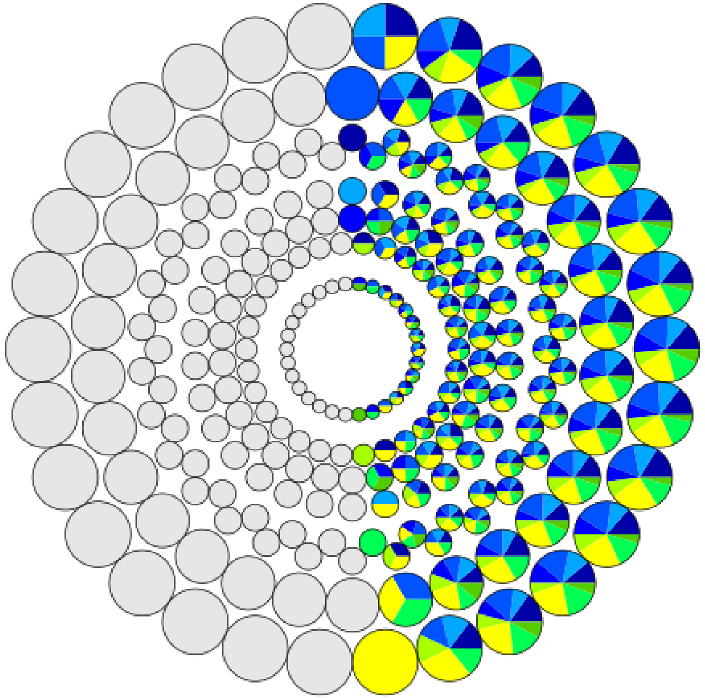
"There is always room for a new theory in mathematics"

MARYNA VIAZOVSKA AND HER MATHEMATICS



Daniel Mathews, August 30, 2022

Maryna Viazovska

5 July 2022:

International
Mathematical
Union

The Fields Medal 2022

Short citation:

Maryna Viazovska is awarded the Fields Medal 2022 for the proof that the E_8 lattice provides the densest packing of identical spheres in 8 dimensions, and further contributions to related extremal problems and interpolation problems in Fourier analysis.

Long citation:

A very long-standing problem in mathematics is to find the densest way to pack identical spheres in a given dimension. It has been known for some time that the hexagonal packing of circles is the densest packing in 2 dimensions, while in 1998 Hales gave a computer assisted proof of the Kepler conjecture that the faced centered cubic lattice packing gives the densest

Brief biography

2 December 1984: Born in Kyiv, Ukraine (USSR)

“The late 1980s were a difficult time in the Soviet Union. It took people many, many hours to buy even basic things… The Soviet Union fell apart when I was 6.”

“When everyone goes to sleep, she has her notepad and she draws some formulas”

1998-2001: Attended specialised school in Kyiv for high-achieving students in science and technology

- Taught by professional mathematician Andrii Knyazyuk

“In the 11th grade… I didn't qualify for the International Mathematics Olympiad and felt discouraged from becoming a mathematician. That was a huge disappointment.”

Source: Nature, Credit: EPFL/Fred Merz

“Always room for a new theory”

“We had a neighbor who died a long time ago. He was an old man who had fought in the war and then worked with my grandfather at the same university.

And he had such a huge… collection of different popular books on physics and mathematics at home… he gave them to me… And there, I found an astronomy book that really impressed me.

There were stories about different theories in which everything fit together really well… And then they would find a new star. And the whole theory would go in the trash… Some years go by, this more complicated theory works, and then another discovery happens, forcing it to be thrown out as well…

This really impressed me, because this was so unlike what we were taught at school. The teacher would say: ‘Here is the theory, you should learn it, and it will work’. But it turns out you can invent new theories!”

But that's not how it works in math, right?

“Times change, discoveries are made, maybe some previously overlooked aspects become important, maybe new ideas come from physics or astronomy. There is always room for a new theory in mathematics.”

Brief biography

2001-5: Bachelor Degree in Mathematics, KNU (Taras Shevchenko National University of Kyiv)

“I was not interested in anything but mathematics.”

“I lived a ‘double life’ between algebra and analysis”

2002-5: Competed in International Mathematics Competition for University Students
- First place 2002, 2005.

“The next crisis happened when those student olympiads came to an end — because I became too old for them. But then, luckily, I realized that there is such a thing as research in mathematics, where one can solve really hard problems and write papers about them.”

2005: First research paper published – (joint w Andriy Bondarenko)
“Bernstein type inequality in monotone rational approximation”

Brief biography

2007: Masters degree, University of Kaiserslautern (TUK), Germany

2010: Candidate degree, Institute of Mathematics of National Academy of Sciences of Ukraine
- Thesis "Inequalities for polynomials and rational functions and quadrature formulas in the sphere"

2013: Doctorate, University of Bonn. Supervisors: Don Zagier, Werner Müller.
- Studied modular forms - *"I realized this is where my two passions meet"*
- Thesis "Modular Functions and Special Cycles"
- Paper "Optimal asymptotic bounds for spherical designs" (w A Bondarenko, D Radchenko), published in *Annals of Mathematics*

2014-17: Postdocs – IHES (France), Berlin Mathematical School, Humboldt University

2018-present: Professor at École Polytechnique Fédérale de Lausanne (EPFL), Switzerland

February 2022: Russia invades Ukraine - Some family stay, some flee Kyiv for Lausanne

International Mathematical Union

IMU Fields Medal 2022
Maryna Viazovska Interview

Conducted by Andrei Okounkov & Andrei Konyaev

Part of the interview was recorded before February 24, 2022, another part — after. In both cases, the interviewers were Russian mathematicians

"There is a war going on right now, the war to destroy my home country, my nation. This is done by the country of which you are citizens. People from your country are either doing this or supporting these actions…

"I know people in Moscow, educated and well-read people, who at the same time support everything that is going on at the moment, everything that Russia does… Unfortunately, neither education nor profession can prevent people from turning into cannibals...

