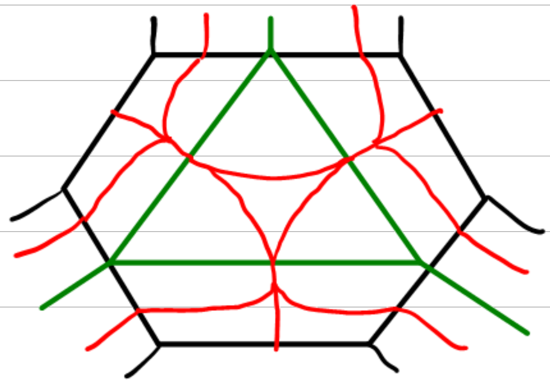
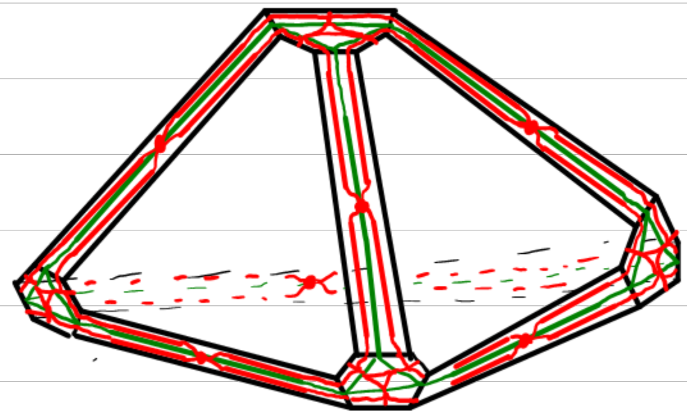
A Symplectic Basis for 3-manifold triangulations





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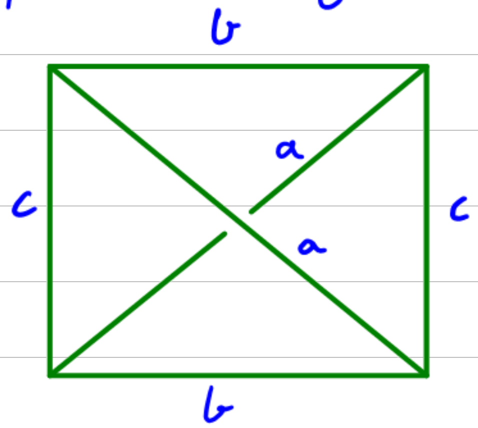
joint work with Jessica Purcell

Combinatorics of 3-manifold Triangulations

Let: * M be a knot / link complement, $M = S^3 \setminus L$

* $\mathcal{T}$ an ideal triangulation of M
with tetrahedra $\Delta_1, \Delta_2, \dots, \Delta_N$
and edges $E_1, E_2, \dots, E_N$

Label opposite pairs of edges of each Δ_j with a, b, c as shown



Let $a_{k,j}$ = # a-edges of Δ_j identified to E_k
$b_{k,j}$ = b - - - - - - .
$c_{k,j}$ = c - - - - - - .

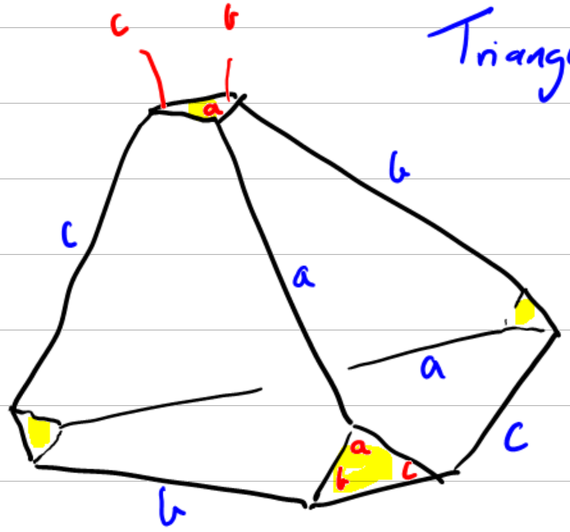
The incidence matrix of $\mathcal{T}$
is the $N \times 3N$ matrix

