a_2} z_{b_2} z_{b_2}^{-1} z_{a_2}^{-1} b_{b_1}^{-1} z_{b_0}^{-1} z_{a_1} z_{c_0}$$

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Theorem (Champanerkar 2003)

$H(l, m)$ is a factor of the $PSL_2\mathbb{C}$ A-polynomial.

Approach #2: Hyperbolic geometry

There's a holonomy map

$$\text{Hol}: P^{\mathcal{T}}(M) \longrightarrow \mathbb{C} \times \mathbb{C}$$

which maps $\mathbf{z}$ satisfying gluing equations to (l, m).

The image $\text{Hol}(P^{\mathcal{T}}(M))$ (taking components with dimension 1) is a curve in $\mathbb{C}^2$ with a defining polynomial $H(l, m)$.

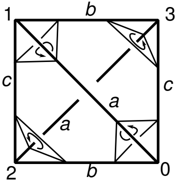
Theorem (Champanerkar 2003)

$H(l, m)$ is a factor of the $PSL_2\mathbb{C}$ A-polynomial.

In other words, from gluing and cusp equations, eliminating the tetrahedron parameters z_i essentially gives the A-polynomial.

In an ideal tetrahedron Δ, consider assigning

$$\begin{aligned}\{\text{Edges of } \Delta\} &\to \mathbb{C}^* \\ (jk) &\mapsto \gamma_{jk}\end{aligned}$$



Approach #3: Ptolemy varieties

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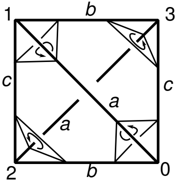

satisfying the Ptolemy equation

$$\gamma_{03}\gamma_{12} + \gamma_{01}\gamma_{23} = \gamma_{02}\gamma_{13}.$$

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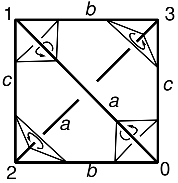

Consider an ideal triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^{n}$ of M.
Take variables $\gamma_{i,jk}$, over each edge (jk) of each tetrahedron Δ_i.

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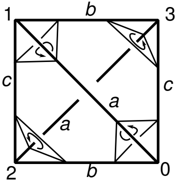

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Take variables $\gamma_{i,jk}$, over each edge (jk) of each tetrahedron Δ_i.

The Ptolemy variety $Pt(\mathcal{T})$ is defined by the Ptolemy equation in each tetrahedron, and identification relations

$$\gamma_{i,jk} = \pm\gamma_{i',j'k'} \quad (\text{sign depends on labelling/orientation})$$

when edges are identified.

Approach #3: Ptolemy varieties

Theorem (Garoufalidis-Thurston-Zickert 2015)

A Ptolemy assignment uniquely determines a boundary-unipotent representation $\pi_1(M) \longrightarrow SL_2\mathbb{C}$*, giving a map* $Pt(\mathcal{T}) \longrightarrow R(M)$.

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(Boundary-unipotent representation: takes peripheral subgroup into a conjugate of $N = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\} \subset SL_2\mathbb{C}$.)

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Theorem (Zickert 2016)

An enhanced Ptolemy assignment uniquely determines a boundary-Borel representation $\pi_1(M) \longrightarrow SL_2\mathbb{C}$*, giving a map* $EPt(\mathcal{T})_{red} \longrightarrow R(M)$*.*

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Boundary-Borel representation: takes peripheral subgroup into a conjugate of $B = \left\{ \begin{bmatrix} x & * \\ 0 & x^{-1} \end{bmatrix} \right\} \subset SL_2\mathbb{C}$.

Approach #3: Ptolemy varieties

The enhanced Ptolemy variety is defined by the same variables and Ptolemy relations, but identification relations

$$\gamma_{i,jk} = \pm L^{\bullet} M^{\bullet} \gamma_{i',j'k'} \quad \left(\text{powers of } L,\, M \text{ depend on cusp triangulation}\right)$$

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Zickert remark: This illustrates a "duality between shapes and Ptolemy coordinates".

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	Coordinates	Equations
Ptolemy var.	1 γ per edge	1 Ptolemy eqn per tetrahedron
Hyp. geom.	1 z per tetrahedron	1 gluing eqn per edge

(Note an Euler χ argument shows # tetrahedra = # edges.)

Approach # 4: Symplectic geometry

Key fact:

The combinatorics of ideal triangulations of 3-manifolds are surprisingly symplectic!

