Congruences
Senior Problems

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1. (1975 IMO) Consider the number 4444^{4444}. Take the sum of its digits and call this number A. Then take the sum of the digits of A and call it B. Find the sum of the digits of B.
2. Show that $75 + 26n^2$ is never a perfect square, for any integer n.
3. Show that $a^2 - 15ab + b^2 = 15$ has no integer solutions.
4. Show that the equation
$$x^8 + 10001y^4 = 13z^4 + 300$$
has no integer solutions.
5. (a) How many times is $3^{128} - 1$ divisible by 2?
   (b) How many times is $3^{2^n} - 1$ divisible by 2?
   (c) How many times is $3^{200} - 1$ divisible by 2?
   (d) For any integer n, how many times is $3^n - 1$ divisible by 2?
6. Show that for every positive integer n not divisible by 2 or 5, there exists a multiple of n all of whose digits are 1s.
7. (1989 Junior ISF) Let m be a positive integer. Prove that there are at least 2000 numbers of the form
$$2^{s+1} + 2^{s+2} + \cdots + 2^{s+r}$$
(with r and s being positive integers) so that m is a factor of each of these numbers.
8. (IMO 1971 Q3) Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \ldots$) contains an infinite subset in which every two members are relatively prime.
9. (1994 AMO) Prove that for every integer x, the number
$$\frac{x^5}{5} + \frac{x^3}{3} + \frac{7x}{15}$$
is an integer.
10. Let $m = (4^p - 1)/3$ where $p > 3$ is prime. Prove that $2^{m-1} \equiv 1 \mod m$.
11. Let n be a given positive integer.
    (a) Prove that the sequence
$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots\ldots$$
    is eventually constant modulo n.
    (b) Prove that the sequence
$$a, a^a, a^{a^a}, a^{a^{a^a}}, \ldots$$
    is eventually constant modulo n.
12. Suppose that a positive integer a and prime p satisfy $p | a^p - 1$.
    (a) Show that $p^2 | a^p - 1$.
    (b) Suppose further that p is odd. Show that $p^3 | a^p - 1$ if and only if $a \equiv 1 \mod p^2$.
13. (a) If p is a prime, show that $\binom{2p}{p} - 2$ is divisible by p.
    (b) If p is a prime, show that $\binom{2p}{p} - 2$ is divisible by p^2.
    (c) If $p > 3$ is a prime, show that $\binom{2p}{p} - 2$ is divisible by p^3.
14. (1970 IMO Q4) Find the set of all positive integers n with the property that the set
$$\{n, n+1, n+2, n+3, n+4, n+5\}$$
can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

15. Find all pairs of positive integers (m, n) such that

$$n | 1 + m^{3^n} + m^{2 \cdot 3^n}.$$

16. (a) Let n be a positive integer. Prove that there are infinitely many perfect squares of the form $2^n a - 7$, where a is a positive integer.

    (b) Let n be a positive integer. Prove that there are infinitely many perfect cubes of the form $2^n a - 7$, where a is a positive integer.

17. Show that the cyclotomic polynomial $\Phi_n(x)$ has degree $\phi(n)$ given by the Euler phi function.

    Relatedly, show that the Euler phi function satisfies

$$n = \sum_{d|n} \phi(d).$$

18. Prove that there is no integer $n > 1$ for which $n | 2^n - 1$.

19. (1994 IMO Q4) Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

    is an integer.

20. (1990 IMO Q3) Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

    is an integer.