# Ptolemy

vs

# Thurston

in . . .

# Hyperbolic Geometry & Topology


Daniel Mathews

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– AustMS Meeting – 9/12/20

# Head to Head...

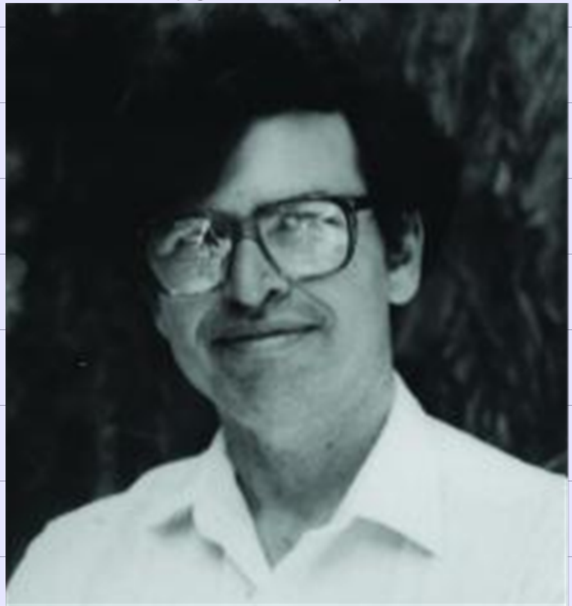
## Claudius Ptolemaeus
(c. 100 – 170)

**Strengths:** Star catalogues,
Geocentric models

**Greatest Work:** Almagest

**Weaknesses:** Epicycles

## William P Thurston
(1946 – 2012)

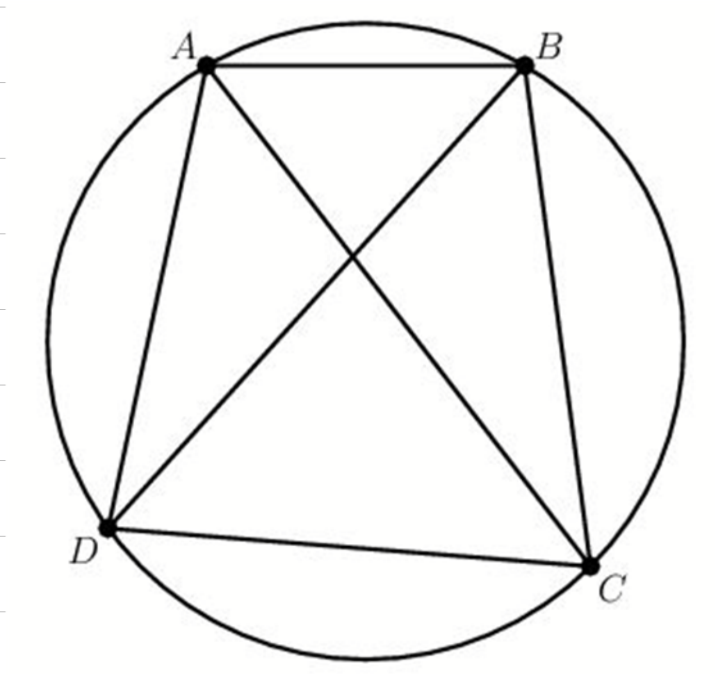


**Strengths:** Geometrisation,
Imagination

**Greatest Work:** Three-dimensional geometry and topology

**Weaknesses:** Publishing

# Ptolemy's Theorem




ABCD is a cyclic quadrilateral

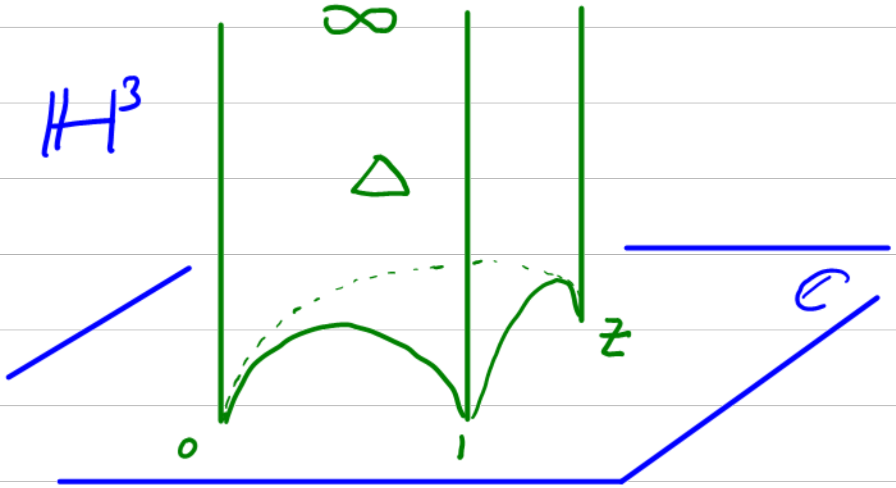
Theorem : $AB \cdot CD + BC \cdot AD = AC \cdot BD$

