A symplectic approach to 3-manifold triangulations and hyperbolic structures

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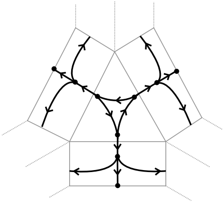
20 September 2022



Overview

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Paper on arxiv:

- A symplectic basis for 3-manifold triangulations
  2208.06969 (joint w Purcell)

Also:

- A-polynomials, Ptolemy equations and Dehn filling
  2002.10356 (joint w Howie, Purcell)

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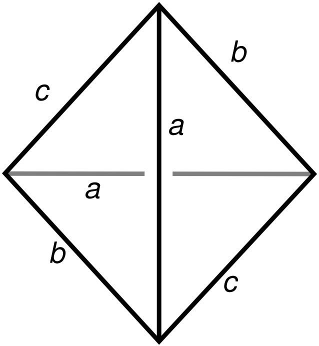

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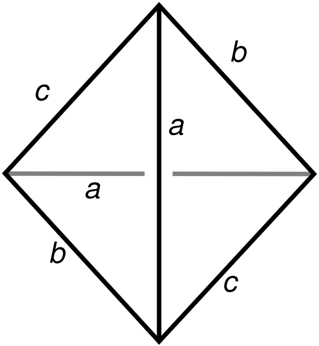
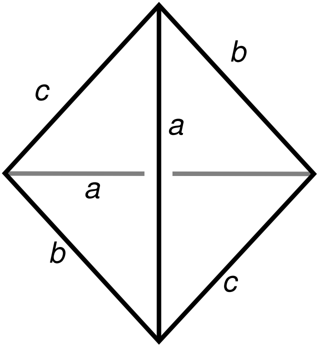
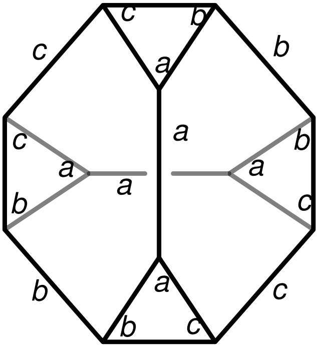
(Results apply more generally, but for convenience...)



Then:

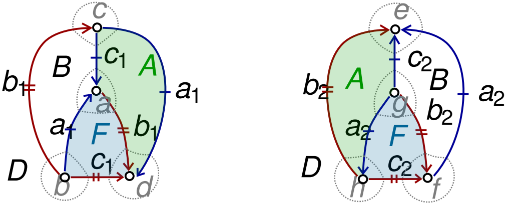
- Truncating $\Delta_j \rightsquigarrow$ polyhedra decomposing $\overline{M} = S^3 \setminus N(L)$.
- Triangular faces of polyhedra triangulate boundary tori T_i.
- Each vertex of each triangle has an a, b or c label.

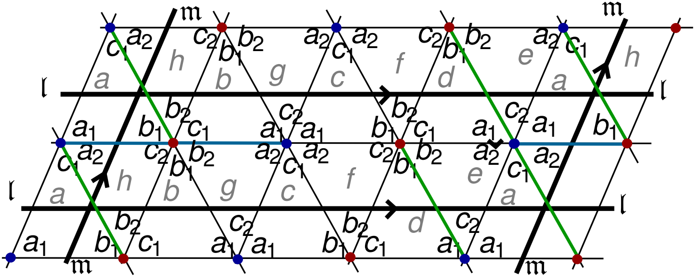




Example: figure-8 knot complement

When $K =$ figure-8 knot, M decomposes into two ideal tetrahedra, with cusp triangulation shown.





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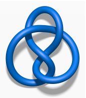

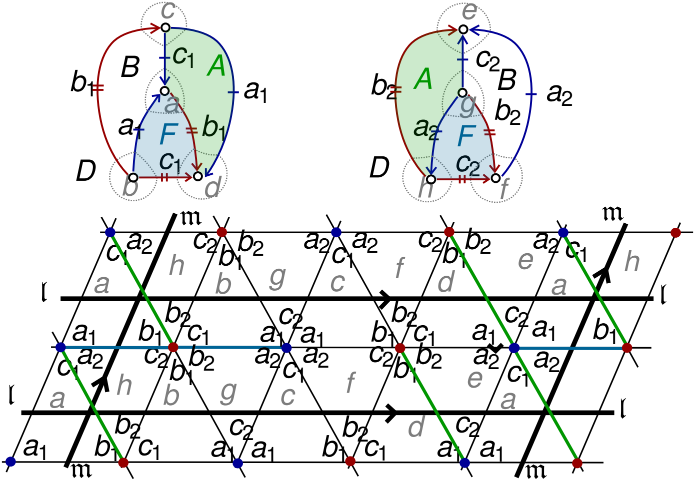

The Neumann–Zagier vector space

Neumann–Zagier associated to $\mathcal{T}$ a vector space V.

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Definition (Neumann–Zagier 1985)

Let V be the $2N$-dimensional $\mathbb{R}$-vector space generated by

$$a_1, b_1, c_1, \ldots, a_N, b_N, c_N$$

subject to relations

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It follows that

$$\omega(a_i, b_i) = \omega(b_i, c_i) = \omega(c_i, a_i) = 1$$
$$\omega(b_i, a_i) = \omega(c_i, b_i) = \omega(a_i, c_i) = -1$$
$$\omega = 0 \quad \text{on all other pairs of generators.}$$

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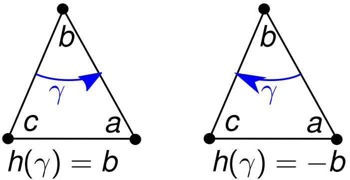

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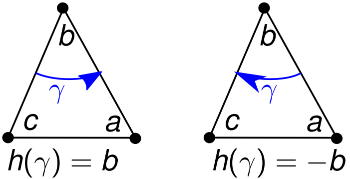
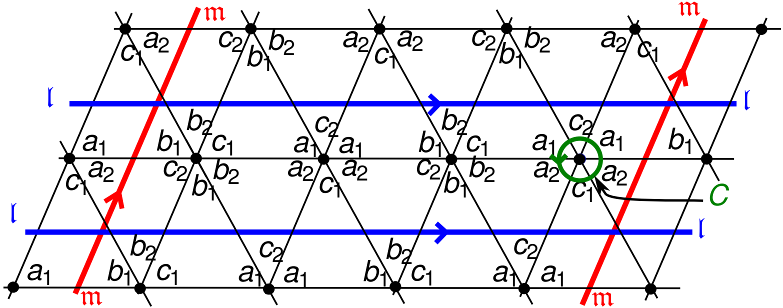


Definition

The combinatorial holonomy $h(\gamma) \in V$ is the sum of contributions $\pm a_j, b_j, c_j$ for each arc of γ.