International
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"If we talk about what can and should be done, then we need to talk about the plight of the refugees. The crisis created a huge humanitarian problem…

"Let me speak only about one aspect close to my personal experience - the plight of refugees and the effect of war on Ukrainian education. For example, Kyiv did not suffer as much as cities in the East, but 25% of students left Kyiv University. A huge number of children from Ukraine have now left for Europe, and they have to adapt to a completely different education system in a different language. And if school education is free almost everywhere, the situation with university students is more difficult… It is especially hard for those who have just graduated from school, or who are in their first year of university…

"it is impossible to replace what is lost and we will suffer the consequences of this war for generations.

"I would like to thank everyone who helps refugees."

On mathematics education

“It always surprises me — if you look at the math curriculum, it changes all the time.

“When my son started learning geometry, I bought the textbook that I used. Written by Pogorelov back in the 60s. It's a great textbook, why would you need another one? …

“The plane geometry taught at school, it is still the same plane geometry as invented by Euclid, nothing has changed… A maximum is beautiful, why change it? …

“Maybe I don't have the same needs as others. And it is obvious that the textbook must be understandable to everyone… But with the old textbooks, I like that there are really a lot of words and explanations in them. There are definitions, there are theorems…

“Modern textbooks are just an outline, a cheat sheet. Lots of pictures, not much text. Literally like in children's books…

“Perhaps one expects the teacher in class to fill in the voids, but it seems to me that the children in general rarely remember… having a book that says it all is actually a very good thing.”

Sphere packing

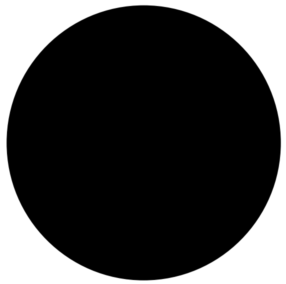
The sphere packing problem:

What is the densest packing of space with unit balls?

May be asked in any dimension: pack $\mathbb{R}^n$ with n-dimensional unit balls.

Unit ball: centred at $x \in \mathbb{R}^n$

$$B(x) = \{p \in \mathbb{R}^n : \text{the distance from } p \text{ to } x \text{ is less than } 1\}$$
$$= \{p \in \mathbb{R}^n : |p - x| < 1\}$$



Higher dimensions

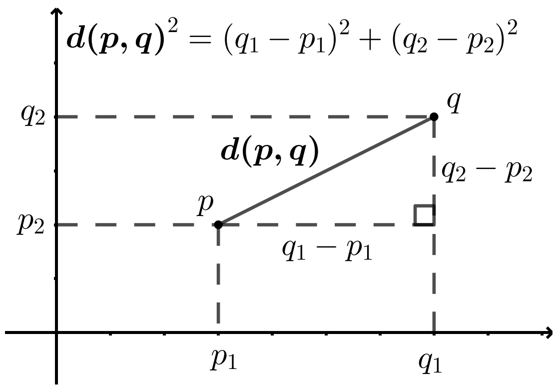
Do not be afraid! Just don't expect to visualise!

A point in $\mathbb{R}^2$ is given by

$$p = (p_1, p_2)$$

And the distance $|p - q|$ between two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ is given by

$$|p - q|^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2$$



Source: Wikipedia, Kmhkmh

Higher dimensions

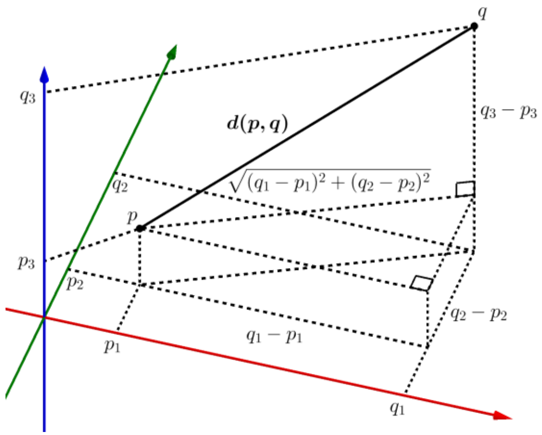
Do not be afraid! Just don't expect to visualise!

A point in $\mathbb{R}^3$ is given by

$$p = (p_1, p_2, p_3)$$

And the distance $|p - q|$ between two points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ is given by

$$|p - q|^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2$$



Source: Wikipedia, Kmhkmh

Higher dimensions

Do not be afraid! Just don't expect to visualise!

