A symplectic approach to 3-manifold triangulations and hyperbolic structures

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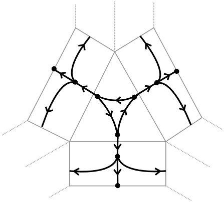
Monash topology seminar
14 September 2022



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Paper on arxiv:

- A symplectic basis for 3-manifold triangulations
  2208.06969 (joint w Purcell)

Also:

- A-polynomials, Ptolemy equations and Dehn filling
  2002.10356 (joint w Howie, Purcell)

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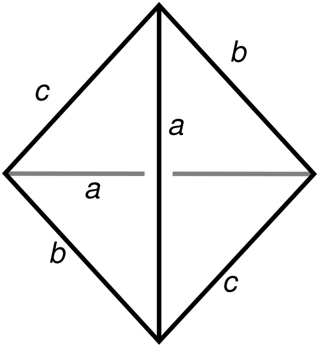

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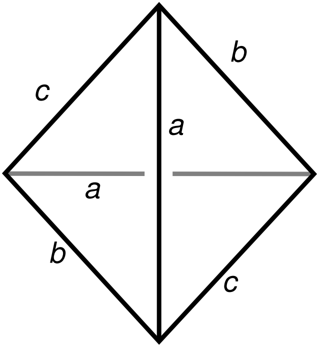
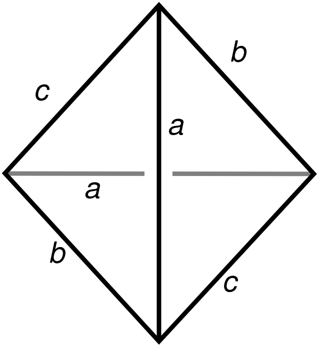
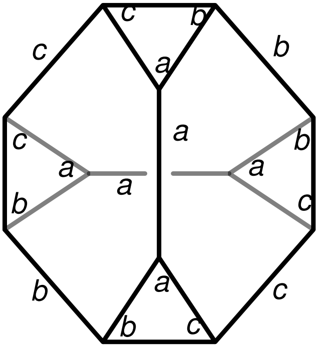
(Results apply more generally, but for convenience...)



Then:

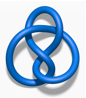
- Truncating $\Delta_j \rightsquigarrow$ polyhedra decomposing $\overline{M} = S^3 \setminus N(L)$.
- Triangular faces of polyhedra triangulate boundary tori T_i.
- Each vertex of each triangle has an a, b or c label.

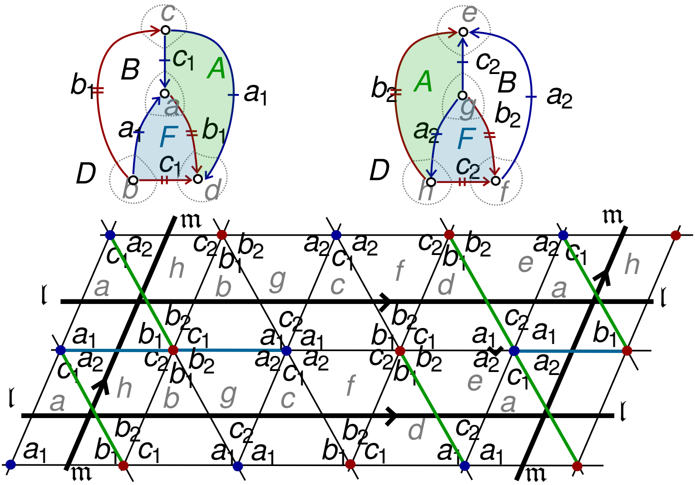




Example: figure-8 knot complement

When $K =$ figure-8 knot, M decomposes into two ideal tetrahedra, with cusp triangulation shown.





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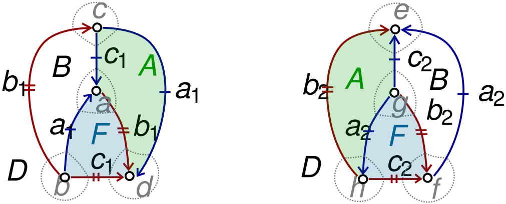

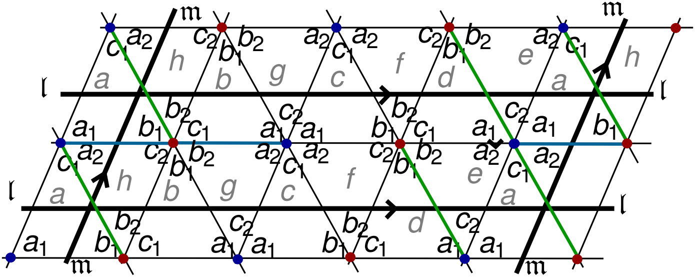

The Neumann–Zagier vector space

Neumann–Zagier associated to $\mathcal{T}$ a vector space V.

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Definition (Neumann–Zagier 1985)

Let V be the $2N$-dimensional $\mathbb{R}$-vector space generated by

$$a_1, b_1, c_1, \ldots, a_N, b_N, c_N$$

subject to relations

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It follows that

$$\omega(a_i, b_i) = \omega(b_i, c_i) = \omega(c_i, a_i) = 1$$
$$\omega(b_i, a_i) = \omega(c_i, b_i) = \omega(a_i, c_i) = -1$$
$$\omega = 0 \quad \text{on all other pairs of generators.}$$

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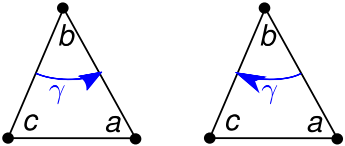

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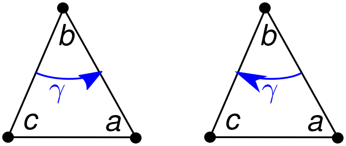

