Symplectic structures in hyperbolic 3-manifold triangulations

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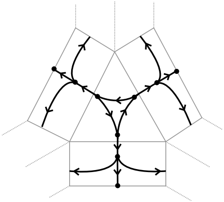
Oklahoma State Topology seminar
8/9 November 2022



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Paper on arxiv:

- A symplectic basis for 3-manifold triangulations
  2208.06969 (joint w Purcell)

Also:

- A-polynomials, Ptolemy equations and Dehn filling
  2002.10356 (joint w Howie, Purcell)

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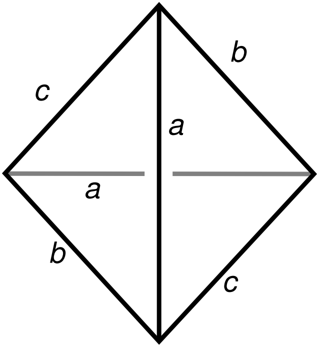

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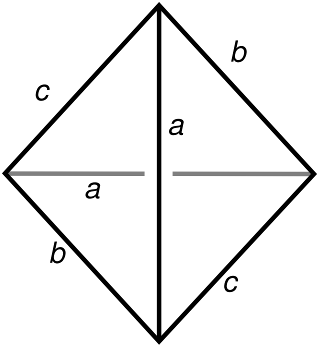
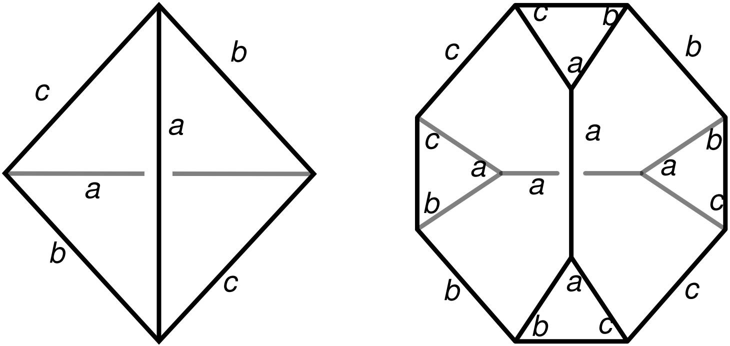
(Results apply more generally, but for convenience...)



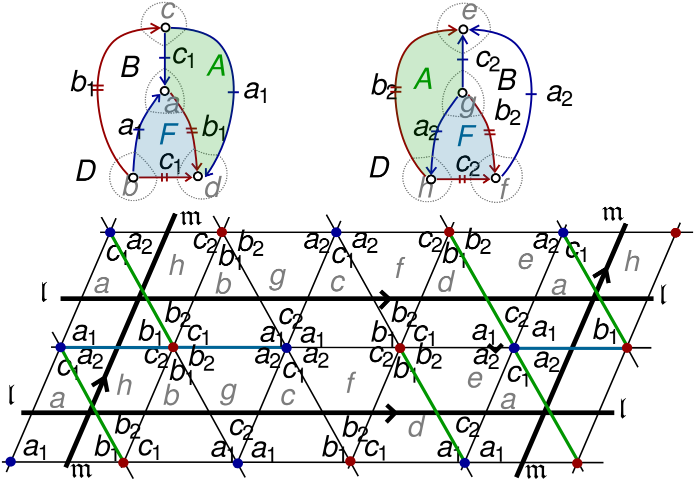
Then:

- Truncating $\Delta_j \rightsquigarrow$ polyhedra decomposing $\overline{M} = S^3 \setminus N(L)$.
- Triangular faces of polyhedra triangulate boundary tori T_i.
- Each vertex of each triangle has an a, b or c label.



Example: figure-8 knot complement

When K = figure-8 knot, M decomposes into two ideal tetrahedra, with cusp triangulation shown.



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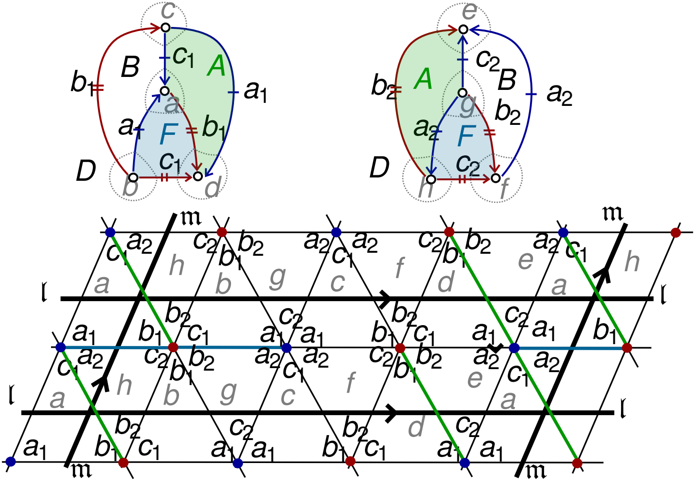

Neumann–Zagier associated to $\mathcal{T}$ a vector space V.

The Neumann–Zagier vector space

Neumann–Zagier associated to $\mathcal{T}$ a vector space V.

Definition (Neumann–Zagier 1985)

Let V be the $2N$-dimensional $\mathbb{R}$-vector space generated by

$$a_1, b_1, c_1, \ldots, a_N, b_N, c_N$$

subject to relations

$$a_i + b_i + c_i = 0, \quad i = 1, \ldots, N.$$

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It follows that

$$\omega(a_i, b_i) = \omega(b_i, c_i) = \omega(c_i, a_i) = 1$$
$$\omega(b_i, a_i) = \omega(c_i, b_i) = \omega(a_i, c_i) = -1$$
$$\omega = 0 \quad \text{on all other pairs of generators.}$$

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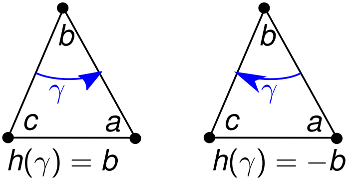