Such equations arise all over mathematics!

→ Cluster algebras

Good for star charts too!

# Thurston's hyperbolic gluing equations

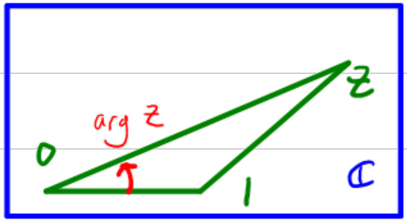



$$\{\text{Hyperbolic ideal tetrahedra } \Delta\} \cong \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

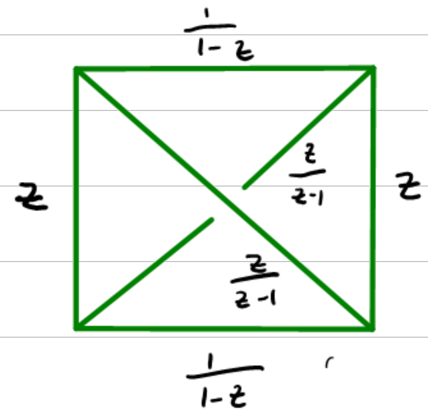
$\exists!$ orientation-preserving isometry taking 3 vertices to $0, 1, \infty$ in upper half space model.

$4^{th}$ vertex $\mapsto z$ (cross ratio)

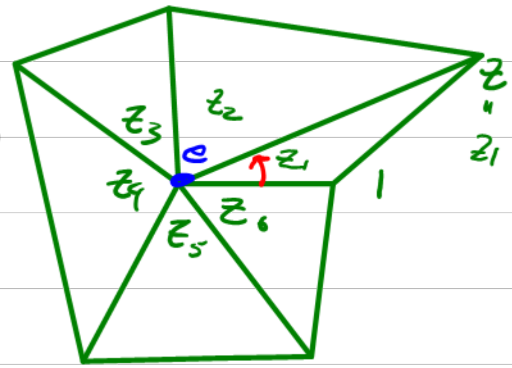
$\operatorname{Arg} z =$ Dihedral angle




Each edge of $\Delta$ has its own parameter, a function of $z$. Opposite edges have the same parameter.




In a hyperbolic ideal triangulation of a 3-manifold, tetrahedra fit around each edge $e$.



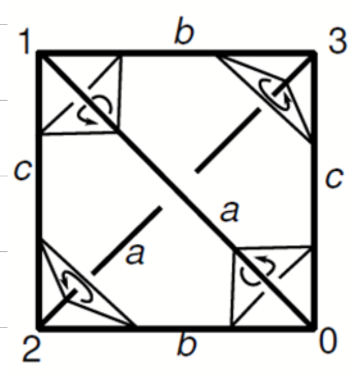

$$\prod_{\substack{\text{parameters } z_i \\ \text{around } e}} z_i = 1.$$

$\rightsquigarrow$ Solutions describe hyperbolic structures!

# Ptolemy equations for representations

(Zickert, together with Garoufalidis, Goerner, D Thurston... 2014 – 2018)

Consider an ideal triangulation $\mathcal{T}$ of a hyperbolic 3-manifold $M$




Assign a variable $\gamma_e$ to each edge $e$ of $\mathcal{T}$

For each ideal tetrahedron $\Delta$ with oriented labelling of vertices $0, 1, 2, 3$ form a Ptolemy equation*

$$\gamma_{03}\gamma_{12} + \gamma_{01}\gamma_{23} = \gamma_{02}\gamma_{13}$$

Solutions describe representations* $\pi_1(M) \to SL_2\mathbb{C}$

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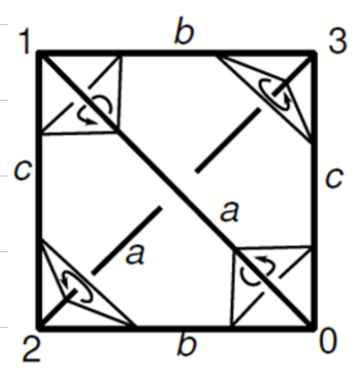


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* Technicalities!
- labels in different $\Delta$
- signs in the equations

* Reps of a specific type!
$\pi_1(\partial M) \to$ unipotent matrices $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

Many variations on equations & corresponding reps!

