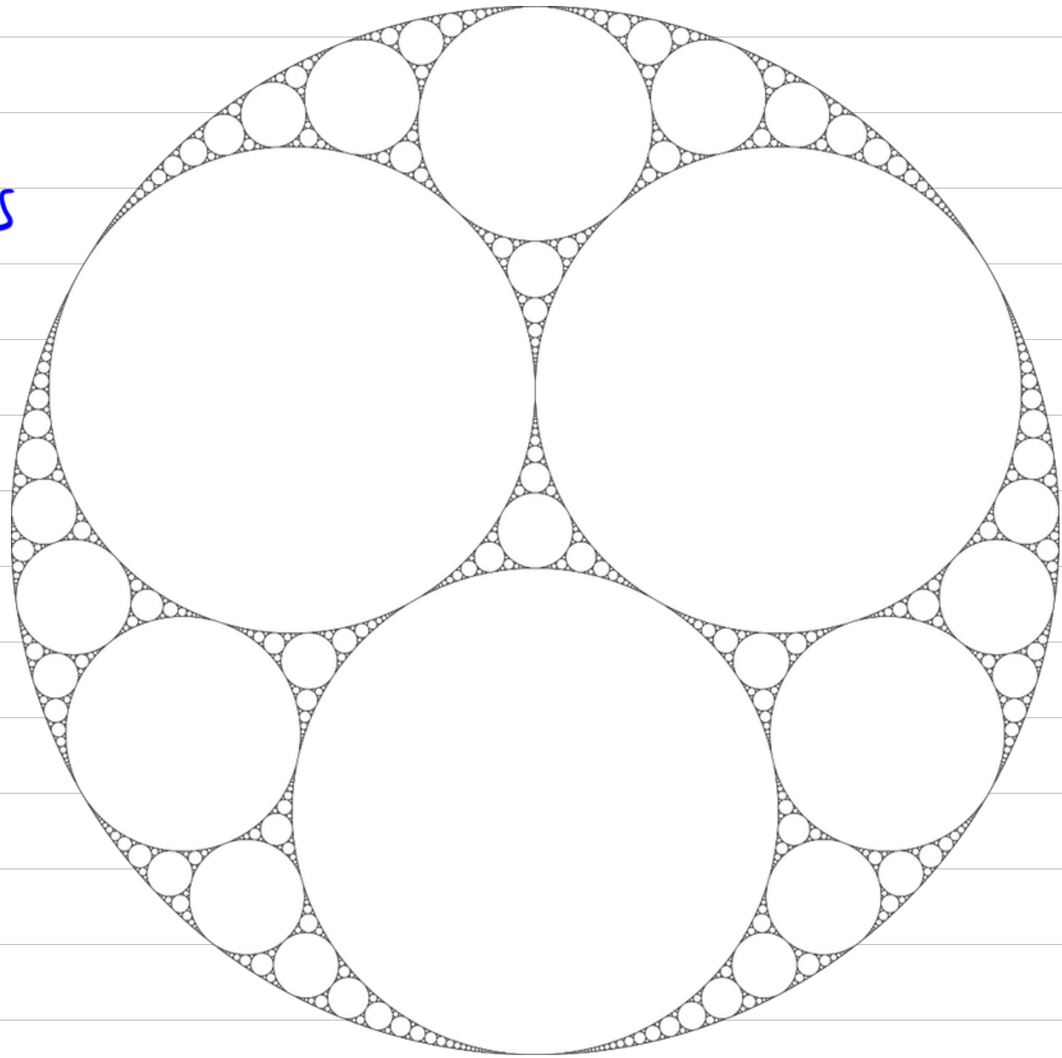
Geometry & Physics of Circle Packings



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Aust MS meeting, 4/12/19

Overview

A few ideas from the theory of circle packings & recent work in progress
→ relations to spinors, symplectic geometry, Grassmannians & scattering theory.

1. (Koebe, Andreev, Thurston, Stephenson, ...) There is a wonderful theory of packing circles in the plane

   → like "discrete complex analysis"
   → related to geometry of hyperbolic polyhedra (Purcell, Hodgson, ...)

2. (M.) One can understand & compute circle packings (& more general arrangements) as Lagrangian plane arrangements in a symplectic and complex vector space of spinors.

3. (M.) The resulting mathematics is closely related to recent developments in $\mathcal{N}=4$ SUSY Yang-Mills theory (on-shell diagrams, amplituhedron, ...).

Highlights from the Theory of Circle Packings

Fix a 2-complex K, a triangulation of a surface (w/ or w/o ∂).