Combinatorial holonomy example: figure-8 knot



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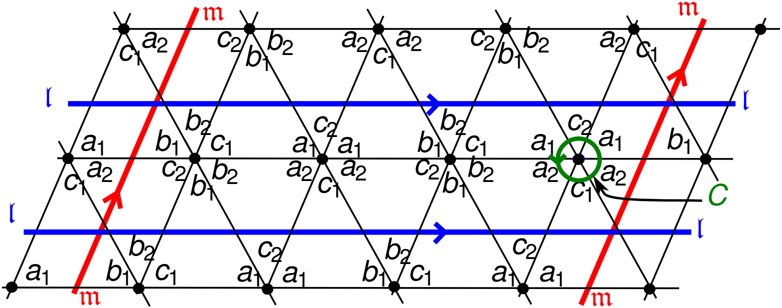

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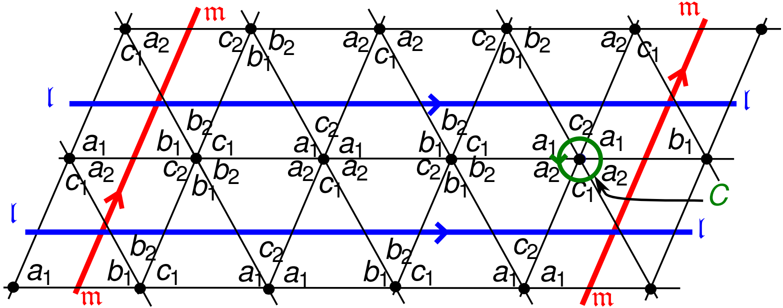

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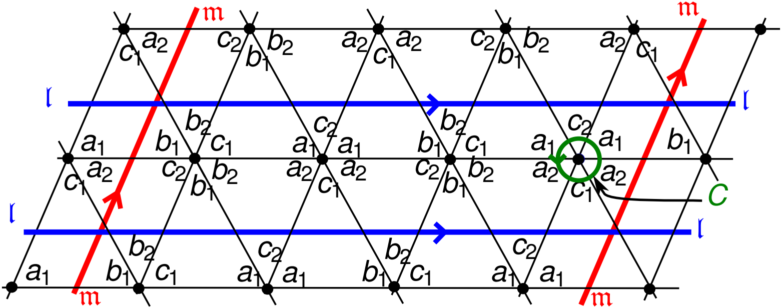

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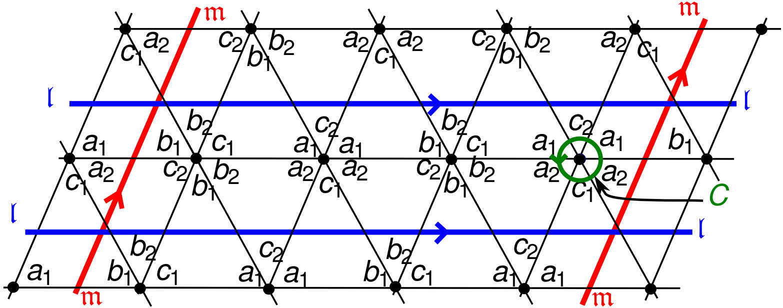

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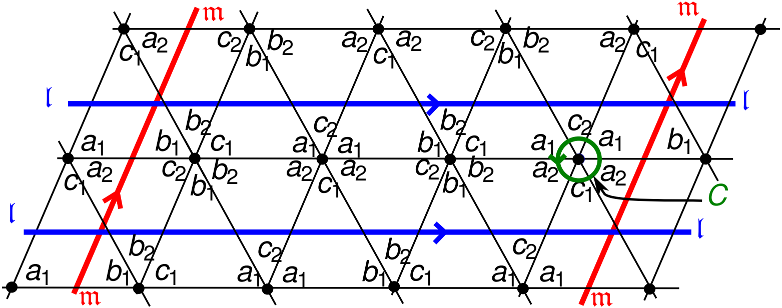

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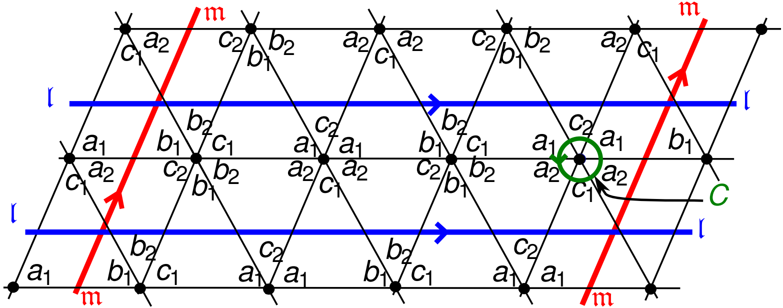

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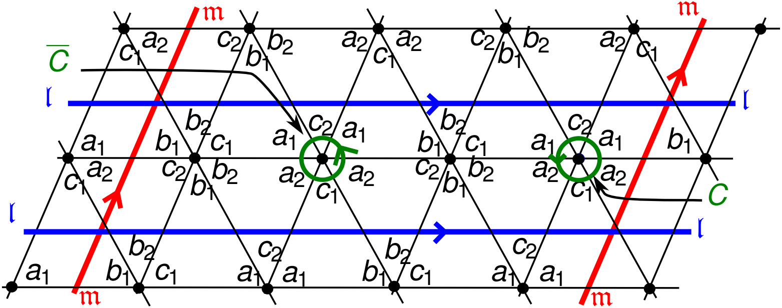