# Ptolemy and Thurston are Dual, Somehow...

Garoufalidis – Goerner – Zickert :

"Our observations suggest that there is a fundamental duality between the two sets of coordinates, which is interesting in its own right."

Thurston :

* One variable $z_i$ for each $\Delta$
* One equation $\prod z_i = 1$ for each edge

Ptolemy :

* One variable $\gamma_e$ for each edge
* One Ptolemy equation for each $\Delta$.

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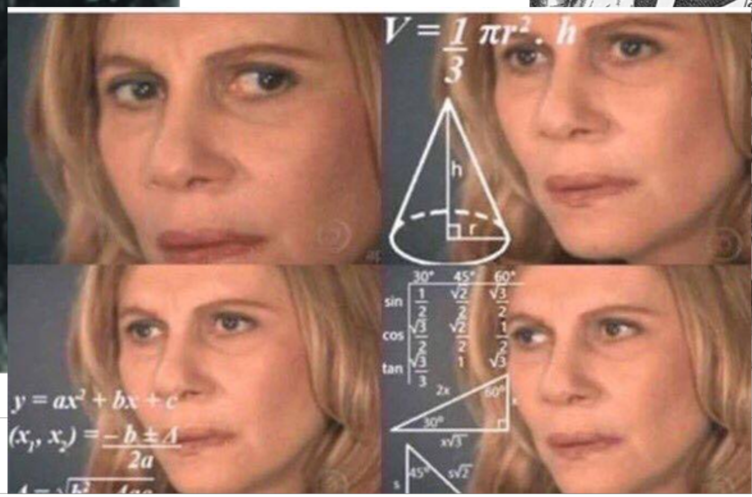

# Hyperbolic Geometric meaning of Ptolemy coordinates

Garoufalidis – D Thurston – Zickert 2014:

Ptolemy coordinates $\{\gamma_e\}$ and the associated representation $\rho : \pi_1(M) \to SL_2\mathbb{C}$ are related by...

* assigning cosets of the subgroup $N = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$ to the vertices of each tetrahedron $\tilde{\Delta}$ of $\tilde{T}$

  ($\Delta$ with vertices $0,1,2,3 \rightsquigarrow g_0N, g_1N, g_2N, g_3N$)

* then for an edge $e$ of $T$ on $\Delta$ with endpoints $i,j \in \{0,1,2,3\}$

$$\gamma_e = \det\left( \begin{Bmatrix} \text{first column} \\ \text{of } g_i \end{Bmatrix} \begin{Bmatrix} \text{first column} \\ \text{of } g_j \end{Bmatrix} \right) \qquad (?!)$$

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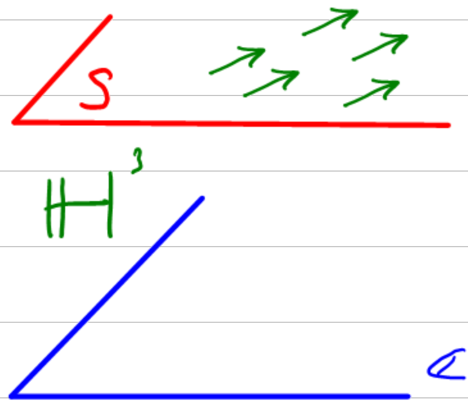


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Hyperbolic interpretation: (M-Purcell, forthcoming)

– projecting to $PSL_2\mathbb{C} \cong \mathrm{Isom}^+ \mathbb{H}^3$ (neglecting signs etc...)


