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A–infinity algebras, strand algebras, and contact categories

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In previous work we showed that the contact category algebra of a quadrangulated surface is isomorphic to the homology of a strand algebra from bordered sutured Floer theory. Being isomorphic to the homology of a differential graded algebra, this contact category algebra has an A–infinity structure, allowing us to combine contact structures not just by gluing, but also by higher-order operations.

We investigate such A–infinity structures and higher-order operations on contact structures. We give explicit constructions of such A–infinity structures, and establish some of their properties, including conditions for the vanishing and nonvanishing of A–infinity operations. Along the way we develop several related notions, including a detailed consideration of tensor products of strand diagrams.

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1 Introduction

1.1 Overview

In previous work [22] we demonstrated an isomorphism of two unital \mathbb{Z}_2 –algebras, the first arising from contact geometry, the second from bordered Floer theory:

$$(1) \quad CA(\Sigma, Q) \cong H(\mathcal{A}(\mathcal{Z})).$$

Here (Σ, Q) is a *quadrangulated surface*, a useful object in TQFT-type structures in contact geometry (see Mathews [19; 20]), and \mathcal{Z} is an *arc diagram*, an equivalent object used in bordered sutured Floer theory (see Zarev [30]). The left-hand side $CA(\Sigma, Q)$ is the algebra of a *contact category*, with objects and morphisms given by certain contact structures on $\Sigma \times [0, 1]$. The right-hand side $H(\mathcal{A}(\mathcal{Z}))$ is the homology of the strand algebra $\mathcal{A}(\mathcal{Z})$, a differential graded algebra (DGA) generated by *strand diagrams* on \mathcal{Z} , which encode Reeb chords arising as asymptotics of certain holomorphic curves. The isomorphism (1) therefore allows us to interpret (homology classes of) strand diagrams as contact structures.

Of particular interest, (1) expresses the contact category algebra as the homology of a DGA. The homology of a DGA is known to have the structure of an A_∞ algebra. This A_∞ structure provides a sequence of higher-order operations X_n on the homology, extending from multiplication X_2 and satisfying relations which provide a homotopy-theoretic form of associativity; see Stasheff [26; 27].

While A_∞ structures are well known to arise in Floer theory (see eg [25]), it is perhaps surprising that an A_∞ structure should arise directly out of contact structures. The A_∞ operations allow us to combine contact structures not just by gluing, but also by higher-order operations. A natural question arises: what are the higher A_∞ operations on contact structures, and what do they mean geometrically?

This paper essentially consists of an investigation of A_∞ structures on this contact category algebra. This investigation is carried out through the use of strand diagrams, which are more general objects, and easier to work with algebraically than contact structures. Therefore, more accurately, this paper consists of an investigation of A_∞ structures on $H(\mathcal{A}(\mathcal{Z}))$, from a contact-geometric perspective.

Throughout this paper we work with \mathbb{Z}_2 coefficients; signs are always irrelevant.

1.2 Main results

Our first main result is the explicit construction of A_∞ structures on $H(\mathcal{A}(\mathcal{Z}))$.

Theorem 1.1 *A pair ordering of \mathcal{Z} can be used to define an explicit A_∞ structure X on $H(\mathcal{A}(\mathcal{Z}))$, together with a morphism of A_∞ algebras $f: H(\mathcal{A}(\mathcal{Z})) \rightarrow \mathcal{A}(\mathcal{Z})$. These consist of maps*

$$X_n: H(\mathcal{A}(\mathcal{Z}))^{\otimes n} \rightarrow H(\mathcal{A}(\mathcal{Z})), \quad f_n: H(\mathcal{A}(\mathcal{Z}))^{\otimes n} \rightarrow \mathcal{A}(\mathcal{Z}),$$

where X extends the DGA structure of $H(\mathcal{A}(\mathcal{Z}))$, and $\mathcal{A}(\mathcal{Z})$ is regarded as an A_∞ algebra with trivial n -ary operations for $n \geq 3$.

By (1), [Theorem 1.1](#) provides A_∞ structures on the contact category algebra $CA(\Sigma, Q)$.

We will discuss *pair orderings* as we proceed ([Section 3.6](#)); they consist of a total order on the matched pairs of \mathcal{Z} , along with an ordering of the two points in each pair. In fact the full statement ([Theorem 4.2](#)) allows for a slightly more general A_∞ structures, using certain types of “choice functions” to parametrise the various choices involved in the construction. A pair ordering allows A_∞ operations to be computed relatively straightforwardly, but we know of no direct contact-geometric meaning.

The second main result provides necessary conditions under which these A_∞ maps are nontrivial, and under those conditions gives an explicit description of the results. The idea is that certain “local” conditions at the matched pairs of \mathcal{Z} are necessary to obtain nonzero output from the A_∞ maps.

Theorem 1.2 *Let $M = M_1 \otimes \cdots \otimes M_n \in H(\mathcal{A}(\mathcal{Z}))^{\otimes n}$ be a tensor product of nonzero homology classes of strand diagrams. The maps f_n and X_n of [Theorem 1.1](#) have the following properties:*

- (i) *If $\bar{f}_n(M) \neq 0$, then M has l twisted and m critical matched pairs, where $l + m \geq n - 1$ and $m \leq n - 2$, and all other matched pairs are tight. In this case $\bar{f}_n(M)$ is a sum of strand diagrams, where each diagram D is tight at all matched pairs where M is critical or tight, and has $n - 1 - m$ crossed and $l + m - n + 1$ twisted matched pairs.*
- (ii) *If $X_n(M) \neq 0$, then M has precisely $n - 2$ critical matched pairs, and all other matched pairs tight. In this case, $X_n(M)$ is the unique homology class of tight diagram with the appropriate gradings.*

All the terminology will be defined in due course. Very roughly, \bar{f}_n is the projection of f_n into a useful quotient algebra; matched pairs are objects which appear in the arc diagrams on which strand diagrams are drawn; and the words “tight”, “twisted”, and “critical” are descriptions of types of configurations of strands in strand diagrams (and their homology classes and their tensor products).

The other main results involve the notion of *operation trees*. These will be defined in due course ([Section 7.1](#)). They consist of rooted plane binary trees with vertices labelled by strand diagrams or contact structures; they encode the way in which contact structures can be combined by the various A_∞ operations. Trees have commonly been used to encode A_∞ operations; see eg Keller [9], Kontsevich and Soibelman [10] and Seidel [25].

Certain trees of this type are required to obtain nonzero output from an A_∞ operation.

Proposition 1.3 *If $X_n(M) \neq 0$ or $\bar{f}_n(M) \neq 0$, there is a valid distributive operation tree for M .*

Our final main result gives sufficient conditions on diagrams and trees which ensure a nonzero result; this result is again described explicitly.

Theorem 1.4 (i) *Suppose M has no on-on doubly occupied matched pairs. If every valid distributive operation tree for M is strictly f -distributive, and at least one such tree exists, then $\bar{f}_n(M) \neq 0$. Moreover, $\bar{f}_n(M)$ is given by a single diagram D , which can be described explicitly.*

(ii) *Suppose M has no twisted or on-on doubly occupied matched pairs. If every valid distributive operation tree for M is strictly X -distributive, and at least one such tree exists, then $X_n(M) \neq 0$. Moreover, $X_n(M)$ is given by the homology class of unique tight diagram with appropriate gradings.*

Very roughly, “on-on” and “doubly occupied” refer to particular configurations of strand diagrams at a matched pair; an operation tree is “valid” if the labels are “nonsingular” in an appropriate sense; and it is “distributive” if the contact structures labelling the tree have their “twistedness” spread across its various leaves in an appropriate sense.

As we will explain, these results are quite partial. The necessary conditions of [Theorem 1.2](#) are far from sufficient, and the sufficient conditions of [Theorem 1.4](#) are far from necessary. Since there are many A_∞ structures on $H(\mathcal{A}(\mathcal{Z}))$, we cannot expect a complete characterisation of diagrams which yield zero and nonzero results; still, we hope these results can be improved.

As is already clear, there is a *lot* of terminology to define. Simply stating these results requires us to describe precisely many aspects of strand diagrams, and their tensor products and homology classes. We must name this world in order to understand it.

1.3 Construction of A-infinity structures

In a certain sense, the A_∞ structures on $CA(\Sigma, Q)$ or $H(\mathcal{A}(\mathcal{Z}))$ are already understood. In the 1980 paper [\[8\]](#), Kadeishvili showed how to define an A_∞ structure on the homology H of any DGA A (provided H is free, which is always true with

\mathbb{Z}_2 coefficients). Indeed, in this paper we follow this construction, and [Theorem 1.1](#) can be regarded as fleshing out its details when $A = \mathcal{A}(\mathcal{Z})$. The only thing possibly new in [Theorem 1.1](#) is the level of explicitness in the construction.

We briefly recall some facts about A_∞ algebra; we refer to Keller [\[9\]](#) for an introduction to A_∞ algebra, or to Seidel [\[25\]](#) for further details. An A_∞ structure m on a \mathbb{Z} -graded \mathbb{Z}_2 -module A is a collection of operations $m_n: A^{\otimes n} \rightarrow A$ for each $n \geq 1$, where each m_n has degree $n - 2$. We call m_n the n -ary or level n operation. The operations m_i satisfy, for each $n \geq 1$,

$$\sum_{i+j+k=n} m_{i+1+k}(1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = 0.$$

This identity for $n = 1$ says that $m_1^2 = 0$, so m_1 is a differential; then the identity for $n = 2$ is the Leibniz rule, with m_2 regarded as multiplication. Indeed an A_∞ algebra with all $m_n = 0$ for $n \geq 3$ is precisely a DGA. A morphism f of A_∞ algebras $A \rightarrow A'$ (where the operations on A and A' are denoted by m_i and m'_i , respectively) is a collection of \mathbb{Z}_2 -module homomorphisms $f_n: A^{\otimes n} \rightarrow A'$, where each f_n has degree $n - 1$. We call f_n the level n map. The maps f_i satisfy, for each $n \geq 1$,

$$\sum_{i+j+k=n} f_{i+1+k}(1^{\otimes i} \otimes m_j \otimes 1^{\otimes k}) = \sum_{i_1+\dots+i_s=n} m'_s(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_s}).$$

Kadeishvili's construction in [\[8\]](#) produces an A_∞ structure X on H , consisting of operations $X_n: H^{\otimes n} \rightarrow H$, and a morphism f of A_∞ algebras $H \rightarrow A$, consisting of maps $f_n: H^{\otimes n} \rightarrow A$. The DGA A is regarded as an A_∞ algebra with trivial n -ary operations for $n \geq 3$. The A_∞ structure constructed on H begins with trivial differential $X_1 = 0$, and X_2 is the multiplication on H inherited from A . If H is free then there is a map $f_1: H \rightarrow A$ (possibly many) which is an isomorphism in homology, sending each homology class to a cycle representative. The constructed f_n can be taken to begin with any such f_1 . Moreover, the f_n form a quasi-isomorphism and this A_∞ structure is unique up to (nonunique) isomorphism of A_∞ algebras; see also [\[9, Section 3\]](#).

The construction proceeds inductively and uses auxiliary maps $U_n: H^{\otimes n} \rightarrow A$ of degree $n - 2$, starting from $U_1 = 0$. Once U_i, X_i, f_i are defined for $i < n$, U_n is given by an explicit expression in previous f_i and X_i , and X_n is the homology class of U_n . The map f_n is constructed as the solution of a particular equation. There is a choice for f_n at each stage, but no choice for U_n or X_n . We discuss the construction in detail in [Section 4.1](#). The choice for f_n is roughly a choice of inverse for the differential ∂ .

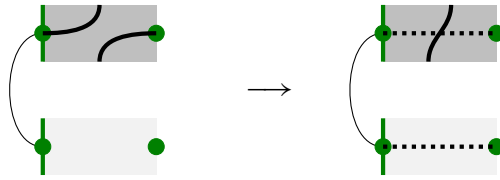


Figure 1: The action of a creation operator.

In Section 4.1 we give an explicit way to choose an f_n at each stage. This choice is made by maps which we call *creation operators*. We regard the differential in $\mathcal{A}(\mathcal{Z})$ as an “annihilation operator”, destroying crossings between strands by resolving them. Creation operators, on the other hand, insert crossings in a controlled way. The idea is shown in Figure 1. We introduce creation operators in Section 3. Creation operators satisfy Heisenberg relations (Proposition 3.16); this amounts to a chain homotopy from the identity to zero. In a certain sense, creation operators are the only operators obeying such Heisenberg relations; however they only form a very small subspace of the space of operators inverting the differential as required in Kadeishvili’s construction (Section 3.4). Similar “creation operators” have been put to use elsewhere in contexts related to contact geometry and Floer homology; see eg Mathews [21] and Mathews and Schoenfeld [23].

However, there is still choice involved in where to apply creation operators, ie where to insert crossings. There is also a choice for the initial cycle selection homomorphism f_1 . We parametrise such choices through notions of *creation choice functions* and *cycle choice functions* respectively. Our construction in general (Theorem 4.2) produces an A_∞ structure on $H(\mathcal{A}(\mathcal{Z}))$ or $CA(\Sigma, Q)$ from a given cycle choice function and creation choice function. A pair ordering can be used to obtain such choice functions, leading to the formulation of Theorem 1.1.

In order to define the A_∞ structure X on $H(\mathcal{A}(\mathcal{Z}))$, it turns out to be sufficient to work in a particular *quotient* of $\mathcal{A}(\mathcal{Z})$. This simplifies details considerably. We define a two-sided ideal \mathcal{F} in Section 2.14. The maps \tilde{f}_n appearing in Theorems 1.2 and 1.4 are the images of f_n in the quotient by \mathcal{F} . Related ideas appeared in Lipshitz, Ozsváth and Thurston [13].

Algorithmically, the calculation of an A_∞ map $X_n(M_1 \otimes \cdots \otimes M_n)$, where M_1, \dots, M_n are homology classes of strand diagrams (or contact structures) by the method described above requires the computation, for $1 \leq i \leq j \leq n$, of each $f_{j-i+1}(M_i \otimes \cdots \otimes M_j)$ and $X_{j-i+1}(M_i \otimes \cdots \otimes M_j)$. The algorithm therefore has complexity $O(n^2)$ (where we

regard each computation of expressions such as (7) as constant time, and the complexity of the arc diagram \mathcal{Z} , as constant). The contrapositive of [Theorem 1.2](#) provides a set of conditions which imply $X_n(M) = 0$, which are easily checked in constant time. On the other hand, [Proposition 1.3](#) and [Theorem 1.4](#) provide conditions which are much more difficult to check, as the number of operation trees grows much faster with n . We regard these results as interesting not because of algorithmic usefulness, but because they perhaps provide some insight into A_∞ operations.

1.4 Classifications of diagrams, and the many types of twisted

As already mentioned above, there are many features of strand diagrams which are relevant for our purposes, but which have not been given names in the existing literature. A large part of this paper, especially [Section 2](#), is devoted to defining and classifying these features, and establishing some of their properties. These are all required for our main theorems.

Therefore, some of the work here is an exercise in taxonomy. We briefly explain what we need to define and why, and the resulting classifications.

Contact structures naturally come in two types: tight and overtwisted. This dichotomy goes back to Eliashberg's work [\[2\]](#) in the 1980s. In the present work, consideration of the relationship between strand diagrams and contact structures naturally leads to further distinctions. Roughly speaking, when we look at strand diagrams from a contact-geometric perspective, there are *many* types of “twisted”.

According to the isomorphism [\(1\)](#) of [\[22\]](#), tight contact structures correspond to strand diagrams which are nonzero in homology. Such diagrams are characterised by certain conditions; roughly speaking, they must have an appropriate grading, no crossings, and must not have any matched pair that looks like the left of [Figure 1](#). A strand diagram which fails one or more of these conditions can be regarded as “overtwisted” in some sense.

The simplest way for a diagram to fail to represent a tight contact structure is by grading: it may lie in a summand of $\mathcal{A}(\mathcal{Z})$ which has no homology. This leads to the notion of *viability* ([Section 2.3](#)). Only viable strand diagrams can possibly represent tight contact structures.

It is essential for our purposes to have precise terminology relating to these gradings and summands. We introduce a notion of *H-data*, which combines homological grading

and idempotents (Section 2.1). We also introduce notions of *on/off* or 1/0 to describe idempotents locally, and *occupation* of various parts of a strand diagram, such as *places* and *steps*, to describe homological grading locally (Section 2.6). Some of this terminology was used in [22].

A viable diagram can still fail to represent a tight contact structure for multiple reasons. The mildest case is shown in Figure 2, which shows both strand diagrams and contact cubes. The strand diagram is the product of strand diagrams corresponding to tight contact structures, but the full contact structure is overtwisted. (In fact, stacking only the two relevant cubes yields a tight contact structure; when combined with adjacent cubes however the structure is overtwisted.) It can also be described in terms of bypasses. In a future paper we hope to describe the relationship between strands and bypasses systematically. We define such “minimally overtwisted” diagrams as *twisted* in Section 2.9.

Viable strand diagrams can also fail to represent tight contact structures because they have *crossings*. Thus, the natural tight/overtwisted classification of contact structures naturally becomes a 3-fold classification of viable strand diagrams into tight/twisted/crossed. This classification is, in a precise sense (Lemma 2.24), in ascending order of degeneracy.

Proceeding to *tensor products* of diagrams, our notion of viability still applies. Diagrams can represent contact structures, and their tensor products can be regarded as “stacked” contact structures on $\Sigma \times [0, 1]$. Viability then incorporates the natural contact-geometric condition that such stacked structures agree along their common boundaries.

Tensor products of diagrams again have a natural “tight/twisted” classification (see Section 2.7), but now there are *six* types, which we call *tight*, *sublime*, *twisted*, *crossed*, *critical*, and *singular*, again in an ascending scale of degeneracy.

When we then arrive at tensor products of *homology classes* of diagrams in Section 2.11, only *four* of these types of tightness/twistedness remain.

Throughout, it is necessary to consider strand diagrams *locally* at matched pairs; this corresponds to considering contact structures locally at individual cubes of a cubulated contact structure. Indeed, we show that $H(\mathcal{A}(\mathcal{Z}))$ decomposes into a *tensor product* over matched pairs; and we have *local* strand algebras, each with their *local* homology at each matched pair.

Here again, we encounter phenomena not yet given a name in the literature. The observed local diagrams, described as “fragments” of strand diagrams in [22], are not

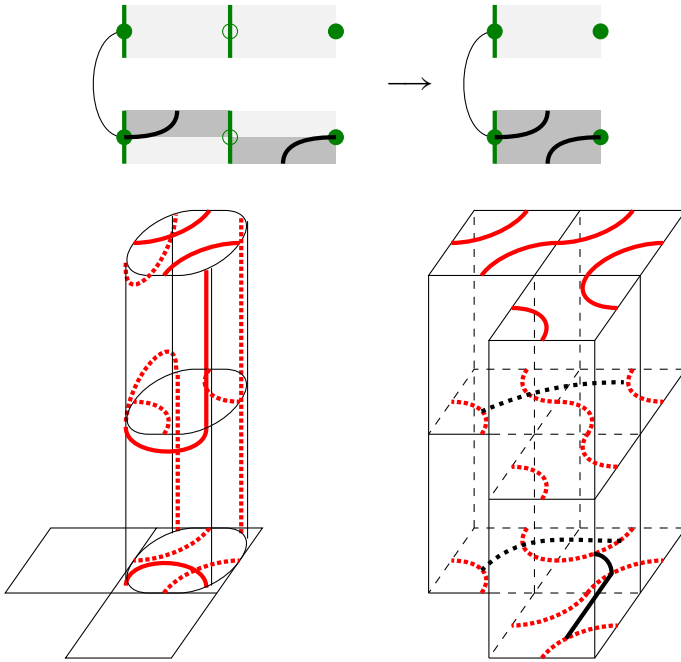


Figure 2: Top: A twisted diagram at a matched pair, the product of two tight diagrams. Bottom left: the corresponding contact cubes. Stacking the cubes yields a contact structure which remains tight, but combined with adjacent cubes the contact structure is overtwisted. Bottom right: the same contact structure described in terms of bypasses. A bypass is first attached to the bottom dividing set along the solid arc, yielding the intermediate dividing set; then a bypass is added along the dotted attaching arc. The overtwisted disc can be seen by attaching the latter bypass first.

strand diagrams in the usual sense of bordered Floer theory (eg Lipshitz, Ozsváth and Thurston [12]) or bordered sutured Floer theory (eg Zarev [30]), since strands may “run off the top of an arc”. Therefore, before we can even start our investigations, we must broaden the usual definition of strand diagrams. In Section 2 we therefore introduce a notion of *augmented strand diagram*.

Tensor products of strand diagrams (ie elements of $\mathcal{A}(\mathcal{Z})^{\otimes n}$), or their homology classes (ie elements of $H(\mathcal{A}(\mathcal{Z}))^{\otimes n}$) thus have a tensor decomposition over matched pairs of \mathcal{Z} , into local diagrams, in addition to their obvious decomposition into tensor factors. We regard these two types of decomposition as “vertical” and “horizontal”, respectively, and draw pictures accordingly. Contact-geometrically these two types of

tensor decomposition correspond to two types of geometric decomposition of stacked contact structures. A tensor product in $CA(\Sigma, Q)^{\otimes n} \cong H(\mathcal{A}(\mathcal{Z}))^{\otimes n}$ can be regarded as a stacking of n cubulated contact structures on $\Sigma \times [0, 1]$: this can be cut “horizontally” into n slices, each containing a contact structure on $\Sigma \times [0, 1]$; or it can be cut “vertically” to obtain stacked contact structures on $\square \times [0, 1]$, over each square \square of the quadrangulation.

We give a complete classification of viable local strand diagrams in [Section 2.6](#), summarised in [Table 1](#). We also give a complete classification of viable local tensor products of strand diagrams in [Section 2.10](#), summarised in [Table 2](#). We show ([Proposition 2.30](#)) that any viable tensor product of diagrams, observed locally at a single matched pair, must appear as one of the tensor products in the table, up to a notion of *extension* and *contraction*, which provide ways, trivial in a contact-geometric sense, to grow or shrink a tensor product. This also yields ([Proposition 2.33](#)) a complete classification of viable local tensor products of homology classes of strand diagrams.

Having made such definitions and classifications, we also establish several basic properties of these notions. In order to prove our main theorems, we need to answer questions such as which types of tightness/twistedness can occur within others, in various ways.

1.5 Contact meaning of A-infinity operations

We now attempt to give some idea of what the A_∞ operations X_n “mean” in terms of contact geometry. For details and background on the precise correspondence between contact structures and strand diagrams, we refer to our previous paper [\[22\]](#). We also intend to expand on the contact-geometric meaning of strand diagrams, particularly in terms of bypasses, in a future paper.

As discussed in [\[22\]](#), a strand diagram D on an arc diagram \mathcal{Z} with appropriate grading (each step of \mathcal{Z} covered at most once; no crossings) can be interpreted as a contact structure on $\Sigma \times [0, 1]$. Each matched pair of \mathcal{Z} corresponds to a square of the quadrangulation Q , or a cube in the cubulation $Q \times [0, 1]$ of $\Sigma \times [0, 1]$.

A strand diagram D containing a single moving strand going from one point (“place” in our terminology) of \mathcal{Z} to the next can be regarded as a *bypass*: in passing from one strand to the next, the strand affects two places, and the corresponding contact structure is a *bypass addition*, where the bypass is placed along the two cubes. Bypass addition is a basic operation in 3-dimensional contact geometry [\[4\]](#), and in a certain

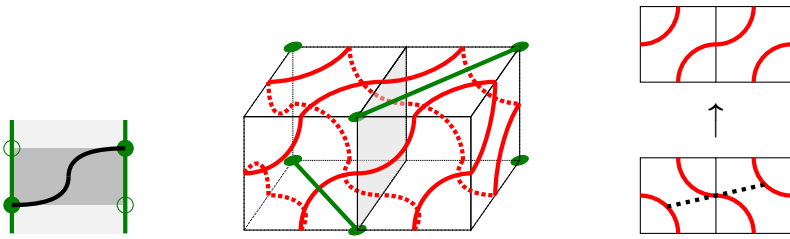


Figure 3: Left: A portion of a strand diagram consisting of a single strand from one place to the next. Centre: The corresponding cubulated contact structure. Right: This contact structure is given by a bypass attachment.

sense is the “simplest” modification one can make to a contact manifold [5]. The result is shown in Figure 3.

A strand diagram consisting of a longer strand can sometimes be regarded as a product of diagrams with shorter strands, each covering a single step of \mathcal{Z} as above. (However, other restrictions may get in the way; for instance arcs of an arc diagram may prevent factorising a longer strand into smaller ones. See eg the example of [13, Figure 11].) The corresponding contact structure is given a sequence of bypass additions very closely related to the *bypass systems* of [15; 16]. See Figure 4.

However, when we have a *tensor product* of strand diagrams corresponding to contact structures, the various steps of \mathcal{Z} may not be covered in the order in which they would be covered by single strands. If the various diagrams in the tensor product cover the various steps in a matched pair in a “correct” order, the factors in the tensor product

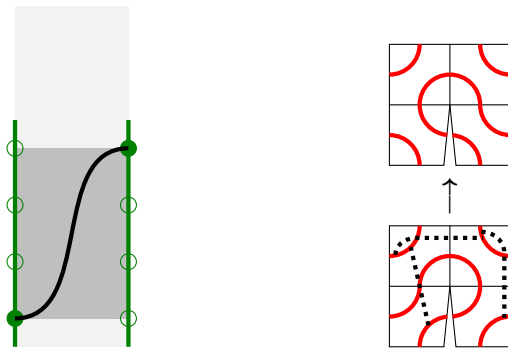


Figure 4: Left: A portion of a strand diagram consisting of a single strand. Right: The corresponding contact structure is given by a sequence of bypass attachments.

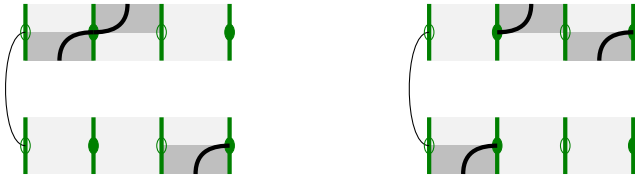


Figure 5: Left: This tensor product (tight in our classification) has a nonzero product in $\mathcal{A}(\mathcal{Z})$ or $H(\mathcal{A}(\mathcal{Z}))$. Right: This tensor product (critical in our classification) covers the same steps in a different order, and has zero product in $\mathcal{A}(\mathcal{Z})$ or $H(\mathcal{A}(\mathcal{Z}))$, but an A_∞ operation may reorder the bypasses and give a nonzero result.

may multiply (using the standard multiplication in $\mathcal{A}(\mathcal{Z})$) to give a diagram which is nonzero in homology. This corresponds to a contact structure built out of bypasses as described above. But if the various diagrams cover the various steps in a different order, then they will not multiply to give something nonzero in homology. Moreover, the *Maslov index* at the matched pair will be lower by 1 from the “correct” order.

The simplest example of this phenomenon is shown in [Figure 2](#). The product of two diagrams, corresponding to tight contact structures, gives an overtwisted contact structure. But if they were multiplied in the opposite order, the result would be tight. For a slightly more complicated example, still “localised” at a single matched pair, see [Figure 5](#).

In general, the A_∞ operation X_n , when it produces a nonzero result, will effectively reorder the bypasses at $n - 2$ matched pairs (since it has grading $n - 2$) so as to make their product tight. This is the rough meaning of [Theorem 1.2](#); the statement is simply an elaboration of this idea, being precise about the various types of tightness/twistedness at each matched pair.

We can also say a little about how this “reordering” is achieved; it seems to be unique to strand algebras. As mentioned above, our construction in [Section 4.1](#) of the A_∞ structure on $CA(\Sigma, Q)$ or $H(\mathcal{A}(\mathcal{Z}))$, following Kadeishvili’s method of [\[8\]](#), uses creation operators, whose operation is described locally by [Figure 1](#). A creation operator acts on a local diagram which is *twisted*, ie represents a “minimally overtwisted” contact structure, and makes it *crossed*.

We may then observe a phenomenon which is rather curious from a contact-geometric point of view. Starting from a tensor product which is twisted (or worse), applying a creation operator yields a tensor product of diagrams including crossings—the most degenerate type of “twistedness”. Yet multiplying out this tensor product may yield a

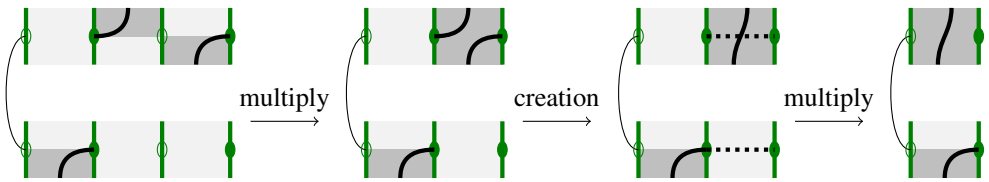


Figure 6: Mechanics of A_∞ operations, effectively reordering bypasses. Multiplying the last two factors of a critical tensor product yields a twisted diagram. A creation operator turns the twisted diagram into a crossed one, and the tensor product becomes sublime. Multiplication then yields a tight diagram.

diagram corresponding to a tight contact structure! After multiplication, no crossings remain, nor any twistedness. The result is as if the original diagrams were reordered into the “correct” order at that matched pair. See Figure 6 for an example based on the “badly ordered” tensor product of Figure 5 (right).

In this way, strand diagrams may pass from being crossed to tight without being twisted along the way. We call this process *sublimation* because of its “phase-skipping” behaviour. We call a tensor product in which the diagrams are not all tight, but their product is tight, *sublime*.

However, it is not the case that X_n always performs reorderings and sublimate in this way; it simply *may* do so. Depending on the various choices involved in the construction, the result may or may not be nonzero on various tensor products. Theorem 1.2 tells us what the answer must be, if it is nonzero; and gives necessary conditions for it to be nonzero. Theorem 1.4 does however provide a guarantee that for any A_∞ structures produced by our construction, certain (highly restricted) tensor products always yield a nonzero result.