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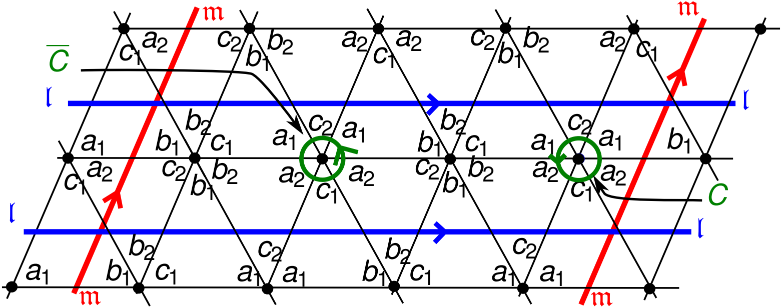

$$h(\overline{C}) = h(C)$$

The NZ symplectic form as a 2D intersection form

Theorem (Neumann–Zagier 1985)

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Let γ, δ be oriented curves on $\partial \overline{M}$. Then

$$\omega\left(h(\gamma), h(\delta)\right) = 2\gamma \cdot \delta,$$

where $\cdot$ denotes algebraic intersection number.

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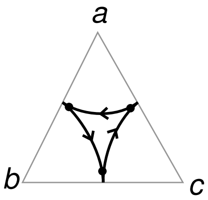
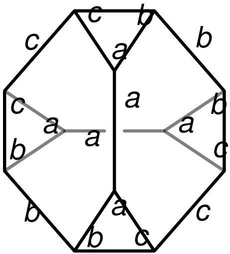
Note ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!





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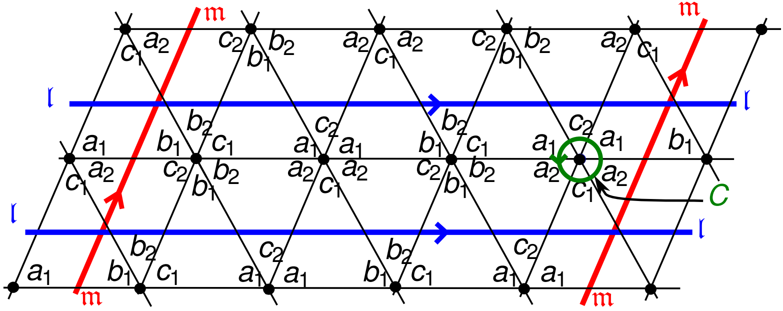

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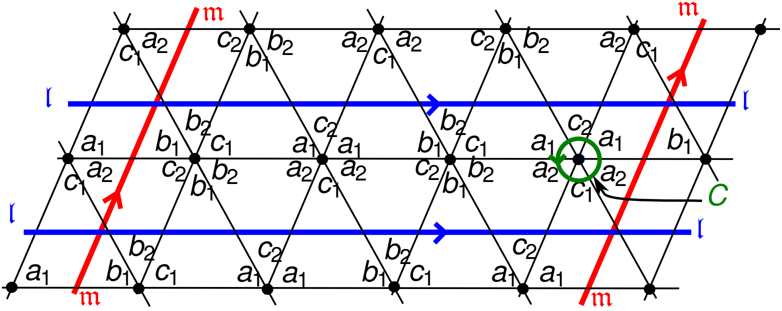

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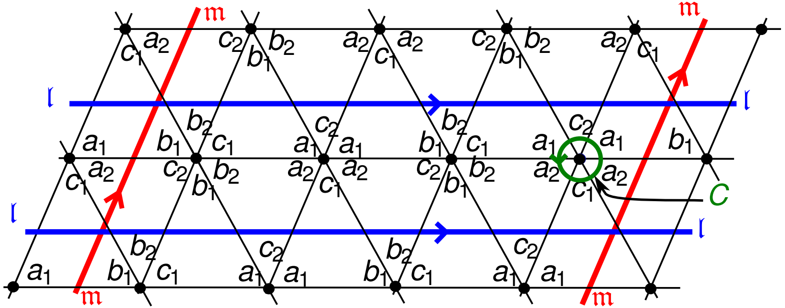
$\omega(h(\mathfrak{m}), h(\mathfrak{l}))$



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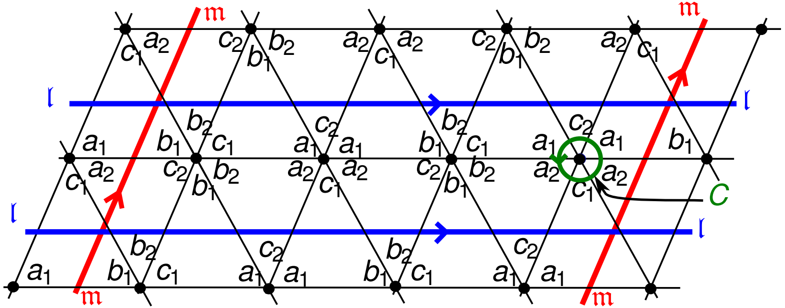

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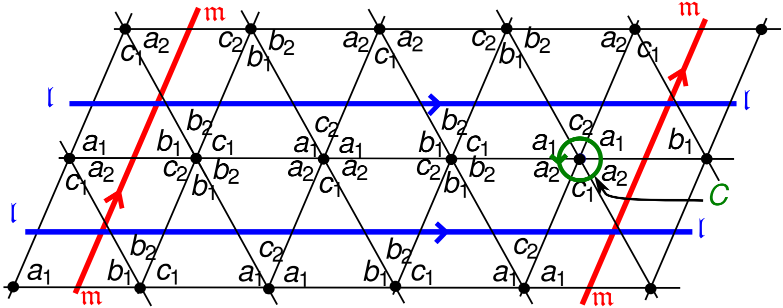