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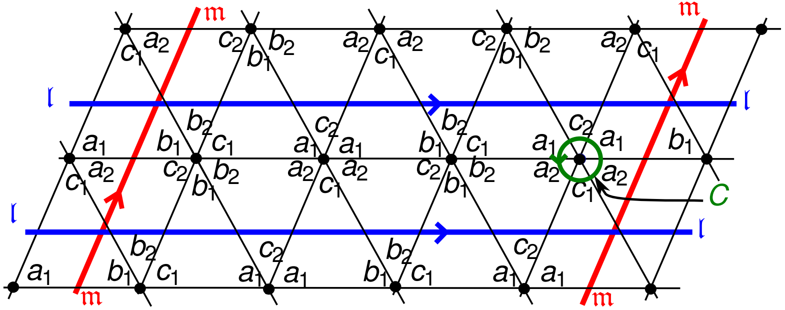
Definition

The combinatorial holonomy $h(\gamma) \in V$ is the sum of contributions $\pm a_j, b_j, c_j$ for each arc of γ.

Combinatorial holonomy example: figure-8 knot



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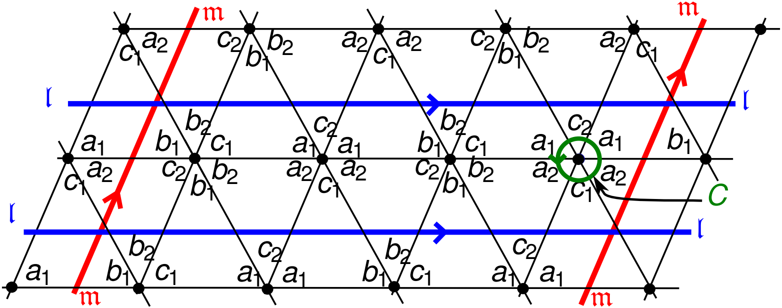

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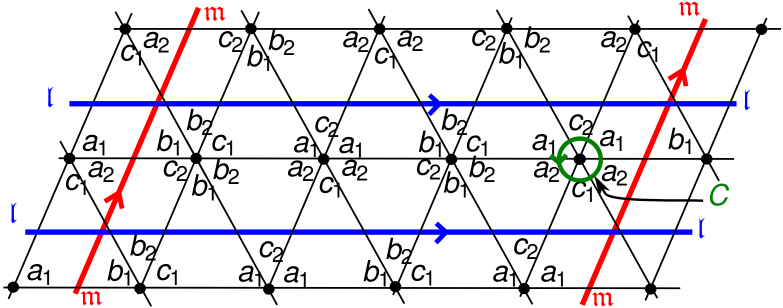

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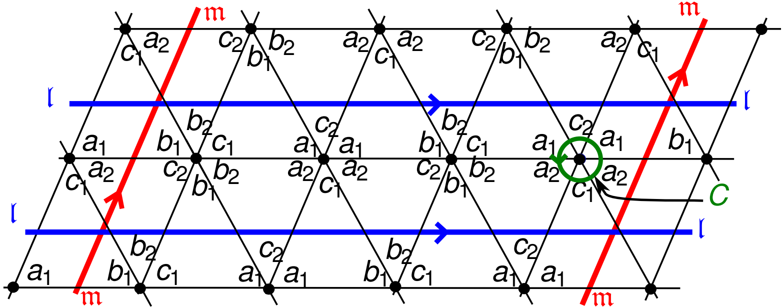

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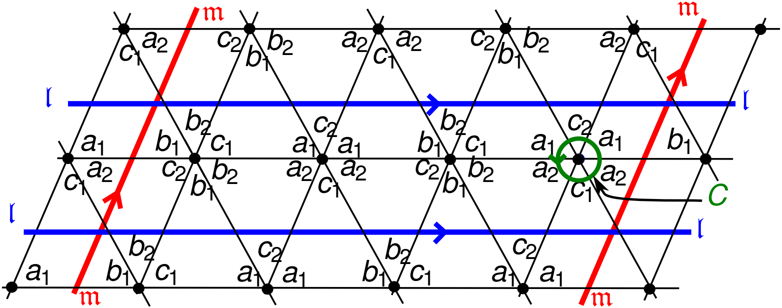

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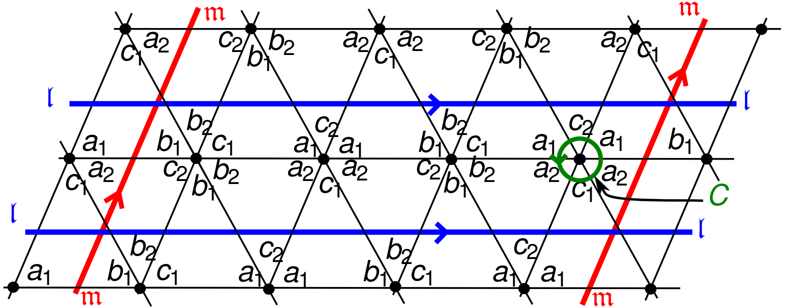

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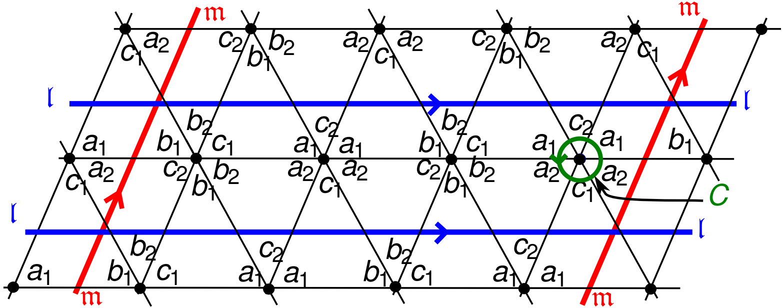