A point in $\mathbb{R}^8$ is given by

$$p = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$$

The distance $|p - q|$ between two points $p = (p_1, p_2, \dots, p_8)$ and $q = (q_1, q_2, \dots, q_8)$ is given by

$$|p - q|^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots + (p_8 - q_8)^2$$

Sphere packing in 1 dimension

Doable!

100% density!

Solved by nature in practice > 90 Mya

Source: Wikipedia, Bill Ebbesen

Sphere packing in 2 dimensions

Solved in practice by nature ~63 Mya

Harder to prove than it looks!

Thue 1910 gave a proof of optimal packing.

Density: $\frac{\pi}{2\sqrt{3}} \sim 0.9069$

Source: Matthew T Rader

Sphere packing in 3 dimensions

Solved in practice by…

- Greengrocers everywhere
- Any crystal with a face-centred cubic lattice structure (eg aluminium, copper, gold, silver…)

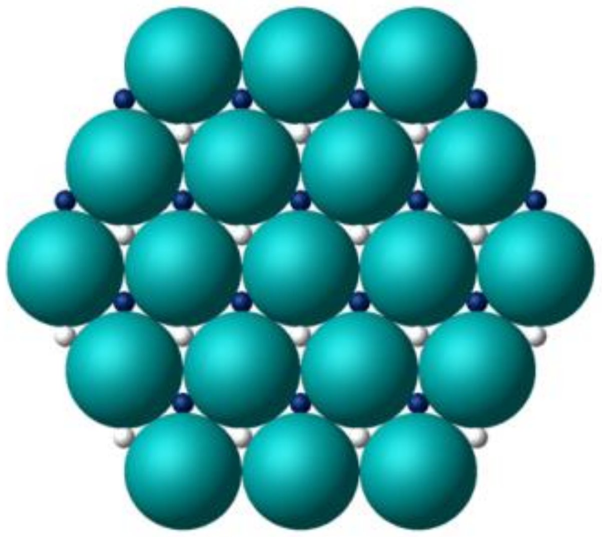
Asserted as best possible by Kepler 1611: **Kepler's conjecture**

Infinitely many packings with equal density!

Rogers 1958: "many mathematicians believe,
and all physicists know" the answer

Extremely difficult to prove! Many attempts!

Density: $\frac{\pi}{3\sqrt{2}} \sim 0.7405$



Source: Okounkov, The magic of 8 and 24

Sphere packing in 3 dimensions

Proved by T Hales – raising philosophical issues

- Reduced problem to “100,000 linear programming problems”
- 1998 proof announcement: 339 pages, 50,000 lines of code
- Referees “99% certain” of correctness
- Published (abridged) in *Annals of mathematics* in 2005
- Formal proof using computerised assistants completed 2014

Source: Quanta.

Current status of the problem

Best sphere packings are currently known in dimensions:

- 1
- 2 (Thue 1910)
- 3 (Hales 1998-2014)
- 8 (Viazovska 2016)
- 24 (Cohn, Kumar, Miller, Radchenko, Viazovska 2017)

No proof in any other dimension is known.

There are widely believed conjectures for optimal packing in dimensions 4-7.

For all dimensions 9 and above (except 24), all bets are off.

Best known upper & lower bounds in large dimension differ by large factors.

Lattices

Consider a set of linearly independent vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.

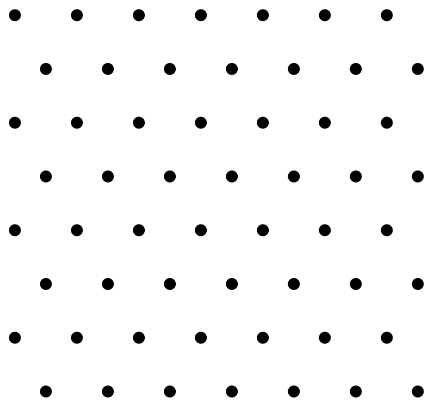
Taking integer combinations of them yields a discrete set of points called a *lattice* Λ.

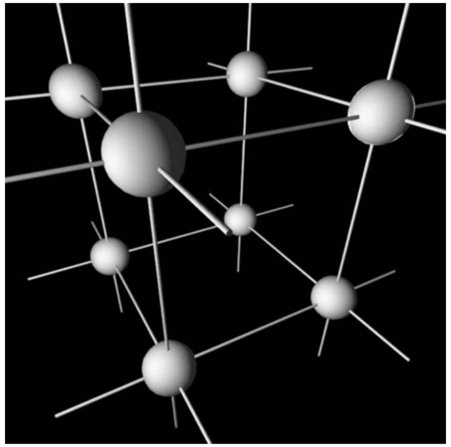
$$\Lambda = \mathbb{Z}\, v_1 + \mathbb{Z}\, v_2 + \cdots + \mathbb{Z}\, v_n = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n : a_1, a_2, \ldots, a_n \in \mathbb{Z}\}$$

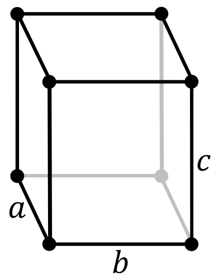
(A discrete additive subgroup of $\mathbb{R}^n$.)

Points are arranged regularly! Space is divided into fundamental cells!

If points are all distance $\geq 2r$ apart then we obtain a lattice sphere packing of radius r.