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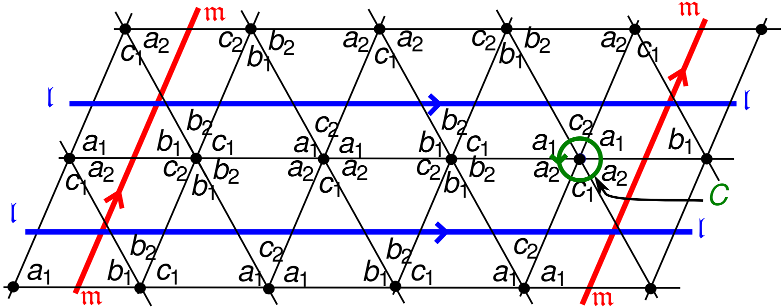

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&= \omega(-b_1, -2a_1) + \omega(a_2, 2a_2) = -2 = 2\mathfrak{m} \cdot \mathfrak{l}
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$$\omega(h(\mathfrak{m}_j), h(\mathfrak{l}_j)) = 2, \quad j = 1, 2, \ldots, \mathfrak{c}.$$

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Observation (Dimofte 2013)

We can form a partial symplectic basis* of V, consisting of

the $h(\mathfrak{m}_j), h(\mathfrak{l}_j)$, for $j = 1, \ldots, \mathfrak{c}$

and the $h(C_j)$ (some of them)

(Symplectic basis:
vectors $e_1, f_1, \cdots, e_N, f_N$ such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, $\omega(e_i, f_j) = \delta_{ij}$.)

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* Up to factors of 2...

Questions

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Can we find more curves to obtain a full symplectic basis* of V?

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Neumann-Zagier's results are 2-dimensional, only about curves on $\partial\overline{M}$.
Can these results be extended into 3D?
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Yes!

The NZ symplectic form as a 3D intersection form

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Theorem 1 (M–Purcell)

Let ζ, ζ' be oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

$$\omega\left(h(\zeta), h(\zeta')\right) = 2\zeta \cdot \zeta'$$

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I.e. $\omega(\Gamma_j, C_k) = 2\delta_{jk}, \quad \omega(\Gamma_j, \mathfrak{m}_k) = \omega(\Gamma_j, \mathfrak{l}_k) = \omega(\Gamma_j, \Gamma_k) = 0.$

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Idea of Oscillating curves

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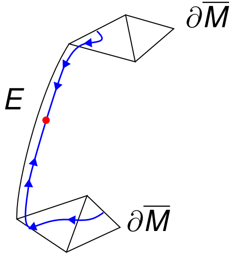
- May dive from $\partial\overline{M}$ into the interior of $\overline{M}$, along an edge E, and come out on $\partial\overline{M}$ at the other end of E.



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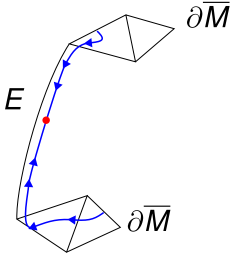

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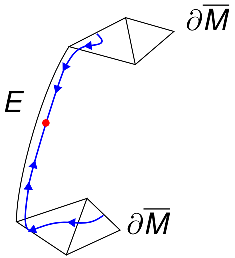

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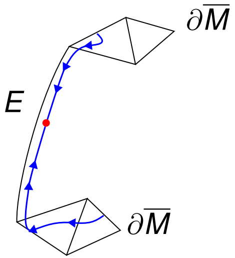

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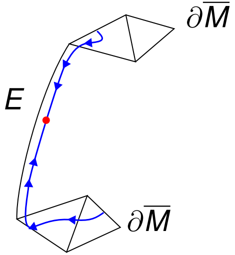

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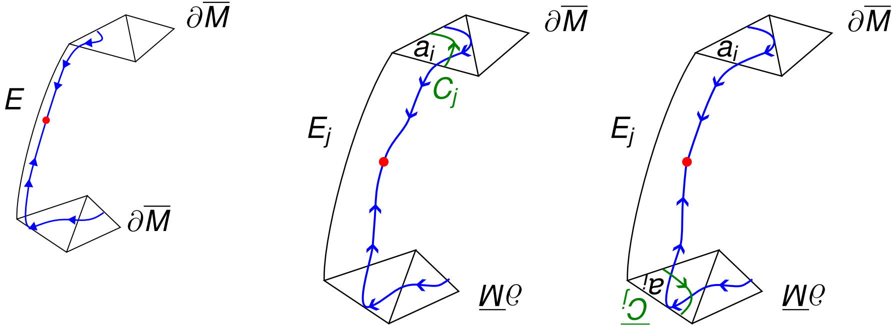

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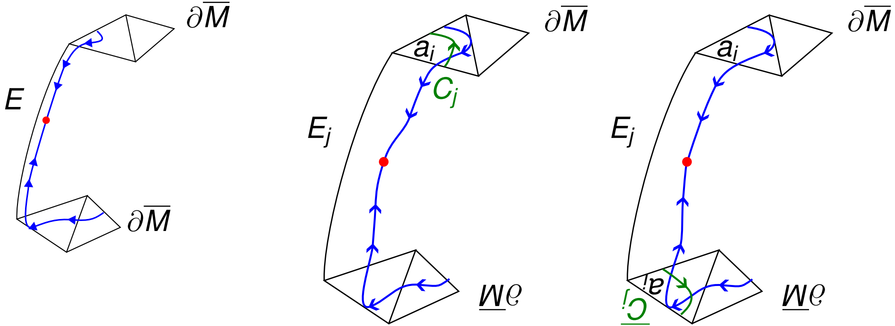
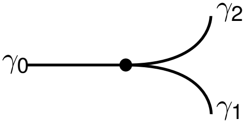


Definition

A train track is a smoothly embedded graph on a surface such that at each vertex, incident edges are all tangent, with at least one edge on each side.

- Edges are called branches; vertices called switches.