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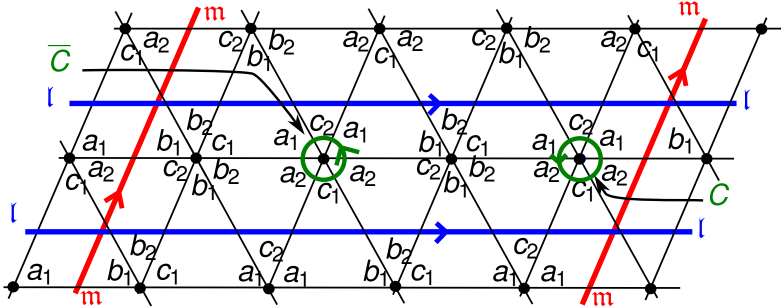

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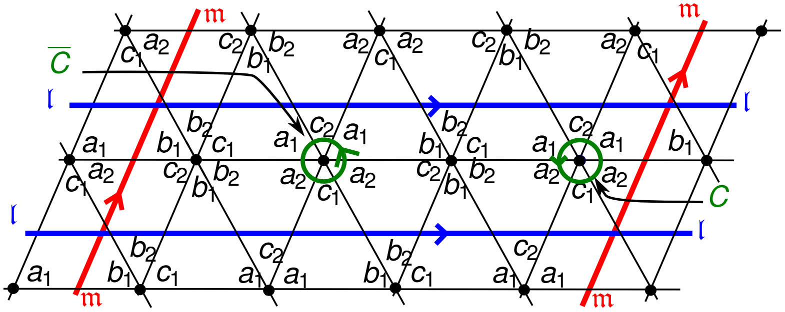

$$h(\overline{C}) = h(C)$$

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Let γ, δ be oriented curves on $\partial\overline{M}$. Then

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where $\cdot$ denotes algebraic intersection number.

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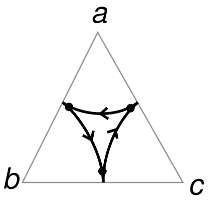
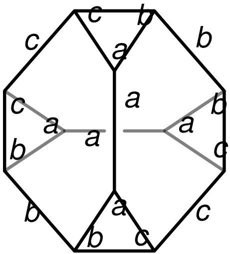
Note ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!





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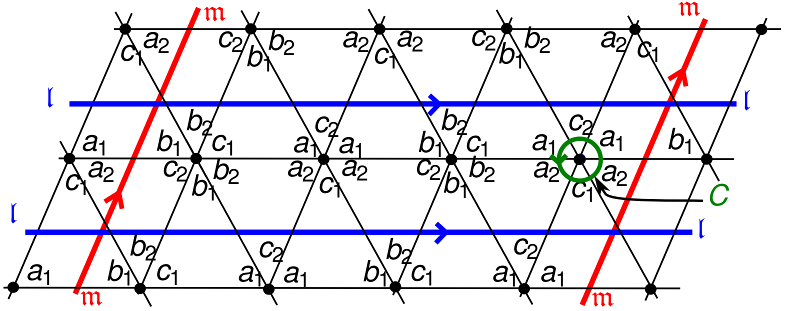

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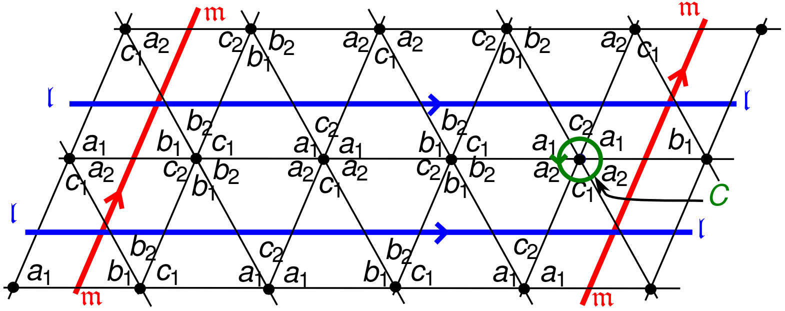

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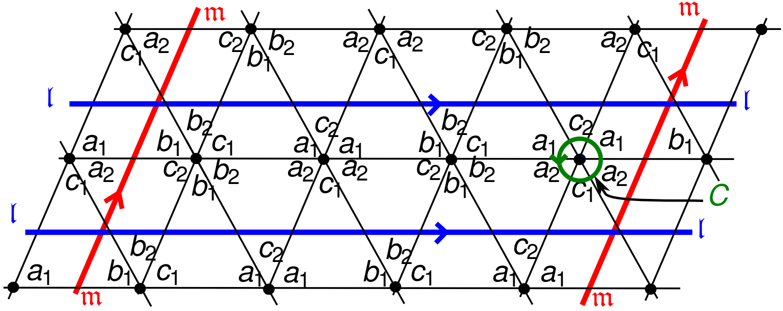

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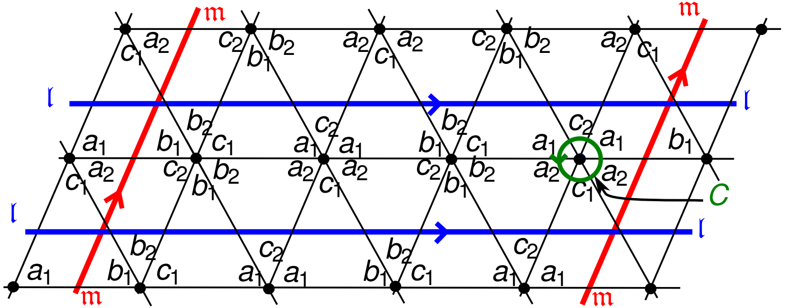