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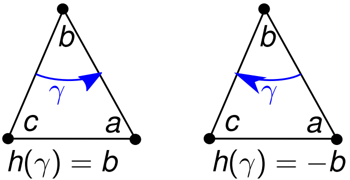
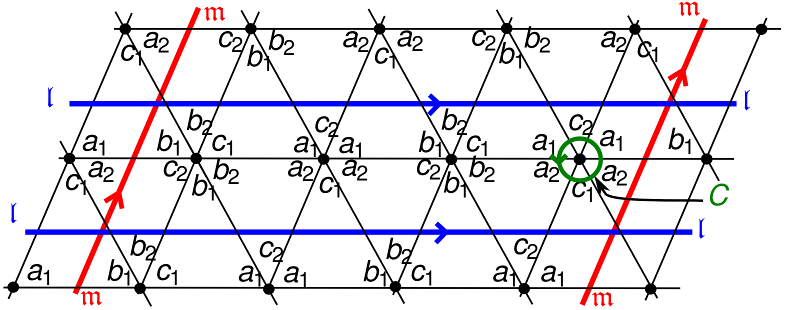


Definition

The combinatorial holonomy $h(\gamma) \in V$ is the sum of contributions $\pm a_j, b_j, c_j$ for each arc of γ.

Combinatorial holonomy example: figure-8 knot



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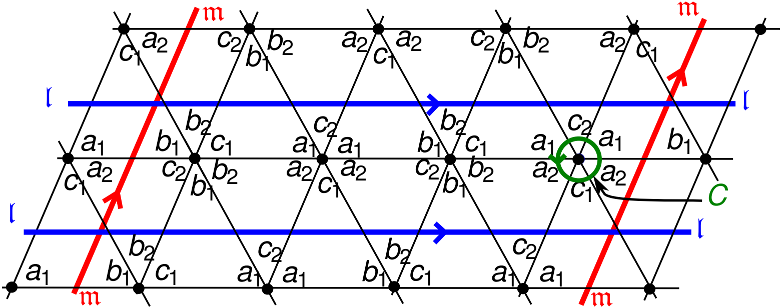

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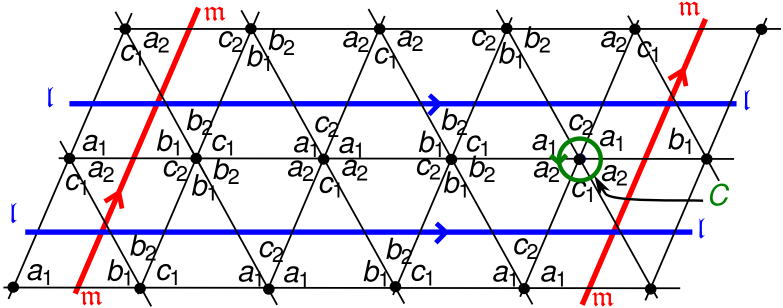

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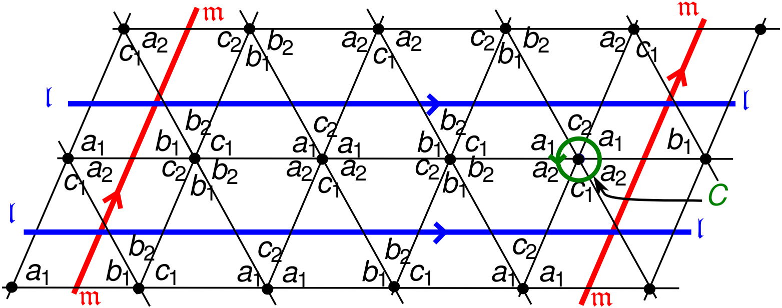

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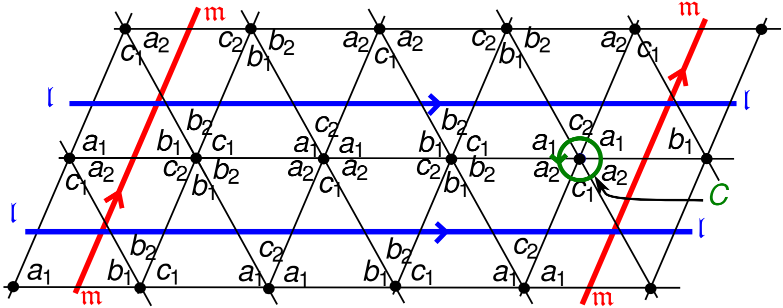

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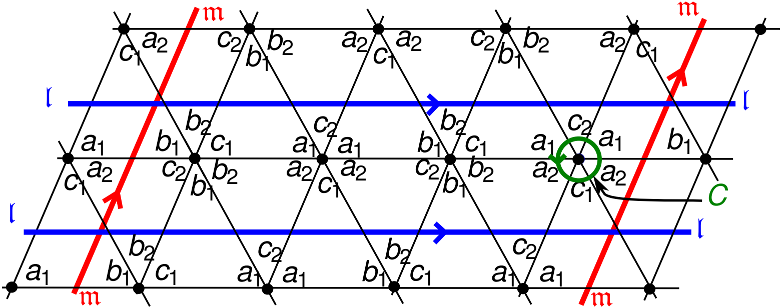

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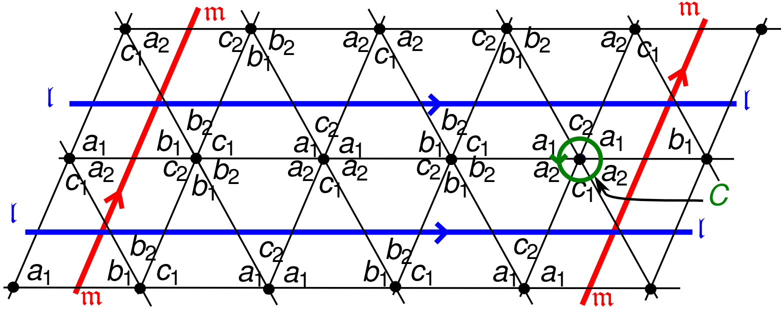