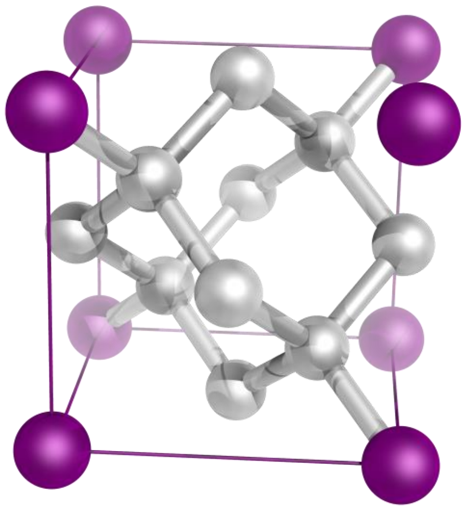
Sources: Wikipedia; Melcom; Wikimedia, Daniel Mayer, Stannered

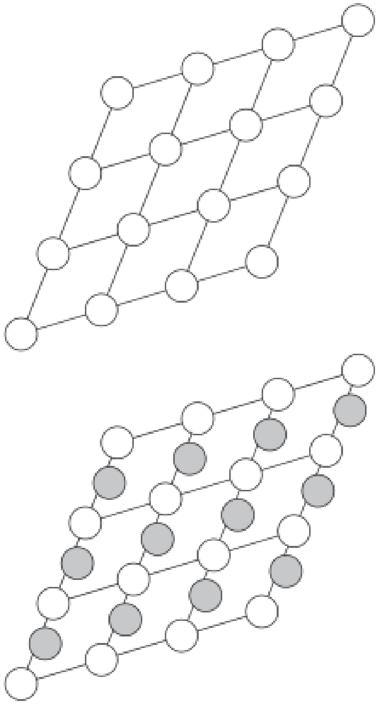
Lattices vs periodic packings

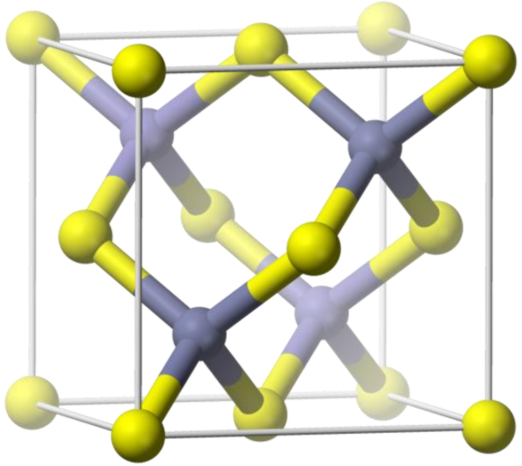
A *periodic* sphere packing is obtained by placing spheres in the fundamental cell of a lattice, and translating them through space by the lattice.

$$\{ \text{Lattice packings} \} \subset \{ \text{Periodic packings} \}$$

Efficient sphere packings can sometimes be obtained from lattice packings by placing spheres in *interstitial sites / holes.*







Sources: GYassineMrabet, Wikimedia; Cohn, *A conceptual breakthrough;* Benjah-bmm27, Wikimedia

The D_n lattice

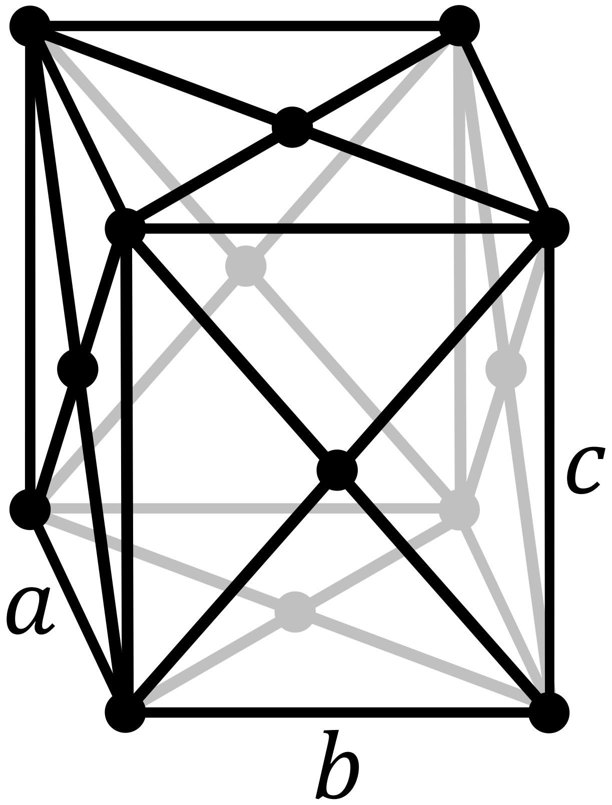
$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n : \textstyle\sum_{i=1}^{n} x_i \text{ is even}\}$$

E.g. $D_3 = \{(\text{even}, \text{even}, \text{even}), (\text{even}, \text{odd}, \text{odd}), (\text{odd}, \text{even}, \text{odd}), (\text{odd}, \text{odd}, \text{even})\}$

$$= \{(0,0,0), (0,1,1), (1,0,1), (1,1,0) \text{ mod } 2\}$$

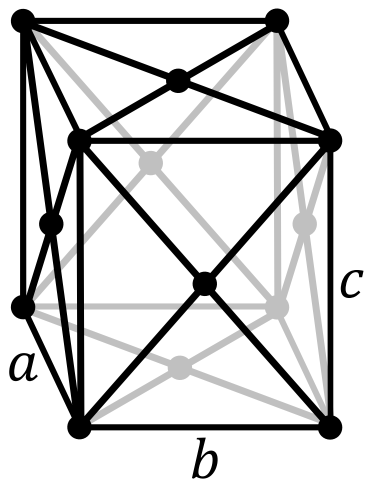
= Face-centred cubic lattice! Optimal!