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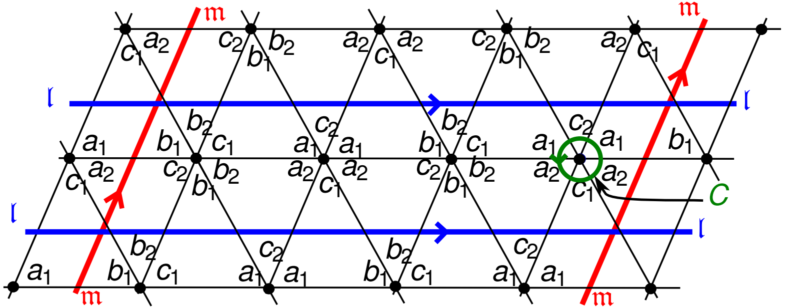

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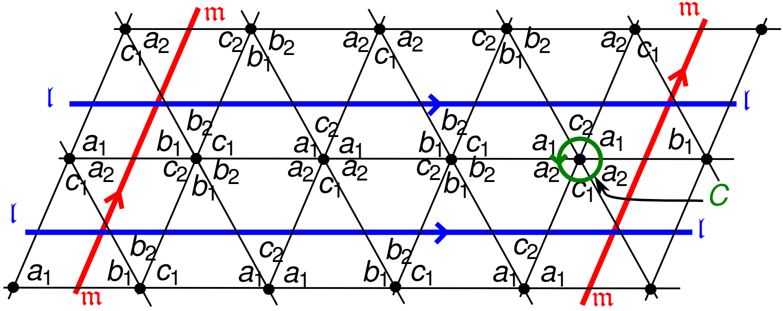

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the $h(\mathfrak{m}_j), h(\mathfrak{l}_j)$, for $j = 1, \ldots, \mathfrak{c}$

and the $h(C_j)$

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Then $\omega = 0$ on holonomies of all such pairs except

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Observation (Dimofte 2013)

We can form a partial symplectic basis* of V, consisting of

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vectors $e_1, f_1, \cdots, e_N, f_N$ such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, $\omega(e_i, f_j) = \delta_{ij}$.)

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* Up to factors of 2...

Questions

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Can we find more curves to obtain a full symplectic basis* of V?

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Neumann-Zagier's results are 2-dimensional, only about curves on $\partial\overline{M}$.
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Yes!

The NZ symplectic form as a 3D intersection form

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Theorem 1 (M–Purcell)

Let ζ, ζ' be oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

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I.e. $\omega(\Gamma_j, C_k) = 2\delta_{jk}, \quad \omega(\Gamma_j, \mathfrak{m}_k) = \omega(\Gamma_j, \mathfrak{l}_k) = \omega(\Gamma_j, \Gamma_k) = 0.$

Idea of Oscillating curves

- Generalise oriented curves in $\partial\overline{M}$

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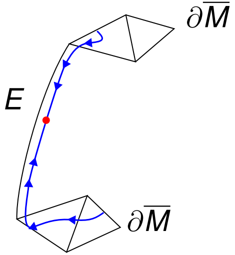

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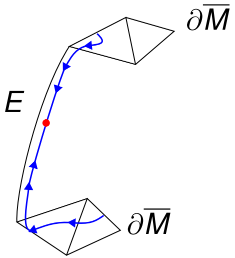

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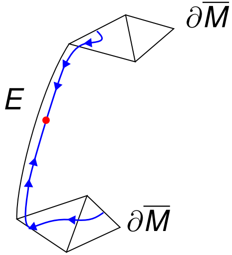

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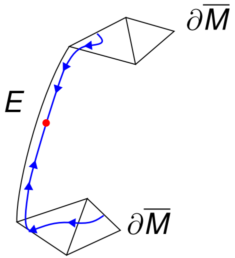

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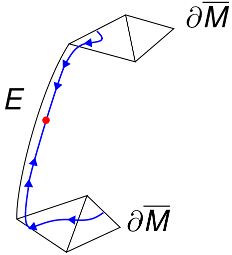

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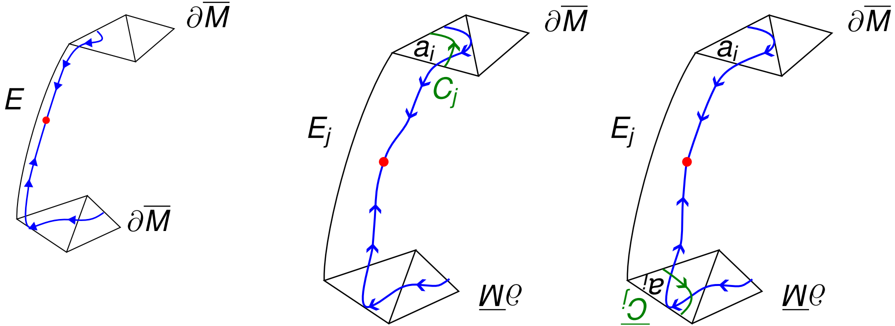

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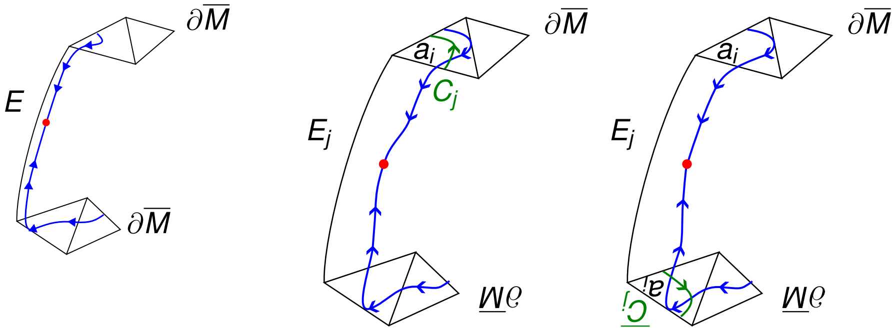
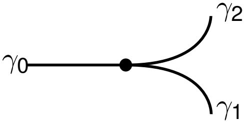


Definition

A train track is a smoothly embedded graph on a surface such that at each vertex, incident edges are all tangent, with at least one edge on each side.

- Edges are called branches; vertices called switches.