For lower-level operations, we can say more. We know $X_1 = 0$ and X_2 is just multiplication, and we can in fact give an explicit description of X_3 (Proposition 5.9). Beyond that, the multiplicity of choices makes specific statements unwieldy, and Theorem 1.4 is the strongest guarantee of nonzero results that we could find, for now.

For the rest of this paper, we work primarily with strand diagrams. But our approach is heavily influenced by contact geometry, and we regularly comment on the contact-geometric significance of our definitions and results. For these comments, we assume some familiarity with the correspondence between strand algebras and contact structures in [22], and refer there for further details.

1.6 Relationship to other work

The strands algebra is a crucial object in bordered Floer theory, appearing in the work of Lipshitz, Ozsváth and Thurston [14; 11; 12; 13]. The slightly more general arc diagrams we use here appeared in Zarev’s work [30; 31]. Its homology was explicitly computed in Section 4 of [13]. This description was reformulated in [22], where the isomorphism (1) was proved. In [13, Section 4.2], Lipshitz–Ozsváth–Thurston considered Massey products on the homology of a strands algebra.

The general construction of A_∞ structures on DGAs by Kadeishvili in [8] is part of a much larger subject, not one in which the author claims much expertise. There are other methods, such as those of Kontsevich and Soibelman [10], Nikolov and Zahariev [24] and Huebschmann [7]. We do not know of examples where Kadeishvili’s construction has been made as absolutely explicit as by the “creation” operators here. In previous work we have found several roles for objects like creation and annihilation operators in contact geometry [15; 16; 17; 18; 19; 20; 21; 23].

The various contact-geometric interpretations appearing here derive not only from our previous work [22] but also from work on quadrangulated surfaces and their connections to contact geometry, Heegaard Floer theory and TQFT [19; 20]. Some of these ideas are also implicit in Zarev’s work cited above. Constructions with bypasses go back to Honda’s [4].

The contact category was introduced by Honda in unpublished work. It has been studied by Cooper [1]. Related categorifications have been studied by Tian [28; 29]. The case of discs was considered in our [15] and in detail by Honda and Tian in [6].

1.7 Structure of this paper

As discussed above, there is some work required before we can even properly state our main theorems. First we must define the relevant notions and establish the properties we need.

We begin in Section 2 by considering the algebra and anatomy of strand diagrams. We recall existing definitions in Section 2.1, and generalise them to *augmented* diagrams and tensor products in Section 2.2. We can then define the notion of viability in Section 2.3. We discuss subtensor-products, and the associated notions of extension and contraction, in Section 2.4. We consider how augmented diagrams can be cut into local diagrams, and the associated algebra, in Section 2.5. In Section 2.6 we establish terminology for

strand diagrams and their tensor products, including occupation of places and pairs for homological grading, and on/off or 1/0 for idempotents; then (Section 2.7) we define the six types of tightness/twistedness. In Section 2.8 we consider the various possibilities “locally” at each matched pair, discussing local strand algebras and their homology, and the homology of strand algebras in general. In Section 2.9 we study properties of variously twisted diagrams. We can then give a full enumeration of all possible viable local tensor products in Section 2.10. We consider the implications of these results for homology in Section 2.11, and then in Section 2.12 we consider how tightness of tensor products and subtensor-products are related. In Section 2.13 we calculate the dimensions of various vector spaces related to strand algebras, and in Section 2.14 we introduce the ideal \mathcal{F} and a quotient which simplifies our calculations.

In Section 3 we then consider objects parametrising the choices involved in constructing A_∞ structures. We discuss cycle selection homomorphisms in Section 3.1. We discuss how different cycle selection maps can differ in Section 3.2. We then introduce creation operators in Section 3.3, and discuss how they can invert the differential in Section 3.4. We put them together into global creation operators in Section 3.5, and discuss how they can be obtained from a pair ordering in Section 3.6.

We then have everything we need to construct A_∞ structures explicitly in Section 4. The construction itself is given in Section 4.1, proving Theorem 1.1. In Section 4.2 we establish a shorthand notation for tensor products of strand diagrams. In Section 4.3 we calculate some examples at low levels of the A_∞ structure.

In Section 5 we then discuss some properties of the A_∞ structures we have constructed, and in fact slightly more general A_∞ structures from Kadeishvili’s construction. In Section 5.1 we discuss how A_∞ operations relate to viability. In Section 5.2 we discuss how the various choices made in Kadeishvili’s construction affect the result. Then in Section 5.3 we establish some of the elementary properties of the constructed A_∞ operations, and in Section 5.4 prove some necessary conditions for nontrivial A_∞ operations, including those of Theorem 1.2. In Section 5.5 we establish general properties of the A_∞ maps at levels up to 3.

In Section 6 we calculate some further examples, illustrating some of the complexities which arise.

Finally in Section 7 we consider higher A_∞ operations and when they are nontrivial. We introduce the notion of operation trees in Section 7.1, and notions of *validity* and *distributivity* in Section 7.2. In Section 7.3 we discuss some constructions we need on

trees (*joining* and *grafting*). Then in Section 7.4 we show how certain trees are required for nonzero results, proving Proposition 1.3. In Section 7.5 we discuss the operation trees local to a matched pair, and classify them in Section 7.6. In Section 7.7 we introduce a stronger notion of validity necessary for our results, and after discussing the further operations of *transplantation* and *branch shifts* in Section 7.8, and introducing a stronger notion of distributivity in Section 7.9, we prove Theorem 1.4 in Section 7.10.

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2 Algebra of strand diagrams and their tensor products

2.1 Strand diagrams and their tensor products

We recall the definition of strand diagrams, before proceeding in Section 2.2 to augment them. We follow our previous paper [22], which in turn is based on Zarev [30], as well as Lipshitz–Ozsváth–Thurston [14; 13]. We refer to these papers for further details.

An *arc* diagram consists of a triple $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$, where $\mathbf{Z} = \{Z_1, \dots, Z_l\}$ is a set of oriented line segments (intervals), $\mathbf{a} = (a_1, \dots, a_{2k})$ is a sequence of distinct points in the interior of the line segments of \mathbf{Z} , ordered along the intervals, and $M: \mathbf{a} \rightarrow \{1, 2, \dots, k\}$ is a 2-to-1 function. As in [22], performing oriented surgery on \mathbf{Z} at all the 0–spheres $M^{-1}(i)$ is required to yield an oriented 1–manifold consisting entirely of arcs (no circles). We say \mathcal{Z} is *connected* if the graph obtained from \mathbf{Z} by identifying each pair $M^{-1}(i)$ is connected.

We call the points of \mathbf{a} *places*. If $M(a_i) = M(a_j)$ we say a_i and a_j are *twins*; then a_i, a_j form a *matched pair* (or just *pair*). The function M partitions \mathbf{a} into k such pairs. There is a partial order on \mathbf{a} where $a_i \leq a_j$ if a_i, a_j lie on the same oriented interval, and are in order along it.

An *unconstrained strand diagram* over \mathcal{Z} is a triple $\mu = (S, T, \phi)$, where $S, T \subseteq \{a_1, \dots, a_{2k}\}$ with $|S| = |T|$ and $\phi: S \rightarrow T$ is a bijection, which is increasing with respect to the partial order on \mathbf{a} in the sense that $\phi(x) \geq x$ for all $x \in S$. There is a standard way to draw an unconstrained strand diagram in the plane (in fact in $[0, 1] \times \mathbf{Z}$), with $|S| = |T|$ *strands*. The strands *begin* at S (drawn at $\{0\} \times S$), *end* at T (drawn at $\{1\} \times T$), and move to the right (in the positive direction along $[0, 1]$), never going down,

and meeting efficiently without triple crossings. We say μ goes from S to T . The points of \mathbf{a} split \mathcal{Z} into intervals called *steps*, of two types: *interior* to an interval Z_i , and *exterior*, ie at the boundary of a Z_i . The product $\mu\nu$ of two strand diagrams $\mu = (S, T, \phi)$ and $\nu = (U, V, \psi)$ is given by $(S, V, \psi \circ \phi)$, provided that $T = U$ and the composition $\psi \circ \phi: S \rightarrow V$ satisfies $\text{inv}(\psi \circ \phi) = \text{inv}(\phi) + \text{inv}(\psi)$; otherwise it is zero. Here $\text{inv}(\mu)$ is the number of inversions, or crossings, in μ . Equivalently, the product $\mu\nu$ is given by concatenating strand diagrams, provided that there are no “excess inversions”, ie crossings which can be simplified by a Reidemeister II-type isotopy of strands relative to endpoints. There is a differential ∂ which resolves crossings in strand diagrams; $\partial\mu$ is the sum of all strand diagrams obtained from μ by resolving a crossing so that the number of crossings decreases by exactly 1. This structure makes the free \mathbb{Z}_2 -module on strand diagrams over \mathcal{Z} into a DGA over \mathbb{Z}_2 , which we denote by $\tilde{\mathcal{A}}(\mathcal{Z})$. For each subset $S \subseteq \mathbf{a}$ there is an idempotent $I(S)$.

A \mathcal{Z} -constrained, or just *constrained*, strand diagram takes into account also the matching M of \mathcal{Z} . For each $s \subseteq \{1, \dots, k\}$ we define $I(s) = \sum_S I(S)$, where the sum is over sections S of s under M . Here a *section* of s means an $S \subseteq \mathbf{a}$ such that $M|_S$ is a bijection $S \rightarrow s$. The $I(s)$ generate a \mathbb{Z}_2 -subalgebra of $\tilde{\mathcal{A}}(\mathcal{Z})$. A strand diagram which begins at a section of s and ends at a section of t , for $s, t \subseteq \{1, \dots, k\}$, is said to be \mathcal{Z} -constrained. We say it *begins* at s and *ends* at t , or *goes from s to t* ; we also say $I(s)$, or by abuse of notation just s , is the *initial idempotent*, and $I(t)$ or t is the *final idempotent*. Thus a constrained strand diagram begins and ends at subsets of \mathbf{a} which contain at most one place of each matched pair. If $I(s)\tilde{\mathcal{A}}(\mathcal{Z})I(t)$ is nonzero then $|s| = |t| = i$, in which case it is freely generated as a \mathbb{Z}_2 -module by \mathcal{Z} -constrained strand diagrams of i strands from s to t .

Finally, we symmetrise our strand diagrams with respect to the matched pairs. If $\mu = (S, T, \phi)$ is an unconstrained strand diagram on \mathcal{Z} without horizontal strands (ie ϕ has no fixed points) then we consider adding horizontal strands to μ at some places $U \subseteq \mathbf{a} \setminus (S \cup T)$, ie adding fixed points to ϕ to obtain a function $\phi_U: S \cup U \rightarrow T \cup U$, which is still a bijection with $\phi(x) \geq x$. We define $a(\mu)$ to be the sum of all strand diagrams that can be obtained from μ by adding horizontal strands,

$$a(\mu) = \sum_U (S \cup U, T \cup U, \phi_U) \in \tilde{\mathcal{A}}(\mathcal{Z}),$$

and then for each $s, t \subseteq \{1, \dots, k\}$, $I(s)a(\mu)I(t)$ is the sum of all \mathcal{Z} -constrained strand diagrams from s to t obtained from μ by adding horizontal strands (possibly zero). (Left-multiplying by $I(s)$ filters for diagrams which start at s ; right-multiplying

by $I(t)$ filters for diagrams which start at t ; multiplying by both ensures the result is \mathcal{Z} -constrained.) Note that if it is possible to add a horizontal strand to μ at a place a of a matched pair $\{a, a'\}$ to obtain a strand diagram in $I(s)a(\mu)I(t)$, then it is also possible to add a horizontal strand at the twin place a' . In this case every diagram in $I(s)a(\mu)I(t)$ contains a horizontal strand at precisely one of a or a' ; further, for every diagram with a horizontal strand at a appearing in $I(s)a(\mu)I(t)$, the corresponding diagram with a horizontal strand at a' and otherwise identical will also appear. If there are j such pairs $\{a, a'\}$, then $I(s)a(\mu)I(t)$ is a sum of 2^j terms, one for each choice of a or a' in each pair.

We denote such a sum $I(s)a(\mu)I(t)$ as a single diagram D by drawing the $2j$ horizontal strands dotted, and call it a *symmetrised \mathcal{Z} -constrained strand diagram*. In such a diagram, the dotted strands are precisely the horizontal ones, and dotted strands come in pairs. So a symmetrised \mathcal{Z} -constrained strand diagram with j pairs of dotted strands is in fact a sum of 2^j \mathcal{Z} -constrained strand diagrams.

The *strand algebra* $\mathcal{A}(\mathcal{Z})$ is the subalgebra of $\tilde{\mathcal{A}}(\mathcal{Z})$ generated by symmetrised \mathcal{Z} -constrained strand diagrams. It is preserved by ∂ and hence forms a DGA. This algebra has several gradings.

The *homological grading*, also known as the *spin-c* or *Alexander grading*, we abbreviate to *H-grading*. It is valued in $H_1(\mathbf{Z}, \mathbf{a})$. Given a strand map $\mu = (S, T, \phi)$ on \mathcal{Z} , for each $a \in S$, the oriented interval $[a, \phi(a)]$ from a to $\phi(a)$ gives a homology class in $H_1(\mathbf{Z}, \mathbf{a})$, and the H-grading of μ , denoted by $h(\mu)$ or just h , is the sum of such intervals $[a, \phi(a)]$ over all $a \in S$. In other words, h counts how often each step of \mathbf{Z} is covered. Since horizontal strands cover no steps, a symmetrised constrained diagram D has a well-defined H-grading $h(D)$. The H-grading is additive under multiplication of strand diagrams, and preserved by ∂ . We denote by $\mathcal{A}(\mathcal{Z}; h)$ the \mathbb{Z}_2 -submodule of $\mathcal{A}(\mathcal{Z})$ generated by diagrams with H-grading h , so we have a direct-sum decomposition $\mathcal{A}(\mathcal{Z}) = \bigoplus_h \mathcal{A}(\mathcal{Z}; h)$.

Definition 2.1 Let D be a (symmetrised constrained) diagram from s to t (where $s, t \subseteq \{1, \dots, k\}$), with H-grading $h \in H_1(\mathbf{Z}, \mathbf{a})$. The *H-data* of D is the triple (h, s, t) .

In other words, the H-data of D consists of its H-grading together with its initial and final idempotents. By inspecting h , we can sometimes deduce that certain strands must begin or end at certain places, and hence deduce properties of s and t ; but h does not

in general determine s or t . In particular, h gives no information about horizontal strands. Writing $\mathcal{A}(\mathcal{Z}; h, s, t) = I(s)\mathcal{A}(\mathcal{Z}; h)I(t)$, we have a decomposition of $\mathcal{A}(\mathcal{Z})$ as a direct sum of \mathbb{Z}_2 –modules over H–data,

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{h,s,t} \mathcal{A}(\mathcal{Z}; h, s, t) = \bigoplus_{h,s,t} I(s)\mathcal{A}(\mathcal{Z}; h)I(t).$$

The Maslov grading of $\mathcal{A}(\mathcal{Z})$ is valued in $\frac{1}{2}\mathbb{Z}$. If μ is a \mathcal{Z} –constrained strand diagram (not yet symmetrised) from S to T with H–grading h , then its Maslov grading is

$$\iota(\mu) = \text{inv}(\mu) - m(h, S),$$

where the function

$$m: H_1(\mathbf{Z}, \mathbf{a}) \times H_0(\mathbf{a}) \rightarrow \frac{1}{2}\mathbb{Z}$$

counts local multiplicities of strand diagrams around places. Specifically, for a place a and $h \in H_1(\mathbf{Z}, \mathbf{a})$, $m(h, a)$ is the average of the local multiplicities of h on the steps after and before a . It is not difficult to check that all the constrained diagrams in a symmetrised constrained diagram D have the same Maslov grading, so the Maslov grading of D is the grading of any of the constrained diagrams in it.

The differential ∂ does not affect H–data, but lowers the number of crossings in a diagram by 1 (if the result is nonzero), hence has Maslov degree -1 . The Maslov index does not respect multiplication in $\mathcal{A}(\mathcal{Z})$; rather, for symmetrised \mathcal{Z} –constrained strand diagrams D and D' with H–gradings h and h' we have

$$(2) \quad \iota(DD') = \iota(D) + \iota(D') + m(h', \partial h).$$

The homology of $\mathcal{A}(\mathcal{Z})$ was described by Lipshitz–Ozsváth–Thurston [13, Theorem 9]. As ∂ respects H–data, the decomposition $\mathcal{A}(\mathcal{Z}) = \bigoplus_{h,s,t} \mathcal{A}(\mathcal{Z}; h, s, t)$ descends to homology:

$$H(\mathcal{A}(\mathcal{Z})) = \bigoplus_{h,s,t} H(\mathcal{A}(\mathcal{Z}; h, s, t)).$$

Lipshitz–Ozsváth–Thurston showed that the summand $H(\mathcal{A}(\mathcal{Z}; h, s, t))$ is nontrivial if and only if there exists a symmetrised \mathcal{Z} –constrained strand diagram D with H–data (h, s, t) without crossings, satisfying two conditions:

- (i) the multiplicity of h on every step of \mathbf{Z} is 0 or 1; and
- (ii) if $\{a, a'\}$ is a matched pair with a in the interior of the support of h , and a' not in the interior of the support of h , then a does not lie in both s and t .

Such a D , having no crossings, is obviously a cycle and in fact the homology class of any such D generates $H(\mathcal{A}(\mathcal{Z}; h, s, t)) \cong \mathbb{Z}_2$. We use property (i) extensively as a notion of *viability* from Section 2.3 onwards. We discuss and reformulate the second condition in Section 2.8 below; see also [22, Sections 3.5–3.7].

Since we usually work with a single arc diagram \mathcal{Z} , we often leave \mathcal{Z} implicit and write

$$\mathcal{A} = \mathcal{A}(\mathcal{Z}), \quad \mathcal{A}(h, s, t) = \mathcal{A}(\mathcal{Z}; h, s, t), \quad \mathcal{H} = H(\mathcal{A}), \quad \mathcal{H}(h, s, t) = H(\mathcal{A}(h, s, t)).$$

The homology \mathcal{H} inherits multiplication from \mathcal{A} and becomes a DGA with trivial differential. The point of this paper is to extend this DGA structure to A_∞ -structures.

Turning to tensor products, we observe that since \mathcal{A} is freely generated as a \mathbb{Z}_2 vector space by symmetrised constrained augmented strand diagrams on \mathcal{Z} , its tensor power $\mathcal{A}^{\otimes n}$ is freely generated by tensor products $D_1 \otimes \cdots \otimes D_n$ of such diagrams. We have the decomposition

$$\mathcal{A}^{\otimes n} = \bigoplus_{(h_1, s_1, t_1), \dots, (h_n, s_n, t_n)} (\mathcal{A}(h_1, s_1, t_1) \otimes \mathcal{A}(h_2, s_2, t_2) \otimes \cdots \otimes \mathcal{A}(h_n, s_n, t_n)),$$

with a similar decomposition for $\mathcal{H}^{\otimes n}$.

The Maslov and H-gradings naturally carry over to $\mathcal{A}^{\otimes n}$ so that the gradings of $D_1 \otimes \cdots \otimes D_n$ agree with those of the product $D_1 \cdots D_n$ in \mathcal{A} .

Definition 2.2 (gradings for tensor products) (i) The *H-grading* of $D_1 \otimes \cdots \otimes D_n$ is $h(D_1 \otimes \cdots \otimes D_n) = \sum_{i=1}^n h_i \in H_1(\mathbf{Z}, \mathbf{a})$.

(ii) The *Maslov grading* of $D_1 \otimes \cdots \otimes D_n$ is $\iota(D_1 \otimes \cdots \otimes D_n) = \sum_{i=1}^n \iota(D_i) + \sum_{1 \leq j < k \leq n} m(h_k, \partial h_j)$.

Applying (2) repeatedly shows that $\iota(D_1 \otimes \cdots \otimes D_n) = \iota(D_1 \cdots D_n)$; we also have $h(D_1 \otimes \cdots \otimes D_n) = h(D_1 \cdots D_n)$. These gradings naturally descend to tensor powers $\mathcal{H}^{\otimes n}$ of the homology \mathcal{H} .

2.2 Augmented strand diagrams

In a symmetrised \mathcal{Z} -constrained strand diagram, strands run between places in $\mathbf{a} = (a_1, \dots, a_{2k})$. Since places lie in the interior of the intervals Z_i of \mathbf{Z} , no strand ever reaches an endpoint of any interval Z_i . In other words, strand diagrams only cover interior steps of \mathbf{Z} .

In the sequel however we need to consider strand diagrams where strands cover exterior steps of \mathcal{Z} and reach endpoints of the intervals Z_i . We describe this as *flying off* an interval. *Augmented* strand diagrams, which we define presently, extend strand diagrams to allow such behaviour.

To define augmented diagrams formally we again use nondecreasing bijections, but now on sets including the endpoints of each interval. Let the endpoints of the interval Z_i be $-\infty_i$ and $+\infty_i$, at the start and end respectively. A strand flies off the top end of an interval Z_i if some $a_j \neq +\infty_i$ is sent to $+\infty_i$, and a strand flies off the bottom if some $a_j \neq -\infty_i$ satisfies $-\infty_i \mapsto a_j$. A strand may fly off both ends of an interval if $-\infty_i \mapsto +\infty_i$. We also allow horizontal strands at $\pm\infty_i$, but these present a slight subtlety, discussed below: they simply exist for technical reasons.

Let $\mathbf{a}_{\pm\infty} = \mathbf{a} \cup \{-\infty_1, \dots, -\infty_l\}_{i=1}^l \cup \{+\infty_1, \dots, +\infty_l\}_{i=1}^l$. The points of $\mathbf{a}_{\pm\infty}$ are naturally partially ordered by the total order along each interval, extending the partial order on \mathbf{a} .

The definition of the augmented strand algebra follows the definition of the strand algebra, with \mathbf{a} replaced by $\mathbf{a}_{\pm\infty}$ — with a few technicalities.

An *unconstrained augmented strand diagram* over \mathcal{Z} is a triple (S, T, ϕ) , where $S, T \subseteq \mathbf{a}_{\pm\infty}$ and $\phi: S \rightarrow T$ is a bijection such that $\phi(x) \geq x$ for all $x \in S$. As in the nonaugmented case, the product of two such diagrams concatenates diagrams, provided a concatenation exists and has no excess crossings; the differential resolves crossings, provided the number of crossings decreases by exactly 1; so we obtain a DGA $\tilde{\mathcal{A}}^{\text{aug}}(\mathcal{Z})$, which is a DGA over \mathbb{Z}_2 with an idempotent $I(S)$ for each $S \subseteq \mathbf{a}_{\pm\infty}$.

A subtlety arises here because if an (unconstrained) augmented strand diagram μ has a strand (say) flying off the end of an interval to $+\infty_i$, it should still be able to give a nonzero result when composed with another diagram on the right, which does not have any strand at $+\infty_i$. We extend our notion of matching to achieve this effect, but it is no longer a function; rather it is a *partial function* (ie partially defined).

To this end, extend the matching $M: \mathbf{a} \rightarrow \{1, \dots, k\}$ to the partial function

$$M^{\text{aug}}: \mathbf{a}_{\pm\infty} \rightarrow \{1, \dots, k\}$$

which is equal to M on \mathbf{a} and is *not defined* on each $-\infty_i$ or $+\infty_i$. Given $s \in \{1, \dots, k\}$, a *section* of s under M^{aug} is then any set $S \subseteq \mathbf{a}_{\pm\infty}$ such that the restriction of M^{aug} to S is a (possibly partially defined) function mapping surjectively and

injectively to s . Thus a section of s under M^{aug} consists of a section of s under M , together with any subset of $\{-\infty_i, +\infty_i\}_{i=1}^l$.

As in the nonaugmented case, for $s \subseteq \{1, \dots, k\}$, define $I(s) = \sum_S I(S)$, the sum over sections S of s under M^{aug} ; when $I(s)\tilde{\mathcal{A}}^{\text{aug}}(\mathcal{Z})I(t) \neq 0$, there is at least one section S of s under M^{aug} , and at least one section T of t under M^{aug} , such that there exists an (unconstrained) augmented strand diagram from S to T . However, now s and t need not have the same size, because S and T can also contain points of the form $+\infty_i$ or $-\infty_i$.

If $\mu = (S, T, \phi)$ is an unconstrained augmented strand diagram on \mathcal{Z} without horizontal strands, we again consider adding horizontal strands to μ at places $U \subseteq \mathbf{a}_{\pm\infty} \setminus (S \cup T)$ (there can be horizontal strands at $\pm\infty_i$), extending ϕ by the identity to $\phi_U: S \cup U \rightarrow T \cup U$, and defining $a(\mu) = \sum_U (S \cup U, T \cup U, \phi_U) \in \tilde{\mathcal{A}}^{\text{aug}}(\mathcal{Z})$. For $s, t \subseteq \{1, \dots, k\}$, then, $I(s)a(\mu)I(t)$ is the sum of all \mathcal{Z} -constrained augmented strand diagrams obtained from μ by adding horizontal strands, possibly at interval endpoints $\pm\infty_i$. As in the nonaugmented case, if one such diagram has j horizontal strands at the places of \mathbf{a} , these horizontal strands can be swapped with their twins, resulting in 2^j possible arrangements of horizontal strands at these places. Unlike the nonaugmented case, for any point of the form $-\infty_i$ or $+\infty_i$ not in $S \cup T$, a horizontal strand can be added at this point. Thus if $|\bigcup_{i=1}^l \{-\infty_i, +\infty_i\} \setminus (S \cup T)| = n$, then there are 2^n possible arrangements of horizontal strands at these endpoints.

Definition 2.3 With notation as above, $I(s)a(\mu)I(t)$ is a sum of 2^{j+n} \mathcal{Z} -constrained augmented strand diagrams. We can draw such a sum as a single diagram D with $2j$ dotted horizontal strands (leaving the possible horizontal strands at $\pm\infty_i$ implicit) and we call it a *symmetrised \mathcal{Z} -constrained augmented strand diagram* or just *diagram*.

Multiplication of two diagrams D, D' is described as follows. If no strand in D or D' flies off an interval, then their product DD' as augmented diagrams is given by concatenating strands, just as for nonaugmented diagrams. Formally the symmetrised augmented diagram is a sum of 2^n diagrams, involving possible horizontal strands at $\pm\infty_i$, but the augmented diagram DD' is drawn identically to the diagram of the product of nonaugmented diagrams.

If on some interval Z_i , both D and D' fly off the top end, then $DD' = 0$. This is because, for any \mathcal{Z} -constrained augmented strand diagram (S, T, ϕ) in D , and any such diagram (S', T', ϕ') in D' , ϕ has $+\infty_i$ in its image, but ϕ' does not have $+\infty_i$



Figure 7: Multiplication of augmented diagrams.

in its domain, so the functions cannot be composed. Similarly, if both D, D' fly off the negative end, then $DD' = 0$. If D flies off the top end of Z_i but D' does not, then the composition is well defined there: each (S, T, ϕ) in D has $+\infty_i$ in the image of ϕ ; and half of the constrained augmented diagrams (S', T', ϕ') in D' have ϕ' mapping $+\infty_i \mapsto +\infty_i$ (ie a horizontal strand at $+\infty_i$), so such ϕ' compose with ϕ at $+\infty_i$. If D' flies off the top end of Z_i but D does not, then again composition is well defined: each (S', T', ϕ') in D' has $+\infty_i$ in its image, but not in its domain; half the (S, T, ϕ) in D do not have $+\infty_i$ in the domain or image; and these ϕ and ϕ' compose without any problems at $+\infty_i$. Thus, if one of D, D' flies off the top end of Z_i and the other does not, then the product DD' is well defined there. Similarly, if one of D, D' flies off the bottom end of Z_i and the other does not, then the product DD' is well defined there.

Thus, roughly, if we can concatenate strands of D and D' into another augmented diagram, with at most one strand flying off any end of any interval, then the product DD' is given by concatenating strands, just as for (nonaugmented) strand diagrams. Some examples are shown in Figure 7.

The rest of the structure follows the nonaugmented case. The *augmented strand algebra* $\mathcal{A}^{\text{aug}}(\mathcal{Z})$ is the subalgebra of $\tilde{\mathcal{A}}^{\text{aug}}(\mathcal{Z})$ generated by (symmetrised \mathcal{Z} –constrained augmented strand) diagrams. It is preserved by ∂ and forms a DGA with homology $\mathcal{H}^{\text{aug}}(\mathcal{Z})$. H–grading h is given by the sum of oriented intervals $[a, \phi(a)]$, regarded in relative H_1 . However now the endpoints of $[a, \phi(a)]$ may include the $\pm\infty_i$, so $h \in H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty})$. Since $H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty})$ naturally contains $H_1(\mathbf{Z}, \mathbf{a})$ as a subgroup, we regard the H–grading as an extension of H–grading in the nonaugmented case. Diagrams have H–data (h, s, t) ; writing $\mathcal{A}^{\text{aug}}(\mathcal{Z}; h), \mathcal{A}^{\text{aug}}(\mathcal{Z}; h, s, t)$ for submodules of $\mathcal{A}^{\text{aug}}(\mathcal{Z})$ with specific H–grading or H–data, we have decompositions $\mathcal{A}^{\text{aug}}(\mathcal{Z}) = \bigoplus_h \mathcal{A}^{\text{aug}}(\mathcal{Z}; h)$ and $\mathcal{A}^{\text{aug}}(\mathcal{Z}) = \bigoplus_{h,s,t} \mathcal{A}^{\text{aug}}(\mathcal{Z}; h, s, t)$, and similarly for homology.

Maslov grading is given by $\iota(\mu) = \text{inv}(\mu) - m(h, S) \in \frac{1}{2}\mathbb{Z}$, where now

$$m: H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty}) \times H_0(\mathbf{a}) \rightarrow \frac{1}{2}\mathbb{Z}$$

counts local multiplicities of augmented diagrams around places a_i in S . (We use $H_0(\mathbf{a})$ rather than $H_0(\mathbf{a}_{\pm\infty})$ so that Maslov grading is additive when we glue arc diagrams together. The points $\pm\infty_i$ are not places like the a_i .) Maslov grading is well defined, since all the diagrams in a symmetrised diagram have the same Maslov grading. (When we add a horizontal strand at a $\pm\infty_i$, the fact that we can add the strand means that there is no strand at $\pm\infty_i$ for the horizontal strand to cross; moreover the horizontal strand at $\pm\infty_i$ does not contribute to $m(h, S)$.)

