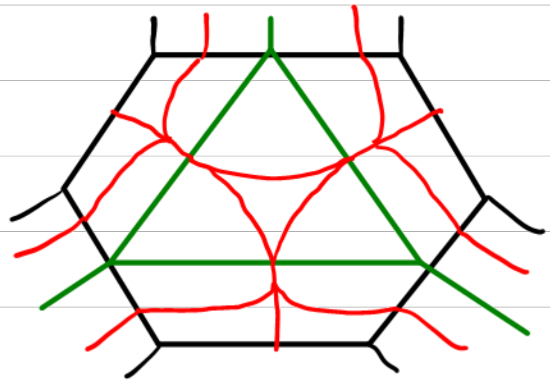
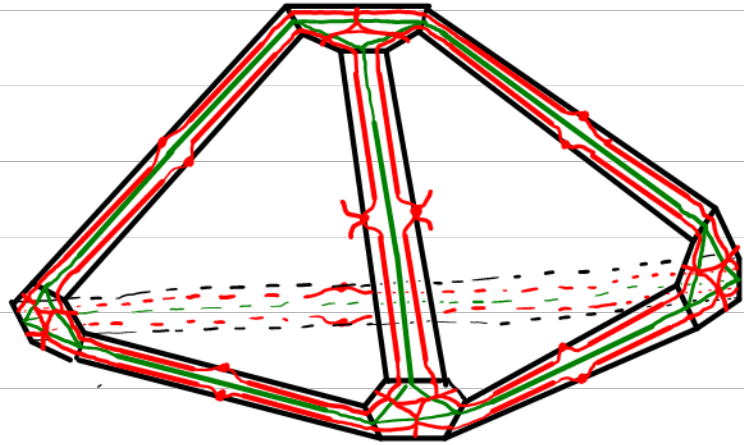
A Symplectic Basis

for

3-manifold triangulations





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(Virtual) Knots in Washington 49.75

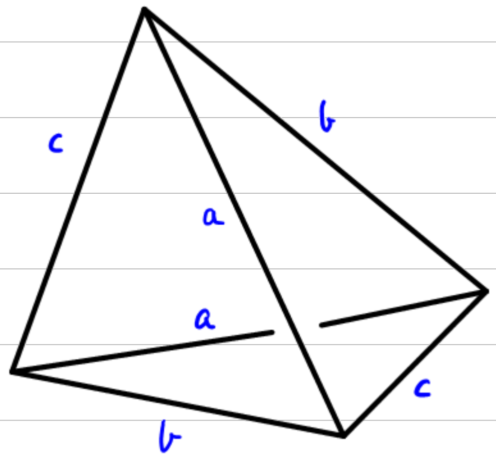
April 22-24, 2022

joint w/ Jessica Purcell

Combinatorics of 3-manifold Triangulations

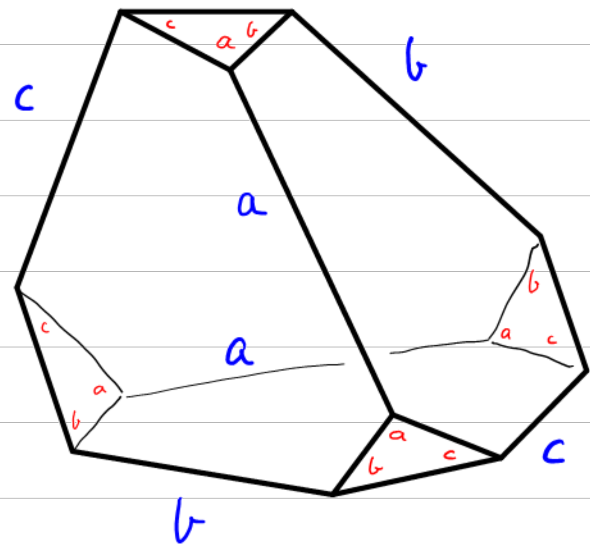
Let: $*$ M be a knot/link complement, $M = S^3 \setminus L$

$*$ $\mathcal{T}$ be an ideal triangulation of M
with tetrahedra $\Delta_1, \Delta_2, \dots, \Delta_N$
and edges $E_1, E_2, \dots, E_N$.