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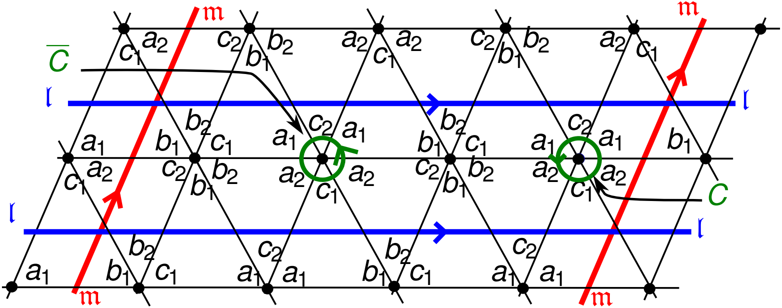

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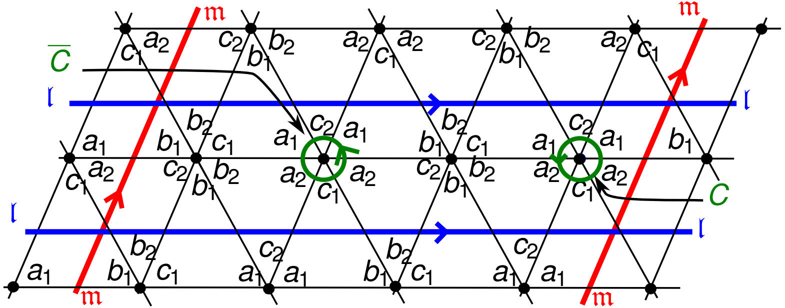

$$h(\overline{C}) = h(C)$$

The NZ symplectic form as a 2D intersection form

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Let γ, δ be oriented curves on $\partial\overline{M}$. Then

$$\omega\left(h(\gamma), h(\delta)\right) = 2\gamma \cdot \delta,$$

where $\cdot$ denotes algebraic intersection number.

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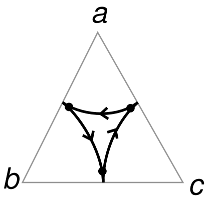
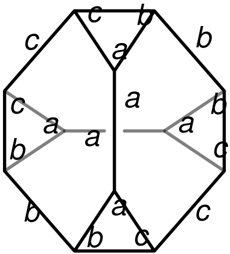
Note ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!





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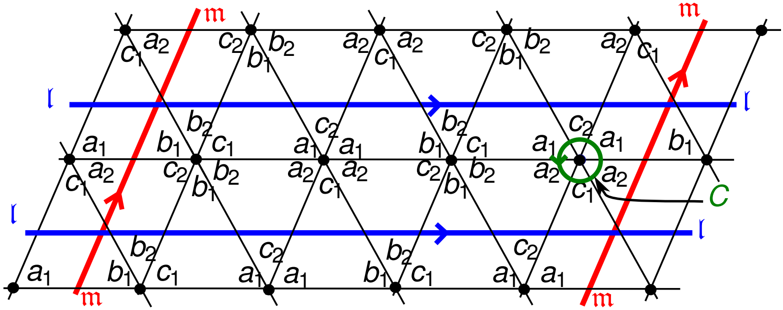

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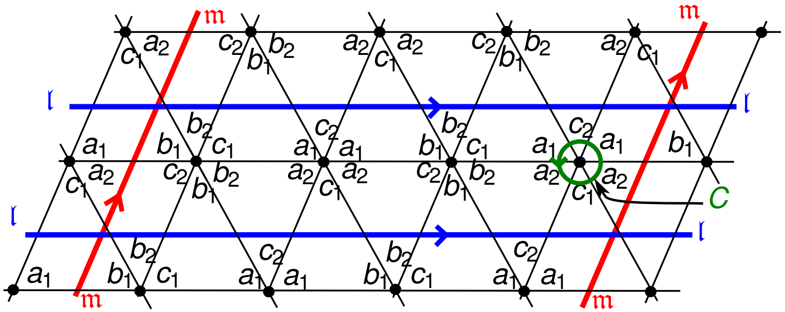

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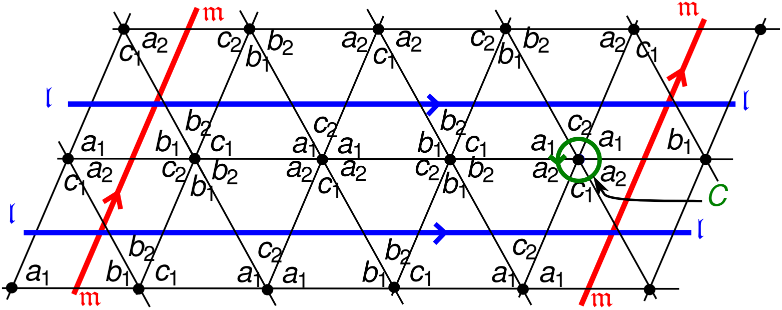

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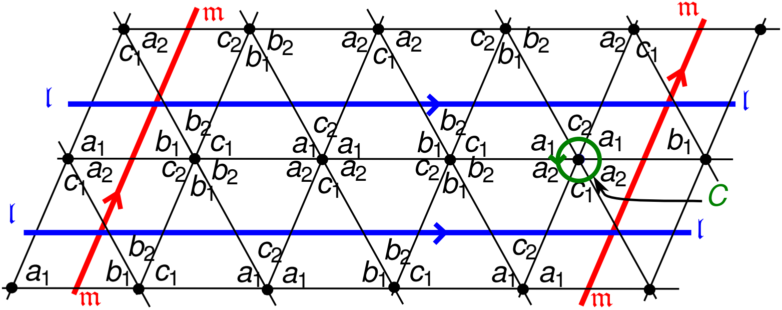

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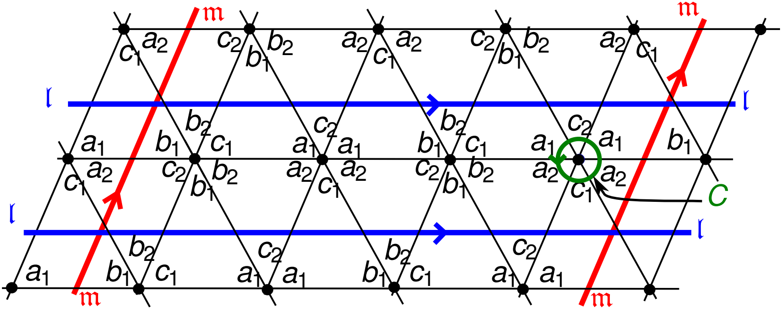

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Now suppose M has $\mathfrak{c}$ cusps and consider particular curves:

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Observation (Dimofte 2013)

We can form a partial symplectic basis* of V, consisting of

the $h(\mathfrak{m}_j), h(\mathfrak{l}_j)$, for $j = 1, \ldots, \mathfrak{c}$

and the $h(C_j)$ (some of them) ($N - \mathfrak{c}$ of them)

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A partial symplectic basis

Now suppose M has $\mathfrak{c}$ cusps and consider particular curves:

- $C_1, \ldots, C_N$ around edges $E_1, \ldots, E_N$
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* Up to factors of 2...