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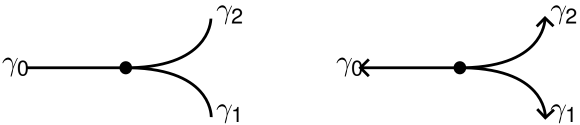

Branches, if oriented, have intersection numbers, defined locally at switches:

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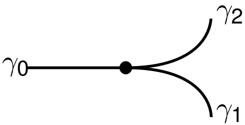

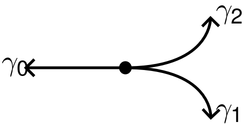

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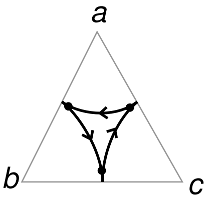
Smooth oriented curves on train tracks then obtain intersection numbers agreeing with usual algebraic intersection number.

A generic curve on a boundary torus intersects triangles in arcs, so can be made to run on train tracks.



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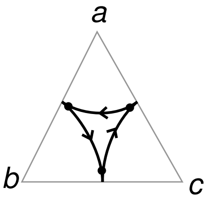

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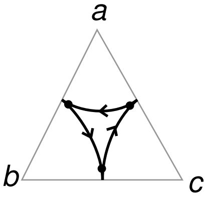

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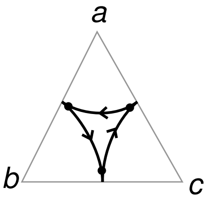

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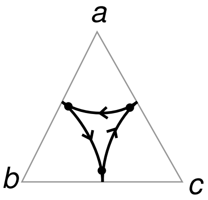

Similarly, oscillating curves can run on train tracks. But...

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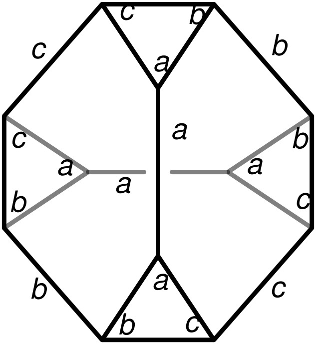
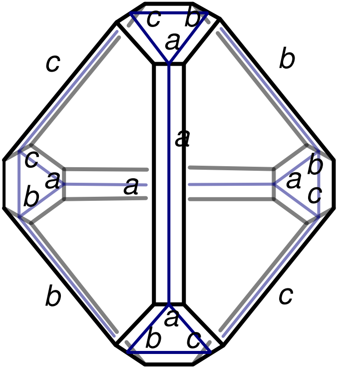
- Special “stations" for each orientation reversal.

Truncated tetrahedra

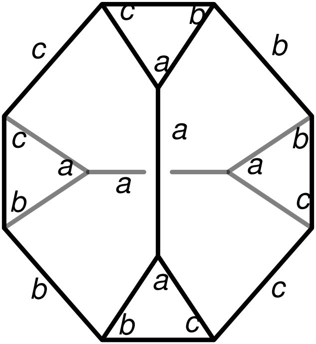
Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra

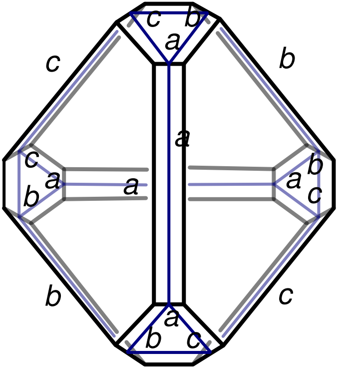




Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra with:

- hexagonal faces glued in pairs





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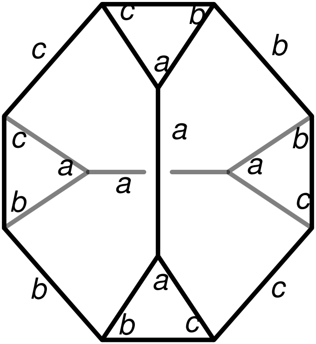
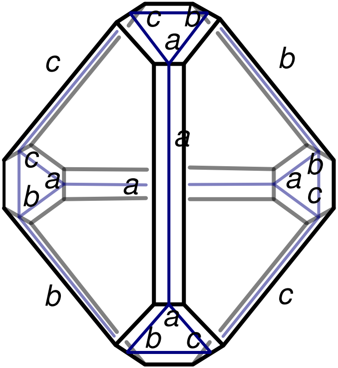
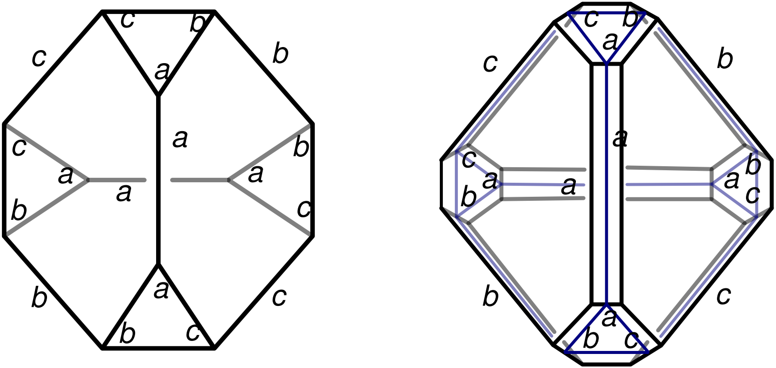




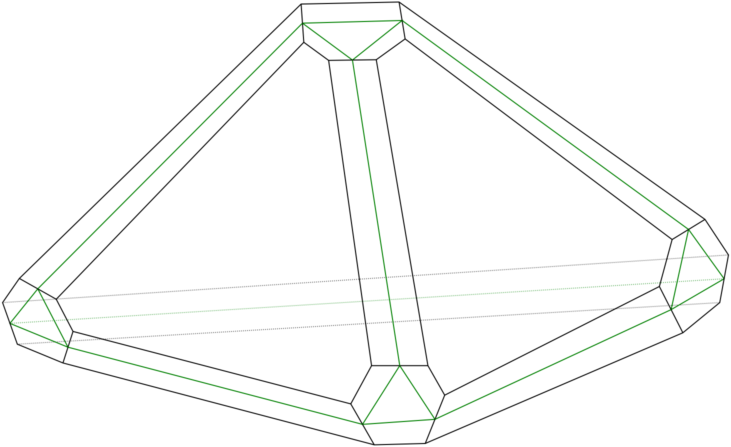
Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra with:

- hexagonal faces glued in pairs
- hexagons on boundary tori (split further into triangle + 3 rectangles)
- rectangles along removed edges (split further into 2 rectangles)

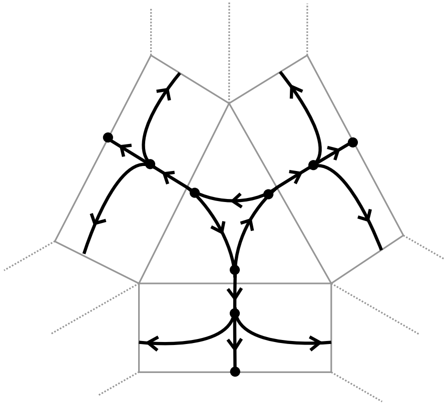


Triangles + rectangles give a decomposition of a Heegaard surface for M.