$$
I_n = \begin{array}{c} \\ E_1 \\ E_2 \\ \vdots \\ E_N \end{array}
\begin{array}{c}
\overbrace{a \;\; b \;\; c}^{\Delta_1} \;\;\; \overbrace{a \;\; b \;\; c}^{\Delta_2} \;\;\; \cdots \;\;\; \overbrace{\qquad\qquad}^{\Delta_N} \\
\left[\begin{array}{ccccccccc}
a_{11} & b_{11} & c_{11} & a_{12} & b_{12} & c_{12} & & a_{1N} & b_{1N} \; c_{1N} \\
a_{21} & b_{21} & c_{21} & a_{22} & b_{22} & c_{22} & \cdots & a_{2N} & b_{2N} \; c_{2N} \\
 & \vdots & & & \vdots & & \ddots & \vdots & \\
a_{N1} & b_{N1} & c_{N1} & & - - - & & & a_{NN} & b_{NN} \; c_{NN}
\end{array}\right]
\end{array}
$$

(Sparse! Each entry 0, 1 or 2!
Column sum to 2!)

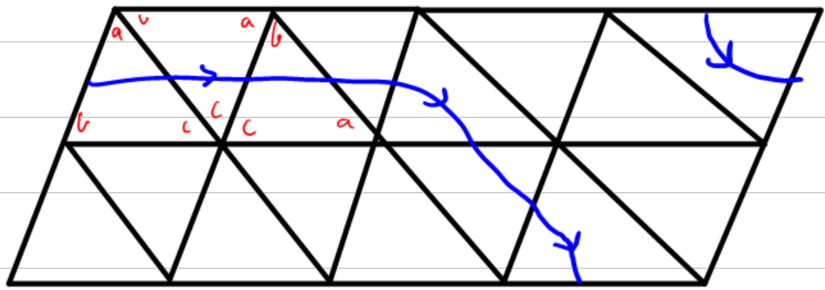
Boundary Combinatorics of 3-manifold Triangulations

Truncating each Δ_j gives a decomposition of a compact manifold $\bar{M} = S^3 \setminus N(L)$ into truncated tetrahedra!

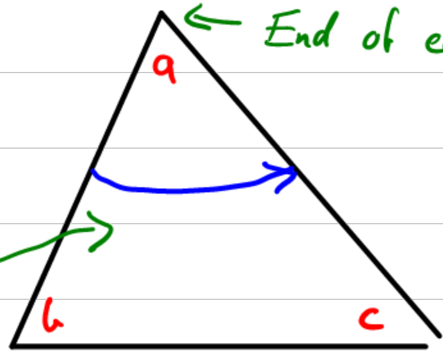


Triangular faces give a triangulation of the boundary tori!

Each vertex of each triangle has an a, b, c label!



A generic curve γ in a boundary torus then has a combinatorial holonomy $\check{h}(\gamma) \in \mathbb{R}^{3N}$



Triangle from Δ_j

+1 to a_j coordinate!

Neumann - Zagier matrix

Reduce! From the $N \times 3N$ incidence matrix $I_n = [a_{kj} \; b_{kj} \; c_{kj}]$

to an $N \times 2N$ matrix with entries $[a_{kj} - c_{kj} \;\; b_{kj} - c_{kj}]$

Add! For each boundary torus T_i take a meridian m_i, longitude l_i

We can naturally add $\tilde{h}(m_i)$, $\tilde{h}(l_i)$ rows (in $\mathbb{R}^{3N}$) to I_n
& reduce them to give rows of length $2N$, $h(m_i), h(l_i) \in \mathbb{R}^{2N}$

The result: the Neumann - Zagier matrix!