Questions

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Can we find more curves to obtain a full symplectic basis* of V?

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Yes!

The NZ symplectic form as a 3D intersection form

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Let ζ, ζ' be oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

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I.e. $\omega(\Gamma_j, C_k) = 2\delta_{jk}, \quad \omega(\Gamma_j, \mathfrak{m}_k) = \omega(\Gamma_j, \mathfrak{l}_k) = \omega(\Gamma_j, \Gamma_k) = 0.$

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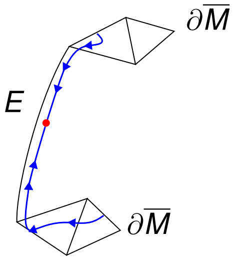
- May dive from $\partial\overline{M}$ into the interior of $\overline{M}$, along an edge E, and come out on $\partial\overline{M}$ at the other end of E.



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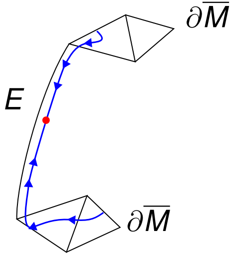

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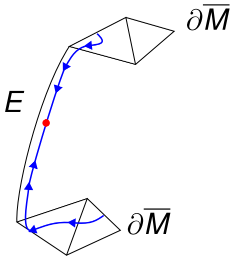

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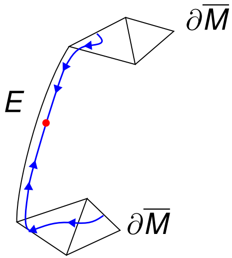

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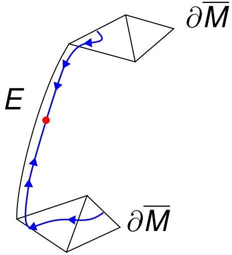

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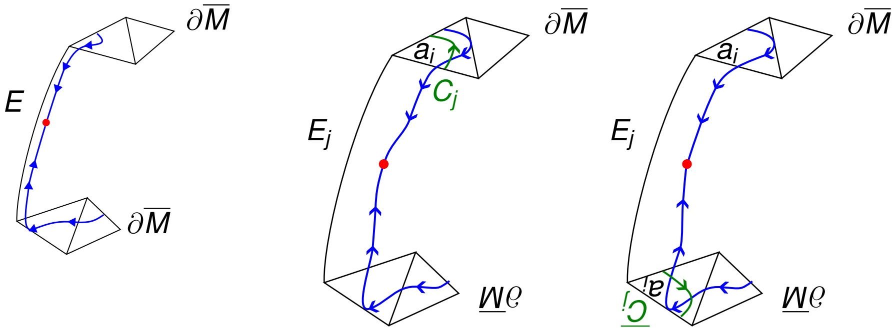

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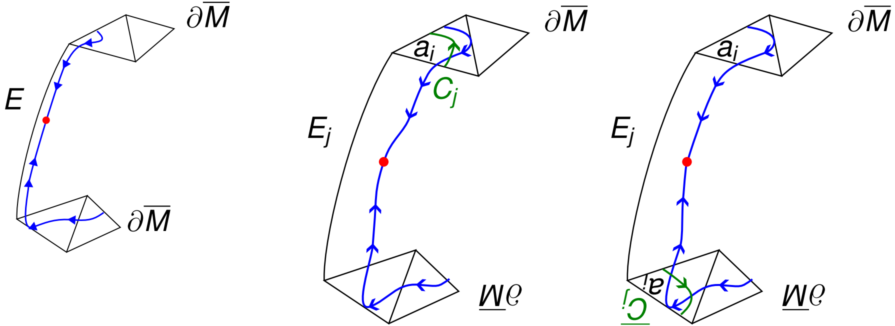
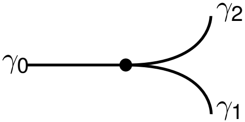


Definition

A train track is a smoothly embedded graph on a surface such that at each vertex, incident edges are all tangent, with at least one edge on each side.

- Edges are called branches; vertices called switches.



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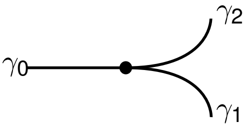

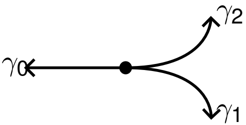

Branches, if oriented, have intersection numbers, defined locally at switches:

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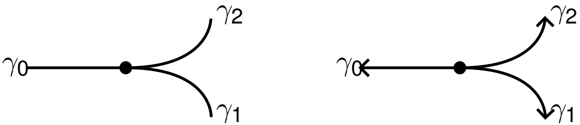

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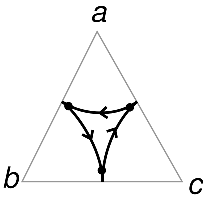
Smooth oriented curves on train tracks then obtain intersection numbers agreeing with usual algebraic intersection number.

Train tracks for oscillating curves: motivation

A generic curve on a boundary torus intersects triangles in arcs, so can be made to run on train tracks.



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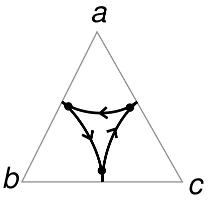

Similarly, oscillating curves can run on train tracks. But...