- $M =$ Handlebody $\cup$ Compresson body
- with handlebody decomposed into truncated tetrahdra



Train tracks for oscillating curves



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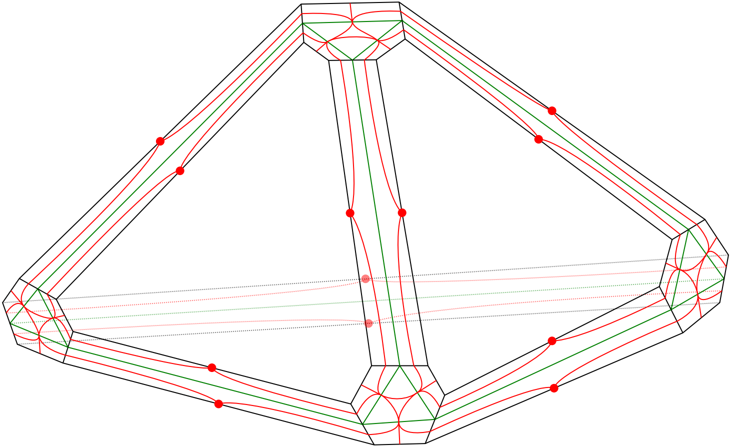

Red dots: stations for reversing direction.

⇝ "Enhanced" train tracks.

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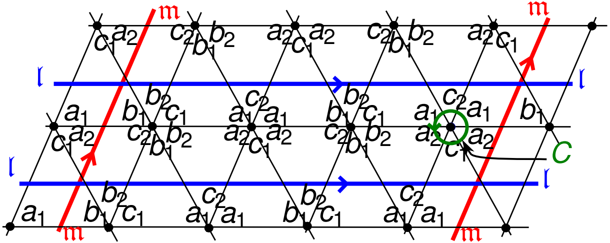

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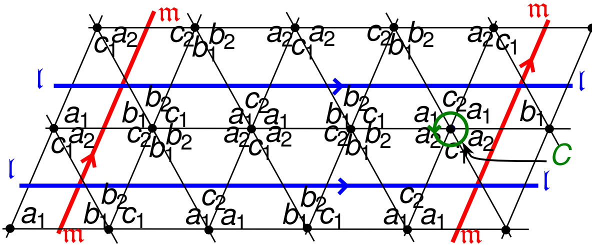

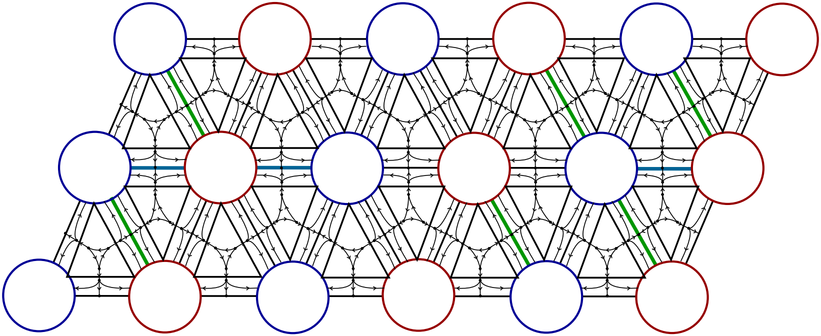

Let τ be the (enhanced) train tracks on the triangulation $\mathcal{T}$

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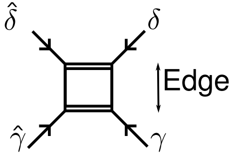

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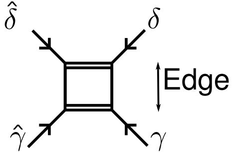
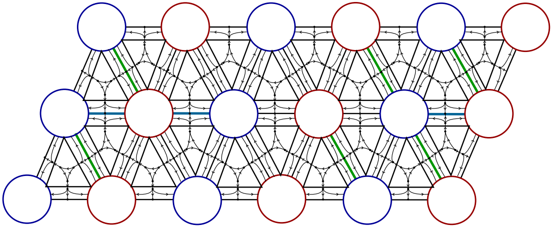
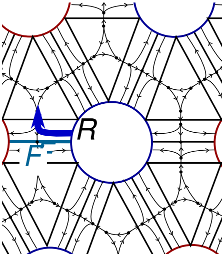
- γ_{j-1} and γ_j have different orientations precisely when they lie at opposite ends of a station.