Again ∂ respects H–data but has Maslov degree -1 . Maslov index behaves under multiplication as in the nonaugmented case. When we have $h \in H_1(\mathbf{Z}, \mathbf{a}) \subset H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty})$ then strands do not fly off intervals and we have an isomorphism of DGAs,

$$\mathcal{A}(\mathcal{Z}; h, s, t) \cong \mathcal{A}^{\text{aug}}(\mathcal{Z}; h, s, t).$$

The isomorphism takes a symmetrised diagram $D \in \mathcal{A}(\mathcal{Z}; h, s, t)$ (formally a sum of 2^j constrained diagrams) to the element of $\mathcal{A}^{\text{aug}}(\mathcal{Z}; h, s, t)$ represented by the same diagram (formally a sum of 2^{j+2l} constrained diagrams, where l is the number of intervals in \mathbf{Z} ; all possible horizontal strands at $\pm\infty_i$ are now included). We draw the same diagrams and treat them the same way in both cases.

Accordingly, throughout this paper we regard augmented diagrams as a generalisation of nonaugmented diagrams, even though the definition is not formally a generalisation. Alternatively we can regard nonaugmented diagrams as augmented diagrams with H–grading zero on exterior steps, in which case augmented diagrams do become a generalisation in a formal sense.

Thus, we drop the “aug” from our notation and simply write $\mathcal{A}(\mathcal{Z})$ or \mathcal{A} for the augmented strand algebra. The tensor product $\mathcal{A}^{\otimes n}$ is again freely generated by tensor products of diagrams, and [Definition 2.2](#) defines gradings in $\mathcal{A}^{\otimes n}$.

To summarise: (symmetrised constrained augmented strand) diagrams are a generalisation of symmetrised constrained strand diagrams — generalising the full DGA structure of strand diagrams, as well as all gradings and idempotents.

2.3 Viability

The following notion of *viability* will be crucial throughout this paper.

Definition 2.4 Let $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$ be an arc diagram.

- (i) An element $h \in H_1(\mathbf{Z}, \mathbf{a})$ or $H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty})$ is *viable* if h has multiplicity 0 or 1 on each step of \mathbf{Z} .

(ii) A sequence of H–data $(h_1, s_1, t_1), \dots, (h_n, s_n, t_n)$ is *viable* if the following conditions hold:

- (a) for each $1 \leq i \leq n - 1$, $t_i = s_{i+1}$; and
- (b) $h_1 + \dots + h_n$ is viable.

(iii) A summand

$$\mathcal{A}(h_1, s_1, t_1) \otimes \dots \otimes \mathcal{A}(h_n, s_n, t_n) \quad \text{of } \mathcal{A}^{\otimes n}$$

or a summand

$$\mathcal{H}(h_1, s_1, t_1) \otimes \dots \otimes \mathcal{H}(h_n, s_n, t_n) \quad \text{of } \mathcal{H}^{\otimes n}$$

is *viable* if the sequence of H–data $(h_1, s_1, t_1), \dots, (h_n, s_n, t_n)$ is viable.

(iv) An element of $\mathcal{A}^{\otimes n}$ or $\mathcal{H}^{\otimes n}$ is *viable* if it lies in a viable summand.

Parts (ii)–(iv) of this definition, when $n = 1$, reduce to notions of viability for H–data, summands $\mathcal{A}(h, s, t)$ and $\mathcal{H}(h, s, t)$, and elements of \mathcal{A} and \mathcal{H} . The H–data (h, s, t) is viable if and only if h is viable; the summand $\mathcal{A}(h, s, t)$ or $\mathcal{H}(h, s, t)$ is viable if and only if h is. An element of \mathcal{A} or \mathcal{H} is viable if and only if it lies in a viable summand; a diagram is viable if and only if its H–grading is viable.

From (iv), a tensor product $D_1 \otimes \dots \otimes D_n$ of diagrams is viable if and only if its sequence of H–data is viable; similarly for a tensor product of homology classes of diagrams $M_1 \otimes \dots \otimes M_n$.

We refer to condition (ii)(a), that all $t_i = s_{i+1}$, as *idempotent matching*. It is vacuous when $n = 1$. When it fails, we say we have an *idempotent mismatch*. We define the H–data of a viable tensor product to include the first and last idempotents.

Definition 2.5 (H–data of tensor product) If a tensor product $D_1 \otimes \dots \otimes D_n$ of diagrams or $M_1 \otimes \dots \otimes M_n$ of homology classes of diagrams is viable, its *H–data* is the triple $(h_1 + \dots + h_n, s_1, t_n)$.

Idempotent matching means that we can draw strand diagrams side by side; the right-hand side of each D_i matches the left-hand side of D_{i+1} . Figure 8 depicts two viable tensor products of diagrams.

We now collect some useful properties of viability.

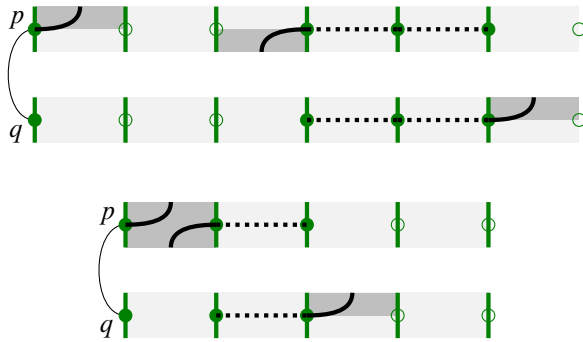


Figure 8: Top: Sesqui-occupied critical tensor product of six diagrams. Bottom: An extension–contraction (Section 2.4).

Lemma 2.6 *Let D_1, \dots, D_n be diagrams and M_1, \dots, M_n homology classes of diagrams on \mathcal{Z} .*

- (i) *If the product $D_1 \cdots D_n$ is a nonzero viable diagram, then $D = D_1 \otimes \cdots \otimes D_n$ is viable.*
- (ii) *If the product $M_1 \cdots M_n$ is a nonzero homology class, then $M = M_1 \otimes \cdots \otimes M_n$ is viable.*

Proof If idempotents don’t match then the product $D_1 \cdots D_n$ or $M_1 \cdots M_n$ is zero. The H–gradings of $D_1 \cdots D_n$ and $D_1 \otimes \cdots \otimes D_n$ are equal; similarly for M_1, \dots, M_n . □

The converses to Lemma 2.6(i) and (ii) are both false: there exist viable D and M with $D_1 \cdots D_n = 0$ and $M_1 \cdots M_n = 0$. In fact, D and M may be viable, yet there may not exist any diagram with its H–data! See eg Figure 9 (right). We introduce notions of “critical” and “singular” to describe these phenomena in Section 2.7.

Lemma 2.7 *In a viable diagram, every crossing is at a horizontal strand.*

Proof A diagram with two nonhorizontal strands crossing covers a step with multiplicity ≥ 2 . □

Thus, when applying ∂ to a viable diagram, any crossing resolved involves a dotted horizontal strand at a particular place; so ∂ acts “locally” on viable diagrams, each resolution at a specific matched pair. We discuss this idea of “locality” in Section 2.5.

2.4 Subtensor-products, extension and contraction

Definition 2.8 Let D_1, \dots, D_n be diagrams, and $D = D_1 \otimes \dots \otimes D_n \in \mathcal{A}^{\otimes n}$. A *subtensor-product* of D is a tensor product $D' = D_i \otimes D_{i+1} \otimes \dots \otimes D_{j-1} \otimes D_j$, where $1 \leq i \leq j \leq n$.

Similarly we can define a subtensor-product of a tensor product of homology classes of diagrams. If D is viable, then any subtensor-product D' is also viable; and similarly for homology classes.

A diagram is an idempotent if and only if all its strands are horizontal. Idempotents can be inserted into a tensor product of strand diagrams to “extend” it, as in the following straightforward statement, which also gives a method to “contract” it.

Lemma 2.9 (extending and contracting tensor products) *Let $D = D_1 \otimes \dots \otimes D_n$ be a viable tensor product of diagrams, where D_i has H–data (h_i, s_i, t_i) . Let D_i^* be the unique idempotent diagram consisting of dotted horizontal strands at all places of $t_i = s_{i+1}$.*

- (i) *The tensor product $D' = D_1 \otimes \dots \otimes D_i \otimes D_i^* \otimes D_{i+1} \otimes \dots \otimes D_n$ is also viable.*
- (ii) *Suppose that for some $1 \leq i < j \leq n$, the product $D_i D_{i+1} \dots D_j$ is nonzero. Then $D'' = D_1 \otimes \dots \otimes D_{i-1} \otimes (D_i D_{i+1} \dots D_j) \otimes D_{j+1} \otimes \dots \otimes D_n$ is also viable. □*

Again, a similar statement applies to tensor products of homology classes of diagrams.

Definition 2.10 In Lemma 2.9, we say D' is obtained from D by *extension* by D_i^* , and D'' is obtained from D by *contraction* of $D_i \otimes \dots \otimes D_j$.

We say a tensor product of diagrams is obtained from another by *extension–contraction* if it is obtained by some sequence of extensions and contractions.

Again, this definition also applies to tensor products of homology classes. Observe that extension and contraction of a tensor product preserve H–data and Maslov grading.

Note that extensions may be reversed by contraction, and contractions of idempotents may be reversed by extension. But a contraction involving more than one factor with nonhorizontal strands (ie more than one nonidempotent factor) cannot be reversed by extension; hence the following definition.

Definition 2.11 If two or more of D_i, D_{i+1}, \dots, D_j are not idempotents, then contraction of $D_i \otimes \dots \otimes D_j$ in D is *nontrivial*. Otherwise, the contraction is *trivial*.

In a trivial contraction, either all of D_i, \dots, D_j are idempotents, as is their product; or precisely one diagram D_k among D_i, \dots, D_j has nonhorizontal strands, in which case $D_i \dots D_j = D_k$. [Figure 8](#) depicts a nontrivial extension–contraction. Since an idempotent is the unique diagram representing its homology class, this definition also applies to homology.

2.5 Local diagrams

In the arc diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$, consider cutting the intervals Z_1, \dots, Z_l of \mathbf{Z} into subintervals, each containing precisely one place. This cuts \mathcal{Z} into disconnected arc diagrams — one connected arc diagram for each matched pair. We call the connected arc diagram so obtained, containing the matched pair P , the *fragment* of \mathcal{Z} at P , and denote it by \mathcal{Z}_P . Clearly any two such fragments are homeomorphic, regardless of where we cut \mathbf{Z} .

Indeed, a fragment \mathcal{Z}_P is the unique arc diagram up to homeomorphism with one matched pair. Under the correspondence between arc diagrams and quadrangulated surfaces of [\[22\]](#), cutting \mathcal{Z} into fragments corresponds to cutting Σ into squares.

Let now D be a diagram on \mathcal{Z} . When we cut \mathcal{Z} into fragments, we consider cutting D into fragments also. Note that the resulting diagrams on fragments may be augmented, even if D is not augmented.

If D has a crossing involving two nonhorizontal strands, then problems arise. Firstly, D could be drawn with the crossing appearing in various possible locations, so that there is no well-defined way to cut D into fragments. Secondly, after cutting, more than one strand may fly off the same end of a fragment, which is not permitted in augmented diagrams.

However, if D is *viable* these problems disappear. By [Lemma 2.7](#) all crossings occur at horizontal strands, so are localised at specific places. Moreover, each interior step of \mathbf{Z} is covered with multiplicity at most 1, so we obtain a well-defined augmented diagram on each fragment.

Definition 2.12 Let P be a matched pair of the arc diagram \mathcal{Z} , and let D be a viable diagram on \mathcal{Z} . The *local diagram* D_P of D at P is the diagram obtained on \mathcal{Z}_P after cutting \mathcal{Z} into fragments. It lies in the *local strand algebra* $\mathcal{A}(\mathcal{Z}_P)$, whose homology is called the *local homology at P* .

We can also extend the notion of a local diagram to tensor products: given a viable tensor product of diagrams $D = D_1 \otimes \cdots \otimes D_n \in \mathcal{A}(\mathcal{Z})^{\otimes n}$, the *local tensor product* D_P is

$$D_P = (D_1 \otimes \cdots \otimes D_n)_P = (D_1)_P \otimes \cdots \otimes (D_n)_P \in \mathcal{A}(\mathcal{Z}_P)^{\otimes n}.$$

Similarly, a tensor product of homology classes of diagrams $M = M_1 \otimes \cdots \otimes M_n \in H(\mathcal{A}(\mathcal{Z}))^{\otimes n}$ has local tensor product

$$M_P = (M_1 \otimes \cdots \otimes M_n)_P = (M_1)_P \otimes \cdots \otimes (M_n)_P \in H(\mathcal{A}(\mathcal{Z}_P))^{\otimes n}.$$

The H–data of each D_P is just a restriction of the H–data of D . Maslov gradings satisfy $\iota(D) = \sum_P \iota(D_P)$. If (h, s, t) denotes the H–data of D , we denote by (h_P, s_P, t_P) the H–data of D_P . When the arc diagram \mathcal{Z} is understood, we abbreviate notation for algebras, summands, and homology:

$$\begin{aligned} \mathcal{A}_P &= \mathcal{A}(\mathcal{Z}_P), & \mathcal{A}_P(h_P, s_P, t_P) &= \mathcal{A}(\mathcal{Z}_P; h_P, s_P, t_P), \\ \mathcal{H}_P &= H(\mathcal{A}_P), & \mathcal{H}_P(h_P, s_P, t_P) &= H(\mathcal{A}_P(h_P, s_P, t_P)). \end{aligned}$$

Diagrams D_P on each \mathcal{Z}_P , which fit together in the sense that strands flying off intervals connect, can be glued together into a viable diagram on \mathcal{Z} , and in fact for viable H–data (h, s, t) ,

$$\mathcal{A}(h, s, t) \cong \bigotimes_{\text{matched pairs } P} \mathcal{A}_P(h_P, s_P, t_P).$$

We regard $\mathcal{A}^{\otimes n}$ as a “horizontal” tensor product, and the above decomposition as a “vertical” tensor product. This is an isomorphism of complexes, or differential \mathbb{Z}_2 –modules. Thus, studying viable diagrams locally is equivalent to studying diagrams on fragments.

This isomorphism also respects multiplication: multiplying two diagrams D and D' on \mathcal{Z} , and then cutting into fragments, yields the same result as cutting D and D' into fragments, and then multiplying the local diagrams — provided that it makes sense to cut all the diagrams D , D' and DD' into fragments, ie they are all viable. In other words, if D and D' are viable diagrams on \mathcal{Z} , with local diagrams D_P and D'_P on each fragment \mathcal{Z}_P , then DD' is nonzero and viable if and only if each $D_P D'_P$ is viable; and $(DD')_P = D_P D'_P$. Thus $D = \bigotimes_P D_P$ and $D' = \bigotimes_P D'_P$ multiply to $DD' = \bigotimes_P D_P D'_P$.

Now for any chain complexes A and B over \mathbb{Z}_2 we have $H(A \otimes B) \cong H(A) \otimes H(B)$ — see eg [21, Section 3.7] or [3, Theorem V.2.1] — giving the following isomorphism, which we often use implicitly in the sequel.

Lemma 2.13 For viable (h, s, t) , there is an isomorphism of graded \mathbb{Z}_2 -algebras

$$\mathcal{H}(h, s, t) \cong \bigotimes_{\text{matched pairs } P} \mathcal{H}_P(h_P, s_P, t_P),$$

respecting H -data and Maslov grading, which is induced by cutting diagrams into fragments. □

Since all local strand algebras are isomorphic, we may speak of *the* local arc diagram or strand algebra, without reference to any specific matched pair. We abusively write \mathcal{Z}_P and \mathcal{A}_P accordingly.

2.6 Terminology for local strand diagrams and their tensor products

We now develop terminology to describe local diagrams. Throughout this section, $P = \{p, q\}$ is a matched pair of an arc diagram $\mathcal{Z} = (\mathbf{Z}, \mathbf{a}, M)$, $h \in H_1(\mathbf{Z}, \mathbf{a}_{\pm\infty})$, D is a diagram, $D_1 \otimes \cdots \otimes D_n$ is a tensor product of diagrams, M is the homology class of a diagram, and $M_1 \otimes \cdots \otimes M_n$ is a tensor product of homology classes of diagrams. Each diagram or homology class or tensor product has H -data (h, s, t) .

Definition 2.14 (occupation of places) If h has multiplicity

- (i) 0 on the steps before and after p , then p is *unoccupied* by h ;
- (ii) 1 on the step before p , and 0 on the step after p , then p is *pre-half-occupied* by h ;
- (iii) 0 on the step before p , and 1 on the step after p , then p is *post-half-occupied* by h ;
- (iv) 1 on both steps before and after p , then p is *fully occupied* by h .

If h is pre-half-occupied or post-half-occupied, then p is *half-occupied* by h .

We equally apply this terminology to diagrams and their homology classes and tensor products via their H -data, saying p is unoccupied (half-occupied, etc) by D or $D_1 \otimes \cdots \otimes D_n$.

Definition 2.15 (occupation of pairs) (i) If both p and q are unoccupied by h , then P is *unoccupied* by h .

- (ii) If p is half-occupied, and q is unoccupied by h , then P is *one-half-occupied at p by h* . Accordingly as p is pre- or post-half-occupied, P is *pre-one-half-occupied* or *post-one-half-occupied*.
- (iii) If both p, q are half-occupied by h , then P is *alternately occupied by h* .
- (iv) If p is fully occupied, and q is unoccupied by h , then P is *once occupied at p* .
- (v) If p is fully occupied and q is half-occupied by h , then P is *sesqui-occupied at p* . Accordingly as p is pre- or post-half-occupied, P is *pre-sesqui-occupied* or *post-sesqui-occupied*.
- (vi) If p, q are both fully occupied by h , then P is *doubly occupied by h* .

Again, we can extend this definition to diagrams and their homology classes and tensor products: P is unoccupied by D or $D_1 \otimes \cdots \otimes D_n$ or $M_1 \otimes \cdots \otimes M_n$, etc.

Definition 2.16 (idempotent terminology) For H–data (h, s, t) and a matched pair P :

- (i) If $P \notin s$ and $P \notin t$, we say P is *off-off* or *all-off* or 00.
- (ii) If $P \notin s$ and $P \in t$, we say P is *off-on* or 01.
- (iii) If $P \in s$ and $P \notin t$, we say P is *on-off* or 10.
- (iv) If $P \in s$ and $P \in t$, we say P is *on-on* or *all-on* or 11.

(We find this terminology awkward, hence offer several equally awkward alternatives.) We can say, for instance, that a pair P is all-on doubly occupied by (h, s, t) , or equivalently that (h, s, t) is 11 doubly occupied at P . Again, this definition extends to diagrams and their homology classes and tensor products. [Figure 8](#) depicts tensor products of diagrams at a sesqui-occupied pair.

H–data (h, s, t) can be described completely by the terminology of occupation (which describes h) and on/off (which describes s, t). As such, we can often deduce properties of a diagram simply from its occupation of places, or its on/off/etc properties.

2.7 Tightness of diagrams and their tensor products

Definition 2.17 (tightness of diagrams and tensor products) Suppose that $D = D_1 \otimes \cdots \otimes D_n$ is a viable tensor product of diagrams, with H–data (h, s, t) .

- (i) If $D_1 \cdots D_n$ is nonzero in homology, and all D_i are nonzero in homology, then D is *tight*.

- (ii) If $D_1 \cdots D_n$ is nonzero in homology, but not all D_i are nonzero in homology, then D is *sublime*.
- (iii) If $D_1 \cdots D_n$ is zero in homology, but is a nonzero diagram with no crossings, then D is *twisted*.
- (iv) If $D_1 \cdots D_n$ is a nonzero diagram with crossings, then D is *crossed*.
- (v) If $D_1 \cdots D_n = 0$, but $\mathcal{A}(h, s, t) \neq 0$, then D is *critical*.
- (vi) If $D_1 \cdots D_n = 0$ and $\mathcal{A}(h, s, t) = 0$, then D is *singular*.

Here when we say an element of \mathcal{A} is “nonzero in homology”, we mean that it represents a nonzero homology class. Since a diagram with crossings does not have a homology class, these cases are disjoint and cover all possibilities.

Examples of each type are shown in Table 2. Figure 9 (right) shows a singular example.

This definition presents tightness as a list of things that go increasingly wrong. First a diagram fails to be nonzero in homology; then the product fails to be nonzero in homology; then it has a crossing (hence does not represent a homology class); then it is zero; and then its existence is nonsensical. Any viable D falls into precisely one of these types. We say that D is *more tight* or *more singular* accordingly as it appears earlier or later in this list, giving the increasing order of singularity

$$\text{tight} < \text{sublime} < \text{twisted} < \text{crossed} < \text{critical} < \text{singular}.$$

We say that D is tight (sublime, twisted, etc) at P if D_P is tight (sublime, twisted, etc) on \mathcal{Z}_P .

The condition that $\mathcal{A}(h, s, t) \neq 0$ is equivalent to the existence of a diagram with H-data (h, s, t) . Thus when D is singular, no diagram exists with its H-data.

When $n = 1$ we obtain notions of tightness for a single viable diagram D . Examples of each type are shown in Table 1. The sublime, critical and singular cases do not arise, and we obtain that D is

- (i) tight if it is nonzero in homology,
- (ii) twisted if it is zero in homology, but has no crossings,
- (iii) crossed if it has crossings.

2.8 Local strand diagrams, local algebras and homology

There are not many possible local diagrams; they are listed in Table 1, by their H–data. Given H–data (h, s, t) at $P = \{p, q\}$, there are at most two diagrams, up to relabelling p and q . There are two diagrams precisely when P is all-on once or doubly occupied, and in this case the two diagrams are distinguished by Maslov grading. Such diagrams are important in the sequel, and so we name them.

- Definition 2.18**
- (i) If $P = \{p, q\}$ is all-on doubly occupied by (h, s, t) ,
 - (a) $b_P = b_{\{p,q\}}$ is the unique crossed diagram;
 - (b) g_p (resp. g_q) is the unique crossingless diagram with strands beginning and ending at p (resp. q).
 - (ii) If $P = \{p, q\}$ is all-on once occupied at p by (h, s, t) ,
 - (a) c_p is the unique crossed diagram;
 - (b) w_p is the unique crossingless diagram.
 - (iii) For any other H–data, denote the unique diagram by u_P .

(Our choice of symbols may seem arbitrary, but there is method in the madness: c for “Crossed”, b for “douBly crossed”, g for “tiGht”, w for “tWisted”, and u for “Unique”.)

Define chain complexes C''_P, C'_P and C_P by

$$\begin{aligned}
 C''_P &: 0 \rightarrow \mathbb{Z}_2\langle b_P \rangle \rightarrow \mathbb{Z}_2\langle g_p, g_q \rangle \rightarrow 0, & \text{where } \partial b_P &= g_p + g_q \text{ and } \partial g_p = \partial g_q = 0, \\
 C'_P &: 0 \rightarrow \mathbb{Z}_2\langle c_p \rangle \rightarrow \mathbb{Z}_2\langle w_p \rangle \rightarrow 0, & \text{where } \partial c_p &= w_p \text{ and } \partial w_p = 0, \\
 C_P &: 0 \rightarrow \mathbb{Z}_2\langle u_P \rangle \rightarrow 0.
 \end{aligned}$$

Up to a shift in Maslov grading, each nonzero summand $\mathcal{A}_P(h, s, t)$ of \mathcal{A}_P is isomorphic to C''_P, C'_P or C_P , accordingly as (h, s, t) is 11 doubly occupied, 11 once occupied, or anything else. These chain complexes have homology given by

- $H(C''_P) \cong \mathbb{Z}_2$, generated by the homology class of g_p or g_q (equal since $\partial b_P = g_p + g_q$),
- $H(C'_P) = 0$,
- $H(C_P) \cong \mathbb{Z}_2$, generated by the homology class of u_P .

From Section 2.5 we then have isomorphisms of chain complexes

$$\begin{aligned} \mathcal{A}(h, s, t) &\cong \bigotimes_P \mathcal{A}_P(h_P, s_P, t_P) \\ &\cong \bigotimes_{P \text{ 11 doubly occupied}} \bigotimes_{P \text{ 11 once occupied}} C''_P \otimes \bigotimes_{\text{other } P} C'_P \otimes C_P, \end{aligned}$$

and on homology we obtain

$$\mathcal{H}(h, s, t) \cong \bigotimes_P \mathcal{H}_P(h_P, s_P, t_P) \cong \begin{cases} 0 & \text{if there is an all-on once occupied pair,} \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Moreover, when there are no all-on once occupied pairs, $\mathcal{H}(h, s, t) \cong \mathbb{Z}_2$ is generated by the homology class of any crossingless diagram. So a diagram D is tight if and only if it has no crossings or all-on once occupied pairs; and D is twisted if and only if it is crossingless with an all-on occupied pair. We then have the following.

Proposition 2.19 (classification of local diagrams) *Let D be a diagram on \mathcal{Z}_P . Then the H -data and tightness of D determine D up to relabelling twins, and D is as shown in Table 1. □*

This recovers the homology calculation of Lipshitz–Ozsváth–Thurston [13], for viable H -data, extended to augmented diagrams. They calculate that $\mathcal{H}(h, s, t)$ is nontrivial if and only if there exists a crossingless diagram D satisfying two conditions (stated in Section 2.1), which we can now translate into our terminology. Condition (i) is that D be viable. Condition (ii) is that if P is once occupied or sesqui-occupied, then P is not all-on. Sesqui-occupied local diagrams are never all-on (eg from Table 1), so condition (ii) simply rules out all-on once occupied pairs.

Because of the above, the following definition makes sense.

Definition 2.20 Let (h, s, t) be viable H -data.

- (i) The *homology class* of (h, s, t) , denoted by $M_{h,s,t}$, is the unique nonzero homology class in $\mathcal{H}(h, s, t)$, if it exists; otherwise $M_{h,s,t} = 0$.
- (ii) The *local homology class* of (h, s, t) at P , denoted by $M_{h,s,t}^P$, is the unique nonzero local homology class in $\mathcal{H}_P(h, s, t)$, if it exists; otherwise $M_{h,s,t}^P = 0$.

The next proposition encapsulates the above discussion.

| H-data | tight | twisted | crossed |
|-----------------------------|---|-------------------------------------|------------------------------------|
| unoccupied | <p>all-off 0</p> <p>all-on 0</p> | | |
| one-half-occupied | <p>pre- 0</p> <p>post- $-\frac{1}{2}$</p> | | |
| alternately occupied all-on | $-\frac{1}{2}$ | | |
| once occupied | <p>all-off 0</p> <p>all-on -1</p> | <p> w_p</p> <p> -1</p> | <p> c_p</p> <p> 0</p> |
| sesqui-occupied | <p>pre- 0</p> <p>post- $-\frac{1}{2}$</p> | | |
| doubly occupied | <p>all-off 0</p> <p>all-on g_p or g_q</p> <p> -1</p> <p> -1</p> | | <p> b_p</p> <p> 0</p> |

Table 1: Local diagrams classified by H-data and tightness. Maslov indices are shown.

Proposition 2.21 *Let (h, s, t) be viable H–data. Then precisely one of the following is true:*

- (i) *There is a tight diagram with H–data (h, s, t) ; (h, s, t) is the H–data of a diagram with no all-on once occupied pairs; $\mathcal{H}(h, s, t) \cong \mathbb{Z}_2$, generated by the homology class $M_{h,s,t}$ of any crossingless diagram with H–data (h, s, t) .*
- (ii) *There is a twisted diagram with H–data (h, s, t) ; (h, s, t) is the H–data of a diagram with an all-on once occupied pair; $\mathcal{H}(h, s, t) = 0$ but $\mathcal{A}(h, s, t) \neq 0$.*
- (iii) *There is no diagram with H–data (h, s, t) ; $\mathcal{A}(h, s, t) = 0$. □*

Definition 2.22 (tightness of H–data) We say the viable H–data (h, s, t) on \mathcal{Z} is *tight*, *twisted* or *singular* according as (i), (ii) or (iii) of [Proposition 2.21](#) applies. The set of all viable tight H–data is denoted by $\mathbf{g}(\mathcal{Z})$, and the set of all viable twisted H–data is denoted by $\mathbf{w}(\mathcal{Z})$.

When the arc diagram is understood we simply write \mathbf{g} or \mathbf{w} rather than $\mathbf{g}(\mathcal{Z})$ or $\mathbf{w}(\mathcal{Z})$. Definitions [2.22](#) and [2.17](#) are consistent: a tight (resp. twisted, singular) tensor product of diagrams has tight (resp. twisted, singular) H–data. (As we will see in [Section 2.10](#), [Table 2](#) shows that a sublime or critical local tensor product has tight H–data, and a crossed local tensor product has tight or twisted H–data.)

When (h, s, t) is tight, there exists a tight diagram with H–data (h, s, t) ; we can ask precisely how many such diagrams exist. Any such diagram has homology class $M_{h,s,t}$ and is determined at all pairs except those which are all-on doubly occupied, where the local diagrams g_p and g_q ([Definition 2.18](#), or see [Table 1](#)) are both tight. We call the operation of replacing $g_p \leftrightarrow g_q$ *strand switching*; see [Figure 9](#) (left). With two choices at each all-on doubly occupied pair, we obtain the following statement.

Lemma 2.23 *Let (h, s, t) be tight viable H–data on the arc diagram \mathcal{Z} . Let L be the number of pairs all-on doubly occupied by (h, s, t) . Then there are precisely 2^L tight diagrams with H–data (h, s, t) ; they are precisely the diagrams representing the homology class $M_{h,s,t}$. Any two such diagrams are related by a sequence of strand switchings. □*

Tightness obeys a “local-to-global” principle, which we now state. Recall that the six tightness types of [Definition 2.17](#) (hence the three types of [Definition 2.22](#)) are arranged in order from tight to singular.