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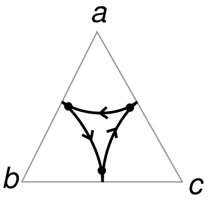

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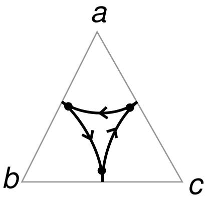

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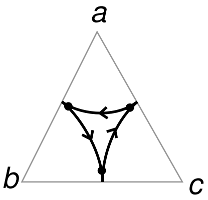

Similarly, oscillating curves can run on train tracks. But...

- To dive into manifold, more tracks & switches required!
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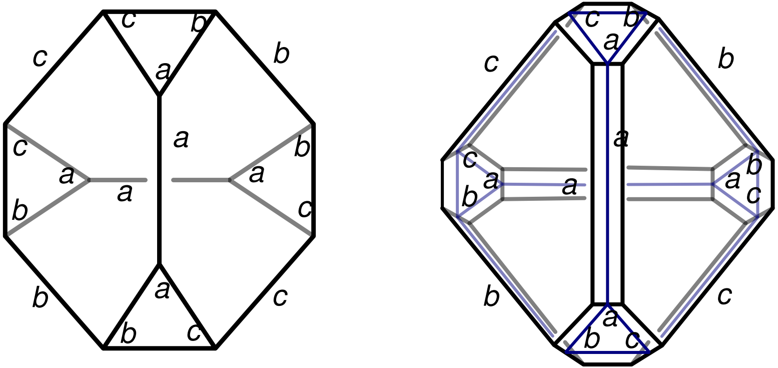
- Special “stations" for each orientation reversal.

Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra



Truncated tetrahedra

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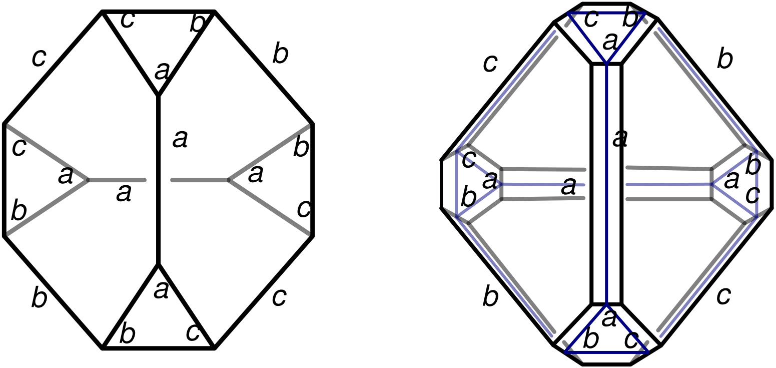
- hexagonal faces glued in pairs



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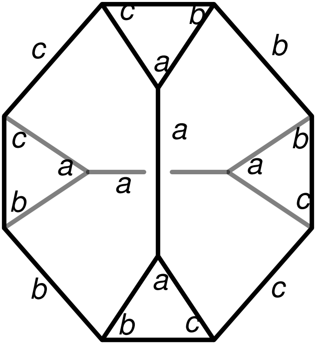

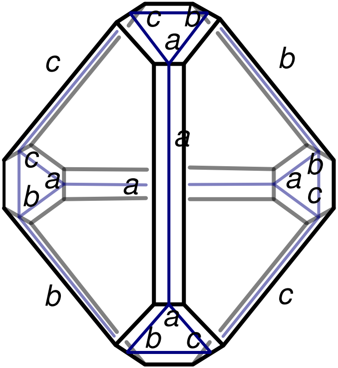

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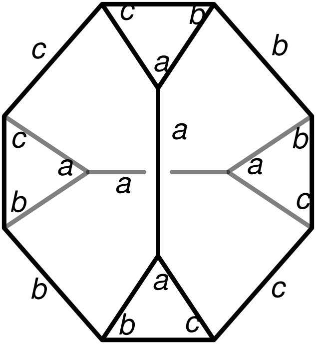
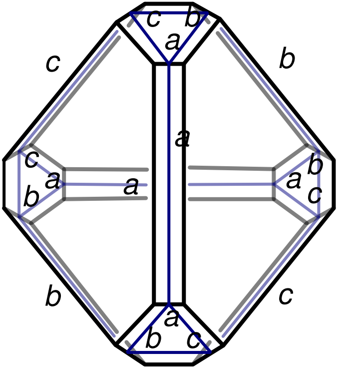
Removing neighbourhood of each edge, tetrahedra are further truncated to polyhedra with:

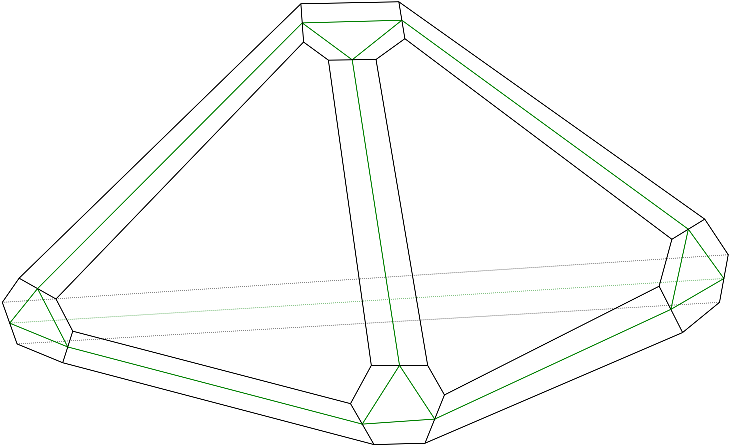
- hexagonal faces glued in pairs
- hexagons on boundary tori (split further into triangle + 3 rectangles)
- rectangles along removed edges (split further into 2 rectangles)





Triangles + rectangles give a decomposition of a Heegaard surface for M.