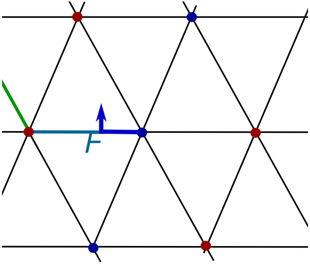


Example: figure-8 knot complement

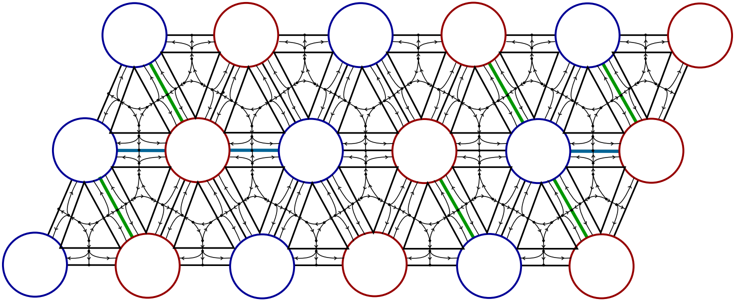


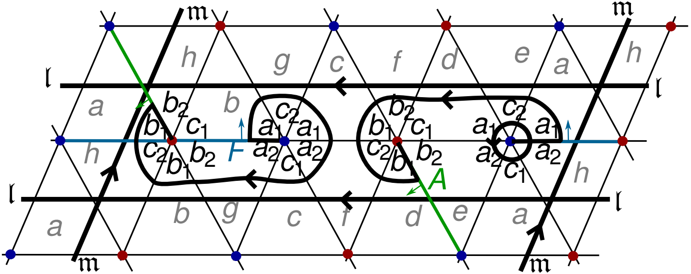
To draw oscillating curves...





Example: figure-8 knot complement





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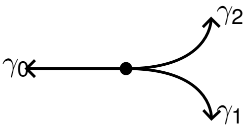

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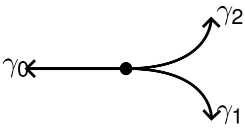

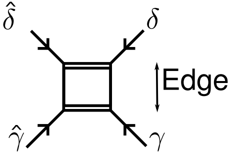

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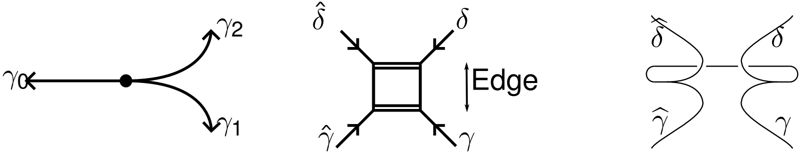
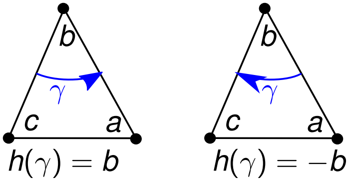


Definition

The <u>combinatorial holonomy</u> $h(\zeta) \in V$ of an (abstract) oscillating curve ζ is the sum of contributions $\pm a_j, b_j, c_j$ for each arc of ζ in a triangle.



Combinatorial holonomy of oscillating curves

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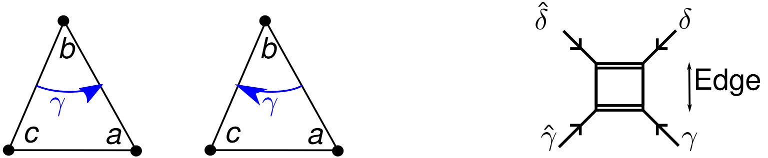
(And nothing from arcs in rectangles / diving into the manifold / passing through stations!)



$h(\gamma) = b$

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The NZ symplectic form as a 3D intersection form

Theorem 1 (M–Purcell)

Let ζ, ζ' be (abstract) oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

$$\omega\left(h(\zeta), h(\zeta')\right) = 2\zeta \cdot \zeta'$$

where $\cdot$ is the intersection form for oscillating curves.

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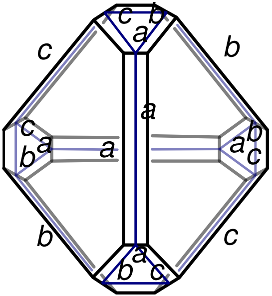

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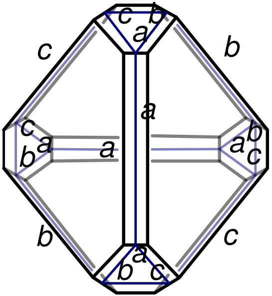
- ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!



Theorem 2 (M–Purcell)

We can construct oscillating curves $\Gamma_1, \ldots, \Gamma_{N-\mathfrak{c}}$ such that

$$h(\mathfrak{m}_j), h(\mathfrak{l}_j) \quad \text{for } j = 1, \ldots, \mathfrak{c}, \text{ and}$$
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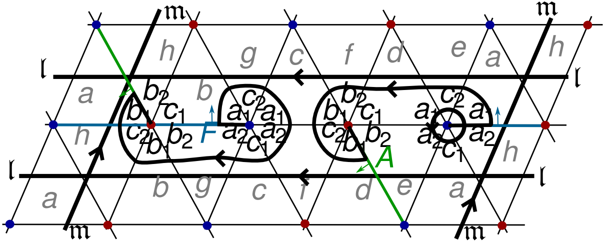
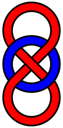
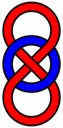


Symplectic basis example: Whitehead link



The Whitehead link complement has a decomposition into 5 ideal tetrahedra.



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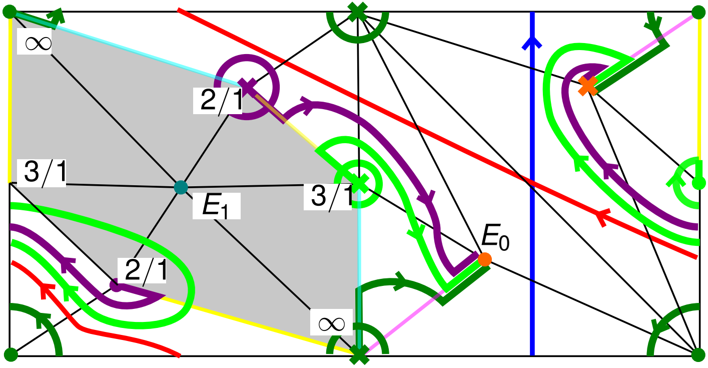
One cusp is triangulated as shown.



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- Longitude and meridian have holonomy L, M which can be expressed as products of z_i, z_i' variables.

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- γ_E (for edges E), L, M (longitude/meridian holonomy).