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Label edges $E_1, \ldots, E_n$, tetrahedra $\Delta_1, \ldots, \Delta_n$.
Let $a_{k,j} =$ # (01) or (23) edges in Δ_j identified to E_k.
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The gluing equation for E_k is then

$$\prod_{j=1}^{n} z_j^{a_{k,j}} \left(z_j'\right)^{b_{k,j}} \left(z_j''\right)^{c_{k,j}} \quad \text{or} \quad \sum_{j=1}^{n} a_{k,j} Z_j + b_{k,j} Z_j' + c_{k,j} Z_j''$$

where $Z_j = \log z_j$, $Z_j' = \log z_j'$, $Z_j'' = \log z_j''$.

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Since $zz'z'' = -1$ then $Z + Z' + Z'' = \pi i$: eliminate each Z''.

Approach # 4: Symplectic geometry

Gluing equation for edge E_k becomes

$$\sum_{j=1}^{n} d_{k,j} Z_j + d'_{k,j} Z'_j = 2\pi i (2 - c_k)$$

where $d_{k,j} = a_{k,j} - c_{k,j}$, $d'_{k,j} = b_{k,j} - c_{k,j}$ and $c_k = \sum_{j=1}^{n} c_{k,j}$ are integers determined by triangulation combinatorics.

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Applying same idea to cusp equations,

$$\sum_{j=1}^{n} \mu_j Z_j + \mu'_j Z'_j = \log m - i\pi c^{\mathfrak{m}}, \quad \sum_{j=1}^{n} \lambda_j Z_j + \lambda'_j Z'_j = \log l - i\pi c^{\mathfrak{l}}$$

for some integers $\mu_j, \mu'_j, c^{\mathfrak{m}}, \lambda_j, \lambda'_j, c^{\mathfrak{l}}$.

Approach # 4: Symplectic geometry

The coefficients form the Neumann-Zagier matrix NZ.

$$NZ = \begin{array}{c} \\ E_1 \\ \vdots \\ E_n \\ \mathfrak{m} \\ \mathfrak{l} \end{array} \begin{array}{c} \begin{array}{ccc} \Delta_1 & \cdots & \Delta_n \end{array} \\ \begin{bmatrix} d_{1,1} & d'_{1,1} & \cdots & d_{1,n} & d'_{1,n} \\ & \vdots & \ddots & \vdots & \\ d_{n,1} & d'_{n,1} & \cdots & d_{n,n} & d'_{n,n} \\ \mu_1 & \mu'_1 & \cdots & \mu_n & \mu'_n \\ \lambda_1 & \lambda'_1 & \cdots & \lambda_n & \lambda'_n \end{bmatrix} \end{array} = \begin{pmatrix} R^G_1 \\ \vdots \\ R^G_n \\ R^{\mathfrak{m}} \\ M^{\mathfrak{l}} \end{pmatrix}$$

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Logarithmic gluing and cusp equations then become

$$NZ \begin{bmatrix} Z_1 \\ Z'_1 \\ \vdots \\ Z_n \\ Z'_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \log m \\ \log l \end{bmatrix} + i\pi \begin{bmatrix} 2 - c_1 \\ \vdots \\ 2 - c_n \\ -c^{\mathfrak{m}} \\ -c^{\mathfrak{l}} \end{bmatrix}$$

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Rows of NZ lie in $\mathbb{R}^{2n}$ and coordinates come in pairs...

Approach # 4: Symplectic geometry

Take $\mathbb{R}^{2n}$ with coords $(x_1, y_1, \ldots, x_n, y_n)$ and standard symplectic form $\omega = \sum dx_i \wedge dy_i$.

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Theorem (Neumann-Zagier 1985)

All rows of NZ are symplectically orthogonal except $\omega(R^{\mathfrak{m}}, R^{\mathfrak{l}}) = 2$. In particular, all $\omega(R_j^G, R_k^G) = 0$.

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E.g. a triangulation of trefoil complement.

$$
NZ = \begin{array}{c} \\ E_{0(23)} \\ E_{3/1} \\ E_{2/1}+E_{1/0} \\ \mathfrak{m} \\ \mathfrak{l} \end{array}
\begin{array}{c} \begin{array}{ccc} \quad\Delta_0 \quad & \quad\Delta_1\quad & \quad\Delta_2\quad \end{array} \\
\left[\begin{array}{cccccc}
1 & 0 & -1 & -1 & -2 & -2 \\
0 & 1 & 1 & 0 & 0 & 1 \\
-1 & -1 & 0 & 1 & 2 & 1 \\
-1 & -1 & 0 & -1 & 0 & 0 \\
-1 & -2 & 1 & -1 & 0 & 0
\end{array}\right] \end{array}
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The rows of NZ extend to a symplectic basis of $\mathbb{R}^{2n}$.