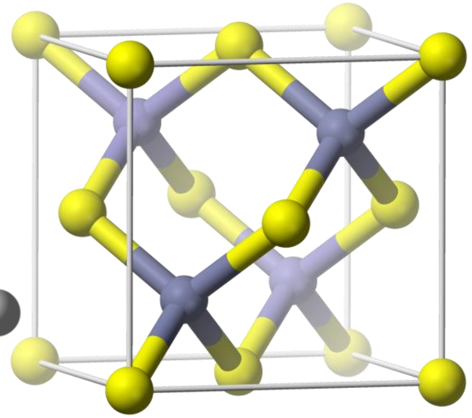
Minimal separation $\sqrt{2}$ - yields sphere packing with radius $\frac{1}{\sqrt{2}}$.



Source: Wikimedia, Rocha, Stannered

E_8 is 8-dimensional diamond





The D_n lattice (separation: $\sqrt{2}$; sphere radius $1/\sqrt{2}$) has two types of holes:

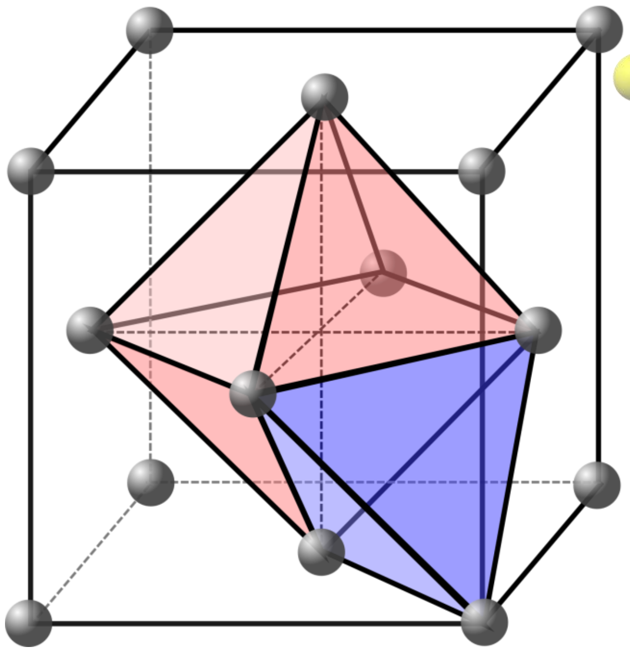
- Shallow holes: $(1,0,0,\dots), (0,1,0,\dots), (1,1,1,\dots)$ etc, distance 1 from other points
- Deep holes: $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots\right)$,

  distance $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{n}{4}}$

Adding $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots\right)$ to the lattice yields the *diamond lattice*.

When $n=8$, spheres at deep holes fit perfectly!

Two copies of D_8 perfectly interleave!



This is the E_8 lattice.

Sources: Benjah-bmm27, via Wikimedia; Wikimedia, Rocha, Stannered; Wikimedia, cdang

The E_8 lattice

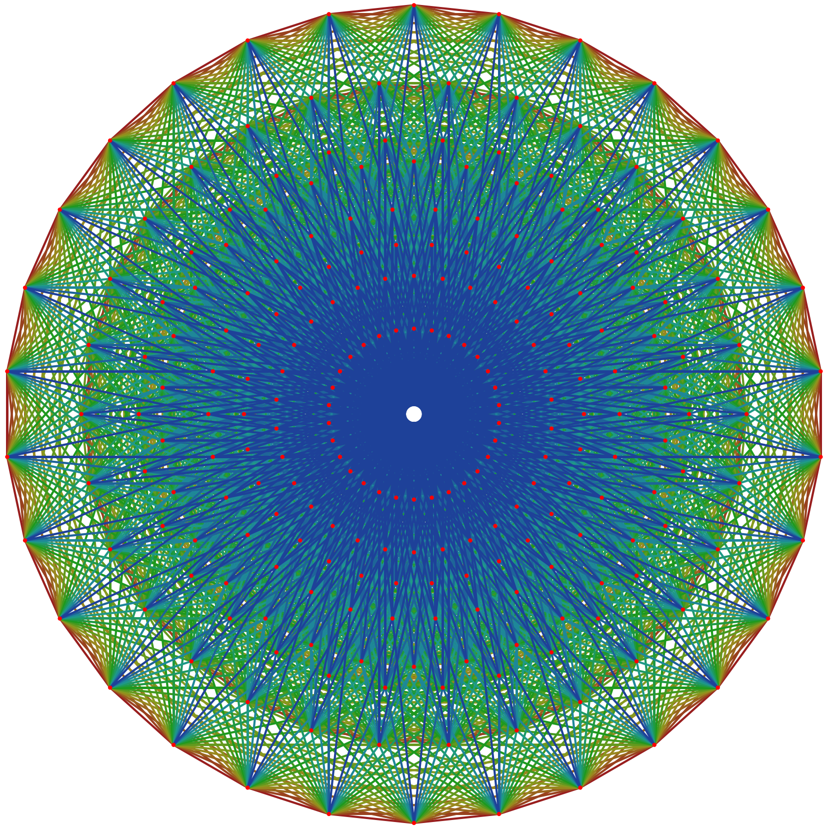
Lattices like D_n, E_8, appear all over mathematics