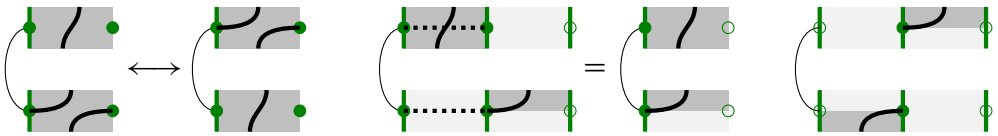


Figure 9: Left: strand switching. Centre: sublimation. Right: a singular tensor product.

Lemma 2.24 (local-to-global principle for tightness)

- (i) Let (h, s, t) be viable. Then the tightness type of (h, s, t) is the most singular tightness type among its local H-data (h_P, s_P, t_P) over matched pairs P .
- (ii) Let $D = D_1 \otimes \cdots \otimes D_n$ be viable. Then the tightness type of D is the most singular tightness type among its local tensor products $D_P = (D_1 \otimes \cdots \otimes D_n)_P$ over matched pairs P .

Proof We deal first with H-data: (h, s, t) is nonsingular if and only if it is the H-data of a diagram, if and only if each (h_P, s_P, t_P) is the H-data of a diagram, if and only if all (h_P, s_P, t_P) are nonsingular. So (h, s, t) is singular if and only if some (h_P, s_P, t_P) is singular. We may then assume (h, s, t) and all (h_P, s_P, t_P) are tight or twisted. Then (h, s, t) is twisted if and only if there is an all-on once-occupied pair P , in which case this (h_P, s_P, t_P) is twisted. Otherwise, (h, s, t) and all (h_P, s_P, t_P) are tight. This proves (i).

Now consider D ; let its H-data be (h, s, t) . This D is singular if and only if $\mathcal{A}(h, s, t) = 0$, if and only if (h, s, t) is singular, if and only if some (h_P, s_P, t_P) is singular (by (i)), if and only if some $\mathcal{A}_P(h_P, s_P, t_P) = 0$, if and only if some D_P is singular. We now assume D and all D_P are nonsingular, hence diagrams exist with H-data (h, s, t) .

Since we have $D_1 \cdots D_n = \bigotimes_P (D_1 \cdots D_n)_P$, we have that $D = 0$ if and only if some $(D_1 \cdots D_n)_P = 0$; that is, D is critical if and only if some D_P is critical. We now assume $D_1 \cdots D_n$ and all $(D_1 \cdots D_n)_P$ are nonzero, ie the tightness type of D and each D_P is crossed or tighter.

If D is crossed then $D_1 \cdots D_n$ has a crossing. By viability (Lemma 2.7), each crossing occurs at some matched pair P , hence some $(D_1 \cdots D_n)_P$ has a crossing, so D_P is crossed. Conversely, if some D_P is crossed then so is D . So D is crossed if and only if some D_P is crossed. We now assume $D_1 \cdots D_n$ and each $(D_1 \cdots D_n)_P$ have no crossings, ie D and each D_P are twisted or tighter.

Since $\mathcal{H}(h, s, t) \cong \bigotimes_P \mathcal{H}_P(h_P, s_P, t_P)$, $D_1 \cdots D_n$ is zero in homology if and only if some $(D_1 \cdots D_n)_P$ is zero in homology; that is, D is twisted if and only if some D_P is twisted. We now assume $D_1 \cdots D_n$ and each $(D_1 \cdots D_n)_P$ are nonzero in homology, ie D and each D_P are sublime or tight.

It remains to show that under these assumptions, D is tight if and only if all D_P are tight. Use $\mathcal{H}(h, s, t) \cong \bigotimes_P \mathcal{H}_P(h_P, s_P, t_P)$: D is tight if and only if $D_1 \cdots D_n$ and all D_i are nonzero in homology, if and only if all $(D_1 \cdots D_n)_P$ and all $(D_i)_P$ are nonzero in homology, if and only if all D_P are tight. □

2.9 Properties of nontight diagrams and tensor products

We now demonstrate various useful properties of various types of nontight diagrams.

Crossed diagrams We note that crossed diagrams cannot arise from crossingless diagrams.

Lemma 2.25 (crossingless subalgebra) *If diagrams D_1 and D_2 are crossingless, then $D_1 D_2$ is zero or crossingless. Hence the submodule of \mathcal{A} generated by crossingless diagrams forms a subalgebra.*

Proof If $D_1 D_2$ has a crossing, then one strand starts below and ends above another. The two strands must change their order either in D_1 or D_2 , so D_1 or D_2 has a crossing. □

Note that this lemma applies to crossingless diagrams in general, not just viable ones. The converse is false: multiplying a crossed diagram by another may yield a crossingless diagram. The result may even be tight, as occurs in sublimation. This occurs repeatedly in A_∞ operations.

Twisted diagrams and tensor products A viable diagram D is twisted if and only if each local diagram D_P is tight or twisted, and at least one D_P is twisted (Lemma 2.24). The only twisted local diagram is w_p (of Definition 2.18; see Table 1), so twisted diagrams are characterised by specific presence of w_p at an all-on once occupied pair $P = \{p, q\}$, where one place p is fully occupied and its twin q is unoccupied.

More generally, a tensor product $D = D_1 \otimes \cdots \otimes D_n$ is twisted if and only if $D_1 \cdots D_n$ is twisted, and so twistedness is characterised by a local diagram w_p in $D_1 \cdots D_n$.

Definition 2.26 (diagram twisted at a place) We say the diagram, or tensor product of diagrams, D is *twisted at the pair* $P = \{p, q\}$, or *twisted at the place* p .

The contact structure corresponding to w_p is “minimally overtwisted”, and can arise from two bypasses passing around a particular corner of a square, as in [Figure 2](#).

If D and D' are viable crossingless diagrams, at least one of which is twisted, then their product DD' (if nonzero and viable) is twisted: DD' is crossingless by [Lemma 2.25](#), and a product with zero in homology is zero. This corresponds to the contact-geometric phenomenon that a contact manifold containing an overtwisted submanifold is overtwisted.

Sublime tensor products Sublime tensor products also contain a specific local diagram c_p at an all-on once occupied pair, as we now show. Thus, sublimation arises by multiplying a crossed diagram by another diagram to undo the crossing and arrive at a tight diagram, as in [Figure 9](#) (centre).

Lemma 2.27 (sublime contains crossed) *If the viable tensor product $D = D_1 \otimes \cdots \otimes D_n$ is sublime, then some D_i is given by c_p at some matched pair $\{p, q\}$.*

Proof If all D_i are crossingless, then they have homology classes. Their product is nonzero since (by definition of sublime) $D_1 \cdots D_n$ is tight, so all D_i are nonzero in homology, ie tight. This contradicts D being sublime; thus some D_i is crossed at some pair P , hence given by b_P or c_P . But b_P is impossible, since it occupies all four steps at P , and by viability then any other D_j is idempotent at P , so that $(D_1 \cdots D_n)_P$, hence $D_1 \cdots D_n$, is not tight. □

Singular tensor products A singular $D = D_1 \otimes \cdots \otimes D_n$ is rather pathological: although viable, its H–data is not the H–data of any single diagram. [Lemma 2.24](#) says D is singular if and only if some D_P is singular. [Figure 9](#) (right) provides an example: there is no diagram with its H–data; there is no such thing as a 00 alternately occupied local strand diagram. We now show that this is essentially the only example.

Lemma 2.28 *Let $P = \{p, q\}$ be a matched pair, and let $D = D_1 \otimes \cdots \otimes D_n$ be a singular tensor product of local diagrams on \mathcal{Z}_P . Then D_P is an extension ([Definition 2.10](#)) of [Figure 9](#) (right).*

Proof Let D have H–data (h, s, t) . We observe (from [Table 1](#) or otherwise) that if a diagram covers an even number of the 4 steps of \mathcal{Z}_P , then P is 00 or 11; and if it covers an odd number of steps, then P is 01 or 10. Applying this observation to each D_i , we see that if h covers an even number of steps of \mathcal{Z}_P , then P is 00 or 11; and if h covers an odd number of steps, then P is 01 or 10.

Moreover, all steps covered by h cannot be covered by a single D_i . For then all other D_j are idempotents, so D_i has the H–data (h, s, t) of D , contradicting D being singular. In particular, h must cover at least two steps of P .

If h covers 2 steps, all possible H–data (h, s, t) satisfying the conditions above appear in [Table 1](#) (hence are nonsingular) except if P is 00 alternately occupied. In this case, P must be 01 one-half-occupied by some D_i , and 10 one-half-occupied by D_j , where $i < j$, giving the structure claimed.

If h covers 3 steps, then the only possible H–data not appearing in [Table 1](#) are where P is 10 pre-sesqui-occupied or 01 post-sesqui-occupied. We consider the first case; the second is similar. Without loss of generality suppose p is pre-half-occupied and q is fully occupied. If the 3 steps are covered by two diagrams D_i and D_j , where D_i covers one step and D_j covers two steps, then P is one-half-occupied by D_i . Moreover, by our initial observation, D_j is 00 or 11, so by viability D_i must be 10, hence P is post-one-half-occupied by D_i . Thus both p and q are pre-half-occupied by D_j , but there is no diagram which does so. If the three steps are covered by three diagrams, then P is pre-one-half-occupied by two diagrams (which must be 01) and post-one-half-occupied by one diagram (which must be 10), and all other diagrams are idempotents. But there is no way to combine the idempotent data 01, 01, 10 of these three diagrams viably so that P is 10 in the tensor product. Hence no such D exists.

If h covers all 4 steps, all possible H–data already appear in [Table 1](#) so D cannot be singular. \square

2.10 Enumeration of local tensor products

We now enumerate all viable local tensor products of diagrams. Let $D = D_1 \otimes \cdots \otimes D_n$ be a viable tensor product of diagrams on \mathcal{Z}_P , where $P = \{p, q\}$. Viability implies that each of the 4 steps of \mathcal{Z}_P is covered at most once. So at most 4 of D_1, \dots, D_n contain nonhorizontal strands; the rest are idempotents. The following lemma describes the tightness of the D_i .

Lemma 2.29 *Let D be a viable tensor product of local diagrams on \mathcal{Z}_P .*

- (i) *If D is tight, then each D_i is tight.*
- (ii) *If D is sublime, then one D_i is crossed 11 once occupied, and for the remaining factors D_j :*
 - (a) *one D_j is twisted, and all other D_j are idempotents; or*
 - (b) *one or two D_j are tight with nonhorizontal strands, and all other D_j are idempotents.*
- (iii) *If D is twisted, then either:*
 - (a) *precisely two D_i are tight nonidempotents, and all other D_j are idempotents; or*
 - (b) *precisely one D_i is twisted, and all other D_j are idempotents.*
- (iv) *If D is crossed, then precisely one or two D_i are crossed, and all other factors D_j are idempotents.*
- (v) *If D is critical, then of the D_i , none are crossed, W are twisted, G are tight nonidempotents, and the rest are idempotents, where $(W, G) = (2, 0), (1, 1), (1, 2), (0, 2), (0, 3)$ or $(0, 4)$.*
- (vi) *If D is singular, then precisely two D_i are tight, and all other D_j are idempotents.*

(In fact in (v) the case $(W, G) = (0, 2)$ never arises; such tensor products turn out to be singular.)

Proof Part (i) is true by definition.

If D is sublime then by [Lemma 2.27](#) some D_i is given by c_p . There are at most two crossed D_i ; if there exactly two, then by viability all other factors are idempotents and $D_1 \cdots D_n$ is crossed, contradicting D being sublime. So there is precisely one crossed diagram D_i , given by c_p . There are then at most 2 other factors D_j with nonhorizontal strands, which are tight or twisted. A twisted D_j would cover both the remaining steps, so (a) and (b) claimed are the only possibilities.

If D is twisted then ([Definition 2.17](#)) $D_1 \cdots D_n$ is twisted, hence ([Table 1](#)) only two steps of \mathcal{Z}_P are covered. Thus at most 2 of the D_i are not idempotents. If one D_i is nonidempotent, then D_i is twisted. If two D_i are nonidempotent, then each must cover one step, hence both are tight.

If D is crossed, then $D_1 \cdots D_n$ has a crossing, hence so does at least one D_i (Lemma 2.25). By viability, there are at most two crossed D_i . If there are two crossed factors, then they cover all steps, so all other factors are idempotents. If only one D_i is crossed, we observe that any viable multiplication of D_i with any tight or twisted diagram results in a tight diagram, so all other factors must be idempotents.

Now suppose D is critical. We claim no D_i are crossed. At most two D_i are crossed; if exactly two, then all other factors are idempotents, so that $D_1 \cdots D_n$ is nonzero crossed; if one D_i is crossed, then any viable product of D_i with a tight or twisted diagram is nonzero; either way D is not critical. Hence no D_i is crossed, so each nonidempotent D_i is twisted or tight. Each twisted factor covers exactly 2 steps; each tight factor covers at least 1 step. These factors altogether cover $2W + G \leq 4$ steps. On the other hand $W + G \geq 2$ since there must be at least 2 nonidempotent factors; otherwise the single nonidempotent $D_i = D_1 \cdots D_n \neq 0$, contradicting criticality. Thus (W, G) lies in the claimed set.

Lemma 2.28 gives the final part. □

Using the structure provided by Lemma 2.29 (or otherwise) we can enumerate viable tensor products of diagrams on \mathcal{Z}_P and obtain the following.

Proposition 2.30 (classification of local tensor products) *Any viable tensor product of local diagrams is an extension–contraction of one shown in Table 2, with H–data and tightness as shown.* □

Note that this tensor product may be an extension–contraction of more than one of the possibilities. For instance, a sublime tensor product and a tight tensor product may have a common contraction.

Table 2 also shows Maslov gradings with each local tensor product. As mentioned in Section 2.4, Maslov grading is preserved under extension and contraction. Observe that, for any given viable H–data, if there is a critical tensor product, then there is also a tight tensor product, and the Maslov grading of the latter is 1 greater than the former.

2.11 Tensor products of homology classes of diagrams

We now turn to $\mathcal{H}^{\otimes n}$. Homology classes of diagrams are illustrated by diagrams of representatives.








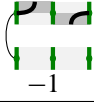
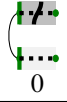
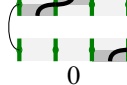
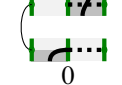
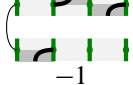
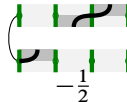
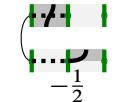
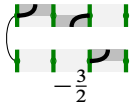
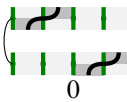
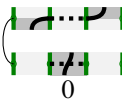
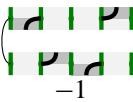
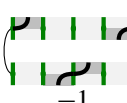
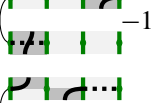
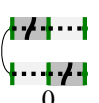
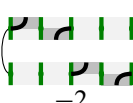
| H-data | tight | sublime | twisted | crossed | critical | singular |
|------------------------------|--|--|--|---|--|---|
| unoccupied 00 |  0 | | | | | |
| unoccupied 11 |  0 | | | | | |
| pre-one-half-occupied 01 |  0 | | | | | |
| post-one-half-occupied 10 |  $-\frac{1}{2}$ | | | | | |
| alternately occupied 00 | | | | | |  $-\frac{1}{2}$ |
| alternately occupied 11 |  $-\frac{1}{2}$ | | | | | |
| once occupied 00 |  0 | | | | | |
| once occupied 11 | | |  -1 |  0 | | |
| pre-sesqui-occupied 01 |  0 |  0 | | |  -1 | |
| post-sesqui-occupied 10 |  $-\frac{1}{2}$ |  $-\frac{1}{2}$ | | |  $-\frac{3}{2}$ | |
| doubly occupied 00 |  0 |  0 | | |  -1 | |
| doubly occupied 11 |  -1 |  -1 | |  0 |  -2 | |

Table 2: Possible local tensor products, by H-data and tightness. Maslov gradings also shown.

Throughout this section, $M = M_1 \otimes \cdots \otimes M_n$ is a viable tensor product of nonzero homology classes of diagrams, where M_i has H-data (h_i, s_i, t_i) . Then each $M_i = M_{h_i, s_i, t_i}$ is represented by a tight diagram D_i , and $D = D_1 \otimes \cdots \otimes D_n$ is viable. There may be multiple choices for the D_i , but they are related by strand switching (Lemma 2.23). We show the tightness of D is independent of these choices.

Lemma 2.31 For $1 \leq i \leq n$, let D_i and D'_i be diagrams representing M_i , and let $D = D_1 \otimes \cdots \otimes D_n$ and $D' = D'_1 \otimes \cdots \otimes D'_n$. Then D and D' have the same tightness.

Proof If D_i and D'_i differ by strand switching at P , then all four steps of \mathcal{Z}_P are covered by D_i , and by D'_i ; so every D_j and D'_j with $j \neq i$ is idempotent at P , and hence D and D' are tight at P . Thus D and D' have the same tightness at each matched pair, and by Lemma 2.24 the result follows. \square

Definition 2.32 The tightness of M is defined as the tightness of any representative D .

Just as for D , we can also speak of M being tight, twisted, critical or singular at a matched pair P , or twisted at a place p .

From Proposition 2.30 and Table 2 we observe that if a local tensor product of diagrams has all diagrams tight, then it is tight, twisted, critical or singular. Thus only four of the six tightness types exist for tensor products of homology classes, and we have the following.

Proposition 2.33 (classification of local tensor products of homology classes) Any viable tensor product of nonzero homology classes of local diagrams is an extension-contraction of one shown in the tight, twisted, critical or singular columns of Table 2, with H-data and tightness as shown. \square

Inspecting Table 2 allows us to make deductions about the tightness of M , merely from H-data.

Lemma 2.34 Let $M = M_1 \otimes \cdots \otimes M_n$ be viable on \mathcal{Z}_P , with H-data (h, s, t) .

- (i) M is tight or critical at P if and only if (h, s, t) is tight at P .
- (ii) M is twisted at P if and only if (h, s, t) is twisted at P .
- (iii) M is singular at P if and only if (h, s, t) is singular at P . \square

We can distinguish tightness in $\mathcal{H}^{\otimes n}$ by the following result.

Lemma 2.35 *Suppose $M = M_1 \otimes \cdots \otimes M_n$ is a viable tensor product of nonzero homology classes of diagrams on \mathcal{Z} , with H –data (h, s, t) , and let D_i be a diagram representing M_i .*

- (i) *M is tight if and only if $M_1 \cdots M_n \neq 0$.*
- (ii) *M is twisted if and only if $M_1 \cdots M_n = 0$ but $D_1 \cdots D_n \neq 0$.*
- (iii) *M is critical if and only if $D_1 \cdots D_n = 0$, but $\mathcal{A}(h, s, t) \neq 0$.*
- (iv) *M is singular if and only if $\mathcal{A}(h, s, t) = 0$.*

Like [Definition 2.17](#), [Lemma 2.35](#) presents tightness as a list of things that go increasingly wrong.

Recalling the isomorphism between \mathcal{H} and the contact category, $M = M_1 \otimes \cdots \otimes M_n$ describes the stacking of tight cubulated contact structures on a thickened surface $\Sigma \times [0, 1]$. Cases (ii) through (iv) describe overtwisted structures, in increasing order of degeneracy. In case (ii) the stacked contact cubes above each individual square remain tight, but the overall contact structure is overtwisted (as in [Figure 2](#)); in case (iii) the contact cube above some square becomes overtwisted; in case (iv) the contact cube above some square is overtwisted, even when restricted to the boundary of the cube.

Proof Let $D = D_1 \otimes \cdots \otimes D_n$, so by [Lemma 2.31](#) and [Definition 2.32](#), M and D have the same tightness.

If D is tight then $D_1 \cdots D_n$ is tight, so $M_1 \cdots M_n \neq 0$. Conversely, if $M_1 \cdots M_n \neq 0$ then all $M_i \neq 0$, and, being represented by the tight diagrams $D_1 \cdots D_n$ and D_i , D is tight.

If D is twisted then $D_1 \cdots D_n \neq 0$ but $M_1 \cdots M_n = 0$. Conversely, if $M_1 \cdots M_n = 0$ but $D_1 \cdots D_n \neq 0$, then $D_1 \cdots D_n$ is not tight; it is also not crossed, by [Lemma 2.25](#), hence it, and D , are twisted.

The characterisations of critical and singular follow directly from [Definition 2.17](#). \square

Just as for tensor products of diagrams, tensor products of homology classes obey a “local-to-global” principle for tightness. Lemmas [2.24](#) and [2.31](#) and [Definition 2.32](#) immediately give the following.

Lemma 2.36 (local–global tightness in $\mathcal{H}^{\otimes n}$) *For viable $M = M_1 \otimes \cdots \otimes M_n$, the tightness type of M is the most singular tightness type among the local tensor products $M_P = (M_1 \otimes \cdots \otimes M_n)_P$.* □

We also consider contractions and extensions: as we now show, contractions are always possible when M is tight, and otherwise nontrivial contractions (Definition 2.11) are impossible.

Lemma 2.37 *Suppose $M = M_1 \otimes \cdots \otimes M_n$ is viable on \mathcal{Z}_P .*

- (i) *If M is tight, then for all $1 \leq i \leq j \leq n$, the product $M_i \cdots M_j$ is nonzero, so $M_1 \otimes \cdots \otimes M_{i-1} \otimes (M_i \cdots M_j) \otimes M_{j+1} \otimes \cdots \otimes M_n$ is a contraction of M .*
- (ii) *If M is twisted, critical or singular, then any contraction of M is trivial (in the sense of Definition 2.11). Moreover, M is an extension of a tensor product of homology classes shown in the twisted, critical or singular columns of Table 2.*

Proof If M is tight, then (Lemma 2.35) $M_1 \cdots M_n \neq 0$; so any $M_i \cdots M_j \neq 0$.

If M is twisted, critical or singular, then by Proposition 2.33, M is an extension–contraction of a tensor product shown in the appropriate column of Table 2. We observe that multiplying any two consecutive diagrams in any of these tensor products yields a twisted or zero diagram, which is zero in homology. Thus no nontrivial contraction exists. □

The following fact about critical tensor products will be useful in the sequel.

Lemma 2.38 (“it takes 3 to be critical”) *If $M_1 \otimes \cdots \otimes M_n$ is viable and critical on an arc diagram \mathcal{Z} , then $n \geq 3$.*

Proof By Lemma 2.36, some local tensor product M_P is critical. By Lemma 2.37, M_P is an extension of a critical diagram in Table 2, and all such diagrams have at least 3 factors. □

2.12 Tightness of subtensor products, extensions and contractions

In the sequel we need to understand tightness of subtensor-products, extensions and contractions. “Local-to-global” principles (Lemmas 2.24 and 2.36) show that when we decompose locally (“vertically”), tightness is well behaved. However, “horizontal” decomposition is more complicated.

By Propositions 2.30 and 2.33, a viable local tensor product of diagrams, or their homology classes, is an extension–contraction of one shown in Table 2. We can then enumerate the tightness of subtensor-products in each case, and obtain the following result.

Lemma 2.39 (tightness of local subtensor-products) *Let D (resp. M) be a viable tensor product of diagrams (resp. homology classes of diagrams) on \mathcal{Z}_P , and let D' (resp. M') be a subtensor-product.*

- (i) *The possible tightness types of D and D' are as shown in Table 3.*
- (ii) *The possible tightness types of M and M' are as shown in the shaded part of Table 3.* □

| | | D', M' | | | | | |
|--------|----------|----------|---------|---------|---------|----------|----------|
| | | tight | sublime | twisted | crossed | critical | singular |
| D, M | tight | X | | | | | |
| | sublime | X | X | X | X | | |
| | twisted | X | | X | | | |
| | crossed | X | | | X | | |
| | critical | X | | X | | X | X |
| | singular | X | | | | | X |

Table 3: Possible tightness types of a viable local tensor product D (or homology class M) and a subtensor-product D' (or M' in homology). Shaded rows and columns refer to homology.

Thus, for instance, if D is tight then D' is also tight; if D' is sublime then D is also sublime; and if D' is critical then D is also critical. Similarly, if M is tight, then M' is tight; in this case M corresponds to a tight contact manifold and M' to a contact submanifold.

We also have a similar “global” result about the possible tightness types of tensor products of diagrams or their homology classes, on a general arc diagram \mathcal{Z} .

Lemma 2.40 (tightness of subtensor-products) *Let D (resp. M) be a viable tensor product of diagrams (resp. homology classes of diagrams) on \mathcal{Z} , and let D' (resp. M') be a subtensor-product.*

- (i) *If D (resp. M) is tight, then D' (resp. M') is tight.*

(ii) If D' (resp. M') is critical or singular, then D (resp. M) is critical or singular.

Every combination of tightness types not ruled out by these implications is possible.

Proof If D is tight, then $D_1 \cdots D_n$ is nonzero in homology (Definition 2.17), hence any $D_i \cdots D_j$ is nonzero in homology, hence tight, hence D' is tight. If D' is critical or singular then $D_i \cdots D_j = 0$ (Definition 2.17), so $D_1 \cdots D_n = 0$, so D is critical or singular.

We show some examples of the remaining possibilities in Figure 10. The small number remaining are omitted. The statements about homology classes then follow straightforwardly. □

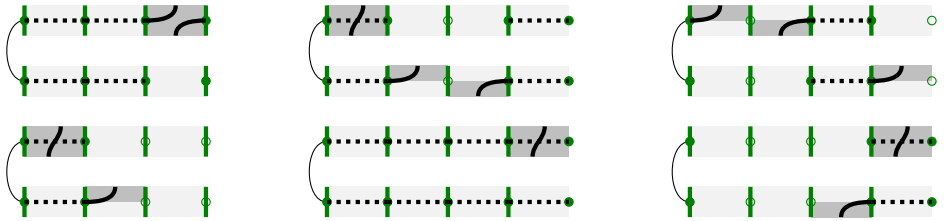


Figure 10: Left: a twisted tensor product $D_1 \otimes D_2 \otimes D_3$ containing tight (D_2), sublime ($D_1 \otimes D_2$), twisted (eg D_3) and crossed (D_1) subtensor-products. Centre: a crossed tensor product $D_1 \otimes D_2 \otimes D_3 \otimes D_4$ containing tight (D_2, D_3), sublime (eg $D_1 \otimes D_2$), twisted ($D_2 \otimes D_3$) and crossed (eg D_1) subtensor-products. Right: a critical tensor product $D_1 \otimes D_2 \otimes D_3 \otimes D_4$ containing tight (eg D_1), sublime ($D_3 \otimes D_4$), twisted (eg $D_1 \otimes D_2$), crossed (D_4), critical ($D_1 \otimes D_2 \otimes D_3 \otimes D_4$) and singular ($D_2 \otimes D_3 \otimes D_4$) subtensor-products.

Note the contrapositive of (ii) in homology: if M is tight or twisted, then M' is tight or twisted.

We now show that extension–contraction preserves tightness, with one exception: sublimation.

Lemma 2.41 Suppose D' is obtained from $D = D_1 \otimes \cdots \otimes D_n$ by extension–contraction. Then D and D' have the same tightness, or D is sublime and D' is tight.

Proof Under extension or contraction, the product $D_1 \cdots D_n$ remains invariant, as does H–data. Singularity of a tensor product is defined by reference only to H–data; hence D is singular if and only if D' is singular. We thus assume D and D' are not singular.

The tightness properties “tight or sublime”, “twisted”, “crossed” and “critical” of D are defined by the properties of the product $D_1 \cdots D_n$ (ie whether $D_1 \cdots D_n$ is tight, twisted, crossed or zero, respectively); hence these tightness properties are preserved under extension–contraction.

It remains to prove that if D is tight then D' is tight. In this case, any subtensor-product of D is tight (Lemma 2.40), and hence for any $1 \leq i \leq j \leq n$ the product $D_i \cdots D_j$ is tight. Thus in any extension–contraction D' of D , the product of any subtensor product is tight; so D' is tight. \square

It is useful to generalise the notion of contraction. Let $M = M_1 \otimes \cdots \otimes M_n$ be a viable tensor product of nonzero homology classes of diagrams. A contraction of M replaces a subtensor-product $M' = M_i \otimes \cdots \otimes M_j$ with $M_i \cdots M_j$ provided that this product is nonzero. Recalling (Proposition 2.21) that any tight H–data (h, s, t) has a unique nonzero homology class, we observe $M_i \cdots M_j$ is the unique homology class of diagram with the H–data of M' . This leads to the following generalisation.

Definition 2.42 Let $M = M_1 \otimes \cdots \otimes M_n$ be a viable tensor product of nonzero homology classes of diagrams. Suppose a subtensor-product $M_i \otimes \cdots \otimes M_j$ has tight H–data, and let M^* be the unique nonzero homology class of a diagram with this H–data.

Then we say $M_1 \otimes \cdots \otimes M_{i-1} \otimes M^* \otimes M_{j+1} \otimes \cdots \otimes M_n$ is obtained from M by *H–contraction*.

If M' is obtained from M by H–contraction, then M' is viable, and has the same H–data as M .

Tightness locally behaves rather nicely under H–contraction.

Lemma 2.43 *Let M be a viable tensor product of nonzero homology classes of diagrams on \mathcal{Z}_P . Suppose M' is obtained from M by H–contraction.*

- (i) M is (tight or critical), twisted, or singular, if and only if the same is true for M' .

- (ii) If M is tight, then M' is tight.
- (iii) If M' is critical, then M is critical.

Proof The H–data of M is tight, twisted or singular accordingly as M is respectively (tight or critical), twisted or singular (Lemma 2.34). Since H–contraction preserves H–data, (i) follows.

If M is tight, then we replace $M_i \otimes \cdots \otimes M_j$ with $M_i \cdots M_j$ (Lemma 2.37), so we have a bona fide contraction, and M' is tight: the product of the factors in both M and M' is $M_1 \cdots M_n$ (Lemma 2.35).