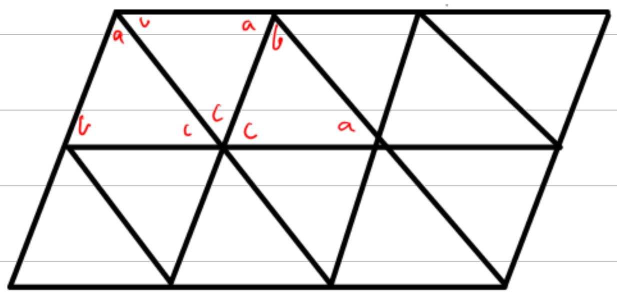
Label opposite pairs of edges with a, b, c.

Truncating each Δ_j gives a decomposition of $\bar{M} = S^3 \setminus N(L)$.



Triangular faces give a triangulation of boundary tori!

Each vertex of each triangle has an a, b or c label



The Neumann - Zagier Symplectic Vector Space

Let V be the $2N$-dimensional $\mathbb{R}$-vector space generated by $3N$ elements

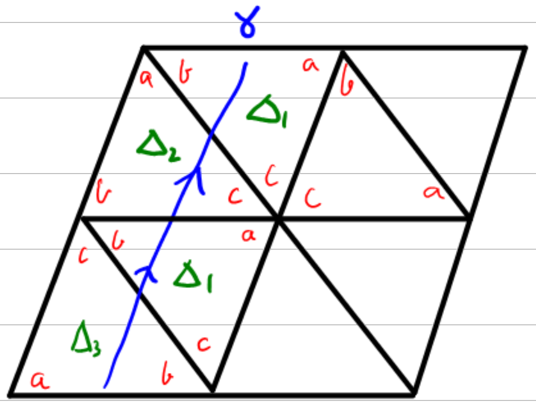
$$a_1, b_1, c_1, \ldots, a_N, b_N, c_N$$

subject to relations $a_i + b_i + c_i = 0$, $\quad i = 1, \ldots, N$.

There's a symplectic form ω on V given by

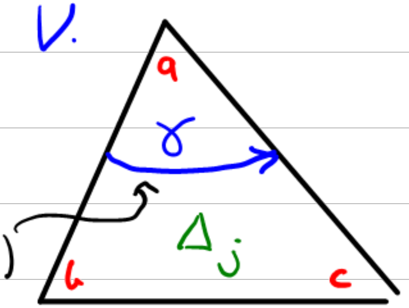
$$\omega(a_i, b_i) = \omega(b_i, c_i) = \omega(c_i, a_i) = 1$$
$$\omega(b_i, a_i) = \omega(c_i, b_i) = \omega(a_i, c_i) = -1$$

& $\omega = 0$ on all other pairs of generators.



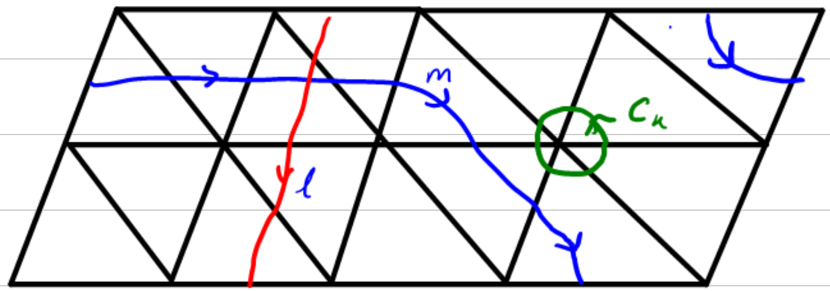
A generic curve γ in a boundary torus obtains a combinatorial holonomy $h(\gamma) \in V$.

$h(\gamma)$ is a sum of contributions from each triangle it passes through: $+\, a_j$ to $h(\gamma)$



The NZ symplectic form as a boundary intersection form

Theorem (NZ 1985): Let γ, δ be generic curves on the boundary tori of $\bar{M}$.
Then
$$\omega\big(h(\gamma), h(\delta)\big) = 2\, \gamma \cdot \delta$$
where $\cdot$ denotes algebraic intersection number on $\partial \bar{M}$.



In particular, on each boundary torus there's a longitude ℓ, meridian m, and loops C_k around various edges E_k

Then $\omega\big(h(m), h(\ell)\big) = 2$
$$\omega\big(h(m), h(C_k)\big) = \omega\big(h(\ell), h(C_k)\big) = \omega\big(h(C_j), h(C_k)\big) = 0.$$

Neumann-Zagier: Use this to understand change in hyperbolic volume, Chern-Simons, ..

Dimofte (2013): $h(m), h(\ell), h(C_1), \ldots h(C_\cdot)$ is an incomplete symplectic basis for V!