* Cosets of $N$ $\cong$ Horospheres $S$ in $\mathbb{H}^3$ decorated with $PSL_2\mathbb{C}/\pm N$ a Euclidean direction

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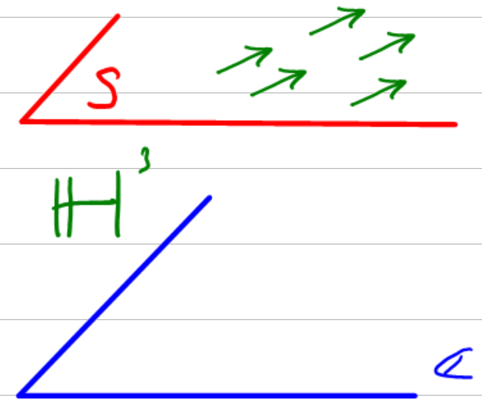
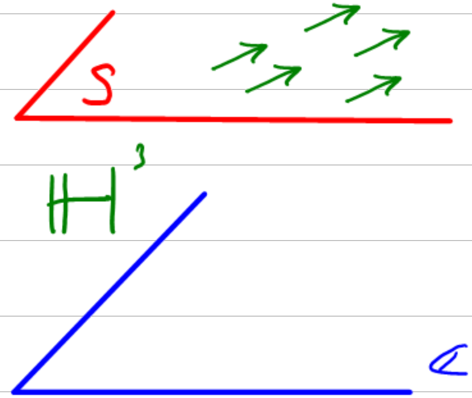
  horospheres $S, S'$ & directions $v, v'$ at each end



# Hyperbolic Geometric meaning of Ptolemy coordinates

* Cosets of $N$ in $PSL_2\mathbb{C}/\pm N$ $\cong$ Horospheres $S$ in $\mathbb{H}^3$ decorated with a Euclidean direction

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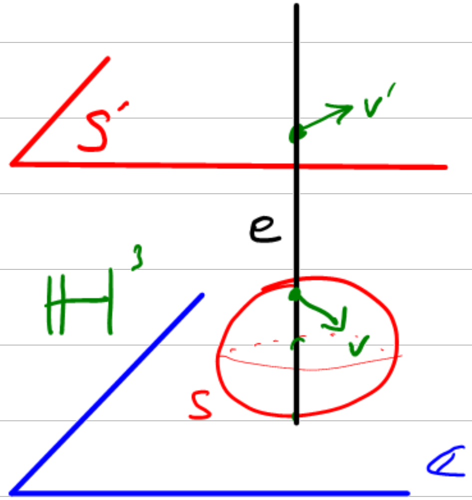


$$\gamma_e \sim (-i)\, e^{\frac{1}{2}(\text{complex translation distance } v \to w)}$$

(Complex translation = Translation + $i \cdot$ Rotation)

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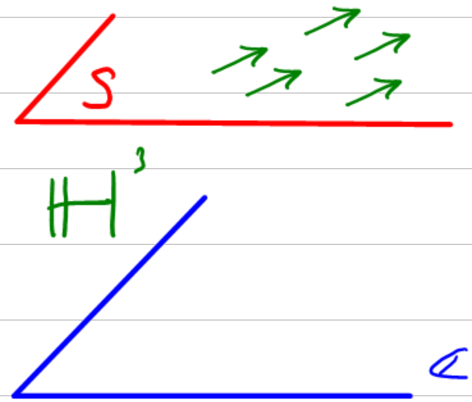


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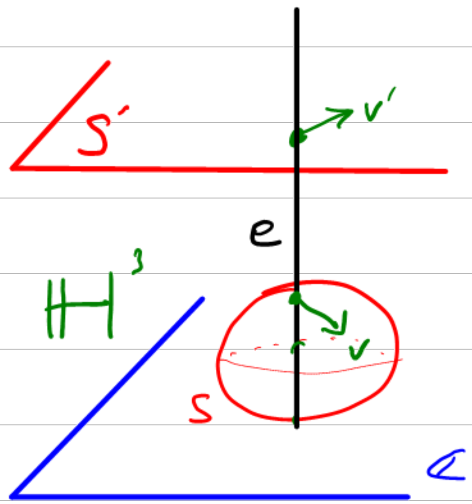


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3D complex $\lambda$-lengths!

(Complex translation
= Translation + $i \cdot$ Rotation)

"Ptolemaic" hyperbolic geometry!