$$
NZ = \begin{array}{ll} & \\ \text{Edges} \left\{ \begin{array}{l} E_1 \\ E_2 \\ \vdots \\ E_N \end{array} \right. \\ \text{Tori} \left\{ \begin{array}{l} m_1 \\ l_1 \end{array} \right. \end{array}
\overbrace{\begin{array}{c} \\ \end{array}}^{\text{Tetrahedra}}
\begin{array}{c} \begin{array}{ccccc} \overbrace{\hphantom{a_{11}-c_{11} \quad b_{11}-c_{11}}}^{\Delta 1} & & \cdots & & \overbrace{\hphantom{a_{1N}-c_{1N} \quad b_{1N}-c_{1N}}}^{\Delta N} \end{array} \\
\left[\begin{array}{ccccc} a_{11}-c_{11} & b_{11}-c_{11} & \cdots & a_{1N}-c_{1N} & b_{1N}-c_{1N} \\ \vdots & & \ddots & & \vdots \\ a_{N1}-c_{N1} & b_{N1}-c_{N1} & \cdots & a_{NN}-c_{NN} & b_{NN}-c_{NN} \\ \hline & & h(m_1) & & \\ & & h(l_1) & & \end{array} \right] \end{array}
= \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \\ R_1^m \\ R_1^l \end{bmatrix}
$$

Properties of the NZ matrix

Let V be a $2N$-dimensional vector space generated by $3N$ elements

$$a_1, b_1, c_1, \dots, a_N, b_N, c_N$$

subject to relations $a_i + b_i + c_i = 0$

Regard the row space of I_n as $\mathbb{R}^{3N}$
and the row space of NZ as V! $\Big\}$ So rows $R_K, R_K^m, R_K^\ell \in V$

<u>Defn</u>: There is a natural <u>symplectic form</u> ω on V given by (antisymmetric nondegenerate bilinear)

$$\omega(a_i, b_i) = \omega(b_i, c_i) = \omega(c_i, a_i) = 1$$
$$\omega(b_i, a_i) = \omega(c_i, b_i) = \omega(a_i, c_i) = -1$$

& ω on all other pairs of generators $= 0$

<u>Theorem</u> (NZ 1985):

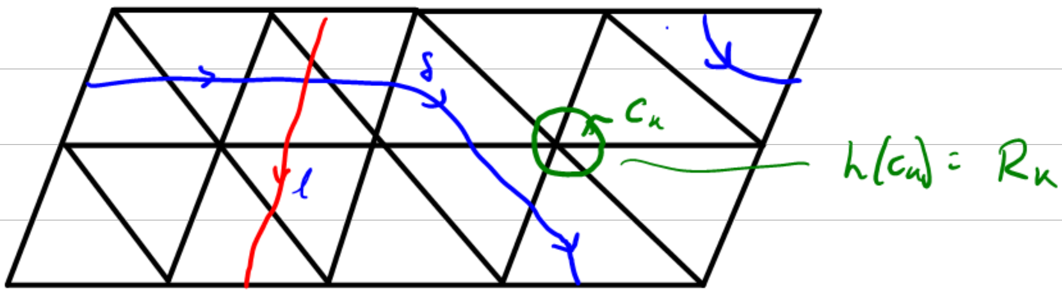
All rows of NZ are ω-orthogonal except $\omega(R_K^m, R_K^\ell) = 2$.

Geometric meaning of the NZ theorem

NZ showed in fact that for curves on boundary tori
w gives algebraic intersection number

$$\omega(h(\gamma), h(\delta)) = 2\,\gamma\cdot\delta.$$

Edge rows R_k can be interpreted as holonomy around an edge of T



However, NZ showed that

$$\mathrm{rank}(NZ) = N + N_c < 2N$$

#edges/tetrahedra #cusps/boundary tori

So $R(NZ) \subsetneq V$

Qn: Can we give a geometric interpretation for V, ω more generally?