The NZ symplectic form as a 3D intersection form

Theorem (M-Purcell): Let ς, ς' be oscillating curves on $\overline{M}$.

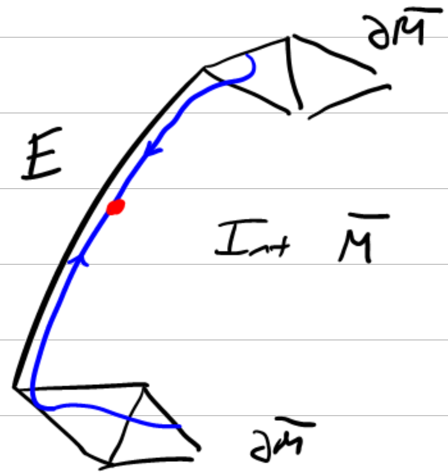
Then
$$\omega\left(h(\varsigma), h(\varsigma')\right) = 2\, \varsigma \cdot \varsigma'.$$

Corollary: $\exists$ curves Γ_k "dual" to the C_k such that holonomies of
$m_1, \ell_1, \ldots, m_T, \ell_T, \quad C_1, \Gamma_1, \ldots, C_{N-T}, \Gamma_{N-T}$
form a symplectic basis for V!

$$\omega(h(m_i), h(\ell_j)) = 2\,\delta_{ij} \qquad \omega(h(C_i), h(\Gamma_j)) = 2\,\delta_{ij}, \qquad \omega = 0 \text{ otherwise.}$$

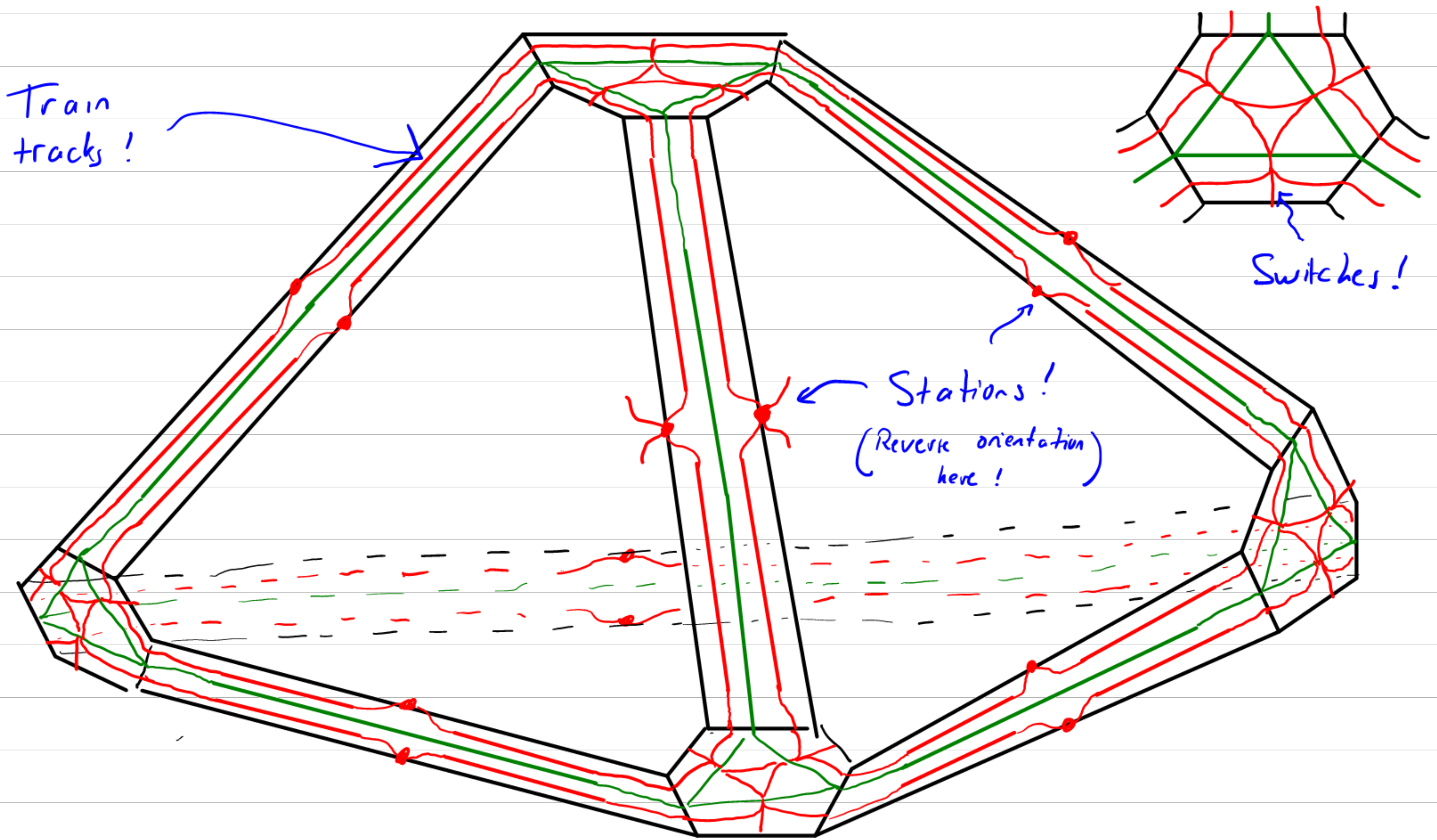
Oscillating curves: generalise oriented curves in $\partial \overline{M}$