* Garoufalidis - Thurston - Zickert showed that $z = \dfrac{\gamma_{03}\,\gamma_{12}}{\gamma_{02}\,\gamma_{13}}$,
  which implies Ptolemy equations reduce to tetrahedron parameter eqn $z + z'^{-1} = 1$.
* A similar idea for skew polygons appeared in work
  of Thistlethwaite - Tsvietkova
  (other places too?)

# Thurston - Ptolemy Duality in the Neumann - Zagier Matrix

Another way Ptolemy is dual to Thurston!

The Neumann - Zagier matrix encodes combinatorics of an ideal triangulation + boundary data

For an ideal triangulation of a knot complement in $S^3$, with $N$ edges & $N$ tetrahedra, it looks like

$(N+2) \times 2N$

$$
\begin{array}{c|c}
 & \overbrace{z_1 \quad z_1'}^{\Delta_1} \quad \overbrace{z_2 \quad z_2'}^{\Delta_2} \quad \cdots \quad \overbrace{z_N \quad z_N'}^{\Delta_N} \\
\begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_N \\ M \\ L \end{matrix} &
\begin{bmatrix}
(\text{Gluing eqn for edge } e_1) \\
\vdots \\
\\
(\text{Gluing eqn for edge } e_N) \\
(\text{Holonomy of meridian}) \\
(\text{Holonomy of longitude})
\end{bmatrix}
\end{array}
$$

Neumann - Zagier: Rows behave like part of a symplectic basis for $\mathbb{R}^{2N}$.

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\hline
e_1 & \multicolumn{5}{l|}{(\text{Gluing eqn for edge } e_1)} \\
e_2 & \multicolumn{5}{l|}{\vdots} \\
\vdots & & & & & \\
e_N & \multicolumn{5}{l|}{(\text{Gluing eqn for edge } e_N)} \\
M & \multicolumn{5}{l|}{(\text{Holonomy of meridian})} \\
L & \multicolumn{5}{l|}{(\text{Holonomy of longitude})} \\
\hline
\gamma_1 & & & & & \\
\vdots & & & & & \\
\gamma_{N-1} & & & & &
\end{array}
$$

<u>Neumann - Zagier</u>: Rows behave like part of a <u>symplectic basis</u> for $\mathbb{R}^{2N}$.

<u>Dimofte</u> (2015): Consider extending to a full symplectic basis! Gluing eqns!

<u>Howie - M - Purcell</u> (2020): In dual gluing equations become Ptolemy equations!

$\rightarrow$ E. Thompson forthcoming

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\hline
e_1 & \multicolumn{4}{c}{(\text{Gluing eqn for edge } e_1)} \\
e_2 & \multicolumn{4}{c}{\vdots} \\
\vdots & & & & \\
e_N & \multicolumn{4}{c}{(\text{Gluing eqn for edge } e_N)} \\
M & \multicolumn{4}{c}{(\text{Holonomy of meridian})} \\
L & \multicolumn{4}{c}{(\text{Holonomy of longitude})} \\
\hline
\gamma_1 & \multicolumn{4}{c}{\text{M- Purcell (forthcoming)}} \\
\vdots & \multicolumn{4}{c}{\text{Topological description of these}} \\
\gamma_{N-1} & \multicolumn{4}{c}{\text{rows in terms of triangulation!}}
\end{array}
$$

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M- Purcell (forthcoming): Symplectic form = intersection form for appropriate curves!

# Thanks for Listening !

Refs:


* Neumann, Zagier, Volumes of hyperbolic three-manifolds, Topology (1985)
* Dimofte, Quantum Riemann surfaces in Chern-Simons theory, Adv Theor Math Phys (2013)
* Thistlethwaite, Tsvietkova, An alternative approach to hyperbolic structures, AGT (2014)
* Garoufalidis, Goerner, Zickert, Gluing equations for PGL(n, $\mathbb{C}$)-representations of 3-manifolds, AGT (2015)
* Garoufalidis, Zickert, The symplectic properties of the PGL(n, $\mathbb{C}$)-gluing equations, QT (2016)
* Zickert, Ptolemy coordinates, Dehn invariant and the A-polynomial, Math Z (2016)
* Goerner, Zickert, Triangulation independent Ptolemy varieties, Math. Z. (2018)
* Howie, Mathews, Purcell, A-polynomials, Ptolemy varieties and Dehn filling, arXiv: 2002.10356


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