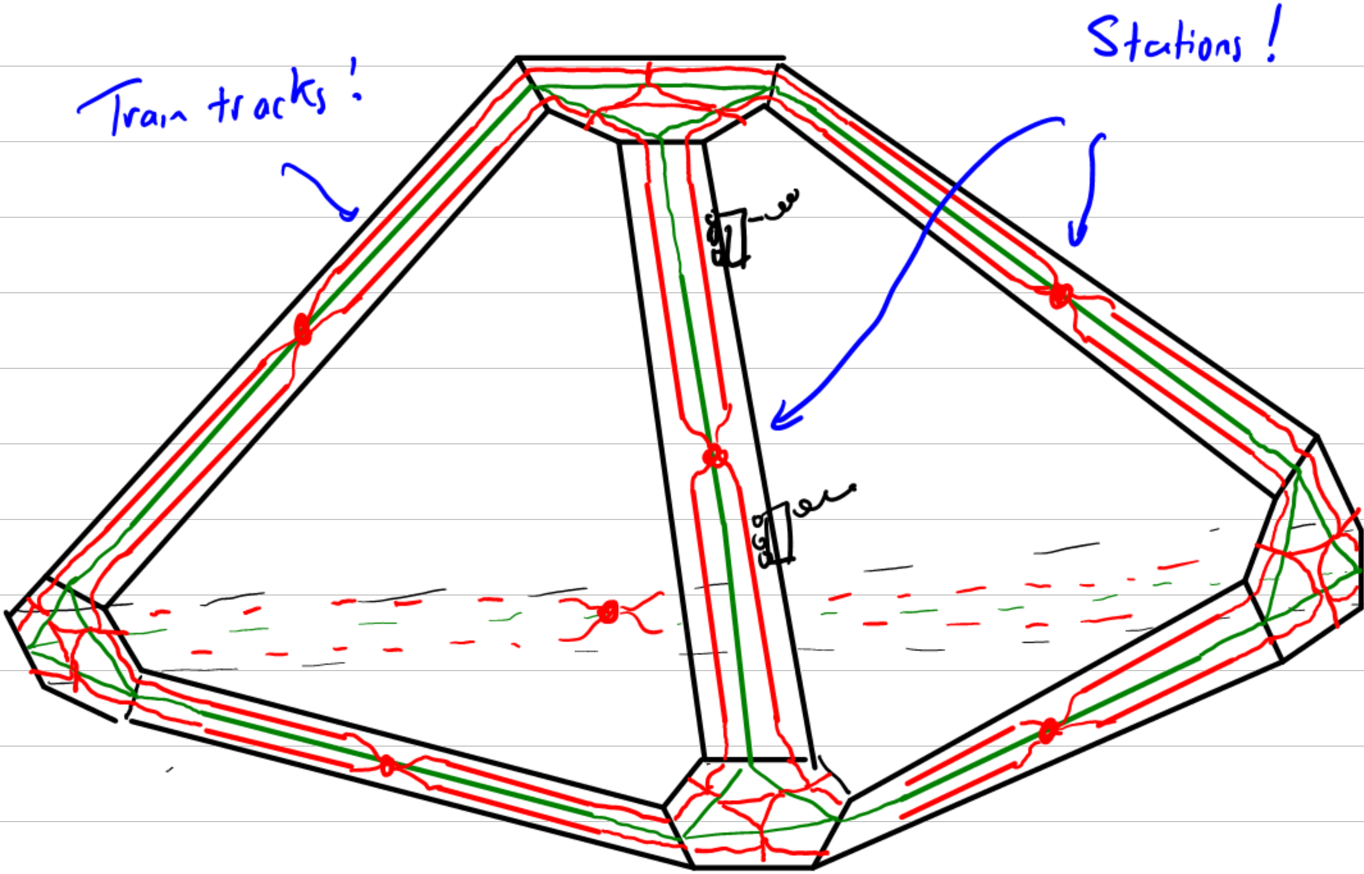
Dimofte (2013): By adding rows, NZ can be made into a symplectic matrix.
But many choices ... what to choose?

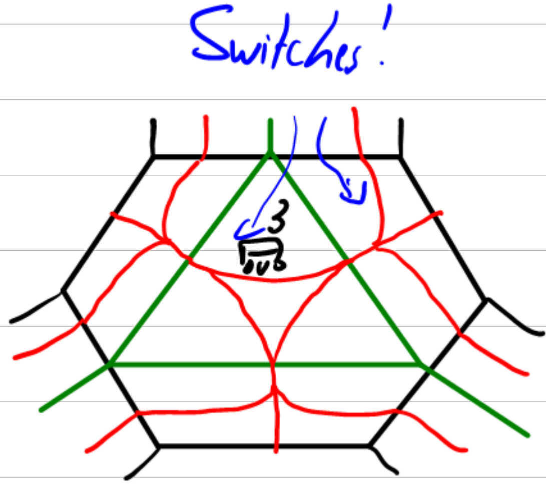
Also.. Isn't this supposed to be 3 dimensional?