- $M =$ Handlebody $\cup$ Compresson body
- with handlebody decomposed into truncated tetrahdra



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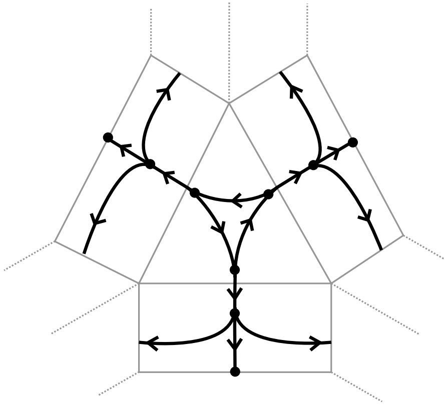

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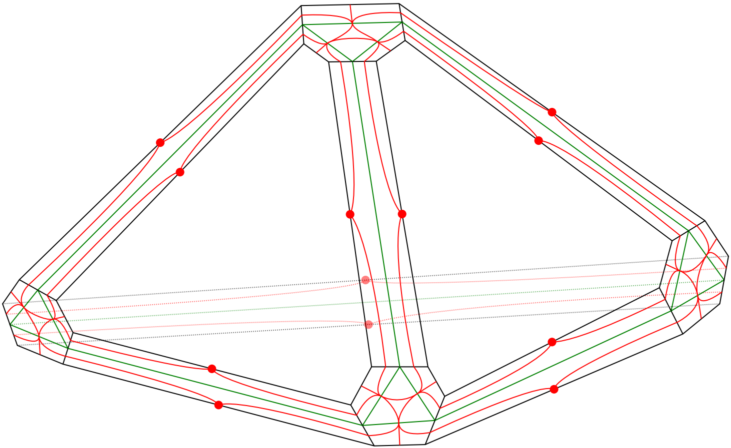
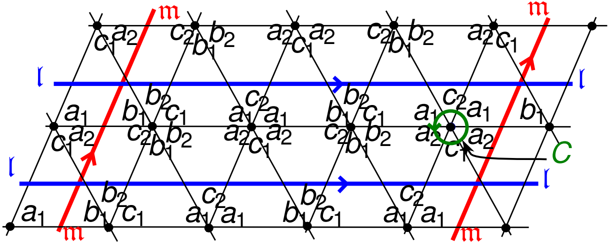


Red dots: stations for reversing direction.

⇝ "Enhanced" train tracks.

Train tracks for oscillating curves



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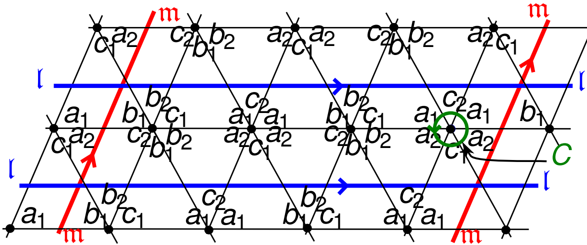

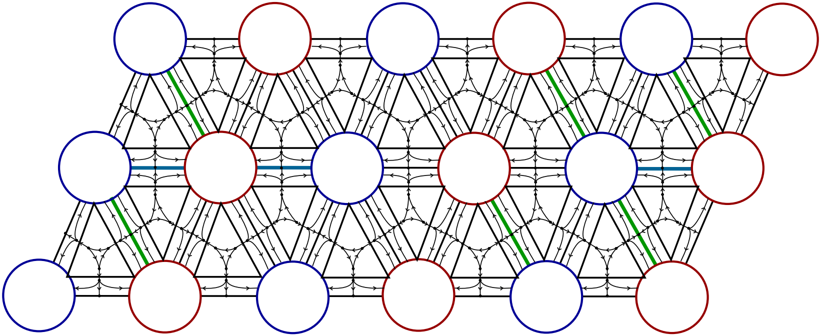

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- at each switch *v*, $\sum_\gamma \epsilon_\gamma n_\gamma = 0$ (i.e. $\#in = \#out$)
- at each station, $\epsilon_\gamma n_\gamma + \epsilon_{\widehat{\gamma}} n_{\widehat{\gamma}} = \epsilon_\delta n_\delta + \epsilon_{\widehat{\delta}} n_{\widehat{\delta}}$.

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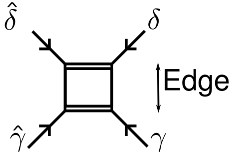

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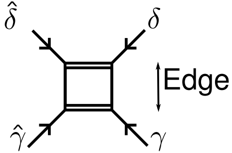
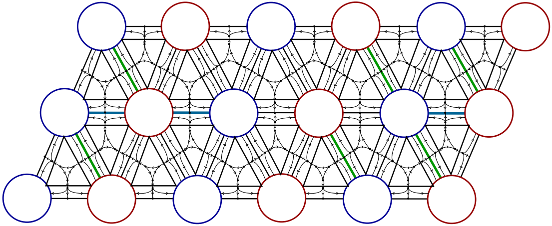
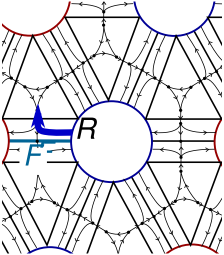
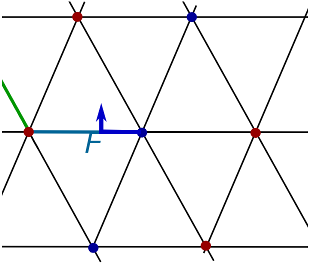


Example: figure-8 knot complement



To draw oscillating curves...





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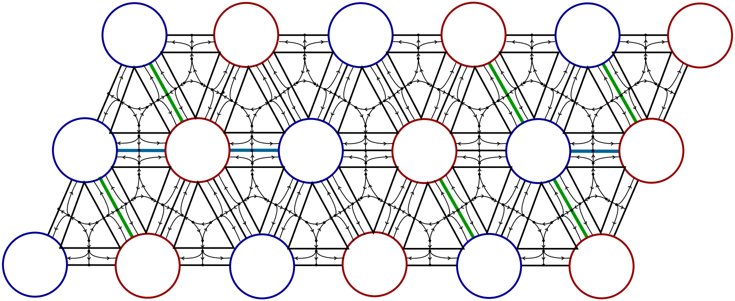