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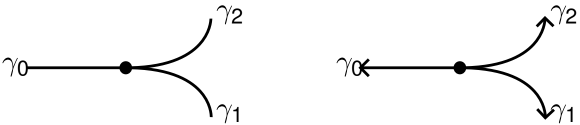

Branches, if oriented, have intersection numbers, defined locally at switches:

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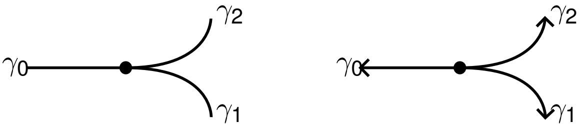

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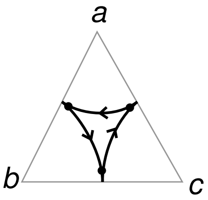
Smooth oriented curves on train tracks then obtain intersection numbers agreeing with usual algebraic intersection number.

Train tracks for oscillating curves: motivation

A generic curve on a boundary torus intersects triangles in arcs, so can be made to run on train tracks.



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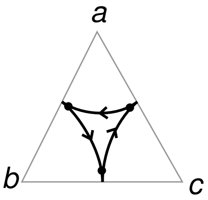

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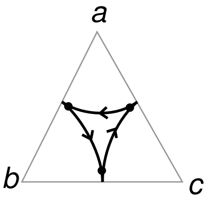

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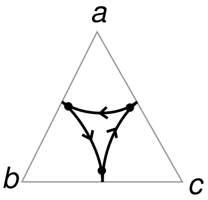

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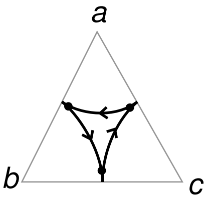

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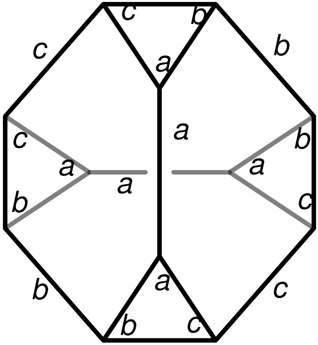
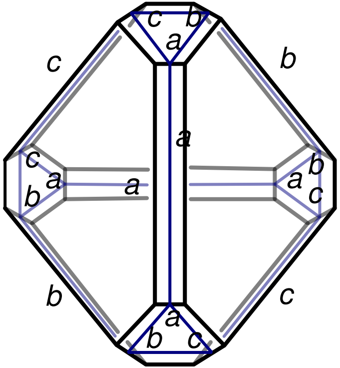
- Special “stations" for each orientation reversal.

Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra





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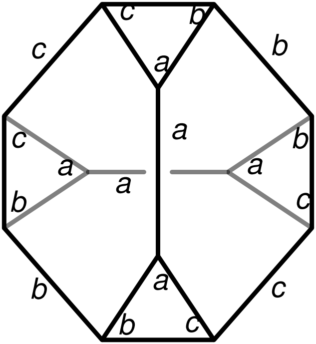

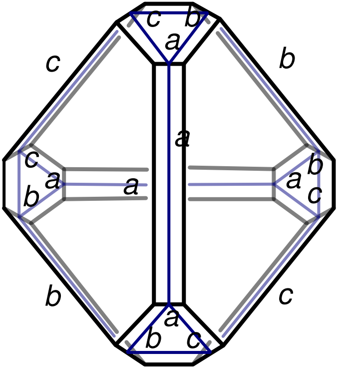

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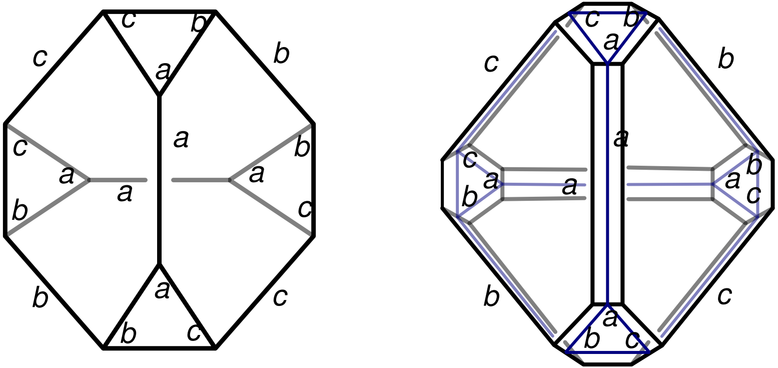
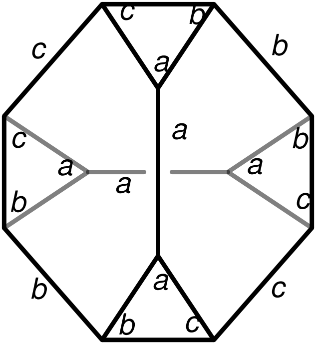
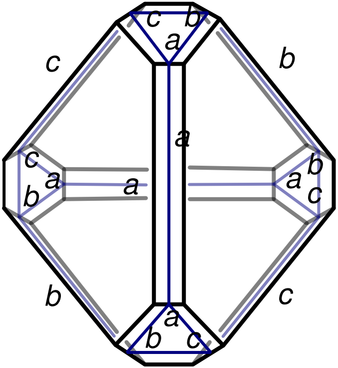


Truncated tetrahedra

Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra with:

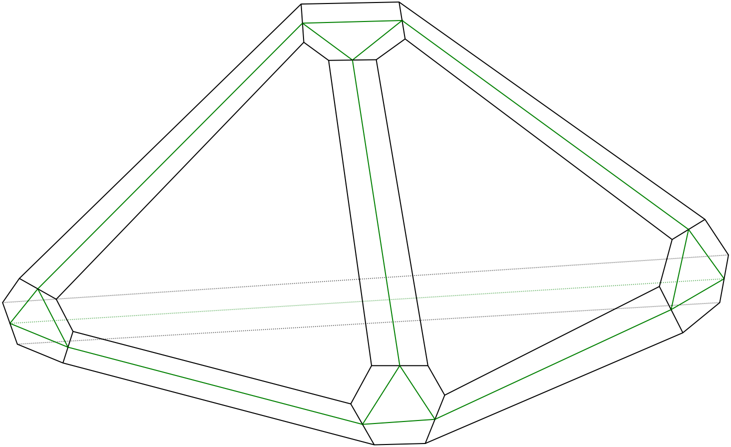
- hexagonal faces glued in pairs
- hexagons on boundary tori (split further into triangle + 3 rectangles)
- rectangles along removed edges (split further into 2 rectangles)





Triangles + rectangles give a decomposition of a Heegaard surface for M.