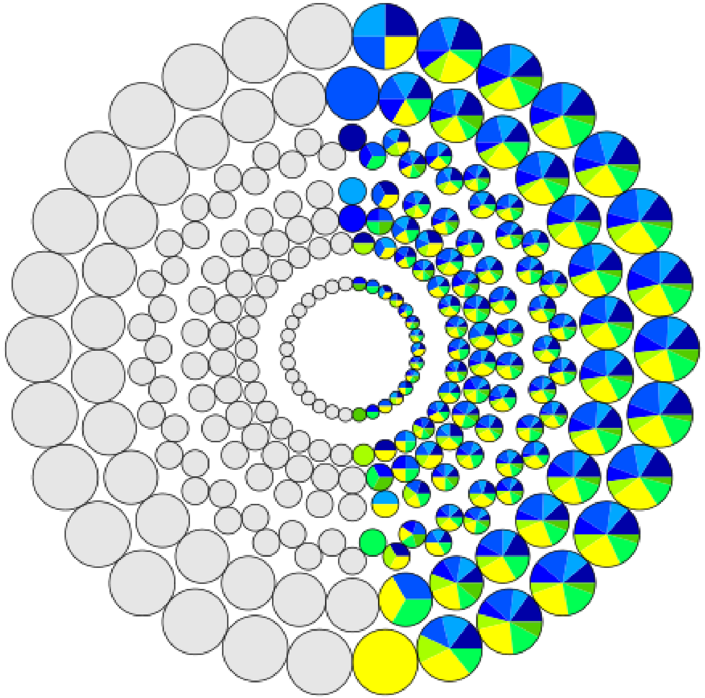
- Classification of finite groups, Lie algebras ($A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$)
- Differential topology, number theory, modular forms, cluster algebras, superconformal gauge quiver theories…

E_8 has many interesting / amazing properties:

- Fundamental cell has volume 1 (*unimodular*)
- The dot product $x \cdot y$ of any $x, y \in E_8$ is an integer (*integral*)
- Any $x = (x_1, \dots, x_8) \in E_8$ has $x \cdot x = |x|^2$ even (*even lattice*), and all even numbers appear this way
- The number of $x \in E_8$ such that $x \cdot x = 2n$ is $240 \sum_{d|n} d^3$ (?!!!!!!!)
- There are 240 shortest nonzero vectors x in E_8, i.e. with $x \cdot x = 2$, forming a semiregular polytope
- When projected onto a certain *Coxeter* plane, it has 30-fold symmetry

The E_8 lattice





Sources: Wikimedia, Claudio Rocchini; Okounkov, *The magic of 8 and 24*

A sharp upper bound?

In 2003, Henry Cohn and Noam Elkies developed a technique for *upper bounds* on sphere packing density, using *harmonic analysis / Fourier transforms.*

$f : \mathbb{R}^n \to \mathbb{R}$ has Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$. If the functions are sufficiently nice then

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\, x \cdot y}\, dy,$$

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y)\, e^{2\pi i\, x \cdot y}\, dy$$