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We can form a symplectic matrix

$$NZ = \begin{pmatrix} R_1^G \\ \vdots \\ R_n^G \\ R^{\mathfrak{m}} \\ M^{\mathfrak{l}} \end{pmatrix} \rightsquigarrow SY = \begin{pmatrix} R_1^\Gamma \\ R_1^G \\ \vdots \\ R_{n-1}^\Gamma \\ R_{n-1}^G \\ R^{\mathfrak{m}} \\ \frac{1}{2} R^{\mathfrak{l}} \end{pmatrix}$$

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where

$$\omega(R_j^\Gamma, R_k^G) = \delta_{j,k}$$

$$\omega(R^{\mathfrak{m}}, \frac{1}{2} R^{\mathfrak{l}}) = 1$$

all other rows ω-orthogonal.

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Note the R_j^Γ are very not unique!

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$$\begin{bmatrix} \Gamma_1 \\ G_1 \\ \vdots \\ \Gamma_{n-1} \\ G_{n-1} \\ M \\ \frac{1}{2}L \end{bmatrix} = SY.Z = SY \begin{bmatrix} Z_1 \\ Z_1' \\ \vdots \\ Z_n \\ Z_n' \end{bmatrix} \quad \text{i.e.} \left\{ \begin{array}{rcl} G_k & = & R_k^G Z \\ \Gamma_k & = & R_k^\Gamma Z \\ M & = & R^{\mathfrak{m}} Z \\ \frac{1}{2}L & = & \frac{1}{2}R^{\mathfrak{l}} Z \end{array} \right.$$

Then change variables from Z_j, Z_j' to Γ_j, G_j, L, M! One Γ_j for each edge (except one).

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Gluing & cusp equations now become very simple

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However numerous difficulties remain.

Approach # 4: Symplectic geometry

Difficulties eliminating variables to obtain A-polynomial:

- Implement the equation $z_j' = \frac{1}{1-z_j}$, i.e. $e^{Z_j'} = \frac{1}{1-e^{Z_j}}$.
- Need to write Z_j, Z_j' in terms of Γ_j, G_j, L, M, i.e. invert SY.
- Half the entries of SY (i.e. R_j^Γ) require work to calculate.
- The R_j^Γ are not unique.

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Claim (HMP)

All these issues can be resolved simultaneously and systematically, “inverting without inverting".

Each equation $z_j' = \frac{1}{1-z_j}$ becomes a Ptolemy equation in $\gamma_j = e^{\Gamma_j}$, up to signs and powers of l, m.

Approach # 4: Symplectic geometry

Write $\gamma_{i(jk)}$ for variable of edge (jk) of Δ_i (one edge has $\gamma = 1$).
The Ptolemy equation for Δ_i is

$$\pm l^{\bullet} m^{\bullet} \gamma_{i(01)} \gamma_{i(23)} \pm l^{\bullet} m^{\bullet} \gamma_{i(02)} \gamma_{i(13)} - \gamma_{i(03)} \gamma_{i(12)} = 0$$

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$$(-1)^{B'_j} l^{-\frac{\mu_i}{2}} m^{\frac{\lambda_i}{2}} \gamma_{i(01)}\gamma_{i(23)} + (-1)^{B_j} l^{-\frac{\mu'_i}{2}} m^{\frac{\lambda'_i}{2}} \gamma_{i(02)}\gamma_{i(13)} - \gamma_{i(03)}\gamma_{i(12)} = 0.$$

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Theorem (HMP)

Eliminating the γ variables from the Ptolemy equations results in a polynomial which is a factor of the PSL$(2,\mathbb{C})$ A-polynomial. These equations are equivalent to Champanerkar's equations.

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Write $\gamma_{i(jk)}$ for variable of edge (jk) of Δ_i (one edge has $\gamma = 1$).
The Ptolemy equation for Δ_i is

$$\pm l^\bullet m^\bullet \gamma_{i(01)}\gamma_{i(23)} \pm l^\bullet m^\bullet \gamma_{i(02)}\gamma_{i(13)} - \gamma_{i(03)}\gamma_{i(12)} = 0$$

$$(-1)^{B_j'} l^{-\frac{\mu_i}{2}} m^{\frac{\lambda_i}{2}} \gamma_{i(01)}\gamma_{i(23)} + (-1)^{B_j} l^{-\frac{\mu_i'}{2}} m^{\frac{\lambda_i'}{2}} \gamma_{i(02)}\gamma_{i(13)} - \gamma_{i(03)}\gamma_{i(12)} = 0.$$

Theorem (HMP)

Eliminating the γ variables from the Ptolemy equations results in a polynomial which is a factor of the $PSL(2, \mathbb{C})$ A-polynomial. These equations are equivalent to Champanerkar's equations.