- $M =$ Handlebody $\cup$ Compresson body
- with handlebody decomposed into truncated tetrahdra



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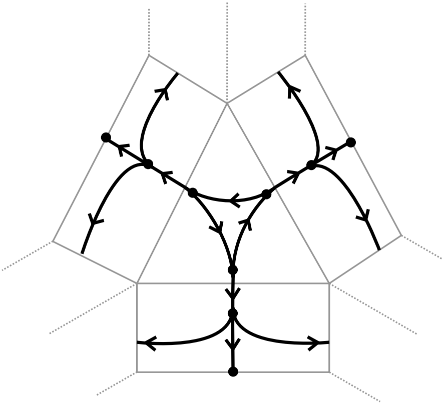
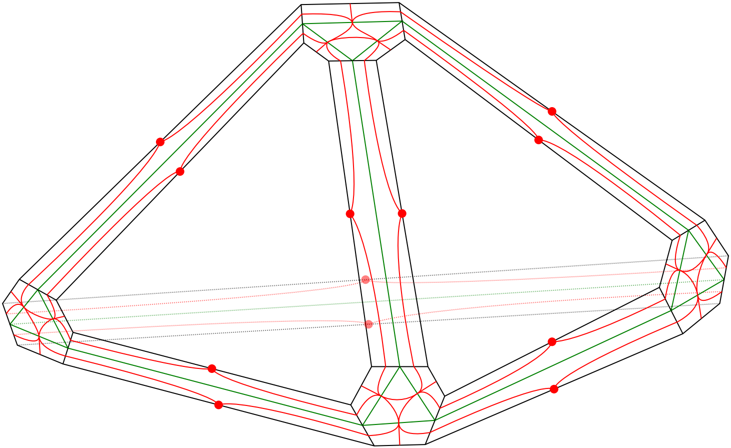
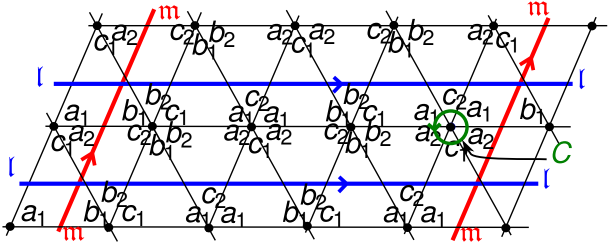




Red dots: stations for reversing direction.

⇝ “Enhanced" train tracks.



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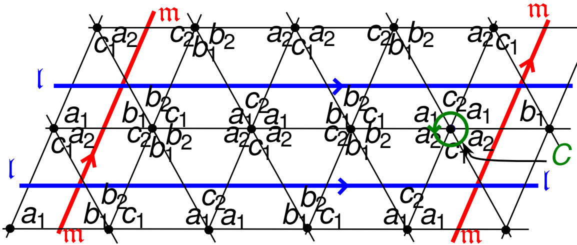

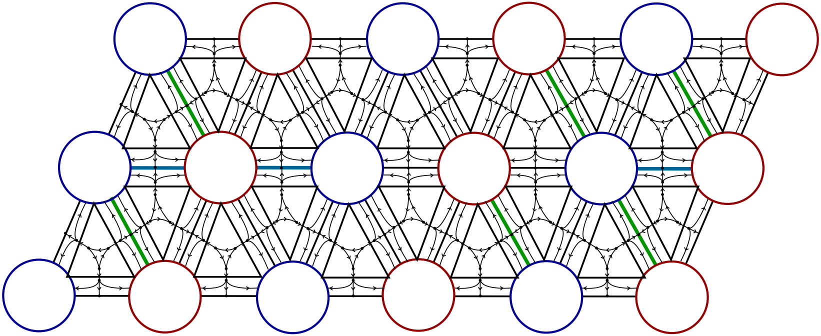

Let τ be the (enhanced) train tracks on the triangulation $\mathcal{T}$

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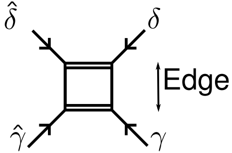

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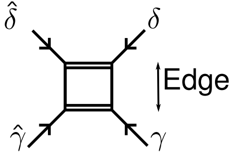
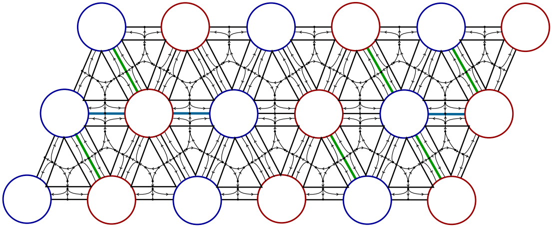
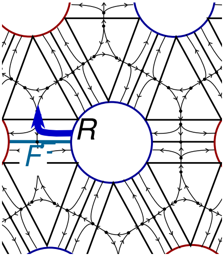
- γ_{j-1} and γ_j have different orientations precisely when they lie at opposite ends of a station.