If M' is critical, then, by Proposition 2.33 and Lemma 2.37, M' is an extension of one of the tensor products shown in the critical column of Table 2. Thus each tensor factor of M' covers at most one step of \mathcal{Z}_P . Since M is obtained from M' by replacing a tensor factor of M' with $M_i \otimes \cdots \otimes M_j$, in a way that preserves H–data, M is an extension of M' . So M is critical. □

2.13 Dimensions of strand algebras

We now consider the dimension of $\mathcal{A}(h, s, t)$, and some related subspaces. Throughout this section let \mathcal{Z} be an arc diagram and (h, s, t) be viable nonsingular H–data on \mathcal{Z} , with L all-on doubly occupied pairs, and N all-on once occupied pairs. Dimension always refers to the dimension of a \mathbb{Z}_2 vector space.

From Table 1, we observe that given (h, s, t) , there are 3 choices of local diagram at a pair P which is all-on once occupied; 2 choices if P is all-on doubly occupied; and otherwise a unique choice. Thus

$$(3) \quad \dim \mathcal{A}(h, s, t) = 3^L 2^N.$$

Now we refine $\mathcal{A}(h, s, t)$ by Maslov grading. With H–data fixed, the Maslov grading of a diagram D is given, up to a constant, by the number of matched pairs at which D is crossed. Denote by $\mathcal{A}_n(h, s, t)$ the \mathbb{Z}_2 vector subspace of $\mathcal{A}(h, s, t)$ spanned by diagrams with crossings at precisely n matched pairs.

Once each all-on once or doubly occupied pair is selected to contain a crossed or noncrossed diagram, all local diagrams are uniquely determined, except that at each noncrossed all-on doubly occupied pair, there are 2 possible diagrams. Thus there are

$2^{L-i} \binom{L}{i} \binom{N}{n-i}$ diagrams with crossings at n matched pairs, and i crossed all-on doubly occupied pairs, and we have the first equality in

$$(4) \quad \dim \mathcal{A}_n(h, s, t) = \sum_i 2^{L-i} \binom{L}{i} \binom{N}{n-i} = \sum_k \binom{L}{k} \binom{N+k}{n}.$$

For the second equality, fix a reference diagram D_0 with H–data (h, s, t) and no crossings. (Such a diagram always exists locally, by Table 1, and the local diagrams glue together.) Consider a diagram D in $\mathcal{A}_n(h, s, t)$ and let k be the number of all-on doubly occupied pairs at which D and D_0 differ. There are $\binom{L}{k}$ ways in which we can choose these k pairs. Now the n pairs with crossings must come from the k all-on doubly occupied pairs just chosen, together with the N all-on once occupied pairs. There are $\binom{N+k}{n}$ ways to choose which of these $N + k$ pairs will be crossed. The equality now follows from the observation that once such choices are made, the diagram D is uniquely determined.

We remark that it is also possible to prove directly that the two summations are equal.

Next, we consider the dimension of the spaces of *boundaries* and *cycles* in $\mathcal{A}_n(h, s, t)$. Let $B_n(h, s, t)$ and $Z_n(h, s, t)$ (or just B_n and Z_n) respectively denote the \mathbb{Z}_2 vector subspaces of $\mathcal{A}_n(h, s, t)$ generated by boundaries and cycles. In other words, for any $n \geq 0$, the map $\partial: \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n$ has image B_n and kernel Z_{n+1} . When (h, s, t) is twisted, $\mathcal{A}(h, s, t)$ has trivial homology, so $B_n = Z_n$ for all n . When (h, s, t) is tight, since homology is 1–dimensional and supported in $n = 0$, we have $B_n = Z_n$ for all $n \geq 1$, and $\dim Z_0 = \dim B_0 + 1$.

Lemma 2.44 $\dim B_n(h, s, t) = \sum_i 2^{L-i} \binom{L}{i} \binom{N-1}{n-i} = \sum_k \binom{L}{k} \binom{N+k-1}{n}.$

Proof For all $n \geq 0$ we have $Z_{n+1} = B_{n+1}$, so $\dim B_n = \dim \mathcal{A}_{n+1} - \dim B_{n+1}$, so that

$$\dim B_n = \dim \mathcal{A}_{n+1} - \dim \mathcal{A}_{n+2} + \dim \mathcal{A}_{n+3} - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \dim \mathcal{A}_{n+k}.$$

From (4) and the identity $\sum_{k=1}^{\infty} (-1)^{k+1} \binom{a}{b+k} = \binom{a-1}{b}$, the above is equal to

$$\sum_{k=1}^{\infty} \sum_i (-1)^{k+1} 2^{L-i} \binom{L}{i} \binom{N}{n+k-i} = \sum_i 2^{L-i} \binom{L}{i} \binom{N-1}{n-i},$$

giving the first claimed equality; the second follows from the identity of equation (4). \square

2.14 An ideal in the strand algebra

We now introduce an ideal \mathcal{F} in $\mathcal{A} = \mathcal{A}(\mathcal{Z})$, which will be useful for computations, as we will see especially in Sections 3.2 and 5.2. A related notion appears in [13, Section 4.3].

Definition 2.45 The \mathbb{Z}_2 -submodule of \mathcal{A} generated by diagrams which are not viable, or have at least one doubly occupied crossed pair b_P , is denoted by \mathcal{F} .

Lemma 2.46 \mathcal{F} is a two-sided ideal of \mathcal{A} .

Proof First we observe that if D and D' are diagrams where D is not viable, then DD' and $D'D$ are zero or nonviable. For D then has some step covered by two or more strands, so DD' is either zero, or has a step covered by two or more strands, hence is not viable; similarly for $D'D$.

Now suppose D is viable and has a b_P . After multiplication on either side by D' the result may become nonviable, in which case it lies in \mathcal{F} . If the result is viable, then it still has a b_P . □

The quotient \mathcal{A}/\mathcal{F} is freely generated as a \mathbb{Z}_2 -module by viable diagrams without crossed doubly occupied pairs. Products can then be taken as in \mathcal{A} , unless the result is nonviable or has a crossed doubly occupied pair, in which case the result is zero.

The decomposition $\mathcal{A} \cong \bigoplus_{h,s,t} \mathcal{A}(h, s, t)$ descends to the quotient \mathcal{A}/\mathcal{F} . However, the differential ∂ does not, as it does not preserve \mathcal{F} ; so \mathcal{A}/\mathcal{F} is not naturally a DGA. We make the following definitions.

- Definition 2.47**
- (i) The \mathbb{Z}_2 -algebra $\bar{\mathcal{A}}$ is the quotient algebra \mathcal{A}/\mathcal{F} .
 - (ii) The \mathbb{Z}_2 vector space $\bar{\mathcal{A}}(h, s, t)$ is the (h, s, t) graded summand of $\bar{\mathcal{A}}$.
 - (iii) For $x \in \mathcal{A}$, we denote by \bar{x} its image in $\bar{\mathcal{A}}$ under the quotient map $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.
 - (iv) For a homomorphism f with image in \mathcal{A} , we denote by \bar{f} the homomorphism obtained by composing f with the quotient map $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.
 - (v) The *standard form* $x \in \mathcal{A}$ of an $\bar{x} \in \bar{\mathcal{A}}$ is the sum of viable diagrams without crossed doubly occupied pairs whose image under the quotient map $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ is \bar{x} .

The quotient $\bar{\mathcal{A}}$ is useful for our needs. Nonviable diagrams cannot contribute to homology, and although some crossed diagrams can be “salvaged” into tight diagrams (thus contributing to homology) via sublimation, sublimation does not apply to crossed doubly occupied pairs. Thus $\bar{\mathcal{A}}$ is generated by diagrams which are “salvageable” in this sense.

3 Cycle selection and creation operators

3.1 Cycle selection homomorphisms

Throughout this section we fix an arc diagram \mathcal{Z} .

The construction of A_∞ operations on \mathcal{H} begins from the map $f_1 : \mathcal{H} \rightarrow \mathcal{A}$ (Section 1.3) as follows.

Definition 3.1 A cycle selection map is a \mathbb{Z}_2 –module homomorphism $f : \mathcal{H} \rightarrow \mathcal{A}$ which preserves Maslov and H–gradings, and sends each homology class $x \in \mathcal{H}$ to a cycle in \mathcal{A} which represents x .

Constructing such a map finds diagrams (as in Lemma 2.23) representing each homology class.

The following constraint is a natural one to make, avoiding a proliferation of diagrams.

Definition 3.2 (diagrammatically simple homomorphisms) A \mathbb{Z}_2 –module homomorphism $f : \mathcal{A} \rightarrow \mathcal{A}$ (resp. $\mathcal{H} \rightarrow \mathcal{A}$) is *diagrammatically simple* if for each diagram D (resp. each $M \in \mathcal{H}$ that can be represented by a single diagram), $f(D)$ (resp. $f(M)$) is zero, or a single diagram.

Recall from Section 2.8 that a summand $\mathcal{H}(h, s, t)$ of \mathcal{H} is nonzero precisely when (h, s, t) is tight, in which case $\mathcal{H}(h, s, t) \cong \mathbb{Z}_2$, generated by $M_{h,s,t}$. To define a diagrammatically simple $f : \mathcal{H} \rightarrow \mathcal{A}$, we select, for each tight H–data $(h, s, t) \in \mathbf{g}$ (Definition 2.22), a tight diagram with that H–data. There are precisely 2^L diagrams representing $M_{h,s,t}$, where L is the number of 11 doubly occupied pairs in (h, s, t) (Lemma 2.23). Selecting one of these 2^L choices for each $(h, s, t) \in \mathbf{g}$ yields a diagrammatically simple cycle selection map; and all diagrammatically simple cycle selection maps are of this form.

To formally describe all diagrammatically simple cycle selection homomorphisms, recall (as is standard in set theory) that a *set choice function* for a set S (whose elements are sets) assigns to each $x \in S$ an element of x . The set of set choice functions for S is naturally in bijection with the direct product of S (ie the direct product of the elements of S), denoted by $\prod S$. We regard an element of $\prod S$ as a choice function for S . If S is empty, S has a unique choice function, which is the null function.

Definition 3.3 (pair choice function) For $(h, s, t) \in \mathbf{g}$, let $\mathbf{P}_{h,s,t}$ be the set of all-on doubly occupied pairs of (h, s, t) . A *pair choice function* for (h, s, t) is a set choice function for $\mathbf{P}_{h,s,t}$.

Since $\mathbf{P}_{h,s,t}$ is a set of sets, each with two elements, $|\prod \mathbf{P}_{h,s,t}| = 2^{|\mathbf{P}_{h,s,t}|} = 2^L$. If $L = 0$, then (h, s, t) has a unique (null) pair choice function.

Given a pair choice function $\mathcal{C}(h, s, t)$ for $(h, s, t) \in \mathbf{g}$, we draw a tight diagram $D_{\mathcal{C}(h,s,t)}$ with H–data (h, s, t) as follows. At a matched pair $P = \{p, q\} \in \mathbf{P}_{h,s,t}$ (ie all-on doubly occupied), $\mathcal{C}(h, s, t)(P)$ is one of the places p or q . There are two tight local diagrams g_p, g_q (Definition 2.18) with H–data (h_p, s_p, t_p) at P ; we draw $g_{\mathcal{C}(h,s,t)(P)}$, the diagram with strands beginning and ending at $\mathcal{C}(h, s, t)(P)$. At a matched pair $P \notin \mathbf{P}_{h,s,t}$, we draw the unique tight local diagram with H–data (h_p, s_p, t_p) . Putting these local diagrams together gives $D_{\mathcal{C}(h,s,t)}$.

Definition 3.4 (cycle choice function) A *cycle choice function* for \mathcal{Z} is a function which assigns to each $(h, s, t) \in \mathbf{g}(\mathcal{Z})$ a pair choice function for (h, s, t) .

A cycle choice function can be regarded as an element of the set $\prod_{(h,s,t) \in \mathbf{g}} \prod \mathbf{P}_{h,s,t}$.

If \mathcal{C} is a cycle choice function, we write $\mathcal{C}(h, s, t)$ for the pair choice function assigned to $(h, s, t) \in \mathbf{g}$; then $\mathcal{C}(h, s, t)$ determines a tight diagram $D_{\mathcal{C}(h,s,t)}$ with H–data (h, s, t) as described above.

A cycle choice function \mathcal{C} determines a map $f^{\mathcal{C}}: \mathcal{H} \rightarrow \mathcal{A}$ as follows. For $(h, s, t) \in \mathbf{g}$, $\mathcal{H}(h, s, t) \cong \mathbb{Z}_2$ generated by $M_{h,s,t}$, and we set $f^{\mathcal{C}}(M_{h,s,t}) = D_{\mathcal{C}(h,s,t)}$. Combining such maps over $(h, s, t) \in \mathbf{g}$ yields a diagrammatically simple cycle selection map $f^{\mathcal{C}}: \mathcal{H} \rightarrow \mathcal{A}$. Indeed, all diagrammatically simple cycle selection maps are of this form, and distinct \mathcal{C} yield distinct $f^{\mathcal{C}}$, giving the following.

Lemma 3.5 *Let $f : \mathcal{H} \rightarrow \mathcal{A}$ be a cycle selection homomorphism. Then f is diagrammatically simple if and only if $f = f^{\mathcal{C}}$ for a unique cycle choice function \mathcal{C} . \square*

In other words, there is a bijective correspondence between diagrammatically simple cycle selection maps, and cycle choice functions.

A general cycle selection map f (not necessarily diagrammatically simple), for each $(h, s, t) \in \mathbf{g}$, assigns to $M_{h,s,t}$ not necessarily one, but a sum of diagrams representing $M_{h,s,t}$, all of the same H–grading and Maslov grading, hence tight diagrams representing $M_{h,s,t}$. As $f(M_{h,s,t})$ represents $M_{h,s,t}$, $f(M_{h,s,t})$ must be the sum of an odd number of distinct diagrams. Conversely, if for each $(h, s, t) \in \mathbf{g}$ we define $f(M_{h,s,t})$ to be the sum of an odd number of distinct tight diagrams representing $M_{h,s,t}$, we obtain a cycle selection homomorphism.

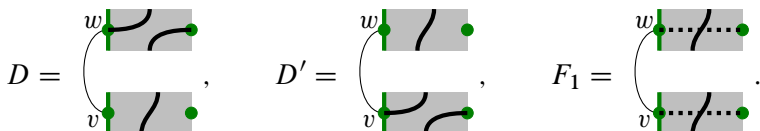
3.2 Differences in cycle selection

The different choices available in cycle selection are related to the ideal \mathcal{F} introduced in Definition 2.45.

Lemma 3.6 *Let $D_1, \dots, D_{2n} \in \mathcal{A}$ be an even number of distinct tight diagrams, all representing the homology class $M \in \mathcal{H}$. Then we have the following:*

- (i) $D_1 + \dots + D_{2n} \in \partial\mathcal{F}$.
- (ii) If $g \in \mathcal{A}$ is homogeneous in Maslov grading and H–data, and $\partial g = D_1 + \dots + D_{2n}$, then $g \in \mathcal{F}$.

Proof We first prove (i) when $n = 1$, so take diagrams D and D' which differ by switching strands at some all-on doubly occupied pairs P_1, \dots, P_k (Lemma 2.23). We proceed by induction on k . When $k = 1$, let F_1 be the diagram all-on doubly occupied crossed at P_1 , and equal to D and D' elsewhere:



Then F_1 is viable, crossed at P_1 , hence lies in \mathcal{F} , and is tight elsewhere; so $\partial F_1 = D + D'$, as desired.

Now consider D, D' differing at k pairs. Switch strands of D at P_1 to obtain D'' . By induction

$$D + D'' = \partial F_1 \quad \text{and} \quad D'' + D' = \partial(F_2 + \cdots + F_k)$$

for some viable diagrams F_1, \dots, F_k , with each F_i crossed at P_i (hence in \mathcal{F}) and tight elsewhere. Thus $D + D' = \partial(F_1 + \cdots + F_k)$, proving (i) when $n = 1$. For general n , simply split the diagrams D_1, \dots, D_{2n} into pairs and apply the $n = 1$ case.

If g is homogeneous in Maslov grading and H–data and $\partial g = D_1 + \cdots + D_{2n}$, then every diagram G in g is viable and has precisely one pair with a crossed local diagram. From Table 1 we see that crossings can only occur in viable diagrams at pairs which are all-on once occupied or all-on doubly occupied. But having tight H–data, G has no all-on once occupied pairs (Proposition 2.21). So G has a crossing at an all-on doubly occupied pair, and $G \in \mathcal{F}$. Hence $g \in \mathcal{F}$. □

3.3 Creation operators

Let (h, s, t) be viable H–data which is all-on once occupied at a pair $P = \{p, q\}$, occupied at p . As in Section 2.8, $\mathcal{A}_P(h_P, s_P, t_P) \cong C'_P$ as a chain complex (Definition 2.18), which has trivial homology:

$$0 \rightarrow \mathbb{Z}_2\langle c_p \rangle \xrightarrow{\partial} \mathbb{Z}_2\langle w_p \rangle \rightarrow 0, \quad \text{where } \partial c_p = w_p.$$

Here c_p is the unique local crossed diagram, w_p is the unique local twisted diagram.

There is a unique chain homotopy $A^*: C'_P \rightarrow C'_P$ from the identity to 0, given as follows.

Definition 3.7 (local creation operator) The creation operator $A^*: C'_P \rightarrow C'_P$ is the \mathbb{Z}_2 –module homomorphism given by $A^*(w_p) = c_p$ and $A^*(c_p) = 0$.

In other words, A^* inserts a crossing, as in Figure 1. The name A^* references creation operators in physics. We have $A^*\partial + \partial A^* = 1$, a ‘‘Heisenberg relation’’ or a chain homotopy from the identity to 0.

Consider $\mathcal{A}(h, s, t) \cong \bigotimes_{P'} \mathcal{A}_{P'}(h_{P'}, s_{P'}, t_{P'})$ (Section 2.5). We can rewrite this as

$$(5) \quad \mathcal{A}(h, s, t) \cong \mathcal{A}_P(h_P, s_P, t_P) \otimes \bigotimes_{P' \neq P} \mathcal{A}_{P'}(h_{P'}, s_{P'}, t_{P'}).$$

A diagram $D \in \mathcal{A}(h, s, t)$ is then $x \otimes y$, where $x \in \mathcal{A}_P(h_P, s_P, t_P) \cong C'_P$ and $y \in \bigotimes_{P' \neq P} \mathcal{A}_{P'}(h_{P'}, s_{P'}, t_{P'})$.

Definition 3.8 (creation operator) Let P be a 11 once occupied pair of viable (h, s, t) . The creation operator $A_P^* : \mathcal{A}(h, s, t) \rightarrow \mathcal{A}(h, s, t)$ is given by $A_P^* = A^* \otimes 1$, in the tensor decomposition (5) above.

In other words, A_P^* inserts a crossing at P . Clearly A_P^* is diagrammatically simple (Definition 3.2). Note that if $D \in \mathcal{F}$ (ie D has a crossed doubly occupied pair: see Definition 2.45), then $A_P^* D \in \mathcal{F}$ also. So A_P^* descends to a map $\bar{A}_P^* : \bar{\mathcal{A}}(h, s, t) \rightarrow \bar{\mathcal{A}}(h, s, t)$.

Lemma 3.9 With P and (h, s, t) as above, $A_P^* \partial + \partial A_P^* = 1$ on $\mathcal{A}(h, s, t)$.

Proof Take a diagram in $\mathcal{A}(h, s, t)$ and write it as $x \otimes y$ according to the decomposition (5) above, so $x = c_P$ or w_P . Recalling that $\partial c_P = w_P$, $\partial w_P = 0$, $A^* w_P = c_P$ and $A^* c_P = 0$, we have

$$\begin{aligned} (A_P^* \partial + \partial A_P^*)(w_P \otimes y) &= A_P^*(w_P \otimes \partial y) + \partial(c_P \otimes y) = w_P \otimes y, \\ (A_P^* \partial + \partial A_P^*)(c_P \otimes y) &= c_P \otimes y. \end{aligned} \quad \square$$

This chain homotopy shows directly that $\mathcal{H}(h, s, t) = 0$ when there is an all-on once occupied pair (Proposition 2.21). In fact, creation operators are the *only* way to obtain a diagrammatically simple (Definition 3.2) chain homotopy to the identity on a summand $\mathcal{A}(h, s, t)$.

Lemma 3.10 Suppose that (h, s, t) is viable and nonsingular, and that $f : \mathcal{A}(h, s, t) \rightarrow \mathcal{A}(h, s, t)$ is a diagrammatically simple \mathbb{Z}_2 -module homomorphism which has pure Maslov degree, satisfying

$$\int \partial + \partial \int = 1.$$

Then $f = A_P^*$ for some all-on once occupied matched pair P of (h, s, t) .

Proof The existence of f implies $\mathcal{H}(h, s, t) = 0$; being nonsingular then (h, s, t) is twisted, so there is an all-on once occupied pair. With (h, s, t) fixed, Maslov degree is given, up to a constant, by the number of pairs at which a diagram is crossed. Since $\int \partial + \partial \int = 1$ and ∂ has Maslov degree -1 , f has Maslov degree 1.

We use the decomposition $\mathcal{A}(h, s, t) \cong \bigotimes_P \mathcal{A}_P(h_P, s_P, t_P)$, noting (Section 2.8) that each $\mathcal{A}_P(h_P, s_P, t_P)$ is isomorphic (as a chain complex) to C_P and C'_P or C''_P (Definition 2.18).

Take an arbitrary crossingless diagram D_0 with H-data (h, s, t) . Then D_0 is twisted at each 11 once occupied pair and $\partial D_0 = 0$. From $\int \partial + \partial \int = 1$ we have $\partial \int D_0 = D_0$. As \int is diagrammatically simple, $\int D_0$ is a diagram whose differential is D_0 . The only such diagrams are those obtained from D_0 by inserting a crossing at an all-on once occupied pair $P = \{p, p'\}$ (say occupied at p), ie $\int D_0 = A_P^* D_0$.

We claim that for any diagram D with H-data (h, s, t) , $\int D = A_P^* D$. The proof is by induction on the number k of pairs at which D is crossed (ie up to a constant, Maslov grading).

Suppose D, D' are distinct crossingless diagrams with H-data (h, s, t) which differ by switching strands at a single all-on doubly occupied pair $Q = \{q, q'\}$. The argument above shows that $\int D = A_R^* D$ for some all-on once occupied pair $R = \{r, r'\}$ (occupied at r), and similarly that $\int D' = A_V^* D'$ for some all-on once occupied pair $V = \{v, v'\}$ (occupied at v). We claim $R = V$. To see why, suppose $R \neq V$ and consider $\mathcal{A}(h, s, t)$ as a tensor product. We may write

$$\begin{aligned} D &= g_q \otimes w_r \otimes w_v \otimes z, & D' &= g_{q'} \otimes w_r \otimes w_v \otimes z, \\ \int D &= g_q \otimes c_r \otimes w_v \otimes z, & \int D' &= g_{q'} \otimes w_r \otimes c_v \otimes z, \end{aligned}$$

where the four tensor factors are given by C''_Q, C'_R, C'_V , and all other matched pairs. Consider the diagram $E = c_Q \otimes w_r \otimes w_v \otimes z$ obtained from D or D' by inserting crossings at Q . We compute

$$\int \partial E = \int (D + D') = g_q \otimes c_r \otimes w_v \otimes z + g_{q'} \otimes w_r \otimes c_v \otimes z,$$

and hence $\int E$ is a single diagram (by diagrammatic simplicity) whose differential is

$$\partial \int E = \left(\int \partial + 1 \right) E = g_q \otimes c_r \otimes w_v \otimes z + g_{q'} \otimes w_r \otimes c_v \otimes z + c_Q \otimes w_r \otimes w_v \otimes z.$$

The three diagrams on the right respectively have crossings at R, V and Q . Hence $\int E$ must have crossings at R, V and Q , contradicting the fact that \int has Maslov degree 1. We conclude that $R = V$.

Thus, if D and D' are crossingless and differ by strand switching at a single matched pair, then $\int D$ and $\int D'$ are both given by applying a creation operator A_P^* at the same matched pair P . Since all crossingless diagrams with H-data (h, s, t) are related by strand switching, repeatedly applying this fact gives $\int D = A_P^* D$ for any crossingless D with H-data (h, s, t) . This proves the result when $k = 0$.

Now take a $k \geq 0$ and suppose that for all diagrams D with H–data (h, s, t) and crossings at $\leq k$ pairs, $\int D = A_P^* D$. Consider a diagram D with H–data (h, s, t) , crossed at $k + 1$ pairs. Then $D = w_p \otimes x$ or $c_p \otimes x$, where the two tensor factors refer to C'_p , and everywhere else.

If $D = w_p \otimes x$ then $\partial D = w_p \otimes \partial x$, which contains diagrams crossed at k pairs. By induction then

$$\int \partial D = A_P^* \partial D = A_P^* (w_p \otimes \partial x) = c_p \otimes \partial x.$$

It follows that $\int D$ is a single diagram (by diagrammatic simplicity) whose differential is

$$\partial \int D = \left(\int \partial + 1 \right) D = c_p \otimes \partial x + w_p \otimes x.$$

There is only one such diagram, namely $c_p \otimes x$. Thus $\int D = c_p \otimes x = A_P^* (w_p \otimes x) = A_P^* D$.

If $D = c_p \otimes x$ then $\partial D = w_p \otimes x + c_p \otimes \partial x$ and so by induction $\int \partial D = A_P^* \partial D = c_p \otimes x = D$. We then have $\partial \int D = \int \partial D + D = 0$, so $\int D$ is a single diagram crossed at $k + 1 \geq 1$ pairs, or zero, whose differential is zero. Thus $\int D = 0 = A_P^* D$.

Thus, in any case, $\int D = A_P^* D$. By induction then $\int = A_P^*$. □

If we drop the requirement that \int be diagrammatically simple, the result no longer holds: there are many \mathbb{Z}_2 –module homomorphisms $\mathcal{A}(h, s, t) \rightarrow \mathcal{A}(h, s, t)$ of pure Maslov degree satisfying $\partial \int + \int \partial = 1$ which are not creation operators. (For instance, take a sum of an odd number of creation operators.)

3.4 Inverting the differential

The following straightforward lemma shows how a creation operator A_P^* finds partial inverses of the differential (hence the notation \int). This is required in constructing an A_∞ structure.

Lemma 3.11 *Suppose the viable H–data (h, s, t) contains an all-on once occupied pair P . If $x \in \mathcal{A}(h, s, t)$ is a cycle, then $x = \partial A_P^* x$.*

Proof As x is a cycle, $\partial x = 0$. Hence $x = (A_P^* \partial + \partial A_P^*)x = \partial A_P^* x$. □

Recall from Section 2.13 the decomposition $\mathcal{A}(h, s, t) = \bigoplus_n \mathcal{A}_n(h, s, t)$ over Maslov grading, where $\mathcal{A}_n(h, s, t)$ contains diagrams with crossings at n pairs, and the subspaces $Z_n(h, s, t)$ of cycles and $B_n(h, s, t)$ of boundaries. We are interested in maps obeying the following property.

Definition 3.12 (inverting differential) A \mathbb{Z}_2 -module homomorphism

$$\int : Z_n(h, s, t) \rightarrow \mathcal{A}_{n+1}(h, s, t)$$

inverts the differential if, for all $x \in Z_n(h, s, t)$, the equation $x = \partial \int x$ holds.

If we have maps inverting the differential on $Z_n(h, s, t)$ for all n , of course these can be combined into a map $Z(h, s, t) \rightarrow \mathcal{A}(h, s, t)$ of Maslov degree 1 such that $\partial \int = 1$.

Lemma 3.11 says that A_p^* — more precisely, its restriction to $Z_n(h, s, t)$ — inverts the differential.

When we have viable H-data (h, s, t) with several all-on once occupied matched pairs P_1, P_2, \dots , there are several creation operators $A_{P_1}^*, A_{P_2}^*, \dots$ on $\mathcal{A}(h, s, t)$, and hence many ways to invert the differential. However, not every operator which inverts the differential is a creation operator.

For one thing, we can simply choose a different creation operator on each Maslov summand. For another, we can also replace a creation operator with a sum of an odd number of creation operators.

More fundamentally, however, not every operator $\int : Z_n(h, s, t) \rightarrow \mathcal{A}_{n+1}(h, s, t)$ inverting the differential is a sum of creation operators. If (h, s, t) is twisted H-data with $L \geq 0$ all-on doubly occupied pairs, and $N \geq 1$ all-on once occupied pairs, then the span of creation operators $Z_n(h, s, t) \rightarrow \mathcal{A}_{n+1}(h, s, t)$ is a \mathbb{Z}_2 vector space of dimension N . But the set \mathcal{S} of maps $Z_n(h, s, t) \rightarrow \mathcal{A}_{n+1}(h, s, t)$ inverting the differential is an affine vector space affine isomorphic to the set \mathcal{T} of \mathbb{Z}_2 -module homomorphisms $Z_n(h, s, t) \rightarrow Z_{n+1}(h, s, t)$. Indeed, we observe that $T \in \mathcal{T}$ if and only if $A_p^* + T \in \mathcal{S}$, where P is any all-on once occupied pair. Thus, by Lemma 2.44,

$$(6) \quad \dim \mathcal{S} = \dim Z_n(h, s, t) \dim Z_{n+1}(h, s, t) \\ = \left[\sum_k \binom{L}{k} \binom{N+k-1}{n} \right] \left[\sum_k \binom{L}{k} \binom{N+k-1}{n+1} \right].$$

The expression (6) is in general much larger than N . For instance, taking $N = 1$, $L = 1$ and $n = 0$ we have $[\sum_k \binom{L}{k} \binom{N+k-1}{n}] [\sum_k \binom{L}{k} \binom{N+k-1}{n+1}] = 2 > 1 = N$; taking

$N = 4$, $L = 0$ and $n = 1$, the dimensions are $9 > 4$. So there exist many more maps inverting the differential than linear combinations of creation operators, as mentioned in [Section 1.3](#).