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- Resulting equations are Ptolemy equations, one for each tetrahedron:

$$\gamma_{03}\gamma_{12} = \pm L^{\bullet}M^{\bullet}\gamma_{01}\gamma_{23} \pm L^{\bullet}M^{\bullet}\gamma_{02}\gamma_{13}.$$

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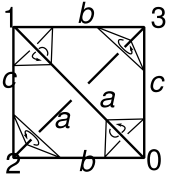

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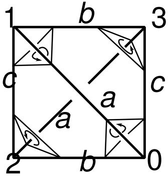

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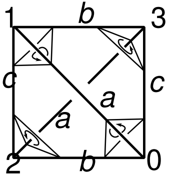

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M–Purcell (in progress):

- The γ variables have an interpretation as complex lambda lengths in spin hyperbolic geometry.

More Applications

Garoufalidis–Le, Dimofte, Gukov...

- Quantising the A-polynomial should produce a non-commutative polynomial annihilating coloured Jones polynomials (AJ conjecture).

Also:

- Space of hyperbolic structures (Neumann-Zagier, Choi)
- Hyperbolic volumes of Dehn fillings (Neumann-Zagier)
- Normal surfaces (Luo, Garoufalidis–Hodgson–Hoffman-Rubinsten)
- Representation theory (Goerner, Zickert, Garoufalidis, ...)
- Chern-Simons theory (Neumann, Dimofte, Garoufalidis, Gukov, ...)

Idea of proof of Theorem 1

To show $\omega(h(\zeta), h(\zeta') = 2\zeta \cdot \zeta'$...
Express both $\omega(h(\zeta), h(\zeta'))$ and $\zeta \cdot \zeta'$ as sums <u>over faces</u>.

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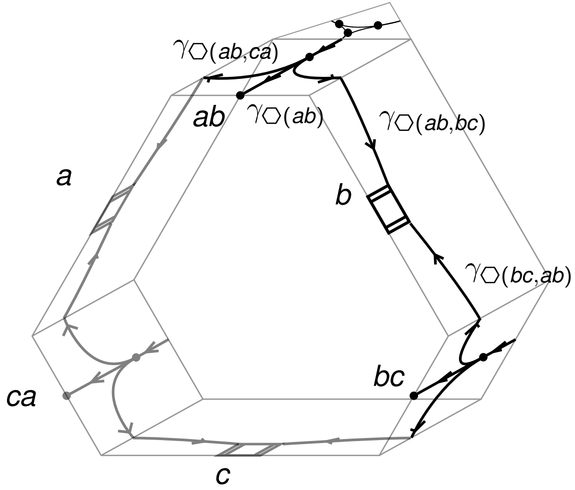

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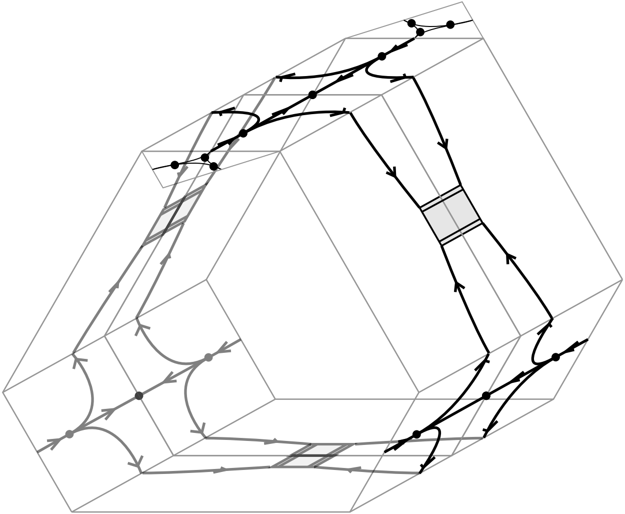

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$$
\begin{aligned}
2\zeta \cdot \zeta' = & \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_k (n'_{kl} - n'_{km} + \widehat{n}'_{kl} - \widehat{n}'_{km}) \\
& \qquad\qquad + n'_k (-n_{kl} + n_{km} - \widehat{n}_{kl} + \widehat{n}_{km}) \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} -n_{kl} n'_{km} + n_{km} n'_{kl} - \widehat{n}_{km} \widehat{n}'_{kl} + \widehat{n}_{kl} \widehat{n}'_{km} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_{kl} n'_{lk} - n_{lk} n'_{kl} - \widehat{n}_{kl} \widehat{n}'_{lk} + \widehat{n}_{lk} \widehat{n}'_{kl}.
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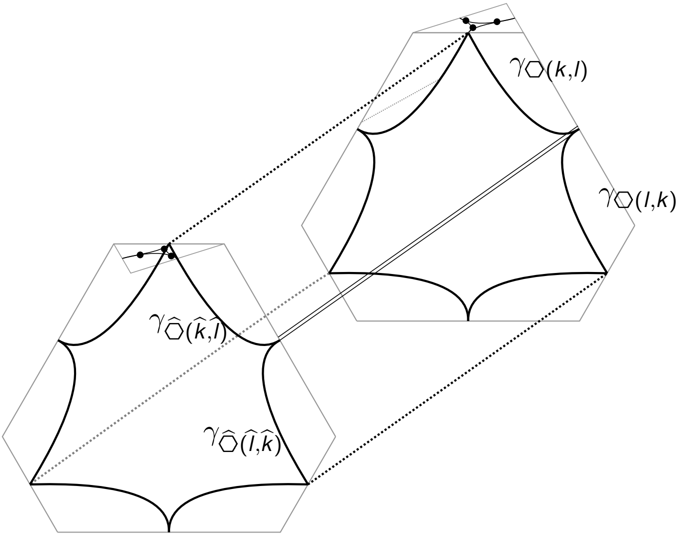
$$

$$\omega\left(h(\zeta), h(\zeta')\right) = \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)}$$
$$+ \sum_{\hexagon} \sum_{(k,l,m)} n_k(n'_{lk} - n'_{mk}) - n'_k(n_{lk} - n_{mk})$$
$$+ \sum_{\hexagon} \sum_{(k,l,m)} (n_{kl} + n_{km})(n'_{lk} + n'_{lm})$$
$$- (n'_{kl} + n'_{km})(n_{lk} + n_{lm}).$$

Show these are equal!

Why the weird intersection form for oscillating curves?



Thanks for listening!