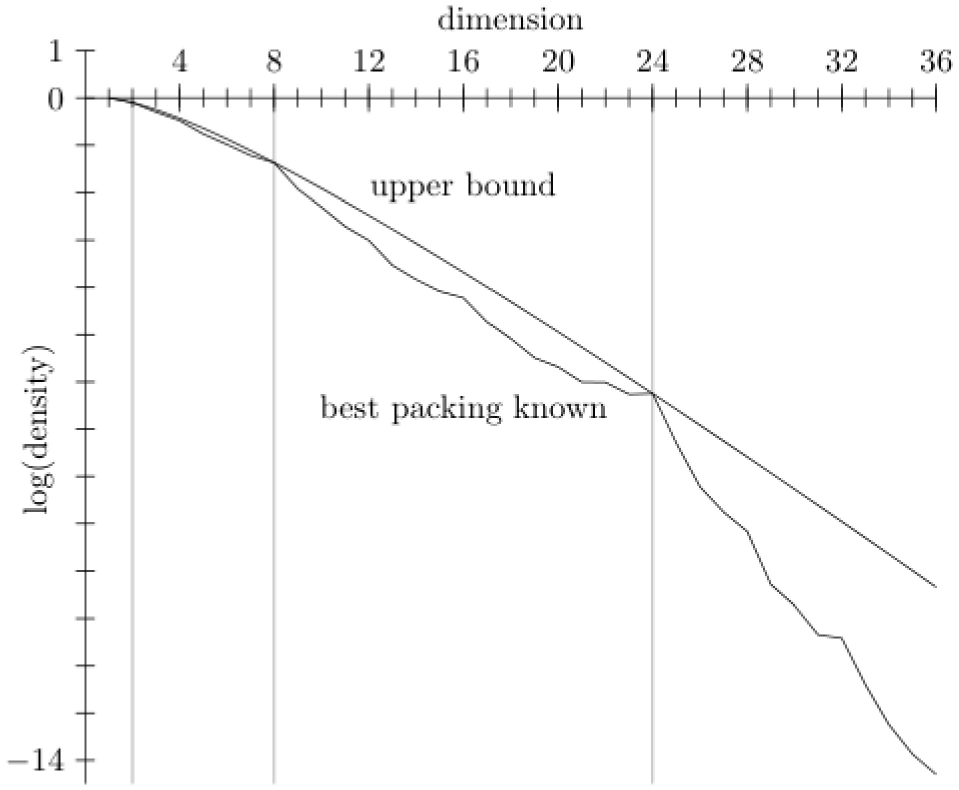
Theorem: If f is sufficiently nice and

1. $f(0) = \hat{f}(0) > 0$
2. $f(x) \le 0$ for $|x| \ge r$
3. $\hat{f}(y) \ge 0$ for all y

Then the sphere packing density in $\mathbb{R}^n$ is at most $\mathrm{vol}(B^n_{r/2})$.

Sources: Wikimedia, Renate Schmid; H Cohn

Search for the magic function



Cohn-Elkies conditions can be investigated numerically!

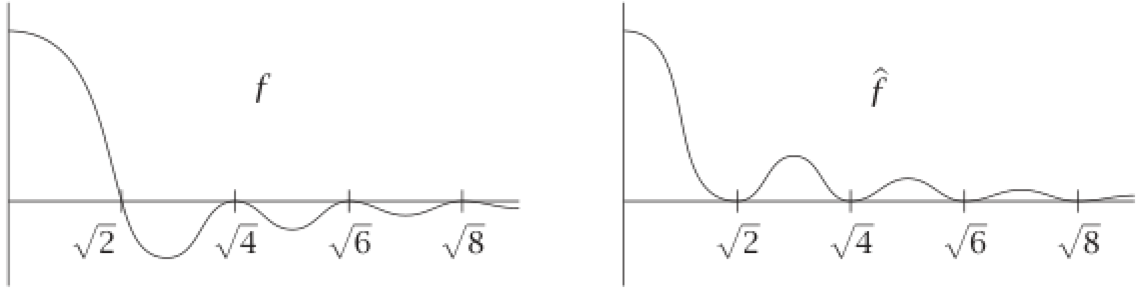
Can find f that matches E_8 packing to 60 decimal places!

Hales: "I felt that it would take a Ramanujan to find it"

Heisenberg uncertainty principle:
It's hard to control both f and $\hat{f}$!

What is the *magic function* in 8 dimensions?

In 2016, Viazovska found it.



Sources: Cohn, *Packing, coding and ground states*; Cohn, *A conceptual breakthrough in sphere packing*

Viazovska's magic function

Theorem (Viazovska 2016). The magic function in $\mathbb{R}^8$ is given by

$$f(x) = \sin^2\left(\frac{\pi\,|x|^2}{2}\right) \int_0^\infty \left(t^2\varphi\left(\frac{i}{t}\right) + \psi(it)\right) e^{-\pi\,|x|^2 t}\, dt$$

where $\varphi = \frac{4\pi\left(E_2E_4 - E_6^2\right)^2}{5\left(E_6^2 - E_4^3\right)}$ and $\psi = -\frac{32\,\Theta_{\mathbb{Z}}^4|_T\,\left(5\Theta_{\mathbb{Z}}^8 - 5\,\Theta_{\mathbb{Z}}^4|_T\,\Theta_{\mathbb{Z}}^4 + 2\Theta_{\mathbb{Z}}^8|_T\right)}{15\pi\,\Theta_{\mathbb{Z}}^8\,\left(\Theta_{\mathbb{Z}}^4 - \Theta_{\mathbb{Z}}^4|_T\right)^2}$

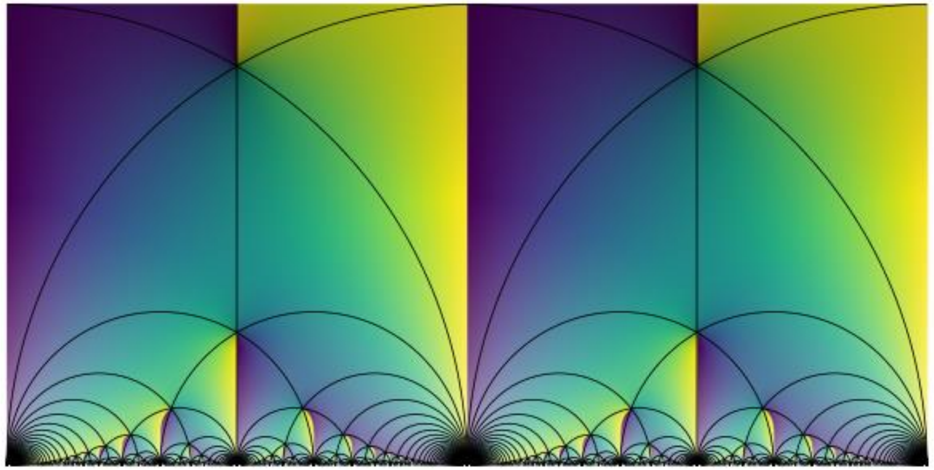
where $E_k = \frac{1}{2\zeta(n)} \sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^k}$

where $\zeta(n)$ is the Riemann zeta function

where $\Theta_{\mathbb{Z}}(z) = \sum_{x\in\mathbb{Z}} e^{\pi i\,|x|^2 z}$

and $f|_T(z) = f(z+1)$.