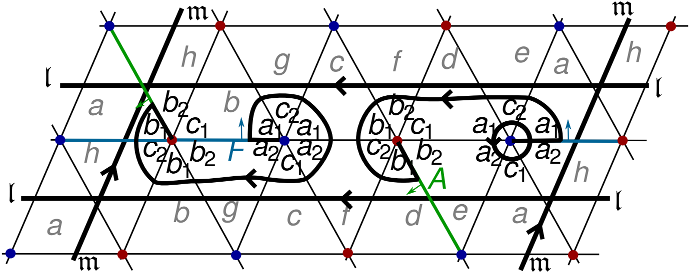

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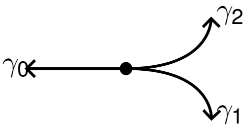

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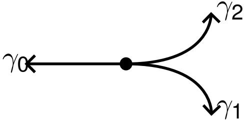
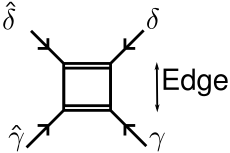
- at stations (unconventional!)

$$\gamma \cdot \delta = -1, \delta \cdot \gamma = 1,$$

$$\widehat{\delta} \cdot \widehat{\gamma} = -1, \widehat{\gamma} \cdot \widehat{\delta} = 1.$$





Intersection number of oscillating curves is defined locally:

- At switches (conventional!)

$$\gamma_1 \cdot \gamma_2 = 1, \gamma_2 \cdot \gamma_1 = -1,$$

$$\gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma_0 = \gamma_0 \cdot \gamma_2 = \gamma_2 \cdot \gamma_0 = 0,$$

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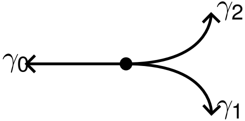
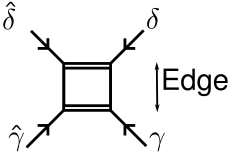
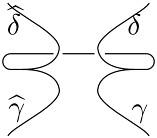


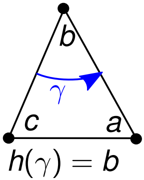


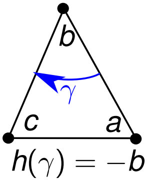


Combinatorial holonomy of oscillating curves

Definition

The <u>*combinatorial holonomy*</u> $h(\zeta) \in V$ *of an (abstract) oscillating curve* ζ *is the sum of contributions* $\pm a_j, b_j, c_j$ *for each arc of* ζ *in a triangle.*





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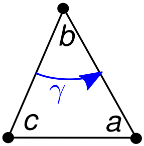
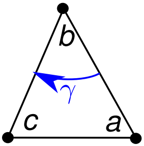
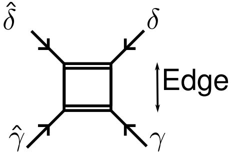
(And nothing from arcs in rectangles / diving into the manifold / passing through stations!)



$h(\gamma) = b$



$h(\gamma) = -b$



$h(\gamma) = h(\widehat{\gamma}) = h(\delta) = h(\widehat{\delta}) = 0$

The NZ symplectic form as a 3D intersection form

Theorem 1 (M–Purcell)

Let ζ, ζ' be (abstract) oscillating curves on $\overline{M}$, with combinatorial holonomies $h(\zeta), h(\zeta')$ respectively. Then

$$\omega\left(h(\zeta), h(\zeta')\right) = 2\zeta \cdot \zeta'$$

where $\cdot$ is the intersection form for oscillating curves.

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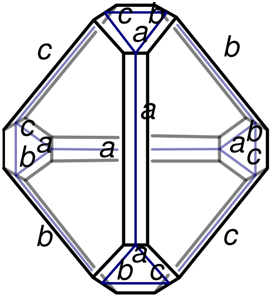

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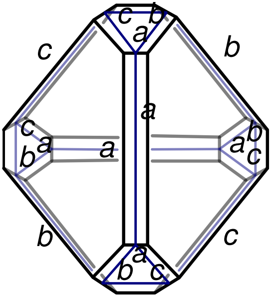
- ω counts intersections in triangles... but also fake intersections in distinct triangles of a polyhedron!



Theorem 2 (M–Purcell)

We can construct oscillating curves $\Gamma_1, \ldots, \Gamma_{N-\mathfrak{c}}$ such that

$$h(\mathfrak{m}_j), h(\mathfrak{l}_j) \quad \text{for } j = 1, \ldots, \mathfrak{c}, \text{ and}$$
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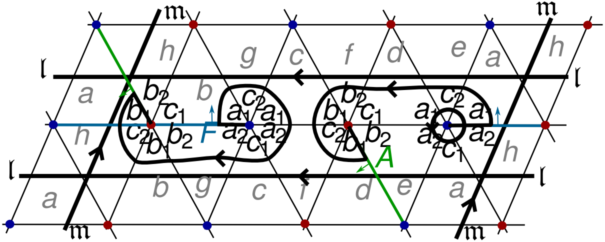
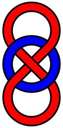
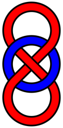




The Whitehead link complement has a decomposition into 5 ideal tetrahedra.



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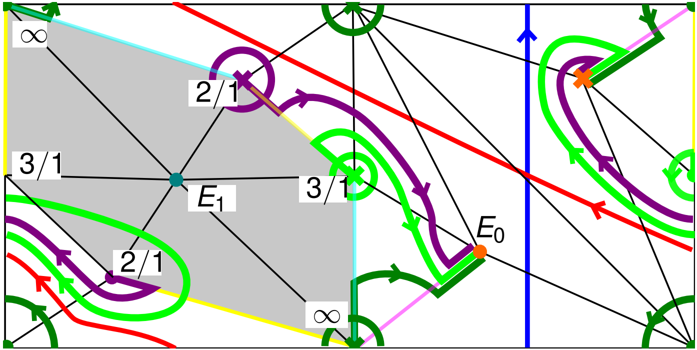
One cusp is triangulated as shown.



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- Longitude and meridian have holonomy L, M which can be expressed as products of z_i, z_i' variables.

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- L, M satisfy a polynomial relation, the A-polynomial $A(L, M) = 0$.

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- γ_E (for edges E), L, M (longitude/meridian holonomy).

