The geometry of spinors in Minkowski space

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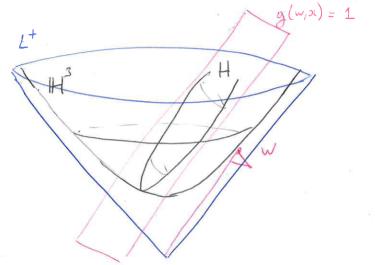

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Overview

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Paper on arXiv soon. Email me if you want an advance copy!

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Penrose–Rindler

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Cast of characters:

- Spinor or spin vectors: elements of $\mathbb{C}^2$.
- Hermitian matrices: $A = A^*$.
- Minkowski space $\mathbb{R}^{3,1}$: coordinates (T, X, Y, Z), metric $dT^2 - dX^2 - dY^2 - dZ^2$.

How to do this?

$$\begin{array}{ccccc} \text{Spinors} & \xrightarrow{\phi_1} & \begin{array}{c}2 \times 2 \text{ Hermitian} \\ \text{matrices}\end{array} & \xrightarrow{\phi_2} & \text{Minkowski space} \\ \mathbb{C}^2 & & \mathcal{H} & & \mathbb{R}^{3,1} \end{array}$$

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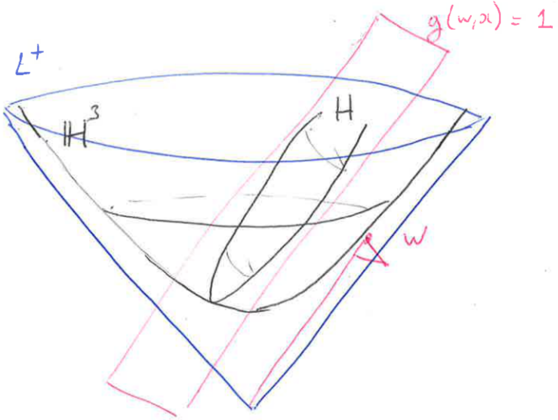
- Linear isomorphism: "$\mathcal{H}$ is $\mathbb{R}^{3,1}$", "det = norm".
- Image ϕ $(= \phi_2 \circ \phi_1)$ = pos. light cone L^+ $(T^2 - X^2 - Y^2 - Z^2 = 0,\ T \geq 0)$

We understand some of the picture now



From $\kappa \in \mathbb{C}^2$, get a point $\phi(\kappa) = w$ on L^+.

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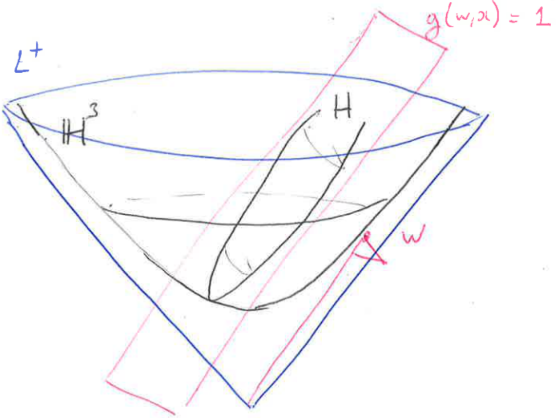
Definition (Penrose–Rindler)

A pointed null flag is a point $p \in L^+$ together with a 2-plane V tangent to L^+ containing $\mathbb{R}p$.

We understand half the picture now



From $\kappa \in \mathbb{C}^2$, get a point on L^+ and a pointed null flag there.

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Spinoriality:

- Take $\kappa \in \mathbb{C}^2$ and consider rotating it: $e^{i\theta}\kappa$.
- $\phi(e^{i\theta}\kappa)$ is constant but $\Phi(e^{i\theta}\kappa)$ is not: plane V rotates.
- As κ rotates by θ, V rotates by 2θ.

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Robert Penner

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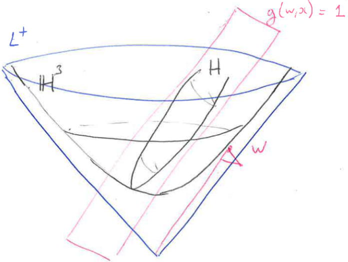

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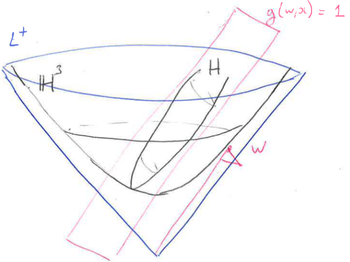

$\kappa \in \mathbb{C}^2 \xrightarrow{\text{Penrose–Rindler}}$ Pointed null flags $(= p \in L^+ \text{ and flag})$ $\xrightarrow{\text{Penner}}$ Horospheres with $\cdots$

New developments

Theorem (M.)

There is a natural ($SL(2,\mathbb{C})$-equivariant) bijection
$\mathbb{C}^2 \setminus \{0\} \longrightarrow \{$*Horospheres in* $\mathbb{H}^3$ *with spin directions*$\}$.

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$$\{\kappa, \omega\} = e^{\frac{d+i\theta}{2}}$$

Consequences:

- Ptolemy theorem for hyperbolic ideal tetrahedra
- New methods to compute hyperbolic structures on 3-manifolds (joint with J. Purcell)
- Equivalence of cluster algebras (Grassmannians / hyp. surfaces)
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Thanks for listening!

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