Therefore E_8 is the optimal sphere packing in $\mathbb{R}^8$!



Source: Cohn, *The work of Maryna Viazovska*

Viazovska's magic function

Viazovska: "It took several steps to reach the solution. The first one, and it gave me the confidence that I will solve it, was when I managed to reduce the problem to a functional equation. I was coming home from a conference in Bonn…

On the train, I thought, since nothing seems to work, let me write the problem out one more time. In school, they taught us that your head is full of rubbish until you write things down to put them in order. So, I am writing it out, and I get this functional equation. I look at it and I think: "I should be able to solve it". And, indeed, I solved it, it only took a couple of months.

…in the summer, [I] wrote long, long formulas on pieces of paper in the evening… Obviously, I made every possible mistake in these notes. But the final time I wrote it, I didn't make any mistakes, and there was the solution."

Peter Sarnak: "It's stunningly simple, as all great things are. You just start reading the paper and you know this is correct."

Akshay Venkatesh: "Wow! How on earth did they find this function?"

Andrei Okounkov: "Viazovska's solution is truly striking. She gives an extremely nontrivial explicit formula for the magic functions in terms of modular forms… I would guess that seeing the solution would have made Ramanujan extremely, extremely happy."

Why pack spheres?

Natural problem!

Has interesting solutions!

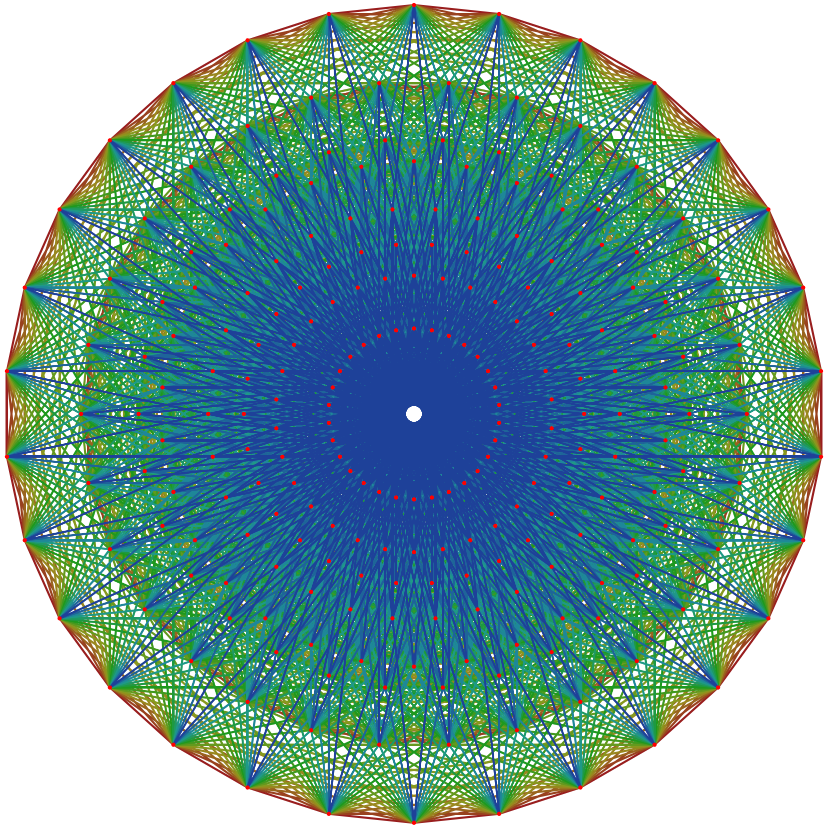
Models for crystalline / granular materials

Most practical application: communication over noisy channels – error correction.

- $\mathbb{R}^n =$ space of possible *signals* / *communications*
- Signal sent may not be the same as signal received! May be perturbed by noise, errors, etc.
- Build a vocabulary of signals $S \subset \mathbb{R}^n$, an *error-correcting code*. Only send signals from S.
- If all points of S are $> 2r$ apart, then even if a signal is perturbed by noise up to distance r, we can tell the original signal.
- Centres of spheres in sphere packings provide error correcting codes!
- Denser sphere packing means more possible signals to send, more accuracy!

E.g. Used in Voyager probes

Thanks for listening!



References:

- Henry Cohn, *A conceptual Breakthrough in Sphere Packing*, Notices of the AMS Feb 2017
- Henry Cohn, *The work of Maryna Viazovska*, Proceedings of the ICM 2022
- Henry Cohn, *The work of Maryna Viazovska*, [youtube] 26/10/20
- John H. Conway and Neil J. A. Sloane, *Sphere Packings, Lattices and Groups* (2nd ed) 1993
- Andrei Okounkov, *The magic of 8 and 24,* Proceedings of the ICM 2022
- Maryna Viazovska, *The sphere packing problem in dimension 8*, Annals of Mathematics 2017