The above deals with inverting the differential when (h, s, t) is twisted, ie there is at least one all-on once occupied pair. When there are *no* all-on once occupied pairs, ie (h, s, t) is tight, the H–data only permits crossings in places which immediately land us in \mathcal{F} .

Lemma 3.13 *Suppose (h, s, t) is tight. If $0 \neq x \in \mathcal{A}(h, s, t)$ has pure Maslov grading, and $x = \partial f$ for some $f \in \mathcal{A}(h, s, t)$ also of pure Maslov grading, then $f \in \mathcal{F}$.*

Proof Since f has pure Maslov grading and $\partial f = x$, f is a nonzero sum of viable diagrams, each crossed at one more matched pair than x . From [Table 1](#), crossings can only occur at all-on once or doubly occupied pairs. But tight (h, s, t) have none of the former ([Proposition 2.21](#)), so $f \in \mathcal{F}$. □

3.5 Global creation operators

For any twisted H–data (h, s, t) (ie $(h, s, t) \in \mathbf{w}(\mathcal{Z})$, see [Definition 2.22](#)), there is an all-on once occupied pair P ([Proposition 2.21](#)), and hence a creation operator A_P^* on $\mathcal{A}(h, s, t)$. We now introduce formalism to piece together such operators into a “global” operator on all twisted summands.

Definition 3.14 *A creation choice function \mathcal{C} for \mathcal{Z} assigns to each $(h, s, t) \in \mathbf{w}(\mathcal{Z})$ one of its all-on once occupied matched pairs $\mathcal{C}(h, s, t)$.*

Hence for each $(h, s, t) \in \mathbf{w}(\mathcal{Z})$, \mathcal{C} selects a creation operator $A_{\mathcal{C}(h,s,t)}^* : \mathcal{A}(h, s, t) \rightarrow \mathcal{A}(h, s, t)$.

Definition 3.15 Let \mathcal{C} be a creation choice function for \mathcal{Z} . The *creation operator* of \mathcal{C} is the \mathbb{Z}_2 –module homomorphism

$$A_{\mathcal{C}}^* : \bigoplus_{(h,s,t) \in \mathbf{w}} \mathcal{A}(h, s, t) \rightarrow \bigoplus_{(h,s,t) \in \mathbf{w}} \mathcal{A}(h, s, t) \quad \text{given by } A_{\mathcal{C}}^* = \bigoplus_{(h,s,t) \in \mathbf{w}} A_{\mathcal{C}(h,s,t)}^*.$$

Putting together what we know on each summand, in particular the Heisenberg relation ([Lemma 3.9](#)), classification of chain homotopies ([Lemma 3.10](#)) and differential inversion ([Lemma 3.11](#)), we immediately obtain the following.

Proposition 3.16 *The creation operator A_C^* of a creation choice function C preserves H -grading, has Maslov degree 1, and satisfies the following:*

$$A_C^* \partial + \partial A_C^* = 1 \quad \text{and} \quad \partial A_C^* x = x \quad \text{for any cycle } x \in \bigoplus_{(h,s,t) \in \mathbf{w}} \mathcal{A}(h, s, t).$$

Conversely, suppose $f: \bigoplus_{(h,s,t) \in \mathbf{w}} \mathcal{A}(h, s, t) \rightarrow \bigoplus_{(h,s,t) \in \mathbf{w}} \mathcal{A}(h, s, t)$ is a diagrammatically simple \mathbb{Z}_2 -module homomorphism which preserves H -data, has pure Maslov degree, and satisfies

$$\int \partial + \partial \int = 1.$$

Then f is the creation operator A_C^ of a creation choice function C . □*

3.6 Cycle selection and creation operators via ordering

In Section 3.1 we defined a diagrammatically simple cycle selection homomorphism $f^C: \mathcal{H} \rightarrow \mathcal{A}$ for any cycle choice function C . Then in Section 3.5 we defined a creation operator A_C^* for any creation choice function C . We now discuss a useful method to obtain such “choice functions”, of both types.

Definition 3.17 Let the pairs of \mathcal{Z} be $P_1 = \{p_1, p'_1\}, \dots, P_k = \{p_k, p'_k\}$. A pair ordering on \mathcal{Z} consists of a total order on each of the sets

$$\{P_1, \dots, P_k\}, P_1, P_2, \dots, P_k.$$

Thus a pair ordering puts the pairs of \mathcal{Z} in some order; and also puts the two places of each pair in some order. We denote a pair ordering by \preceq , and use this symbol for each of the total orders involved.

We note that \mathcal{Z} comes with several naturally ordered sets that can be used to give a pair ordering. Recall that \mathcal{Z} consists of l intervals Z_1, \dots, Z_l . Each interval is naturally totally ordered. Listing them as Z_1, \dots, Z_l orders them. Then \mathcal{Z} is totally ordered, and as places lie on \mathcal{Z} , they inherit a total order. The ordering on places can also be used to obtain an ordering on the set $\{P_1, \dots, P_k\}$, in various reasonable ways: for instance if $P_i = \{p_i, p'_i\}$ and $P_j = \{p_j, p'_j\}$, we could define $P_i < P_j$ when $\min_{\preceq} \{p_i, p'_i\} < \min_{\preceq} \{p_j, p'_j\}$. Thus we obtain a pair ordering. But there is nothing natural about this way to order pairs, just as there is nothing natural about the ordering Z_1, \dots, Z_l of intervals; reordering the Z_i yields a homeomorphic arc diagram, but an entirely different pair ordering.

Nonetheless, from a pair ordering, we naturally obtain cycle choice and creation choice functions.

Definition 3.18 Let \preceq be a pair ordering on \mathcal{Z} .

- (i) The *cycle choice function* of \preceq , denoted by \mathcal{CY}^{\preceq} , assigns to each tight $(h, s, t) \in \mathbf{g}(\mathcal{Z})$ the pair choice function on $\mathbf{P}_{h,s,t}$ which chooses from each all-on doubly occupied pair its \preceq -minimal place.
- (ii) The *creation choice function* of \preceq , denoted by \mathcal{CR}^{\preceq} , assigns to each twisted set of H–data $(h, s, t) \in \mathbf{w}(\mathcal{Z})$ its \preceq -minimal all-on once occupied matched pair.

Note that the definition of \mathcal{CY}^{\preceq} uses the ordering on the P_i , while the definition of \mathcal{CR}^{\preceq} uses the ordering on $\{P_1, \dots, P_k\}$.

Thus, if $P_i = \{p_i, p'_i\}$ is a 11 doubly occupied pair for tight H–data (h, s, t) , with $p_i < p'_i$, then $f^{\mathcal{CY}^{\preceq}}$ always chooses a diagram with strands beginning and ending at p_i rather than p'_i . And if the pairs of \mathcal{Z} are ordered as $P_1 < P_2 < \dots < P_k$, then for twisted H–data, the creation operator $A_{\mathcal{CR}^{\preceq}}^*$ inserts a crossing at P_1 , if it is 11 once occupied; otherwise at P_2 , if it is 11 once occupied; and so on.

Clearly not every cycle choice function arises from a pair ordering, nor does every creation choice function. Nonetheless pair orderings provide a useful method to construct cycle choice functions and creation choice functions, and thus to construct A_∞ structures on \mathcal{H} .

4 Constructing A–infinity structures

4.1 The construction

We now describe Kadeishvili’s construction of [8] (introduced in Section 1.3) in detail, and then adapt it for our purposes.

Let \mathcal{A} be a DGA, regarded as an A_∞ algebra with trivial n –ary operations for $n \geq 3$. Given a cycle selection map $f_1: \mathcal{H} \hookrightarrow \mathcal{A}$, the construction produces an A_∞ structure X on \mathcal{H} with $X_1 = 0$ and X_2 being multiplication, together with a morphism of A_∞ algebras $f: \mathcal{H} \rightarrow \mathcal{A}$, consisting of maps $f_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{A}$. The construction builds maps X_n and f_n and auxiliary maps U_n inductively over n . The maps X_n and U_n have grading $n - 2$ and f_n has grading $n - 1$. At each stage, U_n and X_n are determined; there is only choice in constructing f_n .

First, $U_1 = 0$, $X_1 = 0$, and $f_1: H \rightarrow A$ are given. Once U_i, X_i, f_i are defined for $i < n$, we define U_n by

$$(7) \quad U_n(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^{n-1} m_2(f_j(a_1 \otimes \cdots \otimes a_j) \otimes f_{n-j}(a_{j+1} \otimes \cdots \otimes a_n)) + \sum_{k=0}^{n-2} \sum_{j=2}^{n-1} f_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes X_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n),$$

and X_n is then simply the homology class of U_n ,

$$(8) \quad X_n = [U_n].$$

Since f_1 selects cycles, $f_1 X_n$ and U_n differ by a boundary; f_n is then defined by

$$(9) \quad f_1 X_n - U_n = \partial f_n.$$

From equation (9), we see that the choice for f_n at each stage amounts to a choice of inverse for the differential ∂ . It is shown in [8] that any such f_n and X_n have the desired properties.

Applying this construction to strand algebras, we construct all U_n, X_n, f_n to preserve H–data, and we invert ∂ using creation operators.

As it turns out, we only need to construct maps $\bar{f}_n, \bar{U}_n: \mathcal{H}^{\otimes n} \rightarrow \bar{A}$ (Definition 2.47) into the quotient $\bar{A} = \mathcal{A}/\mathcal{F}$.

To construct the cycle selection homomorphism f_1 , we use a cycle choice function \mathcal{CY} (Section 3.1). A cycle choice function can be constructed from a pair ordering on \mathcal{Z} (Section 3.6). To construct f_n for $n \geq 2$, we need to solve equation (9): $f_1 X_n - U_n = \partial f_n$. This can be done separately on each H–summand. On twisted summands, it amounts to inverting the differential (Section 3.4). We apply creation operators on each summand using a creation choice function (Section 3.5). On other summands, it turns out that no choice is necessary, once we project to \bar{A} , and we can take $\bar{f}_n = 0$.

The f_n in our construction satisfy the following condition. The idea is that if $f_1 X_n - U_n = 0$, then it reasonable to say that f_n should also be zero. (The constant of integration is most naturally zero!)

Definition 4.1 Suppose that for all M , if $(f_1 X_n - U_n)(M) = 0$ then $f_n(M) = 0$. In this case we say f_n is *balanced*.

Theorem 4.2 *Let \mathcal{Z} be an arc diagram and let M_i be nonzero homology classes of diagrams on \mathcal{Z} . Let \mathcal{CY} and \mathcal{CR} , respectively, be cycle choice and creation choice functions for \mathcal{Z} . Then there is an A_∞ structure X on $\mathcal{H}(\mathcal{Z})$ with $X_1 = 0$ and X_2 multiplication, and a morphism of A_∞ algebras $f: \mathcal{H}(\mathcal{Z}) \rightarrow \mathcal{A}(\mathcal{Z})$ with $f_1 = f^{\mathcal{CY}}$, such that the following conditions hold:*

- (i) *If $M = M_1 \otimes \cdots \otimes M_n$ is not viable, then $\bar{f}_n(M) = 0$ and $X_n(M) = 0$; and if M has an idempotent mismatch then $f_n(M) = 0$.*
- (ii) *The maps $X_n: \mathcal{H}(\mathcal{Z})^{\otimes n} \rightarrow \mathcal{H}(\mathcal{Z})$ of X and the maps $f_n: \mathcal{H}(\mathcal{Z})^{\otimes n} \rightarrow \mathcal{A}(\mathcal{Z})$ of f all preserve H-data; moreover X_n has Maslov grading $n - 2$ and f_n has Maslov grading $n - 1$.*
- (iii) *Each map f_n is balanced.*
- (iv) *For $n \geq 2$, on each twisted H-summand, $f_n = A_{\mathcal{CR}}^* \circ (f_1 X_n - U_n)$, where U_n is defined by equation (7) and X_n is defined by equation (8) from Section 1.3.*

The maps X_n satisfying these conditions are unique. The maps f_n are uniquely defined modulo \mathcal{F} .

When M is singular, there are no diagrams with its H-data (h, s, t) , and $\mathcal{A}(h, s, t) = 0$ (Lemma 2.35). So f_n and X_n preserving H-data implies that $f_n(M) = 0$ and $X_n(M) = 0$ for singular M .

The uniqueness statement means that, although the f_n are not uniquely determined, after composing with the quotient $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ to obtain $\bar{f}_n: \mathcal{H}^{\otimes n} \rightarrow \bar{\mathcal{A}}$, the maps \bar{f}_n are uniquely determined.

Since (Section 3.6) a pair ordering \preceq determines cycle choice and creation choice functions \mathcal{CY}^\preceq and \mathcal{CR}^\preceq , we immediately obtain the following corollary.

Corollary 4.3 *Let \preceq be a pair ordering on an arc diagram \mathcal{Z} . Then there is an A_∞ structure X on \mathcal{H} , and a morphism of A_∞ algebras $f: \mathcal{H} \rightarrow \mathcal{A}$ satisfying the conditions of Theorem 4.2, such that $f_1 = f^{\mathcal{CY}^\preceq}$, and on twisted summands for $n \geq 2$, $f_n = A_{\mathcal{CR}^\preceq}^* \circ (f_1 X_n - U_n)$. □*

Corollary 4.3 is a precise form of Theorem 1.1.

Proof of Theorem 4.2 We follow the method described above. At level 1, equations (7), (8) and (9) require $U_1 = 0$, $X_1 = 0$ and $\partial f_1 = 0$. The last equation is satisfied by $f_1 = f^{c\mathcal{Y}}$. Since diagrams with nonviable H–data are zero in homology, $f_1 = 0$ for such diagrams.

Now suppose we have constructed all operations at level $< n$ as required; we construct U_n , X_n and f_n .

We define U_n by equation (7). Then U_n has Maslov grading $n - 2$. As the f_j are not uniquely defined, neither is U_n . However, all the \bar{f}_j are uniquely defined, hence by equation (7), \bar{U}_n is also uniquely defined. Since the f_j (and multiplication in \mathcal{A}) preserve H–data, U_n does also.

We define X_n by (8); then X_n respects gradings as required. As in Kadeishvili [8], U_n is a cycle and X_n is its homology class, so X_n is well defined. Now all diagrams in \mathcal{F} are nonviable or have crossings, and such diagrams do not contribute to homology. Thus $X_n(M)$ is determined completely by $\bar{U}_n(M)$, which is uniquely defined; hence $X_n(M)$ is uniquely defined.

To define f_n , we solve equation (9) for each viable $M = M_1 \otimes \cdots \otimes M_n$:

$$\partial f_n(M) = (f_1 X_n - U_n)(M).$$

We consider various cases.

First, suppose $M = M_1 \otimes \cdots \otimes M_n$ is nonviable because of an idempotent mismatch. Then each term in $U_n(M)$ from (7) is zero: by induction, f_i and X_i for $i < n$ are zero on tensor products with mismatches, and the product of two mismatched diagrams is zero. Thus $U_n(M) = 0$, and by (8) then $X_n(M) = 0$. We set $f_n(M) = 0$ as required by the balanced condition; equation (9) is then satisfied.

Next, suppose M is nonviable but has no idempotent mismatch, hence has some step covered more than once. As U_n preserves H–data then $U_n(M)$ is a sum of nonviable diagrams, so $U_n(M) \in \mathcal{F}$ and $\bar{U}_n(M) = 0$. Then $X_n(M) = 0$, and equation (9) then requires $\partial f_n(M) = U_n(M)$. If $U_n(M) = 0$ then we set $f_n(M) = 0$, satisfying the balanced condition; otherwise we choose $f_n(M)$ arbitrarily to be any solution to this equation with the same H–data as M , and of pure Maslov grading (necessarily 1 greater than M). Then $f_n(M) \in \mathcal{F}$, being a sum of nonviable diagrams. Thus $f_n(M)$ is not uniquely determined, but $\bar{f}_n(M)$ is uniquely determined, indeed $\bar{f}_n(M) = 0$.

If M is singular, then as there are no diagrams with the H–data of M , it follows that $U_n(M)$, $X_n(M)$ and $f_n(M)$ are all zero, and all required conditions are satisfied.

So we may now assume that M is viable and nonsingular; hence its H–data (h, s, t) is tight or twisted.

If (h, s, t) is twisted, then as required we take $f_n = A_{\mathcal{CR}}^* \circ (f_1 X_n - U_n)$. Then f_n is balanced. Since $(f_1 X_n - U_n)(M)$ is a boundary, hence a cycle, by Proposition 3.16,

$$\partial f_n(M) = \partial A_{\mathcal{CR}}^*(f_1 X_n(M) - U_n(M)) = (f_1 X_n - U_n)(M).$$

If (h, s, t) is tight, by Lemma 3.13, any $f_n(M)$ of the required Maslov grading and satisfying $\partial f_n(M) = f_1 X_n(M) - U_n(M)$ lies in \mathcal{F} . We choose $f_n(M)$ to be zero if $f_1 X_n - U_n = 0$ (satisfying the balanced condition), and otherwise to be any solution to this equation with the same H–data as M , and pure Maslov grading. Then $f_n(M)$ is not uniquely determined, but $\bar{f}_n(M) = 0$.

This defines f_n and X_n satisfying the required conditions, with the uniqueness claimed. Having followed Kadeishvili’s construction, the X_n form an A_∞ structure on \mathcal{H} , and the f_n form a morphism of A_∞ algebras $\mathcal{H} \rightarrow \mathcal{A}$. □

It follows from this proof that whenever M has tight H–data, $\bar{f}_n(M) = 0$.

4.2 Shorthand notation

For convenience, we use some shorthand for viable nonzero tensor products in $\mathcal{A}^{\otimes n}$, $\bar{\mathcal{A}}^{\otimes n}$ and $\mathcal{H}^{\otimes n}$. The shorthand is essentially a stylised version of our previous diagrams.

Let $M = M_1 \otimes \cdots \otimes M_n \in \mathcal{H}^{\otimes n}$ be a viable tensor product of nonzero homology classes of diagrams on an arc diagram \mathcal{Z} . A shorthand diagram represents M by an array of data. Each row refers to a matched pair P of \mathcal{Z} . The n columns refer to M_1, \dots, M_n . In the row for $P = \{p, p'\}$ and the column of M_i , we write which of the four steps of \mathcal{Z}_P are covered by M_i . Along the row for P , between the columns we draw a hollow or solid circle indicating whether P is contained in the corresponding idempotent (“on or off”). This is well defined since M is viable.

Such notation specifies M completely, since it specifies the H–data of each M_i .

The step before and after a place p are denoted by p_- and p_+ , respectively; p_\pm indicates that both p_+ and p_- are covered. Figure 11 shows an example.

Occasionally, when the idempotents can be inferred from the H–grading of each M_i , we omit the circles in the notation.

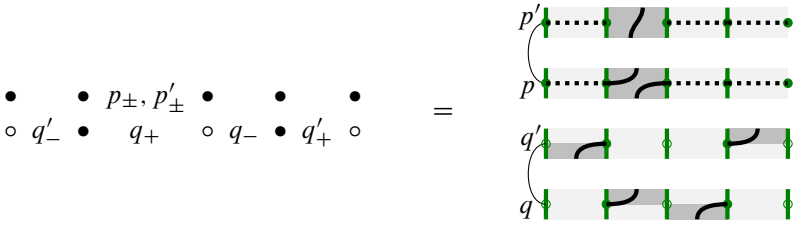


Figure 11: Shorthand notation for viable nonzero tensor products of homology classes of diagrams.

We use a similar notation to write elements of $\mathcal{A}^{\otimes n}$ and $\bar{\mathcal{A}}^{\otimes n}$. For a viable tensor product of diagrams, we write a similar array, however a diagram is not always specified by its H–data, so we use some of the notation of Definition 2.18. When there is a unique tight local diagram with the H–data, we simply write which steps are covered. Otherwise, we use the notation b_p , g_p , c_p and w_p .

When A_∞ operations are defined by a pair ordering, as in Corollary 4.3, we may order the pairs upwards in our array (just as they are ordered along the intervals of \mathbf{Z}). A creation operator then always applies at the all-on once occupied pair which is lowest in our shorthand notation.

We adopt notation where each matched pair is denoted by a capital letter, and its two places by the corresponding lowercase letter, the latter under \leq being primed. Thus we always write pairs as $P = \{p, p'\}$, $Q = \{q, q'\}$, etc, where $p < p'$, $q < q'$, etc. Then a cycle choice function always selects a cycle with strands at a place with an unprimed label.

When a tensor product $M = M_1 \otimes \cdots \otimes M_n$ is twisted at a place p of a pair $P = \{p, p'\}$, it is 11 once occupied at P , with p fully occupied (Table 2), and the two steps p_+ , p_- are covered by some M_i and M_j , with $i < j$. Thus, in shorthand, across the row for P we see p_+ in one column, then p_- in another column, in that order.

Similarly, if P is critical, then it is sesqui-occupied or doubly occupied, and looking across the row corresponding to P we see one of the following sequences, appearing in order, in distinct columns (possibly after relabelling p and p'):

- pre-sesqui-occupied p'_-, p_+, p_- , 00 doubly occupied p_-, p'_+, p'_-, p_+ ,
- post-sesqui-occupied p_+, p_-, p'_+ , 11 doubly occupied p'_+, p'_-, p_+, p_- .

4.3 Low-level maps

We now consider low-level maps in our construction explicitly. We assume A_∞ structures are constructed from a cycle choice function \mathcal{CY} and a creation choice function \mathcal{CR} , as in [Theorem 4.2](#).

Level 1 maps are straightforward ($X_1 = 0$ and $f_1 = f^{\mathcal{CY}}$), as is multiplication X_2 . We consider \bar{f}_2 .

Let $M = M_1 \otimes M_2$ be a tensor product of nonzero homology classes of diagrams, with H–data (h, s, t) . By [Theorem 4.2](#) (and subsequent discussion), if M is nonviable or singular, then $\bar{f}_n(M)$ and $X_n(M)$ are both zero; so we assume M is viable and nonsingular. Then (h, s, t) is tight or twisted.

When (h, s, t) is tight, $\bar{f}_2(M) = 0$. So suppose (h, s, t) is twisted, hence has at least one all-on once occupied pair. Clearly M is then not tight; in fact, M cannot be critical either, since it takes 3 to be critical ([Lemma 2.38](#)). So M is twisted, hence is tight or twisted at each matched pair ([Lemma 2.36](#)). In particular, M is twisted at each all-on once occupied pair, and tight at each other pair.

Since $U_2(M_1 \otimes M_2) = f_1(M_1)f_1(M_2)$, we have

$$(10) \quad f_2(M_1 \otimes M_2) = A_{\mathcal{CR}}^*(f_1(M_1M_2) + f_1(M_1)f_1(M_2)).$$

Since M is twisted, we get $M_1M_2 = 0$ and $f_1(M_1)f_1(M_2) \neq 0$ ([Lemma 2.35](#)). As $f_1(M_1)f_1(M_2)$ is clearly not tight, and it is also not crossed (being the product of two crossingless diagrams — [Lemma 2.25](#)), it is twisted, hence tight or twisted at each pair ([Lemma 2.24](#)). At each 11 once occupied pair $f_1(M_1)f_1(M_2)$ cannot be tight, so must be twisted; and at each other pair, it must be tight. We then have

$$(11) \quad f_2(M_1 \otimes M_2) = A_{\mathcal{CR}}^*(f_1(M_1)f_1(M_2)),$$

where $A_{\mathcal{CR}}^*$ adds a crossing at the pair selected by the creation choice function \mathcal{CR} .

Thus, $f_2(M_1 \otimes M_2)$ is given by a single diagram, which is crossed at the all-on once occupied pair of $M_1 \otimes M_2$ selected by \mathcal{CR} , and is elsewhere given by $f_1(M_1)f_1(M_2)$. The idea is shown in [Figure 12](#). In this way, f_2 turns one pair from twisted to crossed.

When $\mathcal{CR} = \mathcal{CR}^{\preceq}$ is the creation choice function of a pair ordering, $A_{\mathcal{CR}}^*$ adds a crossing at the \preceq –minimal all-on once occupied pair of $f_1(M_1)f_1(M_2)$.

$$f_2(\bullet \ p'_+ \circ \ p'_- \ \bullet) = f_2\left(\begin{array}{c} p' \\ \text{---} \\ p \end{array}\right) = \begin{array}{c} p' \\ \text{---} \\ p \end{array} = \bullet \ c_{p'} \ \bullet$$

Figure 12: The effect of f_2 .

5 Properties of A–infinity structures

We now consider A_∞ structures on \mathcal{H} constructed by Kadeishvili’s method in general — not just those defined by the construction in [Theorem 4.2](#) involving cycle and creation choice functions.

Throughout this section, we consider an A_∞ structure X on \mathcal{H} with operations $X_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}$, and a morphism of A_∞ algebras $f: \mathcal{H} \rightarrow \mathcal{A}$ with maps $f_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{A}$, constructed by Kadeishvili’s method. So there are also auxiliary maps $U_n: \mathcal{H}^{\otimes n} \rightarrow \mathcal{A}$, satisfying equations (7), (8) and (9). We assume all U_n , X_n and f_n preserve H–data and have Maslov degree $n - 2$, $n - 2$ and $n - 1$, respectively. Thus, each f_n inverts the differential in $\partial f_n = f_1 X_n - U_n$, but not necessarily by a creation operator. Throughout, $M = M_1 \otimes \cdots \otimes M_n$ is a tensor product of nonzero homology classes of diagrams.

5.1 Nonviable input

We have seen that if a tensor product of homology classes of diagrams $M_1 \otimes \cdots \otimes M_n$ is not viable, then their product is zero ([Lemma 2.6](#)). We now show that other operations are zero as well.

Lemma 5.1 *Suppose that $M = M_1 \otimes \cdots \otimes M_n$ is not viable.*

- (i) *If M has some step covered more than once, then $\bar{f}_n(M) = 0$ and $X_n(M) = 0$.*
- (ii) *If all f_n are balanced and M has an idempotent mismatch, then $f_n(M) = 0$ and $X_n(M) = 0$.*

In particular, if all f_n are balanced, then $\bar{f}_n(M) = 0$ and $X_n(M) = 0$.

Proof First suppose M has some step covered twice. As X_n preserves H–grading, and there are no tight diagrams with such H–grading, $X_n(M) = 0$. As f_n preserves H–grading, $f_n(M)$ is a sum of nonviable diagrams, hence lies in \mathcal{F} , so $\bar{f}_n(M) = 0$.

We show (ii) by induction on n . When $n = 1$ there is nothing to prove. Suppose the result is true for all f_i with $i < n$; now assume that M is mismatched, and consider $U_n(M)$. In a term of the form $f_j(M_1 \otimes \cdots \otimes M_j) f_{n-j}(M_{j+1} \otimes \cdots \otimes M_n)$, if the mismatch occurs within $M_1 \otimes \cdots \otimes M_j$ or $M_{j+1} \otimes \cdots \otimes M_n$, then by induction the term is zero; otherwise it occurs between M_j and M_{j+1} , in which case the product is zero. In a term of the form $f_{n-j+1}(M_1 \otimes \cdots \otimes M_k \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \otimes \cdots \otimes M_n)$, if the mismatch occurs within $M_{k+1} \otimes \cdots \otimes M_{k+j}$ then the X_j term is zero, hence the whole term is zero; otherwise it occurs within the f_{n-j+1} term and again we have zero. Thus $U_n(M) = 0$, so $X_n(M) = 0$. Then $\partial f_n(M) = (f_1 X_n - U_n)(M) = 0$, and since f_n is balanced then $f_n(M) = 0$ also. \square

5.2 Equivalent choices of maps

In the proof of [Theorem 4.2](#), we saw that although there might be many choices available for the f_n on tight summands, such choices had no effect on the resulting X_n .

In a similar vein, we now show that, in applying Kadeishvili’s construction in general (ie without creation operators), the choices available for the maps \bar{f}_n do not depend on any previous choices.

Lemma 5.2 *Suppose that f_i are defined for all $i < n$, U_i and X_i are defined for all $i \leq n$, and the two functions $a, b: \mathcal{H}^{\otimes n} \rightarrow A$ satisfy*

$$\bar{a} = \bar{b} \quad \text{and} \quad \partial a = \partial b = f_1 X_n - U_n,$$

and are balanced, ie if $(f_1 X_n - U_n)(M) = 0$ then $a(M) = b(M) = 0$. Whether we choose $f_n = a$ or b , for all $N > n$ the choices for each \bar{f}_N are identical.

Let us be more explicit. Taking $f_n = a$ we define U and X maps at level $n + 1$, which we denote by U_{n+1}^a, X_{n+1}^a . Then we have a set of choices

$$\mathcal{S}_{n+1}^a = \{ \bar{f} \mid \partial f = f_1 X_{n+1}^a - U_{n+1}^a \}$$

for \bar{f}_{n+1} . On the other hand, taking $f_n = b$ we define U_{n+1}^b, X_{n+1}^b and have another set of choices

$$\mathcal{S}_{n+1}^b = \{ \bar{f} \mid \partial f = f_1 X_{n+1}^b - U_{n+1}^b \}$$

for \bar{f}_{n+1} . [Lemma 5.2](#) says that $\mathcal{S}_{n+1}^a = \mathcal{S}_{n+1}^b$. Moreover, after taking $f_n = a$ and making arbitrary choices $f_{n+1}^a, \dots, f_{N-1}^a$ using Kadeishvili’s construction, obtaining maps $U_{n+1}^a, X_{n+1}^a, \dots, U_N^a, X_N^a$, we obtain a set of choices

$$\mathcal{S}_N^a = \{ \bar{f} \mid \partial f = f_1 X_N^a - U_N^a \}$$

for \bar{f}_N ; after taking $f_n = b$ and making arbitrary choices $f_{n+1}^b, \dots, f_{N-1}^b$ and obtaining maps $U_{n+1}^b, X_{n+1}^b, \dots, U_N^b, X_N^b$, we have choices $S_N^b = \{\bar{f} \mid \partial f = f_1 X_N - U_N\}$ for \bar{f}_N . Lemma 5.2 says, more generally, that $S_N^a = S_N^b$.