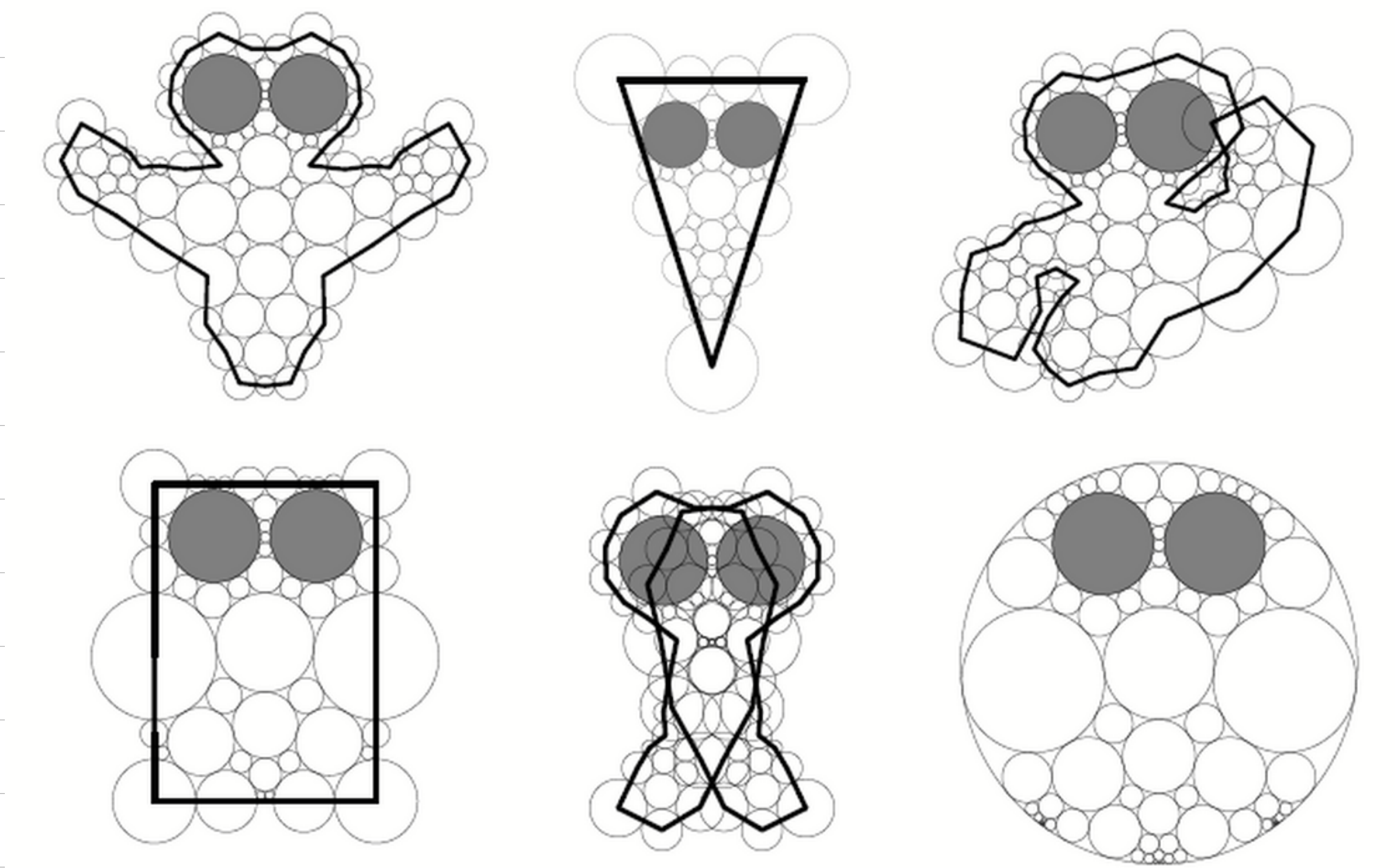
A collection of circles $\{C_v\}$ in a metric space is a circle packing for K if

(1) There is a circle C_v for each vertex v of K
(2) C_u, C_v are externally tangent when u, v are joined by an edge
(3) C_u, C_v, C_w form a positively oriented triple when u, v, w do.

Basically, you can always find a circle packing with spherical / Euclidean / hyp geometry.

<u>Theorem:</u> (Beardon-Stephenson 1990) If K is a disc then $\exists$ a (locally univalent) Euclidean circle packing for K. Boundary radii can be prescribed arbitrarily! It's unique up to Euclidean isometries.

Circle packings of a fixed triangulation K of a disc with varying boundary radii.