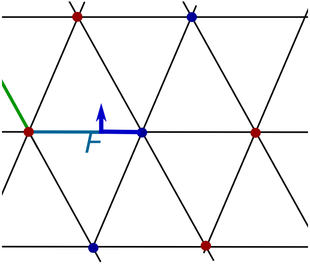


Example: figure-8 knot complement

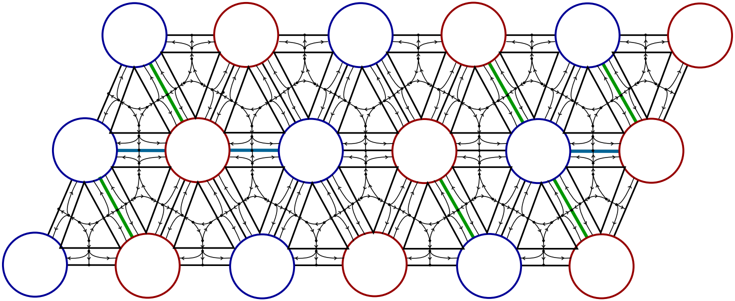


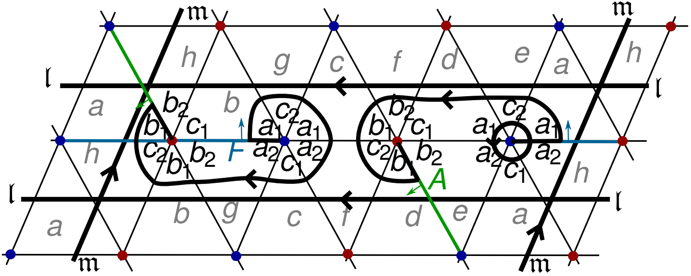
To draw oscillating curves...





Example: figure-8 knot complement





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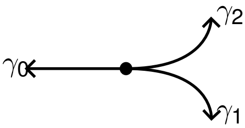

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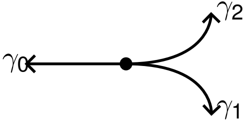

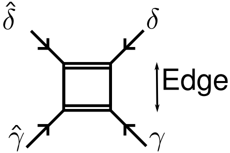

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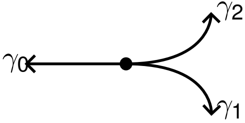

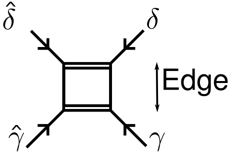

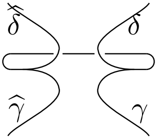

Combinatorial holonomy of oscillating curves

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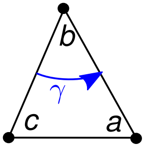
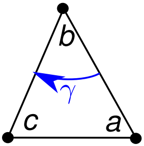
The combinatorial holonomy $h(\zeta) \in V$ *of an (abstract) oscillating curve* ζ *is the sum of contributions* $\pm a_j, b_j, c_j$ *for each arc of* ζ *in a triangle.*



$h(\gamma) = b$



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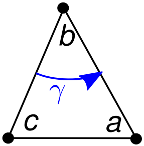
(And nothing from arcs in rectangles / diving into the manifold / passing through stations!)



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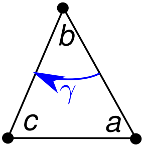

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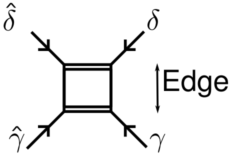

$$h(\gamma) = h(\widehat{\gamma}) = h(\delta) = h(\widehat{\delta}) = 0$$

The NZ symplectic form as a 3D intersection form

Theorem 1 (M–Purcell)

Let ζ, ζ' be (abstract) oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

$$\omega\left(h(\zeta), h(\zeta')\right) = 2\zeta \cdot \zeta'$$

where $\cdot$ is the intersection form for oscillating curves.

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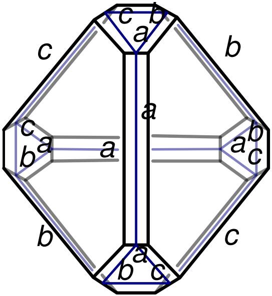

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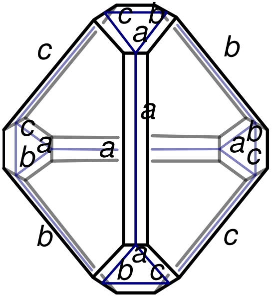
- ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!



Theorem 2 (M–Purcell)

We can construct oscillating curves $\Gamma_1, \ldots, \Gamma_{N-\mathfrak{c}}$ such that

$$\begin{gathered} h(\mathfrak{m}_j), h(\mathfrak{l}_j) \quad \text{for } j = 1, \ldots, \mathfrak{c}, \text{ and} \\ h(\Gamma_k), h(C_k) \quad \text{for } k = 1, \ldots, N - \mathfrak{c} \end{gathered}$$

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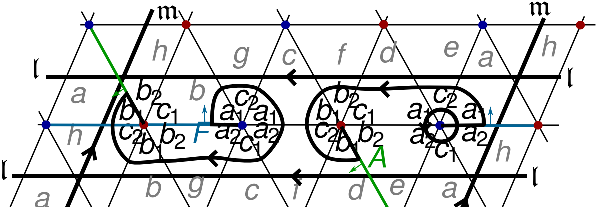
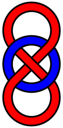
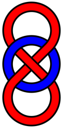


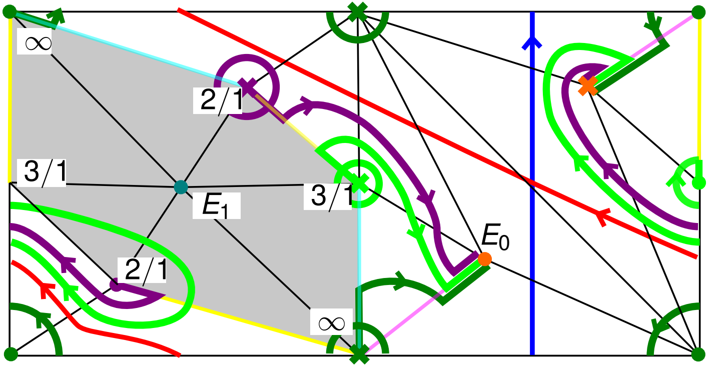
Symplectic basis example: Whitehead link



The Whitehead link complement has a decomposition into 5 ideal tetrahedra.