All aboard!

Welcome to the

M-Purcell 3-manifold train track transport system!

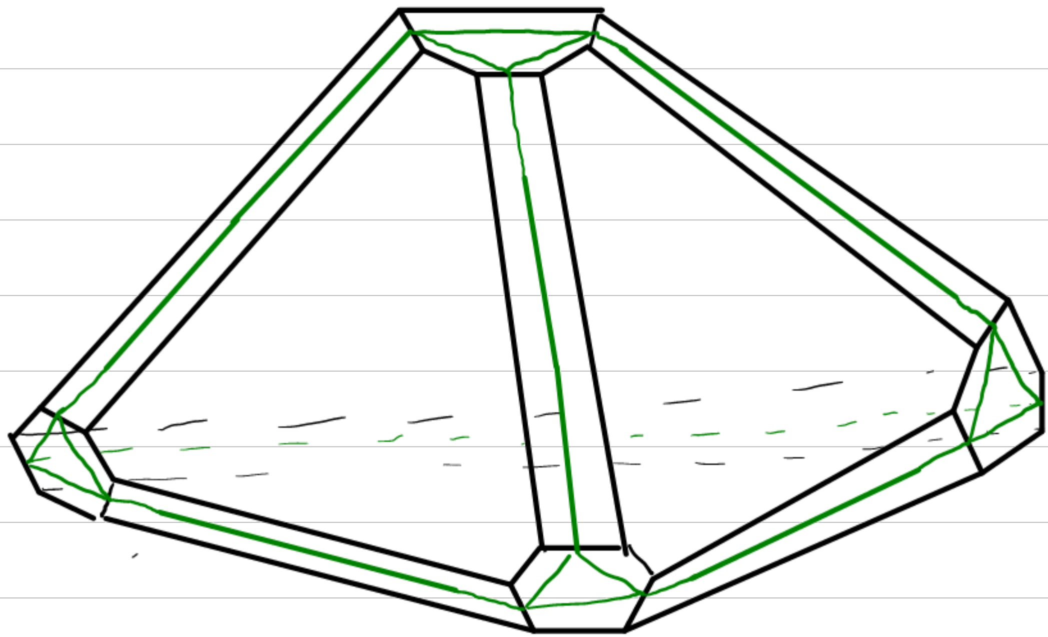




Oscillating Curves

From the triangulation $\mathcal{T}$, truncate tetrahedra as before.

Now also remove a neighbourhood of each edge



$\rightsquigarrow$ Heegaard decomposition of M

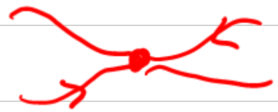
$M =$ Handlebody $\cup$ Compression Body,

with handlebody decomposed into polyhedra

The Heegaard surface is decomposed into rectangles & triangles

We place a system of train tracks on them!

Stations:



when you pass through a station you must reverse orientation (?!)

Geometric meaning & symplectic basis

Defn: An oscillating curve δ on $\mathcal{T}$ is a smooth closed curve train journey, reversing orientation at stations.

(This means you must pass through an even number of stations!)

The segments of δ along triangles have a, b, c labels
$\rightsquigarrow$ δ has a combinatorial holonomy $h(\delta) \in V$.

Theorem (M-Purcell) For any oscillating curves δ, δ',

$$\omega\big(h(\delta), h(\delta')\big) = 2\,\underline{\delta \cdot \delta'}$$

$\leftarrow$ Intersection number on Heegaard surface.

Corollary: By choosing oscillating curves δ_u "dual" to the curves C_u around edges
$\delta_u \cdot C_{u'} = \delta_{uu'}$
we obtain a symplectic basis of V

Example: Figure 8 Knot