Source: K Stephenson, Circle packing & discrete analytic function theory

Spinors & Spacetime

Penrose, Rindler 1984: Think of spinors rather than tangent vectors on $S^2 \cong \mathbb{C}P^1$.

For present purposes, a spinor is just a vector in $\mathbb{C}^2$.

A spinor $\xi = (w, z)$ has a spin vector, a tangent vector to $\mathbb{C}P^1$ given by the vector $1/w^2$ at the point z/w.

Note: $re^{i\theta}\xi$ has spin vector at the same point as ξ but its length is multiplied by $1/r^2$ and it's rotated by -2θ.

There's an antisymmetric bilinear form $\{\cdot,\cdot\} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ on spinors given by $\{\xi_0, \xi_1\} = w_0 z_1 - z_0 w_1$.

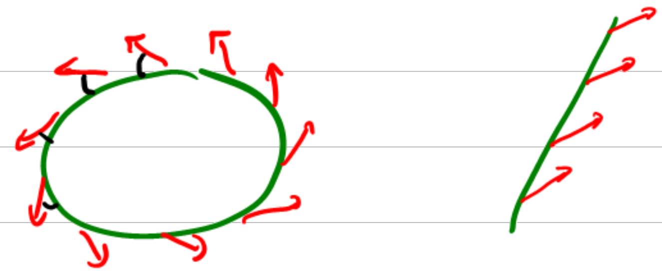
Its imaginary part is a symplectic form ω on $\mathbb{C}^2 \cong \mathbb{R}^4$
$\omega : \mathbb{R}^4 \otimes \mathbb{R}^4 \to \mathbb{R}$, $\qquad \omega(\xi_0, \xi_1) = \mathrm{Im}\{\xi_0, \xi_1\}$.

Spin vectors and Lagrangian subspaces

Let Π be a 2-(real)-dimensional subspace of $\mathbb{C}^2$.

Facts (Penrose, Rindler): Let X denote the set of spin vectors of Π.

* Π is complex $\Leftrightarrow$ All vectors in X lie at a point
* Otherwise, X consists of vectors along a circle in $\mathbb{CP}^1$, making a constant angle.



* The spin vectors are tangent to the circle
  $\Leftrightarrow$ Π is Lagrangian, i.e. ω vanishes on Π.

Indeed, there are natural bijections

Grassmannian $Gr(2,4) = \{2\text{-planes in } \mathbb{R}^4\} \longleftrightarrow \{\text{points and circles with directions in } \mathbb{CP}^1\}$

Lagrangian Grassmannian $LGr(2,4) = \{\text{Lagrangian 2-planes in } \mathbb{R}^4\}$
$\longleftrightarrow \{\text{points and oriented circles in } \mathbb{CP}^1\}$

Lagrangian Plane Arrangements

Thus "Lagrangian planes have centres & radii" (= "distance from being complex")

Note: Π Lagrangian $\Rightarrow$ $i\Pi$ Lagrangian
$\rightarrow$ Corresponds to reversing orientations of circles!

Statements about circle packings now translate into statements about Lagrangian planes
Eg...

Theorem (M): Suppose K triangulates a disc. Then $\exists$:
* for each vertex v of K, a Lagrangian plane Π_v
* for each u, v connected by an edge, $\Pi_u \cap i\Pi_v$ is 1-Dimensional.
* for each u, v, w around a positively oriented triangle,
  Π_u, Π_v, Π_w have centres forming a positively oriented triangle.

Moreover, every boundary Π_v can have any prescribed radius.
These planes are unique up to the action of $GL^+(2,\mathbb{C})$ on $\mathbb{C}^2$
(det real & positive)

Plücker Coordinates & Inversive Vectors

The Grassmannian $Gr(2,4)$ has projective Plücker coordinates

$$[\Delta_{12} : \Delta_{13} : \Delta_{14} : \Delta_{23} : \Delta_{24} : \Delta_{34}]$$

* Given a 2-plane $\Pi \subset \mathbb{R}^4$, form a 2×4 matrix whose rows form a basis
* Δ_{ij} = determinant of 2×2 submatrix formed by i'th & j'th columns

Lemma (M): For $\Pi \in LGr(2,4)$, its Plücker coordinates are

$$[k : -kq+1 : kp : -kp : -kq-1 : \tilde{k}]$$

where Π corresponds to a circle centred at $p+qi$ with curvature k (and $\tilde{k} = k(p^2+q^2) - \frac{1}{k}$)

Thus consider the vector $v_\Pi = (k, -kq, kp, \tilde{k})$ Inversive vector of Π.