Proof Let M have H-data (h, s, t) . When (h, s, t) is not viable, there is only one choice for $\bar{f}_n(M)$, namely 0, by Lemma 5.1. And when (h, s, t) is singular, $\bar{f}_n(M) = 0$ as there are no available diagrams. Hence we need only consider $\bar{f}_n(M)$ when (h, s, t) is viable and nonsingular, hence tight or twisted.

Since $\bar{a} = \bar{b}$, $a - b$ takes values in \mathcal{F} . As \mathcal{F} is an ideal, $U_{n+1}^a(M)$ and $U_{n+1}^b(M)$ differ by values in \mathcal{F} . Diagrams in \mathcal{F} do not contribute to homology, as they have crossings, so $[U_{n+1}^a(M)] = [U_{n+1}^b(M)]$. It follows that $X_{n+1}^a(M) = X_{n+1}^b(M)$; we simply write $X_{n+1}(M)$ in either case. Moreover, $U_{n+1}^a(M) - U_{n+1}^b(M)$ is a boundary; let $U_{n+1}^a(M) - U_{n+1}^b(M) = \partial g_{n+1}$. As $U_{n+1}^a(M) - U_{n+1}^b(M) \in \mathcal{F}$, there is such a g_{n+1} in \mathcal{F} : if (h, s, t) is twisted we can use a creation operator; if (h, s, t) is tight then any crossing occurs at an all-on doubly occupied pair, so any diagram with crossings lies in \mathcal{F} . Then to define $f_{n+1}(M)$ we must solve

$$f_1 X_{n+1}(M) - U_{n+1}^a(M) = \partial f_{n+1}^a(M) \quad \text{or} \quad f_1 X_{n+1}(M) - U_{n+1}^b(M) = \partial f_{n+1}^b(M).$$

Observe that $f_{n+1}^a(M)$ is a solution of the first equation if and only if $f_{n+1}^a(M) + g_{n+1}$ is a solution of the second equation. Since $g_{n+1} \in \mathcal{F}$, the possible $f_{n+1}^a(M)$ and $f_{n+1}^b(M)$ are identical.

Thus, the possible choices for f_{n+1} differ by values in \mathcal{F} . The possible choices for U_{n+2} then differ by values in \mathcal{F} , and the argument proceeds by induction, giving the desired result. □

Thus in the construction of the maps f_n and X_n , it is sufficient to consider \bar{f} rather than f at each level. So we may effectively compute in $\mathcal{A}/\mathcal{F} = \bar{\mathcal{A}}$.

5.3 Preliminary properties of A-infinity operations

Lemma 5.3 For any $n \geq 2$ and any M , $f_n(M)$ is a (possibly empty) sum of crossed diagrams.

Proof For fixed H-data, Maslov grading is (up to a constant) given by the number of pairs with crossings (Section 2.8). As f_n has Maslov grading $n - 1 \geq 1$, all diagrams in $f_n(M)$ have crossings. □

Lemma 5.4 *The homology class of $X_n(M)$ is represented by the sum of all crossing-less diagrams in the following sum, writing elements of $\bar{\mathcal{A}}$ in standard form:*

$$\sum_{j=1}^{n-1} \bar{f}_j(M_1 \otimes \cdots \otimes M_j) \bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes M_n).$$

Recall (Definition 2.47(v)) that the standard form of an element of $\bar{\mathcal{A}}$ is a sum of viable diagrams without crossed doubly occupied pairs. For $n = 1$ the result reduces to $X_1 = 0$.

Proof By construction, $X_n(M) = [U_n(M)]$. Consider the terms of (7) defining $U_n(M)$. Diagrams in $f_\bullet(M_1 \otimes \cdots \otimes X_\bullet(\cdots) \otimes \cdots \otimes M_n)$ are crossed (Lemma 5.3) hence do not contribute to homology. Thus, $X_n(M)$ is represented by the sum of tight diagrams of the form $f_j(M_1 \otimes \cdots \otimes M_j) f_{n-j}(M_{j+1} \otimes \cdots \otimes M_n)$, and diagrams in \mathcal{F} do not contribute to homology. So X_n is represented by the sum claimed. \square

Lemma 5.4 allows us to calculate $X_n(M)$ directly from \bar{f}_j and \bar{f}_{n-j} . Diagrams in \bar{f}_j or \bar{f}_{n-j} usually contain crossings (Lemma 5.3), but the crossings may disappear in a sublimation to give a tight result. Sublimation is therefore ubiquitous in the operations X_n , arising in any nonzero $X_n(M)$.

5.4 Conditions for nontrivial A–infinity operations

Theorem 5.5 *Suppose all f_k are balanced. Let $n \geq 2$, let M_1, \dots, M_n be nonzero homology classes of tight diagrams, and let $M = M_1 \otimes \cdots \otimes M_n$. If $\bar{f}_n(M) \neq 0$, then the following statements hold.*

- (i) *In M there are l matched pairs which are twisted, and m matched pairs which are critical, where $l + m \geq n - 1$ and $m \leq n - 2$. All other matched pairs are tight.*
- (ii) *$\bar{f}_n(M)$ is represented by a sum of diagrams, where each diagram D satisfies the following conditions:*
 - (a) *All m of the critical matched pairs in M become tight in D .*
 - (b) *Precisely $n - m - 1$ of the l twisted matched pairs in M become crossed in D ; the other $l - n + m + 1$ twisted matched pairs in M remain twisted in D .*
 - (c) *All tight matched pairs in M remain tight in D .*

Note in particular that the conditions in (i) imply that $l > 0$, so the H–data of M is twisted. Hence if M has tight H–data then $\bar{f}_n(M) = 0$.

Proof First, by [Lemma 5.1](#), M is viable.

Write $\bar{f}_n(M)$ in standard form ([Definition 2.47\(v\)](#)), as a sum of distinct diagrams without crossed doubly occupied pairs. Let D be one of these diagrams. As \bar{f}_n respects H–grading and has Maslov grading $n - 1$, D is viable, with $h(D) = h(M)$, and $\iota(D) = \iota(M) + n - 1$. From [Tables 1](#) and [2](#), at each matched pair the Maslov grading can increase by at most 1; hence there are precisely $n - 1$ matched pairs at which D has a higher Maslov index than M .

At each matched pair P and D must give a viable local diagram which respects local H–data. There are no such diagrams for singular pairs; hence all matched pairs of M are tight, twisted, or critical.

Consider a matched pair P where M is critical. From [Table 2](#), P is sesqui-occupied or doubly occupied by M . Every all-on doubly occupied pair must remain uncrossed in D (by assumption of standard form). From [Table 1](#), any viable local diagram at a sesqui-occupied or doubly occupied matched pair, which is not crossed all-on doubly occupied, must be tight. So D_P is tight at P , and (again by reference to the table) $\iota(D_P) = \iota(M_P) + 1$.

Now consider a matched pair P where M is tight. Then the local H–data of M at P is tight. We observe from [Table 1](#) that, with crossed all-on doubly occupied local diagrams ruled out, any viable local diagram with tight H–data must be tight. Thus D_P is tight, and hence $\iota(D_P) = \iota(M_P)$.

Now $\iota(D) = \iota(M) + n - 1$, and m of this increase is accounted for at critical matched pairs. The remaining increase of $n - 1 - m$ must arise at the l pairs where M is twisted. From [Table 2](#), we observe that these are precisely the pairs where M is all-on once occupied. At such pairs, two viable local diagrams are possible: a tight and a crossed diagram. Crossings can thus be inserted at such pairs to increase the Maslov index; they must be inserted at $n - 1 - m$ such pairs for D to have the correct Maslov index; so $n - 1 - m \leq l$. The remaining $l + m - n + 1$ pairs must remain twisted in D .

The diagram D thus has precisely $n - m - 1$ crossings. But by [Lemma 5.3](#), D must have at least one crossing. Thus $n - m - 1 \geq 1$. \square

Theorem 5.6 *Suppose that all f_k are balanced. If $X_n(M) \neq 0$, then the following statements hold:*

- (i) *M has precisely $n - 2$ critical matched pairs, and all other matched pairs are tight.*
- (ii) *$X_n(M)$ is the unique homology class of tight diagram with the H–data of M .*

In particular, if $X_n(M) \neq 0$, then M has tight H–data.

For $n = 1$ this result says $X_1 = 0$; the $n = 2$ case follows from Lemmas 2.35 and 2.36.

Proof By Lemma 5.1, M is viable. Since X_n respects H–data, let M and $X_n(M)$ have H–data (h, s, t) . Since there is at most one nonzero homology class with fixed H–data, (ii) follows immediately.

As $X_n(M) \neq 0$, (h, s, t) is tight, hence tight at each matched pair. In particular, M has no 11 once occupied or 00 alternately occupied pairs. From Table 2, M is tight or critical at each pair.

When M is tight at a pair P , by Lemma 2.35 $M_1 \cdots M_n$ is tight at P . As $X_n(M)$ is given at P by the unique tight diagram with the H–data of M , then $X_n(M)_P = (M_1 \cdots M_n)_P$. In this case $X_n(M)$ has the same Maslov index as M at P .

On the other hand, when M is critical at P , $X_n(M)$ must still be given at P by the unique tight diagram with the same H–data. Inspecting Table 2 (and recalling that multiplication in \mathcal{H} has zero Maslov grading), we observe that $\iota(X_n(M)_P) = \iota(M_P) + 1$. Since Maslov index is additive over matched pairs (Section 2.5), and X_n has Maslov index $n - 2$, M has precisely $n - 2$ critical pairs. □

Theorems 5.5 and 5.6 respectively yield parts (i) and (ii) of Theorem 1.2.

Now we show that it’s not possible to have \bar{f}_n nonzero simultaneously with X_n , and more.

Lemma 5.7 *Suppose that f_k is balanced for all $k \geq 1$. Let $n \geq 2$. If $X_n(M) \neq 0$ or $M_1 \cdots M_n \neq 0$, then $\bar{f}_n(M) = 0$.*

Contrapositively, if $\bar{f}_n(M) \neq 0$ then $X_n(M) = 0$ and $M_1 \cdots M_n = 0$.

Proof Let M have H-data (h, s, t) . By the comment right after [Theorem 5.6](#), if $X_n(M) \neq 0$, then (h, s, t) is tight. And if $M_1 \cdots M_n \neq 0$, then M is tight, so again (h, s, t) is tight. But by the comment after the statement of [Theorem 5.5](#), if (h, s, t) is tight then $\bar{f}_n(M) = 0$. \square

[Theorem 5.6](#) places stringent necessary conditions on a tensor product $M_1 \otimes \cdots \otimes M_n$ to yield a nonzero result under X_n . However, we will see in [Section 6.2](#) that these conditions are not sufficient.

5.5 Levels 1, 2 and 3 in general

We now describe some properties of the maps f_n and X_n at levels 1, 2 and 3, still working in general, assuming that Kadeishvili's construction is used, and the maps U_n , X_n and f_n preserve H-data and have appropriate Maslov gradings, but not that creation operators are used (unlike [Section 4.3](#)). We additionally now assume that the f_n are balanced.

Level 1 is again straightforward. By construction $X_1 = 0$, and f_1 is a cycle selection homomorphism. If f_1 is diagrammatically simple then it arises from a cycle choice function ([Lemma 3.5](#)). In general, for each tight (h, s, t) , f_1 maps $M_{h,s,t}$ to the sum of an odd number of tight diagrams representing $M_{h,s,t}$.

For level 2, by construction X_2 is multiplication. As for f_2 , we have the following.

Lemma 5.8 *Suppose that f_1 and f_2 are balanced. Then $\bar{f}_2(M) \neq 0$ if and only if M is viable and has at least one twisted matched pair. Then $\bar{f}_2(M)$ in standard form is the sum of an odd number of diagrams, each with a single crossing at a matched pair where M is twisted, and elsewhere tight or twisted in agreement with M .*

Proof If $\bar{f}_2(M) \neq 0$, then M is viable ([Lemma 5.1](#)). By [Theorem 5.5](#), M has no critical matched pairs and at least one twisted (ie all-on once occupied) matched pair. Moreover, $\bar{f}_2(M)$ is represented by a sum of diagrams, each of which has precisely one crossing at a twisted matched pair.

Conversely, suppose $M = M_1 \otimes M_2$ is viable and has at least one twisted matched pair. Then $X_2(M) = 0$, and $U_2(M) = f_1(M_1)f_1(M_2)$ is the sum of an odd number of diagrams, each tight and twisted at the same matched pairs as M , differing by strand switching at all-on doubly occupied pairs ([Lemma 2.23](#)). Since $\partial f_2(M) =$

$f_1(M_1)f_1(M_2)$ and f_2 respects H–data and has Maslov grading 1, $f_2(M)$ is a sum of diagrams, each of which has a crossing at one matched pair, and each other pair is tight or twisted in agreement with M . In standard form $\bar{f}_2(M)$ is then given by omitting diagrams with crossings at doubly occupied pairs, so that each crossing is at a matched pair where M is twisted. The differential of each remaining diagram is a single diagram, but the differential of each omitted diagram is a sum of two diagrams, so $\bar{f}_2(M)$ in standard form is the sum of an odd number of diagrams. \square

When the A_∞ structure is defined by creation operators, as in Section 4 and Theorem 4.2, then any nonzero $f_2(M)$ is a single diagram, as described in Section 4.3.

We now consider X_3 ; the case is illustrative, showing the role of critical and sublime tensor products. Let $M = M_1 \otimes M_2 \otimes M_3$ and suppose $X_3(M) \neq 0$. Then M is viable (Lemma 5.1, requiring the balanced assumption) and by Theorem 5.6, M has precisely one critical matched pair; all other matched pairs are tight. By Lemma 5.4, $X_3(M)$ is represented by the sum of all crossingless diagrams in

$$\bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3).$$

Each diagram in an $\bar{f}_2(M_i \otimes M_{i+1})$ term has a crossing at precisely one matched pair P , where $M_i \otimes M_{i+1}$ is twisted; since such P cannot be tight in M (Lemma 2.39, Table 3), P is the critical matched pair of M . Multiplying this diagram by the third M_j must then produce a tight diagram. There are two cases for the tightness of the various tensor products:

- $M_1 \otimes M_2$ twisted; each diagram D in $\bar{f}_2(M_1 \otimes M_2)$ crossed; M_3 and each diagram D' in $\bar{f}_1(M_3)$ tight; each $D \otimes D'$ sublime; and $M_1 \otimes M_2 \otimes M_3$ critical.
- $M_2 \otimes M_3$ twisted; each diagram D' in $\bar{f}_2(M_2 \otimes M_3)$ crossed; M_1 and each diagram D in $\bar{f}_1(M_1)$ tight; each $D \otimes D'$ sublime; and $M_1 \otimes M_2 \otimes M_3$ critical. Figure 13 shows the situation at P .

These two cases are mutually exclusive: only one of

$$f_2(M_1 \otimes M_2)f_1(M_3) \quad \text{or} \quad f_1(M_1)f_2(M_2 \otimes M_3)$$

can be nonzero. In the first case $M_2 \otimes M_3$ is singular, and in the second case $M_1 \otimes M_2$ is singular.

Thus, to obtain a nonzero result for X_3 , we start with a critical tensor product $M_1 \otimes M_2 \otimes M_3$; then a twisted subtensor-product (ie $M_2 \otimes M_3$ in Figure 13) combines via f_2

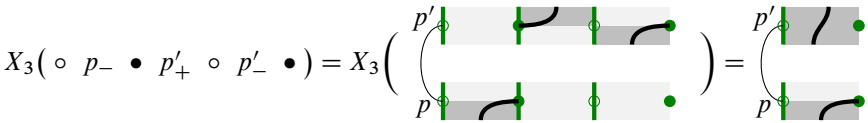


Figure 13: An example of $X_3(M_1 \otimes M_2 \otimes M_3)$, where $M_1 \otimes M_2 \otimes M_3$ is critical, $M_1 \otimes M_2$ is singular, and $M_2 \otimes M_3$ is twisted. Moreover, $\bar{f}_2(M_2 \otimes M_3)$ is crossed, and $\bar{f}_1(M_1) \otimes \bar{f}_2(M_2 \otimes M_3)$ is sublime.

into a crossed diagram, yielding a sublime tensor product (ie $f_1(M_1) \otimes f_2(M_2 \otimes M_3)$ in Figure 13); and then these are multiplied to give a tight result. This process is the process depicted in Figure 6; it occurs generally in Kadeishvili’s construction, without any need for creation operators.

We can prove a converse, and give necessary and sufficient conditions for $X_3 \neq 0$.

Proposition 5.9 *Suppose f_1 and f_2 are balanced. Then $X_3(M)$ is nonzero if and only if M is viable, critical at precisely one matched pair P , and tight at all other matched pairs.*

Proof We only need prove that if the conditions on M hold, then $X_3(M) \neq 0$. By Lemma 2.37, M_P is an extension of a tensor product shown in the critical column of Table 2; but M_P has 3 factors, so M_P is exactly the critical 01 pre-sesqui-occupied or 10 post-sesqui-occupied tensor product shown there. We consider the first case; the second case is similar. So suppose M_P is critical 01 pre-sesqui-occupied.

First, $\bar{f}_1(M_1)$ is the sum of an odd number of tight diagrams, all representing M_1 , and differing by strand switching at all-on doubly occupied pairs (Lemma 2.23).

By Lemma 5.8, $\bar{f}_2(M_2 \otimes M_3)$ is the sum of an odd number of diagrams, each with a single crossing at a pair where $M_2 \otimes M_3$ is twisted. Let D be one of these diagrams; let its crossing be at the pair P' . By Lemma 2.39 and Table 3 then P' cannot be tight in M , so $P' = P$. Thus $\bar{f}_2(M_2 \otimes M_3)$ is represented by the sum of an odd number of diagrams, each of which has a crossing at P and is tight elsewhere, and which differ by strand switching at all-on doubly occupied pairs.

Then $\bar{f}_1(M_1) \otimes \bar{f}_2(M_2 \otimes M_3)$ is the sum of an odd number of sublime tensor products of diagrams, and $\bar{f}_1(M_1) \bar{f}_2(M_2 \otimes M_3)$ is the sum of an odd number of tight diagrams. Moreover, in this case $M_1 \otimes M_2$ is singular so $\bar{f}_2(M_1 \otimes M_2) = 0$.

Thus $\bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3)$ is the sum of an odd number of tight diagrams, related by strand switching. By Lemma 5.4, $X_3(M)$ is the homology class of any one of these diagrams, so $X_3(M) \neq 0$. □

Thus, if there are sufficiently few critical matched pairs in M , we may be able to guarantee that $X_n(M) \neq 0$. In Section 7 we give some results in this direction.

6 Further examples and computations

We now calculate some further examples and prove some further results, for low-level A_∞ maps.

In this section we consider A_∞ operations defined by a pair ordering \preceq , as in Corollary 4.3, and consider maps at level 3 and 4, using the shorthand notation of Section 4.2. This builds upon Section 4.3, where we discussed f_n and X_n for $n \leq 2$, when A_∞ operations are defined by cycle choice and creation choice functions; and Section 5.5 where we again discussed low-level maps, especially \bar{f}_2 and X_3 , for A_∞ operations obtained more generally using Kadeishvili’s method.

As always, let $M = M_1 \otimes \dots \otimes M_n$ denote a tensor product of nonzero homology classes of diagrams. We assume M is viable, necessary for nonzero results (Lemma 5.1); let M have H-data (h, s, t) . We work with \bar{f}_n and \bar{U}_n ; this loses no generality for calculating X_n (Lemma 5.2).

6.1 Level 3

Consider the operation \bar{U}_3 , given by

$$\bar{U}_3(M) = \bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3) + \bar{f}_2(M_1 M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2 M_3).$$

The last two terms cannot contribute to X_3 , as they yield crossed diagrams (Lemma 5.3). But in general all four terms can be nonzero; indeed, some terms may be equal and cancel. An example is shown in Figure 14, with shorthand calculations alongside the standard notation; \bar{f}_2 is calculated using Section 4.3 and equation (11).

Continuing to \bar{f}_3 , we know that when $\bar{f}_3(M) \neq 0$ then M is viable (Lemma 5.1), $X_3(M) = 0$ and $M_1 M_2 M_3 = 0$ (Lemma 5.7). Moreover, M has no singular matched

$$\bar{U}_3(\bullet \ p'_+ \ \circ \ \circ \ p'_- \ \bullet) = \bar{f}_2(\bullet \ p'_+ \ \circ \) + \bar{f}_2(\bullet \ p'_+ \ \circ \) = 0$$

Figure 14: An example of $U_3(M_1 \otimes M_2 \otimes M_3)$. In this case both $f_2(M_1 \otimes M_2)$ and $f_2(M_2 \otimes M_3)$ are zero. The two terms $f_2(M_1 M_2 \otimes M_3)$ and $f_2(M_1 \otimes M_2 M_3)$ are both nonzero, but cancel out.

pairs, l twisted matched pairs, and m critical matched pairs, where $m \leq 1$ and $l + m \geq 2$ (Theorem 5.5). It follows that $l \geq 1$, so (h, s, t) is twisted, so by Theorem 4.2(iv) then $\bar{f}_3(M) = \bar{A}_{\mathcal{CR} \preceq}^* \circ \bar{U}_3(M)$, where $\bar{A}_{\mathcal{CR} \preceq}^*$ is the creation operator of the creation choice function $\mathcal{CR} \preceq$ (Definition 3.15) of the pair ordering \preceq (Definition 3.18):

$$(12) \quad \bar{f}_3(M) = \bar{A}_{\mathcal{CR} \preceq}^*(\bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3) + \bar{f}_2(M_1 M_2 \otimes M_3) + \bar{f}_2(M_1 \otimes M_2 M_3)).$$

Each of the four terms in equation (12) consists of at most one diagram in standard form. Since diagrams in $\bar{U}_3(M)$ may have a crossing, a diagram in $\bar{f}_3(M)$ may have up to two crossings.

Now M has $m \leq 1$ critical matched pairs. If $m = 0$ then all pairs are tight or twisted, and any diagram in $\bar{f}_3(M)$ above has precisely two crossings. If $m = 1$, then the critical pair P must eventually have a tight local diagram to yield a nonzero result, so the diagram at P becomes crossed by an \bar{f}_2 and then sublimates; hence any diagram in $\bar{f}_3(M)$ has one crossing.

We find that, in order to obtain a nonzero result for $\bar{f}_3(M)$, the local diagrams at twisted or critical matched pairs must be “distributed” across M_1, M_2 and M_3 . For twisted pairs we make this precise in the following statement.

Lemma 6.1 Consider an A_∞ structure defined by a pair ordering \preceq .

Suppose $M = M_1 \otimes M_2 \otimes M_3$ is viable, twisted at precisely two places p, q of matched pairs $P = \{p, p'\}$ and $Q = \{q, q'\}$, with all other pairs tight. Moreover, suppose that p_+, q_+ are both covered by the same M_i , and p_-, q_- are both covered by the same M_j .

Then $X_3(M), \bar{U}_3(M)$ and $\bar{f}_3(M)$ are all zero.

We can denote this result for \bar{f}_3 by

$$\bar{f}_3 \left(\begin{array}{ccccccc} \bullet & q_+ & \circ & q_- & \bullet & \bullet & \\ \bullet & p_+ & \circ & p_- & \bullet & \bullet & \end{array} \right) = \bar{f}_3 \left(\begin{array}{ccccccc} \bullet & q_+ & \circ & \circ & q_- & \bullet & \\ \bullet & p_+ & \circ & \circ & p_- & \bullet & \end{array} \right) = \bar{f}_3 \left(\begin{array}{ccccccc} \bullet & \bullet & q_+ & \circ & q_- & \bullet & \\ \bullet & \bullet & p_+ & \circ & p_- & \bullet & \end{array} \right) = 0.$$

Proof There are three possibilities for i and j : $(i, j) = (1, 2), (1, 3)$ or $(2, 3)$. In all cases $X_3(M) = 0$ as there are no critical matched pairs (Theorem 5.6). Suppose without loss of generality that $P \preceq Q$, so creation operators introduce crossings at P in preference to Q .

First suppose $(i, j) = (1, 2)$. Then $M_1M_2 = 0$ (being twisted) and $M_2M_3 \neq 0$ (being tight), so $\bar{f}_2(M_2 \otimes M_3) = 0$ (Lemma 5.7). Thus

$$\bar{U}_3(M) = \bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3) + \bar{f}_2(M_1 \otimes M_2M_3).$$

Now $\bar{f}_2(M_1 \otimes M_2) = \bar{A}_{c\mathcal{R}\preceq}(\bar{f}_1(M_1)\bar{f}_1(M_2))$ is (in standard form) the diagram obtained from $\bar{f}_1(M_1)\bar{f}_1(M_2)$ by inserting a crossing at P . Similarly $\bar{f}_2(M_1 \otimes M_2M_3)$ (in standard form) is obtained from $\bar{f}_1(M_1)\bar{f}_1(M_2M_3)$ by inserting a crossing at P . Since the diagrams $\bar{f}_2(M_1 \otimes M_2M_3)$ and $\bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3)$ have the same H–data, are crossed at P , twisted at Q , elsewhere tight, and have the same strands at all-on doubly occupied pairs (chosen by the same cycle selection function of \preceq), they are equal. Thus $\bar{U}_3(M) = 0$ and $\bar{f}_3(M) = \bar{A}_{c\mathcal{R}\preceq} \circ \bar{U}_3(M) = 0$.

The case $(i, j) = (2, 3)$ is similar.

Finally suppose $(i, j) = (1, 3)$. Then M_1M_2 and M_2M_3 are nonzero, so

$$\bar{f}_2(M_1 \otimes M_2) = \bar{f}_2(M_2 \otimes M_3) = 0$$

(Lemma 5.7). The remaining two terms of $\bar{U}_3(M)$ are

$$\bar{f}_2(M_1M_2 \otimes M_3) \quad \text{and} \quad \bar{f}_2(M_1 \otimes M_2M_3),$$

both of which are crossed at p , twisted at q , and equal elsewhere, so again \bar{U}_3 and \bar{f}_3 are zero. □

The following lemma, together with Lemma 6.1 and the general result of Theorem 5.5, completely calculates $\bar{f}_3(M)$ when M has two nontight matched pairs.

Lemma 6.2 Consider an A_∞ structure defined by a pair ordering \preceq .

Suppose $M = M_1 \otimes M_2 \otimes M_3$ is viable, has two matched pairs

$$P = \{p, p'\} \prec Q = \{q, q'\}$$

which are twisted or critical, and all other matched pairs tight, in one of the arrangements depicted below.

Then $\bar{f}_3(M)$ is zero or nonzero as shown. If nonzero, it is given by a single diagram in \bar{A} , with the H -data of M , which is crossed at each twisted matched pair of M , and elsewhere tight.

Nonzero:

$$\begin{aligned} & \left(\begin{array}{c|c|c} q'_- & q_+ & q_- \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} & q_+ & q_- \\ p_+ & p_- & p'_+ \end{array} \right) \left(\begin{array}{c|c|c} q_+ & & q_- \\ p'_- & p_+ & p_- \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & q'_+ \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & \\ p_+ & p_- & \end{array} \right) \\ & \left(\begin{array}{c|c|c} q_+ & q_- & \\ p'_- & p_+ & p_- \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & q'_+ \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q'_- & q_+ & q_- \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q_+ & & q_- \\ p_+ & p_- & p'_+ \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & \\ p_+ & p_- & \end{array} \right) \\ & \left(\begin{array}{c|c|c} q_+ & q_- & \\ p_+ & p_- & p'_+ \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} & q_+ & q_- \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q_+ & q_- & \\ p'_- & p_+ & p_- \end{array} \right) \end{aligned}$$

Zero:

$$\left(\begin{array}{c|c|c} q_+ & q_- & q'_+ \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q_+ & & q_- \\ p_+ & p_- & \end{array} \right) \left(\begin{array}{c|c|c} q_+ & & q_- \\ & p_+ & p_- \end{array} \right) \left(\begin{array}{c|c|c} q'_- & q_+ & q_- \\ & p_+ & p_- \end{array} \right)$$

(Circles denoting idempotents are omitted; they can be inferred since each nontrivial local diagram covers at most one step.)

The conclusion that, if $\bar{f}_3(M)$ is nonzero, then it is as claimed, follows purely from grading considerations: \bar{f}_3 has Maslov grading 2, but the Maslov index can only be increased at nontight pairs. There are only two nontight matched pairs, so the Maslov index must be increased by 1 at each. A twisted pair must become crossed, and a critical pair must become tight.

Proof In the cases depicted in the first four diagrams in the first two rows above, we have a critical and a twisted pair, and $M_1M_2 = M_2M_3 = 0$. In each of these cases one of $M_1 \otimes M_2$ or $M_2 \otimes M_3$ is singular, and the other is twisted. Then precisely one of $\bar{f}_2(M_1 \otimes M_2)$ or $\bar{f}_2(M_2 \otimes M_3)$ is nonzero, and \bar{f}_2 introduces a crossing at the twisted matched pair. Then the multiplication $\bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3)$ or $\bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3)$ is tight at one pair and twisted at the other; and in fact this diagram is $\bar{U}_3(M)$. Applying a creation operator, we obtain $\bar{f}_3(M)$ as a single diagram with a single crossing.

In the cases depicted at the end of the first and second rows, again $M_1M_2 = M_2M_3 = 0$, and both $\bar{f}_2(M_1 \otimes M_2)$ and $\bar{f}_2(M_2 \otimes M_3)$ are nonzero, each with a single crossed pair. So $\bar{f}_2(M_1 \otimes M_2)\bar{f}_1(M_3)$ and $\bar{f}_1(M_1)\bar{f}_2(M_2 \otimes M_3)$ are both nonzero, one crossed at p and twisted at q , the other crossed at q and twisted at p . The creation operator

$\bar{A}_{\mathcal{CR} \leq}$ sends the former to zero, and introduces a crossing at p into the latter. Thus $\bar{f}_3(M)$ is given by a single diagram, crossed at both p and q , as desired.

The other cases can be calculated by similar reasoning. □

6.2 Level 4

We now compute two examples at level 4, illustrating some interesting phenomena. As usual, let $M = M_1 \otimes \cdots \otimes M_n$ denote a tensor product of nonzero homology classes of diagrams, with H–data (h, s, t) .