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- Resulting equations are Ptolemy equations, one for each tetrahedron:

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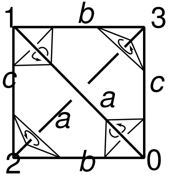

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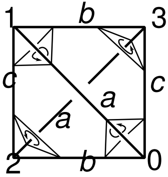

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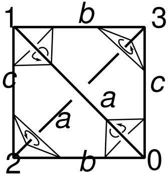

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M–Purcell (in progress):

- The γ variables have an interpretation as complex lambda lengths in spin hyperbolic geometry.

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It can be lifted to an oriented curve $\widetilde{\gamma}$ on a double cover $\widetilde{\Sigma}$.

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- Agrees with work of Dimofte–van der Veen: $H_1^-(\widetilde{\Sigma}) \cong V$.

More Applications

Garoufalidis–Le, Dimofte, Gukov...

- Quantising the A-polynomial should produce a non-commutative polynomial annihilating coloured Jones polynomials (AJ conjecture).

Also:

- Space of hyperbolic structures (Neumann-Zagier, Choi)
- Hyperbolic volumes of Dehn fillings (Neumann-Zagier)
- Normal surfaces (Luo, Garoufalidis–Hodgson–Hoffman-Rubinsten)
- Representation theory (Goerner, Zickert, Garoufalidis, ...)
- Chern-Simons theory (Neumann, Dimofte, Garoufalidis, Gukov, ...)

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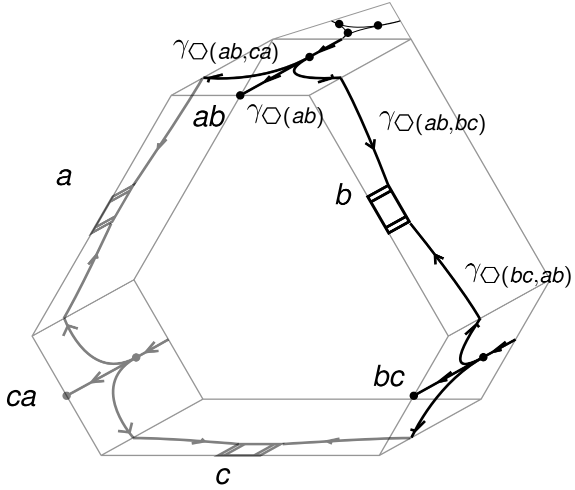

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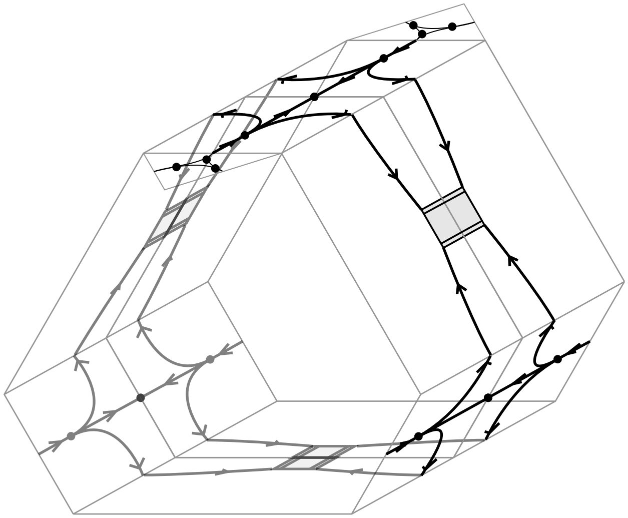

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$$
\begin{aligned}
2\zeta \cdot \zeta' = & \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_k (n'_{kl} - n'_{km} + \widehat{n}'_{kl} - \widehat{n}'_{km}) \\
& \qquad\qquad + n'_k (-n_{kl} + n_{km} - \widehat{n}_{kl} + \widehat{n}_{km}) \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} -n_{kl} n'_{km} + n_{km} n'_{kl} - \widehat{n}_{km} \widehat{n}'_{kl} + \widehat{n}_{kl} \widehat{n}'_{km} \\
& + \sum_{\hexagon, \widehat{\hexagon}} \sum_{(k,l,m)} n_{kl} n'_{lk} - n_{lk} n'_{kl} - \widehat{n}_{kl} \widehat{n}'_{lk} + \widehat{n}_{lk} \widehat{n}'_{kl}.
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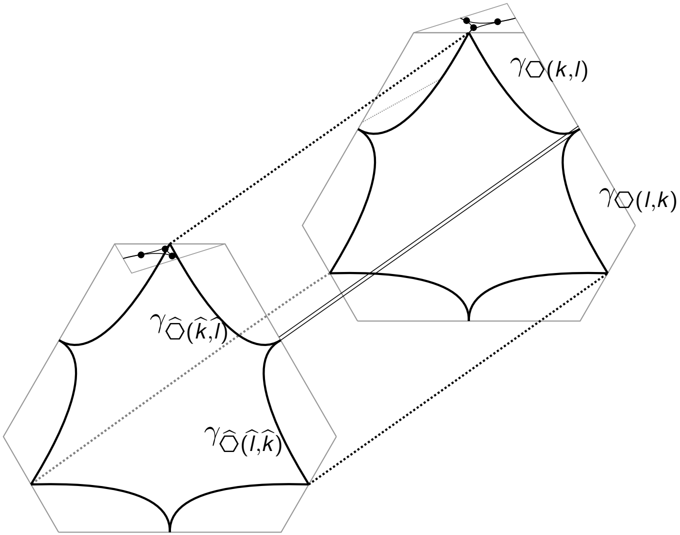
\omega\left(h(\zeta), h(\zeta')\right) = & \sum_{\Delta} n_{\Delta(ab)} n'_{\Delta(bc)} - n_{\Delta(bc)} n'_{\Delta(ab)} \\
& + \sum_{\hexagon} \sum_{(k,l,m)} n_k (n'_{lk} - n'_{mk}) - n'_k (n_{lk} - n_{mk}) \\
& + \sum_{\hexagon} \sum_{(k,l,m)} (n_{kl} + n_{km})(n'_{lk} + n'_{lm}) \\
& \qquad - (n'_{kl} + n'_{km})(n_{lk} + n_{lm}).
\end{aligned}
$$

Show these are equal!

Why the weird intersection form for oscillating curves?



Thanks for listening!