* They may dive into the interior of $\overline{M}$ along edges & come out the other side!
* Have combinatorial holonomy
* Technically easiest described using train tracks
* Must reverse orientation when diving through an edge



All aboard!

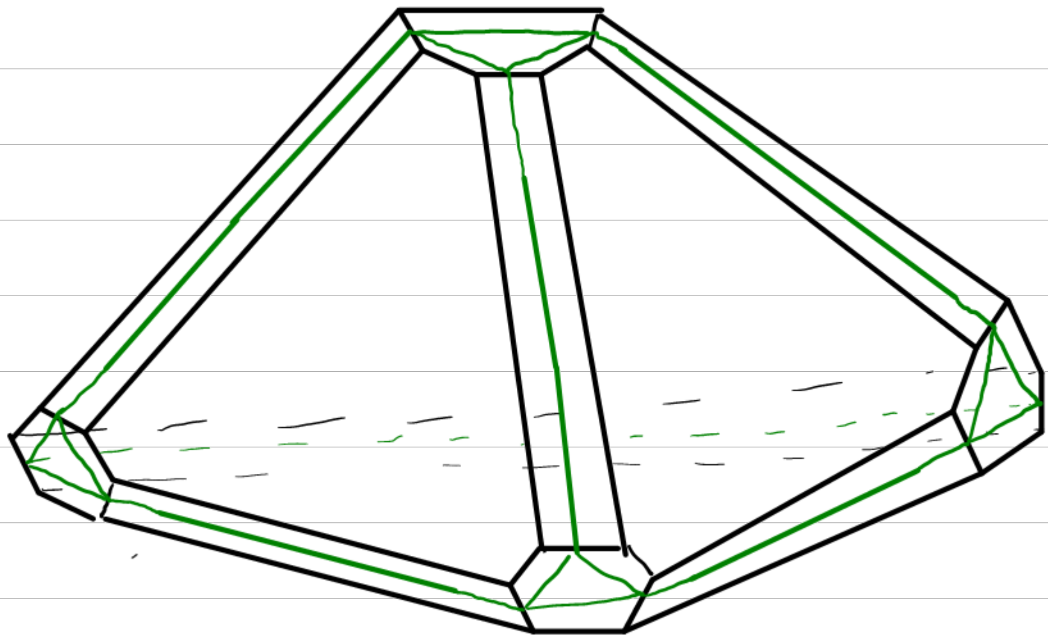
Welcome to the M-Purcell link complement train track transport system!



Oscillating Curves

From the triangulation $\mathcal{T}$, truncate tetrahedra further:

Now also remove a neighbourhood of each edge

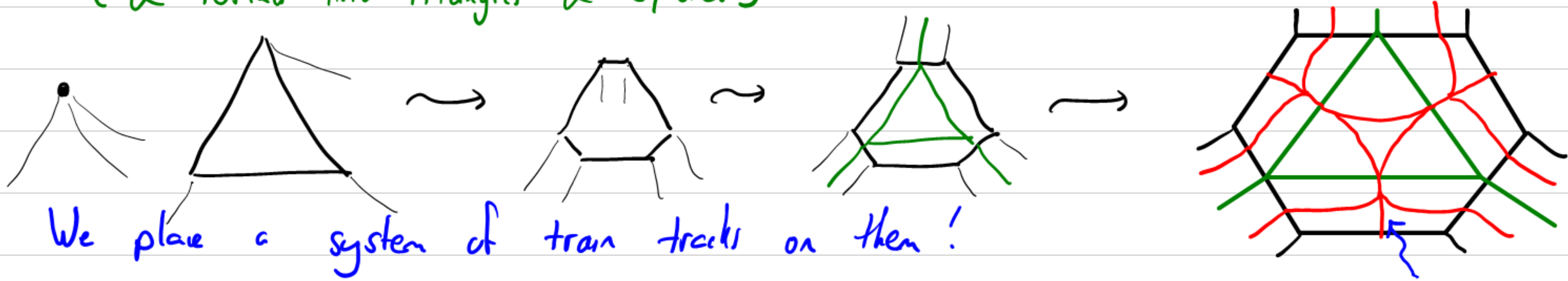


$\rightsquigarrow$ Heegaard decomposition of M

M = Handlebody $\cup$ Compression Body,

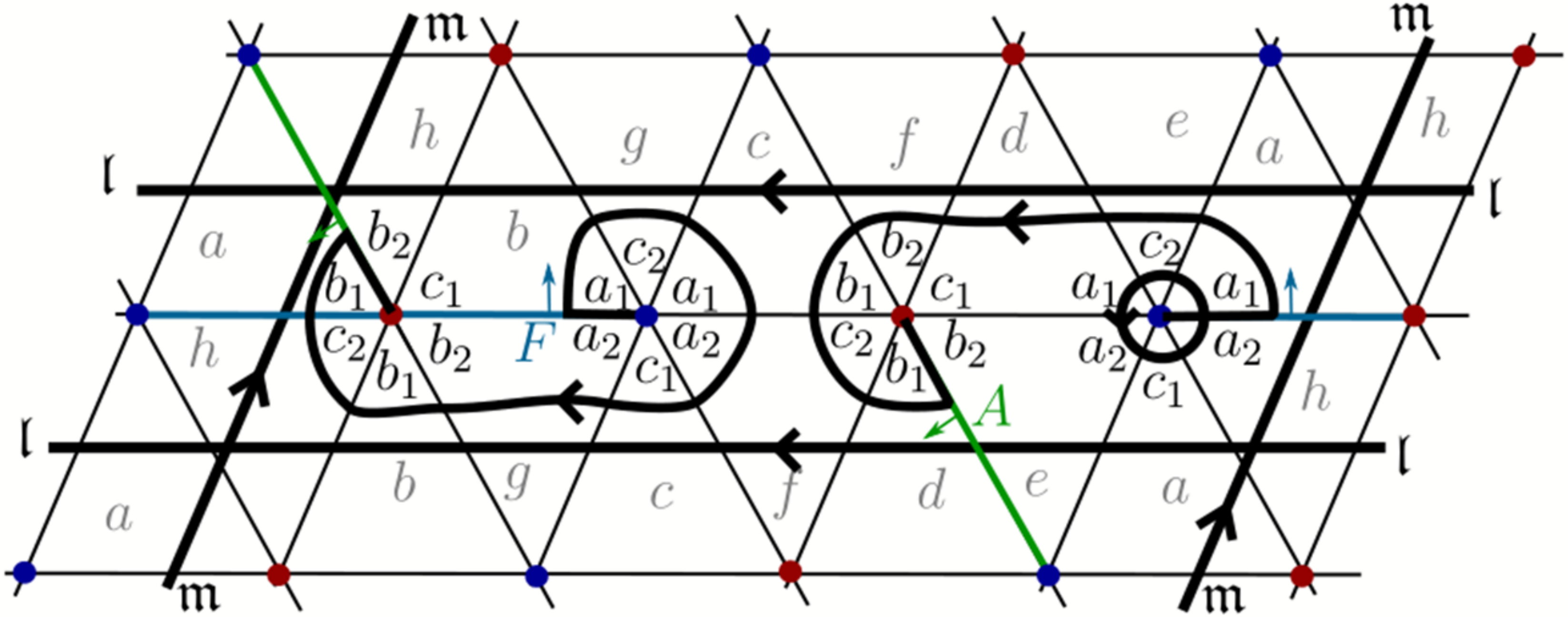
with handlebody decomposed into truncated tetrahedra

The Heegaard surface is decomposed into squares & hexagons
(& further into triangles & squares)



We place a system of train tracks on them!

Example: Figure 8 Knot



Applications:

* Calculate A-polynomials (Chomponerkar, Howie-M-Purcell, Thompson)
* Quantisation (Dimofte)
* Virtual knots ?
* Chern-Simons ?