Note

$$[k \quad -kq \quad kp \quad \tilde{k}] \underbrace{\begin{bmatrix} & & & 1 \\ & -2 & & \\ & & -2 & \\ 1 & & & \end{bmatrix}}_{B} \begin{bmatrix} k \\ -kq \\ kp \\ \tilde{k} \end{bmatrix} = -2$$

so define a bilinear form:

$$J(v,w) = v^T B w.$$

Circle Packing Equations

For any Lagrangian plane Π then $J(v_\Pi, v_\Pi) = -2$

But more...

Proposition (M, Lagarias - Mallows - Wilks 2001) For Lagrangian Π, Π' corresponding to oriented circles C, C'

$$J(v_\Pi, v_{\Pi'}) = -2\cos(\text{angle b/w } C, C') = \text{Inversive distance}(C, C')$$

So we obtain equations for tangency of circles! (Or any prescribed angle!)

One can find a circle packing for a triangulation K by solving quadratic equations of the form $J(v, w) = \pm 2$

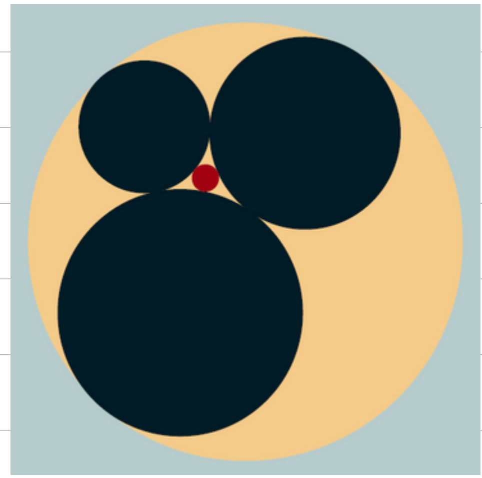
$\rightsquigarrow$ Exercise in elimination theory

Generalisations of Descartes' Circle Theorem

Consider K_n =  (n triangles around a central point)

A circle packing for K_n is a flower with n petals.

$\to$ A central circle C_0, surrounded by tangent circles $C_1, \dots, C_n$.



Source: P Levrie

$n=3$: Flower = 4 mutually externally tangent circles.

Let k_i : curvature of C_i.

Descarte's Circle Theorem (1634):

$$(k_0+k_1+k_2+k_3)^2 = 2(k_0^2+k_1^2+k_2^2+k_3^2)$$

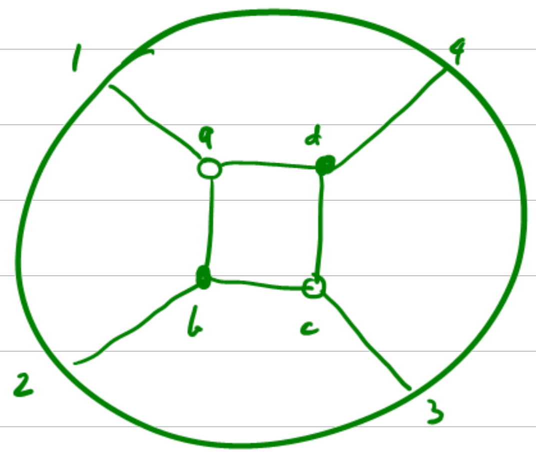
In general, circle packing theory says k_0 is a function of $k_1, \dots, k_n$.
Using our methods we can find generalisations for higher n.

$n=4$ Thm (M):

$$16k_0^4 - 8k_0^2(k_1k_2+k_2k_3+k_3k_4+k_4k_1+2k_1k_3+2k_2k_4)$$
$$+(k_1^2+k_3^2)(k_2^2+k_4^2) - 16k_0k_1k_2k_3k_4\left(\sum_{i=1}^{4}\frac{1}{k_i}\right) - 12k_1k_2k_3k_4 = 0$$
$$-2(k_1k_2+k_3k_4)(k_2k_3+k_4k_1)$$

Circle Packings as Scattering Diagrams

In the scattering theory of $\mathcal{N}=4$ supersymmetric Yang-Mills theory, one considers "on-shell diagrams" which are planar bicoloured graphs with "external" vertices arranged in a circle. (A Postnikov, N Arkani-Hamed, many others...)



Each oriented edge is decorated with a weight $w(U,V)$ satisfying $w(U,V)\,w(V,U)=1$

One seeks spinors $z_{(U,E)}$ associated to each half-edge s.t.
* $w(U,V)\, z_{(V,E)} = z_{(U,E)}$
* At a black V, all spinors are equal $z_{(V,E)} = z_{(V,E')}$
* At a white V, spinors sum to 0, $\sum_E z_{(V,E)} = 0$

These have interpretations as circle arrangements!

Work in progress...