Key ideas:

- Symplectic matrices are easy to invert!
- Symplectic algebra gives freedom to choose nice R_j^Γ.
- Neumann (1990) guarantees integer solutions of $NZ.B = C$.

Thus,

- refining symplectic techniques (Dimofte, approach # 4)
- based on hyp geom approach (# 2, Champanerkar)
- yields Ptolemy varieties very similar to those arising from representation theory approach (#3, Zickert).

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Further questions:

- Are our Ptolemy equations equivalent to those of Zickert?
- The signs are given by a vector $B = (B_1, B'_1, \ldots, B_n, B'_n)$ satisfying $NZ.B = C$. Connections to taut triangulations (Lackenby 2000) or taut angle structures? (Burton, Hodgson, Kang, Rubinstein, Segerman, Tillmann, ...)

Consider a 2-component link in S^3 consisting of knots K_0, K_1.

$M = S^3 \backslash (K_0 \cup K_1)$, Dehn fill K_0 along a slope p/q: obtain 1-cusped $M(p/q)$.

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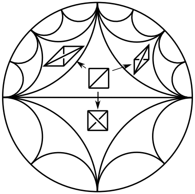
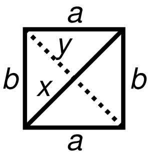


These triangulations look like diagonal flips in a 2D triangulation of the cusp torus — just as in hyperbolic surface cluster algebra.

The A-polynomial Ptolemy equations can be taken to coincide, up to signs, with those in the cusp torus cluster algebra.



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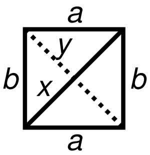

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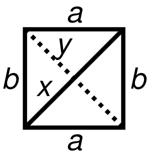

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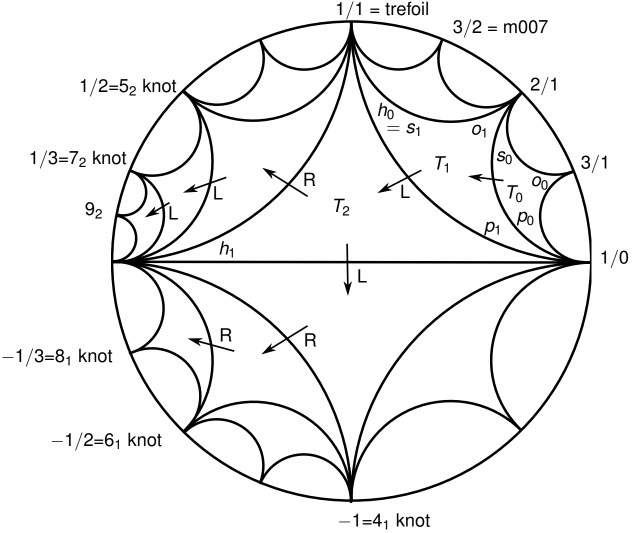
The $+$ or $-$ can be read off from he word in Ls & Rs for the filling.

Example: Whitehead link filling and twist knots



Example: Whitehead link filling and twist knots

The Whitehead link complement has an ideal triangulation with 3 tetrahedra away from Dehn-filled K_0, 3 Ptolemy equations:

$$0 = -LM^{-1}\gamma_a\gamma_2 - LM^{-2}\gamma_{3/1}\gamma_\infty - \gamma_\infty^2, \quad 0 = -M\gamma_3\gamma_\infty - LM^{-1}\gamma_\infty^2 - \gamma_a\gamma_2, \quad 0 = \gamma_\infty^2 - \gamma_\infty\gamma_3 - \gamma_a^2.$$

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Figure-8 (-1 or LL filling): layered solid torus has 2 tetrahedra.

$$0 = \gamma_3\gamma_1 + \gamma_2^2 - \gamma_\infty^2$$
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Fold up the layered solid torus, set $\gamma_0 = \gamma_\infty$.

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9_2 (1/4 or LRLL filling): one more tet, fold identifying $\gamma_{1/3} = \gamma_0$.

$$0 = -\gamma_1\gamma_{1/3} + \gamma_{1/2}^2 - \gamma_0^2$$

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