The Whitehead link complement has a decomposition into 5 ideal tetrahedra.
One cusp is triangulated as shown.



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- Longitude and meridian have <u>holonomy</u> L, M which can be expressed as products of z_i, z_i' variables.

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- L, M satisfy a polynomial relation, the <u>A-polynomial</u> $A(L, M) = 0$.

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- γ_E (for edges E), L, M (longitude/meridian holonomy).

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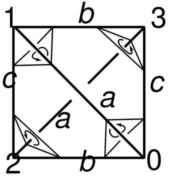
- Resulting equations are Ptolemy equations, one for each tetrahedron:

$$\gamma_{03}\gamma_{12} = \pm L^{\bullet} M^{\bullet} \gamma_{01}\gamma_{23} \pm L^{\bullet} M^{\bullet} \gamma_{02}\gamma_{13}.$$

(Very similar to enhanced Ptolemy variety of Zickert 2016.)



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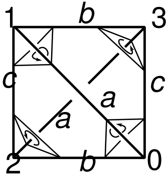

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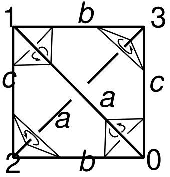

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M–Purcell (in progress):

- The γ variables have an interpretation as complex lambda lengths in spin hyperbolic geometry.

Garoufalidis–Le, Dimofte, Gukov...

- Quantising the A-polynomial should produce a non-commutative polynomial annihilating coloured Jones polynomials (AJ conjecture).

Also:

- Space of hyperbolic structures (Neumann-Zagier, Choi)
- Hyperbolic volumes of Dehn fillings (Neumann-Zagier)
- Normal surfaces (Luo, Garoufalidis–Hodgson–Hoffman-Rubinsten)
- Representation theory (Goerner, Zickert, Garoufalidis, ...)
- Chern-Simons theory (Neumann, Dimofte, Garoufalidis, Gukov, ...)

Idea of proof of Theorem 1

To show $\omega(h(\zeta), h(\zeta') = 2\zeta \cdot \zeta'$...
Express both $\omega(h(\zeta), h(\zeta'))$ and $\zeta \cdot \zeta'$ as sums <u>over faces</u>.

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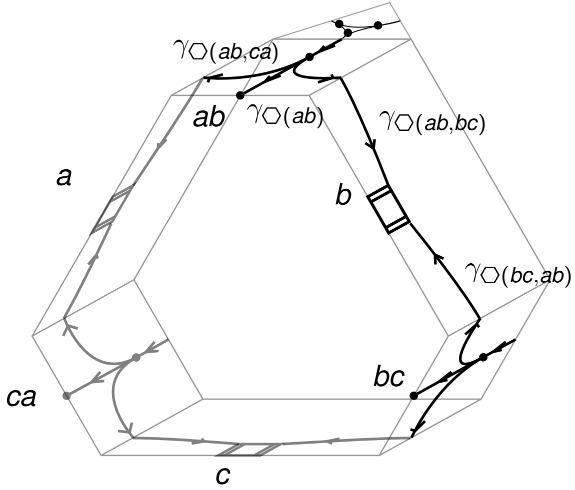

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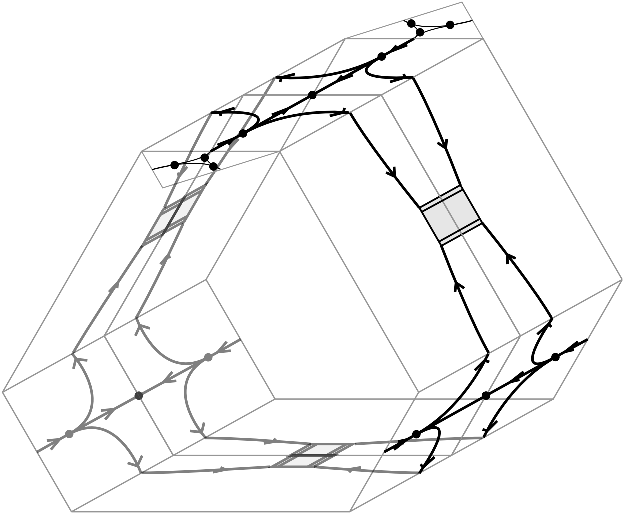

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$$
\begin{aligned}
2\zeta \cdot \zeta' = {} & \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_k (n'_{kl} - n'_{km} + \widehat{n}'_{kl} - \widehat{n}'_{km}) \\
& \qquad\qquad + n'_k(-n_{kl} + n_{km} - \widehat{n}_{kl} + \widehat{n}_{km}) \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} -n_{kl} n'_{km} + n_{km} n'_{kl} - \widehat{n}_{km} \widehat{n}'_{kl} + \widehat{n}_{kl} \widehat{n}'_{km} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_{kl} n'_{lk} - n_{lk} n'_{kl} - \widehat{n}_{kl} \widehat{n}'_{lk} + \widehat{n}_{lk} \widehat{n}'_{kl}.
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$$

$$
\begin{aligned}
\omega\left(h(\zeta), h(\zeta')\right) = & \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)} \\
& + \sum_{\hexagon} \sum_{(k,l,m)} n_k(n'_{lk} - n'_{mk}) - n'_k(n_{lk} - n_{mk}) \\
& + \sum_{\hexagon} \sum_{(k,l,m)} (n_{kl} + n_{km})(n'_{lk} + n'_{lm}) \\
& \qquad - (n'_{kl} + n'_{km})(n_{lk} + n_{lm}).
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$$

Show these are equal!

Thanks for listening!