Our first example shows that the necessary conditions for X_n to be nonzero in [Theorem 5.6](#) are not sufficient. It is an M with precisely 2 critical matched pairs, and all other matched pairs tight—and in fact one can find a tight diagram with the same H–data—but with $X_4(M) = 0$.

Letting $P = \{p, p'\}$ and $Q = \{q, q'\}$ be matched pairs with $P < Q$ as usual, we can compute

$$X_4 \left(\begin{array}{cccccc} \bullet & q_+ & \circ & q_- & \bullet & q'_+ & \circ & \circ \\ \bullet & p_+ & \circ & p_- & \bullet & p'_+ & \circ & \circ \end{array} \right) = 0,$$

since in this case $\bar{f}_3(M_1 \otimes M_2 \otimes M_3) = 0$ ([Theorem 5.5](#); there are two critical pairs), $\bar{f}_3(M_2 \otimes M_3 \otimes M_4) = 0$ (since $M_2 \otimes M_3 \otimes M_4$ is singular), and $\bar{f}_2(M_3 \otimes M_4) = 0$ ([Lemma 5.7](#); as $M_3 M_4 \neq 0$).

One can also compute that the following are zero:

$$\begin{aligned} & X_4 \left(\begin{array}{cccccc} \circ & q'_- & \bullet & q_+ & \circ & q_- & \bullet & q'_+ & \circ \\ \bullet & & \bullet & p_+ & \circ & p_- & \bullet & p'_+ & \circ \end{array} \right), & X_4 \left(\begin{array}{cccccc} \bullet & q_+ & \circ & q_- & \bullet & q'_+ & \circ & \circ \\ \bullet & p_+ & \circ & p_- & \bullet & & \bullet & p'_+ & \circ \end{array} \right), \\ & X_4 \left(\begin{array}{cccccc} \bullet & q_+ & \circ & \circ & q_- & \bullet & q'_+ & \circ \\ \bullet & p_+ & \circ & p_- & \bullet & & \bullet & p'_+ & \circ \end{array} \right), & X_4 \left(\begin{array}{cccccc} \bullet & q_+ & \circ & q_- & \bullet & q'_+ & \circ & \circ \\ \bullet & p_+ & \circ & p_- & \bullet & p'_+ & \circ & p'_- & \bullet \end{array} \right). \end{aligned}$$

Our second example shows that \bar{f}_n is not diagrammatically simple (as might appear from small cases). We have four matched pairs $P < Q < R < S$, with $P = \{p, p'\}$, $Q = \{q, q'\}$, $R = \{r, r'\}$, $S = \{s, s'\}$, and we claim that

$$\bar{f}_4 \left(\begin{array}{cccccc} \bullet & s_+ & \circ & s_- & \bullet & \bullet & \bullet \\ \bullet & r_+ & \circ & \circ & r_- & \bullet & \bullet \\ \bullet & \bullet & \bullet & q_+ & \circ & q_- & \bullet \\ \bullet & \bullet & p_+ & \circ & \circ & p_- & \bullet \end{array} \right) = \begin{pmatrix} c_s \\ w_r \\ c_q \\ c_p \end{pmatrix} + \begin{pmatrix} c_s \\ c_r \\ w_q \\ c_p \end{pmatrix}.$$

Observe that, as there are no critical pairs, any X_k term with $k > 2$ is zero (Theorem 5.6). Moreover, $M_1M_2 = M_3M_4 = 0$. Thus $\bar{f}_4(M) = \bar{A}_P^* \circ \bar{U}_4(M)$, and

$$\begin{aligned} \bar{U}_4(M) &= \bar{f}_1(M_1)\bar{f}_3(M_2 \otimes M_3 \otimes M_4) + \bar{f}_2(M_1 \otimes M_2)\bar{f}_2(M_3 \otimes M_4) \\ &\quad + \bar{f}_3(M_1 \otimes M_2 \otimes M_3)\bar{f}_1(M_4) + \bar{f}_3(M_1 \otimes M_2M_3 \otimes M_4). \end{aligned}$$

Now $M_2 \otimes M_3 \otimes M_4$ is twisted at P and Q , and tight at R and S ; $\bar{f}_3(M_2 \otimes M_3 \otimes M_4)$ is then given by Lemma 6.2 and (in standard form) is a nonzero diagram. The same applies to $\bar{f}_3(M_1 \otimes M_2 \otimes M_3)$. As $M_1 \otimes M_2$ and $M_3 \otimes M_4$ are twisted at a single matched pair, and tight elsewhere, $\bar{f}_2(M_1 \otimes M_2)$ and $\bar{f}_2(M_3 \otimes M_4)$ are also both given by single nonzero diagrams, each with a single crossing.

For the remaining term $\bar{f}_3(M_1 \otimes M_2M_3 \otimes M_4)$, note that $M_1M_2M_3 = M_2M_3M_4 = 0$, so $\bar{U}_3(M_1 \otimes M_2M_3 \otimes M_4) = \bar{f}_1(M_1)\bar{f}_2(M_2M_3 \otimes M_4) + \bar{f}_2(M_1 \otimes M_2M_3)\bar{f}_1(M_4)$. Since $M_2M_3 \otimes M_4$ is twisted at P and Q , the creation operator inserts a crossing at P ; and since $M_1 \otimes M_2M_3$ is twisted at R and S , the creation operator inserts a crossing at R . Hence

$$\begin{aligned} \bar{f}_3(M_1 \otimes M_2M_3 \otimes M_4) &= \bar{A}_P^* (\bar{f}_1(M_1)\bar{f}_2(M_2M_3 \otimes M_4) + \bar{f}_2(M_1 \otimes M_2M_3)\bar{f}_1(M_4)) \\ &= \bar{A}_P^* \left[\begin{pmatrix} \bullet & s_+ & \circ \\ \bullet & r_+ & \circ \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} \circ & s_- & \bullet \\ \circ & r_- & \bullet \\ \bullet & w_q & \bullet \\ \bullet & c_p & \bullet \end{pmatrix} + \begin{pmatrix} \bullet & w_s & \bullet \\ \bullet & c_r & \bullet \\ \bullet & q_+ & \circ \\ \bullet & p_+ & \circ \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \circ & q_- & \bullet \\ \circ & p_- & \bullet \end{pmatrix} \right] \\ &= \bar{A}_P^* \left[\begin{pmatrix} \bullet & w_s & \bullet \\ \bullet & w_r & \bullet \\ \bullet & w_q & \bullet \\ \bullet & c_p & \bullet \end{pmatrix} + \begin{pmatrix} \bullet & w_s & \bullet \\ \bullet & c_r & \bullet \\ \bullet & w_q & \bullet \\ \bullet & w_p & \bullet \end{pmatrix} \right] = \begin{pmatrix} \bullet & w_s & \bullet \\ \bullet & c_r & \bullet \\ \bullet & w_q & \bullet \\ \bullet & c_p & \bullet \end{pmatrix}, \\ \bar{U}_4(M) &= \begin{pmatrix} w_s \\ w_r \\ c_q \\ c_p \end{pmatrix} + \begin{pmatrix} c_s \\ w_r \\ c_q \\ w_p \end{pmatrix} + \begin{pmatrix} c_s \\ c_r \\ w_q \\ w_p \end{pmatrix} + \begin{pmatrix} w_s \\ c_r \\ w_q \\ c_p \end{pmatrix}, \end{aligned}$$

so that, applying \bar{A}_P^* , $\bar{f}_4(M)$ has the claimed form.

7 Nontrivial higher operations

In this section we only consider A_∞ structures arising from a pair ordering \preceq .

Although we have various necessary conditions for X_n or \bar{f}_n to be nonzero (viability, Theorems 5.5 and 5.6, Lemmas 5.7 and 6.1), we do not yet have conditions which are

sufficient to ensure X_n or \bar{f}_n are nonzero — whether the operations are defined via a pair ordering, or by Kadeishvili’s construction more generally.

We have some results at low levels. For instance, $X_2(M_1 \otimes M_2)$ is nonzero if and only if $M_1 \otimes M_2$ is tight, essentially by definition. [Proposition 5.9](#) shows that the necessary conditions of [Theorem 5.6](#) for X_3 to be nonzero are also sufficient. However, the X_4 example of [Section 6.2](#) shows that these conditions are not sufficient for X_4 to be zero.

Indeed, the \bar{f}_3 examples of [Section 6.1](#) (particularly [Lemma 6.2](#)) show that even the question of whether \bar{f}_3 is zero or nonzero can be rather subtle. The \bar{f}_4 example of [Section 6.2](#) there shows that matters do not get simpler at higher levels.

In this section we prove some sufficient conditions for \bar{f}_n and X_n to be nonzero. They are, however, far from being necessary conditions.

As usual, throughout this section $M = M_1 \otimes \cdots \otimes M_n$ always denotes a tensor product of nonzero homology classes of diagrams.

7.1 Operation trees

[Lemma 6.1](#) and some of the level 3 and 4 examples show that, even though a tensor product $M_1 \otimes \cdots \otimes M_n$ might have the right number of critical and twisted matched pairs, the steps of these pairs must be covered by the M_i in a way that is appropriately “horizontally distributed”.

To this end, we study rooted trees describing the order in which operations are performed.

Definition 7.1 (operation tree) An *operation tree* for $\mathcal{H}^{\otimes n}$ is a rooted plane binary tree with n leaves, ordered from left to right, and with each vertex v labelled by a viable tensor product of nonzero homology classes of diagrams M_v , such that the following conditions are satisfied:

- (i) Each leaf is labelled with a nonzero homology class of diagram in \mathcal{H} .
- (ii) Each vertex is labelled with the tensor product of the labels on its ordered children.

If the root vertex is labelled M , we say \mathcal{T} is an *operation tree for M* .

Thus, if the leaves are labelled M_1, \dots, M_n in order, then the root vertex is labelled $M_1 \otimes \dots \otimes M_n$. See [Figure 15](#) for some examples.

It will also be useful to consider a certain type of subtree, as in the following definition.

Definition 7.2 (subtree below v) Let \mathcal{T} be an operation tree, and v a vertex of \mathcal{T} . The *operation subtree of \mathcal{T} below v* is the subtree \mathcal{T}_v of \mathcal{T} , with root vertex v , consisting of all edges and vertices below v , and with all vertex labels inherited from \mathcal{T} .

Clearly \mathcal{T}_v is also an operation tree.

7.2 Validity and distributivity

If \mathcal{T} is an operation tree for M , each vertex of \mathcal{T} is labelled by a subtensor-product M_v of M . The various labels M_v may have different types of tightness, depending on how the various steps around each matched pair are covered.

Singular tensor products should be avoided, and so we make the following definition.

Definition 7.3 Let \mathcal{T} be an operation tree for $\mathcal{H}^{\otimes n}$. A vertex of \mathcal{T} is *valid* if its label is nonsingular. The operation tree \mathcal{T} is *valid* if it is valid at all of its vertices.

Thus, in a valid operation tree for M , each vertex label is tight, twisted or critical. (Note that M may have singular subtensor-products, but they do not appear as vertex labels.) Equivalently, each label is tight, twisted or critical at all matched pairs ([Lemma 2.36](#)).

Lemma 7.4 Let \mathcal{T} be a valid operation tree for M , and v a vertex of \mathcal{T} . Then the operation subtree \mathcal{T}_v of \mathcal{T} below v is valid.

Proof Each label is nonsingular in \mathcal{T} , hence also nonsingular in \mathcal{T}_v . □

Nonzero A_∞ operations require carefully regulated numbers of twisted and critical matched pairs, as required by [Theorems 5.5](#) and [5.6](#). Hence we make the following definition.

Definition 7.5 Let \mathcal{T} be a valid operation tree. A vertex of \mathcal{T} with k leaves, labelled M , is *distributive* if there are at least $k - 2$ matched pairs at which M is twisted or critical. The tree \mathcal{T} is *distributive* if every vertex of \mathcal{T} is distributive.

7.3 Joining and grafting trees

We now consider some methods to combine operation trees into larger trees.

The first operation, *joining*, places two existing operation trees below a new root vertex.

Definition 7.6 Let \mathcal{T}' and \mathcal{T}'' be operation trees for M' and M'' , where $M' \otimes M''$ is viable. Let v' and v'' be the root vertices of \mathcal{T}' and \mathcal{T}'' , respectively. The *join* of \mathcal{T}' and \mathcal{T}'' is the tree \mathcal{T} obtained by placing \mathcal{T}' and \mathcal{T}'' below v_0 , so that v' and v'' are the left and right children of \mathcal{T} . The root vertex v_0 is labelled $M' \otimes M''$, and each other vertex inherits its label from \mathcal{T}' or \mathcal{T}'' .

Clearly, the join of two operation trees is again an operation tree; note that this requires the assumption that $M' \otimes M''$ be viable. Figure 15 shows an example.

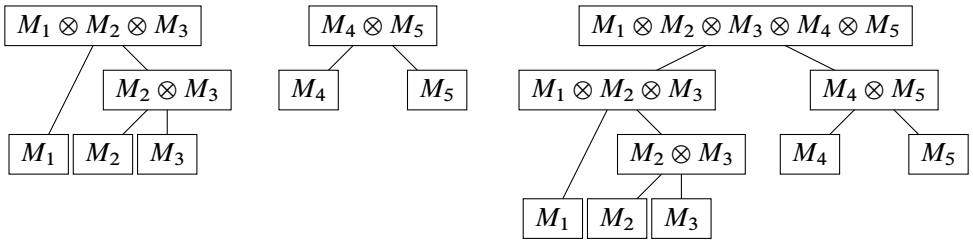


Figure 15: Operation trees \mathcal{T}' , \mathcal{T}'' and \mathcal{T} (left to right), where \mathcal{T} is the join of \mathcal{T}' and \mathcal{T}'' .

Under certain circumstances, joining trees preserves validity and distributivity.

Lemma 7.7 Let $\mathcal{T}', \mathcal{T}''$ be operation trees for

$$M' = M_1 \otimes \cdots \otimes M_j \quad \text{and} \quad M'' = M_{j+1} \otimes \cdots \otimes M_n,$$

and let \mathcal{T} be their join. Suppose that \mathcal{T}' and \mathcal{T}'' are valid and distributive, and that one of the following conditions holds:

- (i) $X_n(M' \otimes M'') \neq 0$;
- (ii) $\bar{f}_n(M' \otimes M'') \neq 0$; or
- (iii) $\bar{f}_j(M') \bar{f}_{n-j}(M'') \neq 0$, and M contains no 11 doubly occupied pairs.

Then \mathcal{T} is also valid and distributive.

Note that if $X_n(M' \otimes M'')$ or $\bar{f}_n(M' \otimes M'')$ is nonzero, then $M' \otimes M''$ is certainly viable, so that \mathcal{T} is a well-defined operation tree.

Proof Each nonroot vertex of \mathcal{T} retains its label from \mathcal{T}' or \mathcal{T}'' . So if \mathcal{T}' and \mathcal{T}'' are valid (resp. distributive), then \mathcal{T} is valid (resp. distributive) at these vertices. So we only need consider the root vertex v_0 of \mathcal{T} , which is labelled with $M = M' \otimes M''$.

If $X_n(M) \neq 0$, then by [Theorem 5.6](#), M has precisely $n - 2$ matched pairs which are critical, and all other matched pairs are tight. If $\bar{f}_n(M) \neq 0$, then by [Theorem 5.5](#), M has at least $n - 1$ matched pairs which are twisted or critical, and all other matched pairs are tight.

If $\bar{f}_j(M')\bar{f}_{n-j}(M'') \neq 0$ then there are at least $j - 1$ matched pairs at which M' is twisted or critical, and at least $n - j - 1$ pairs at which M'' is twisted or critical ([Theorem 5.5](#)). If any of these pairs coincide, then M has a 11 doubly occupied pair; if these are ruled out, then M has at least $(i - 1) + (n - i - 1) = n - 2$ pairs at which it is twisted or critical.

In each case, M is not singular, and the number of critical or twisted matched pairs is $\geq n - 2$. Thus v_0 is valid and distributive, and hence so also is \mathcal{T} . □

The second operation, *grafting*, implants a tree at a leaf of an existing tree.

Definition 7.8 Let \mathcal{T}' and \mathcal{T}'' be operation trees for $M' = M_1 \otimes \cdots \otimes M_n$ and $N' = N_1 \otimes \cdots \otimes N_j$, and suppose N' and M_k have the same H-data.

The *grafting of \mathcal{T}'' onto \mathcal{T}' at position k* is the tree \mathcal{T} obtained by identifying the k^{th} leaf of \mathcal{T}' with the root vertex of \mathcal{T}'' . The vertices of \mathcal{T}' are relabelled by replacing every instance of M_k with the tensor product $N_1 \otimes \cdots \otimes N_j$; other labels are inherited from \mathcal{T}'' .

[Figure 16](#) shows an example. Thus \mathcal{T} is an operation tree for the tensor product

$$M = M_1 \otimes \cdots \otimes M_{k-1} \otimes N_1 \otimes \cdots \otimes N_j \otimes M_{k+1} \otimes \cdots \otimes M_n.$$

The assumption that N' and M_k share the same H-data ensures M is viable.

As with joining, under certain circumstances grafting preserves validity and distributivity.

Lemma 7.9 Let \mathcal{T}' and \mathcal{T}'' be operation trees for $M' = M_1 = M_1 \otimes \cdots \otimes M_n$ and $N' = N_1 \otimes \cdots \otimes N_j$, respectively. Suppose that $X_j(N') = M_k$, and let \mathcal{T} be the grafting of \mathcal{T}'' onto \mathcal{T}' at position k .

If \mathcal{T}' and \mathcal{T}'' are valid and distributive, then \mathcal{T} is also valid and distributive.

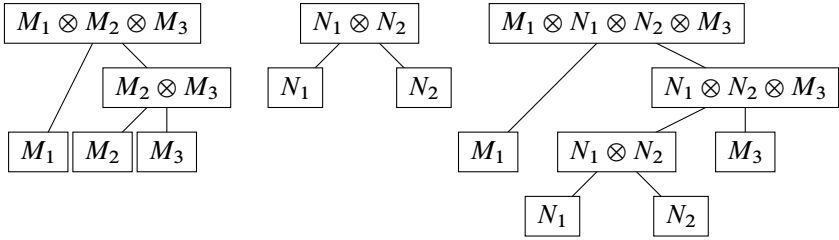


Figure 16: Operation trees \mathcal{T}' , \mathcal{T}'' , \mathcal{T} , where \mathcal{T} is the grafting of \mathcal{T}'' onto \mathcal{T}' at position 2.

Note that $X_j(N') = M_k$ implies N' and M_k have equal H–data, so \mathcal{T} is a well-defined operation tree.

Proof Each vertex of \mathcal{T}'' retains its label, hence validity and distributivity are satisfied. At vertices of \mathcal{T}' which retain their label, the same applies. Thus we only need consider vertices of \mathcal{T}' whose labels are changed in \mathcal{T} , ie those whose label involves M_k .

Let v be a vertex of \mathcal{T}' with l leaves, labelled $M'_v = M_u \otimes \cdots \otimes M_k \otimes \cdots \otimes M_{u+l-1}$; the label in \mathcal{T} is thus $M_v = M_u \otimes \cdots \otimes (N_1 \otimes \cdots \otimes N_j) \otimes \cdots \otimes M_{u+l-1}$. Since \mathcal{T}' is valid, M'_v is nonsingular. Since M_v and M'_v have the same H–data, L is nonsingular; so v is valid.

It remains to show that v is distributive. Since \mathcal{T}' is distributive, M'_v has at least $l - 2$ matched pairs which are twisted or critical. Now $M_k = X_j(N')$ implies that M_k is the unique nonzero homology class of diagram with the H–data of N' , so N' has tight H–data and M'_v is obtained from M_v by an H–contraction (Definition 2.42). By Lemma 2.43, if M'_v is critical at a matched pair P , then M_v is critical at P ; and if M'_v is twisted at P , then M_v is twisted at P . Hence M_v has at least as many twisted and critical matched pairs as M'_v . □

7.4 Nonzero operations require trees

As we now show, a valid distributive operation tree for M is a necessary condition for $X_n(M)$ or $\bar{f}_n(M)$ to be nontrivial.

Proposition 7.10 *Consider an A_∞ structure on \mathcal{H} arising from a pair ordering. If $X_n(M) \neq 0$ or $\bar{f}_n(M) \neq 0$, then there is a valid distributive operation tree for M .*

Proposition 7.10 is a precise version of Proposition 1.3.

Proof First note that as $X_n(M)$ or $\bar{f}_n(M) \neq 0$, M is viable (Lemma 5.1).

When $n = 1$, the valid and distributive conditions are trivial.

Now suppose that the statement holds for all X_k and \bar{f}_k for $k < n$, and consider X_n and \bar{f}_n .

Suppose $X_n(M) \neq 0$. By Lemma 5.4, $X_n(M)$ is represented by the sum of crossingless diagrams in $\bar{f}_j(M_1 \otimes \cdots \otimes M_j) \bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes M_n)$, so some $\bar{f}_j(M_1 \otimes \cdots \otimes M_j)$ and $\bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes M_n)$ are nonzero. By induction there are valid distributive operation trees \mathcal{T}' for $M_1 \otimes \cdots \otimes M_j$ and \mathcal{T}'' for $M_{j+1} \otimes \cdots \otimes M_n$. Now let \mathcal{T} be the join of \mathcal{T}' and \mathcal{T}'' ; this is well-defined as M is viable. Since \mathcal{T}' and \mathcal{T}'' are valid and distributive, by Lemma 7.7 so is \mathcal{T} .

Now suppose $\bar{f}_n(M) \neq 0$. Then $X_n(M) = 0$ (Lemma 5.7), and M has all matched pairs tight, twisted or critical, with at least one matched pair twisted (Theorem 5.5). Thus $\bar{f}_n(M) = A_{\mathcal{CR} \leq}^* \bar{U}_n(M)$, and hence $\bar{U}_n(M) \neq 0$. From equation (7) then some term of the form

$$\bar{f}_j(M_1 \otimes \cdots \otimes M_j) \bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes a_M)$$

or

$$\bar{f}_{n-j+1}(M_1 \otimes \cdots \otimes M_k \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \otimes \cdots \otimes M_n)$$

is nonzero. We consider the two cases separately.

In the first case, by induction, there are operation trees \mathcal{T}' for $M_1 \otimes \cdots \otimes M_j$, and \mathcal{T}'' for $M_{j+1} \otimes \cdots \otimes M_n$, which are valid and distributive. Let \mathcal{T} be the join of \mathcal{T}' and \mathcal{T}'' ; as M is viable, \mathcal{T} is well-defined. By Lemma 7.7 again, \mathcal{T} is valid and distributive.

In the second case induction gives operation trees

$$\mathcal{T}' \quad \text{for } M_1 \otimes \cdots \otimes M_k \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \otimes \cdots \otimes M_n,$$

$$\mathcal{T}'' \quad \text{for } M_{k+1} \otimes \cdots \otimes M_{k+j},$$

which are valid and distributive. Let \mathcal{T} be the grafting of \mathcal{T}'' onto \mathcal{T}' at position $k + 1$. This is clearly a well-defined operation tree, and by Lemma 7.9, \mathcal{T} is valid and distributive. □

7.5 Local trees

Let \mathcal{T} be an operation tree for M . We now consider M at a single matched pair P , and use this to construct “localised” versions of \mathcal{T} . We will define a *local operation tree*, which has the same underlying tree, and a *reduced local operation tree*, whose underlying tree is obtained by contracting “extraneous” vertices.

Recall from Section 2.5 that the local tensor product of $M = M_1 \otimes \cdots \otimes M_n$ at P is given by $M_P = (M_1)_P \otimes \cdots \otimes (M_n)_P$.

Definition 7.11 The local operation tree $\widetilde{\mathcal{T}}_P$ of \mathcal{T} at P is obtained from \mathcal{T} by replacing each M_i with $(M_i)_P$ in each vertex label.

It is straightforward to verify that $\widetilde{\mathcal{T}}_P$ is indeed an operation tree for M_P .

By Proposition 2.33, M_P is an extension–contraction of one of the tensor products shown in the tight, twisted, critical or singular columns of Table 2. So there are at most 4 tensor factors of M which have nonhorizontal strands at P , ie which cover one or more of the 4 steps around P .

Definition 7.12 A tensor factor M_i of M which has a nonhorizontal strand at a matched pair P is called *P–active*. The corresponding leaves of \mathcal{T} are called *P–active leaves*.

For each P , \mathcal{T} has at most 4 *P–active* leaves. These are precisely the leaves of $\widetilde{\mathcal{T}}_P$ labelled by nonidempotent diagrams.

Now we reduce $\widetilde{\mathcal{T}}_P$ to remove nonactive leaves and factors. Consider a non-*P–active* factor M_v of M , and the corresponding leaf v in $\widetilde{\mathcal{T}}_P$. Then $(M_v)_P$ is idempotent, so deleting it as a factor from M_P leaves a tensor product which is still viable. (Indeed, such a deletion is a trivial contraction: Definition 2.11.) We delete $(M_v)_P$ from all labels on $\widetilde{\mathcal{T}}_P$, and we delete the leaf v and its incident edge. This leaves a degree-2 vertex, which we smooth (ie we delete the degree-2 vertex and combine the two adjacent edges into a single edge). We then have a binary planar tree. (If the root vertex is smoothed, precisely one of its children remains; that child becomes the root.) It is an operation tree for $(M_1)_P \otimes \cdots \otimes \widehat{(M_v)_P} \otimes \cdots \otimes (M_n)_P$, where the hat denotes a deleted factor.

Repeating the process for all nonactive factors, we obtain an operation tree \mathcal{T}_P for $(M_{i_1})_P \otimes \cdots \otimes (M_{i_k})_P$, where the M_{i_j} are the *P–active* factors of M . Note that $0 \leq k \leq 4$; if $k = 0$, \mathcal{T}_P is the empty tree.

Definition 7.13 The operation tree \mathcal{T}_P is called the *reduced local operation tree* of \mathcal{T} at P .

The operation tree \mathcal{T}_P does not depend on the order in which the nonactive factors are deleted; in fact it can also be constructed “at once”, as follows. The P -active leaves of $\widetilde{\mathcal{T}}_P$ have a lowest common ancestor v_0 in \mathcal{T} . Take the edges and vertices along shortest paths in \mathcal{T} from each P -active leaf to v_0 . The union of these edges and vertices is a planar subtree of $\widetilde{\mathcal{T}}_P$ with root v_0 and leaves labelled by the M_{i_j} . Smoothing degree-2 vertices in this subtree and labelling vertices appropriately yields \mathcal{T}_P .

Note that the vertices of \mathcal{T}_P can be regarded as a subset of the vertices of \mathcal{T} or $\widetilde{\mathcal{T}}_P$: namely, those vertices which are not deleted or smoothed as we remove non- P -active factors.

Local operation trees are useful because of the following fact, a “local-to-global” law for validity.

Lemma 7.14 *Let \mathcal{T} be an operation tree. The following are equivalent:*

- (i) \mathcal{T} is valid.
- (ii) For all matched pairs P , the local operation tree $\widetilde{\mathcal{T}}_P$ is valid.
- (iii) For all matched pairs P , the reduced local operation tree \mathcal{T}_P is valid.

Proof By Lemma 2.36, a tensor product of homology classes of diagrams is nonsingular if and only if it is nonsingular at all its matched pairs. Since the labels on the operation trees $\widetilde{\mathcal{T}}_P$ are precisely the labels on \mathcal{T} , localised to P , (i) and (ii) are equivalent.

As mentioned above, deleting a non- P -active leaf from $\widetilde{\mathcal{T}}_P$, corresponding to a non- P -active factor M_i , produces a trivial contraction on vertex labels. Thus if all vertex labels were nonsingular in $\widetilde{\mathcal{T}}_P$, then they remain nonsingular. Conversely, if all the “new” vertex labels are nonsingular after deletion, their “old” labels (being obtained by extension from the “new” ones — even at the smoothed vertices) were also nonsingular. The deleted vertex was labelled by a single idempotent diagram, which is nonsingular. After deleting all non- P -active leaves, $\widetilde{\mathcal{T}}_P$ is valid if and only if \mathcal{T}_P is valid. \square

7.6 Climbing a tree

Let \mathcal{T} be a reduced local operation tree. Then \mathcal{T} has no more than 4 leaves, so there are not many possible trees. Indeed, the number of rooted planar binary trees with 1, 2, 3, 4, n leaves is 1, 1, 2, 5, $\frac{1}{n+1} \binom{2n}{n}$.

The tensor products arising in reduced local operation trees are also small in number. If M is the tensor product labelling the root of \mathcal{T} , then M is a viable tensor product of homology classes of diagrams on the arc diagram \mathcal{Z}_P consisting of a single matched pair. As \mathcal{T} is a reduced local operation tree, M has no idempotents, ie every tensor factor of M has nonhorizontal strands. Thus (Proposition 2.33) M is one of the tensor products shown in the tight, twisted, critical or singular columns of Table 2, or (in the tight case) a contraction thereof.

We ask: for each such tensor product M , which of the possible operation trees on M is valid?

If M is tight or twisted, then any subtensor product is tight or twisted (Lemma 2.40 and comment afterward), and in particular nonsingular, so any operation tree for M is valid. And of course if M is singular, then any operation tree for M is invalid, since its root vertex has singular label M .

When M is critical, some but not all operation trees are valid. By examining the possible cases in the critical column of Table 2, we observe the following, illustrated in Tables 4 and 5.

- When M is critical and P is sesqui-occupied, precisely 1 of the 2 operation trees are valid.
- When M is critical and P is 00 doubly occupied, precisely 2 of the 5 operation trees are valid.
- When M is critical and P is 11 doubly occupied, precisely 3 of the 5 operation trees are valid.

Starting from the leaves of \mathcal{T} , which are all tight, we can climb \mathcal{T} , observing how tightness behaves as the (homology classes of) diagrams labelling the vertices are joined into tensor products.

We observe that whenever there is a singular or twisted vertex label, it occurs when two adjacent diagrams are joined into a singular tensor product. Also, we never see both a twisted vertex label and a singular vertex label. This leads to the following statement.

Lemma 7.15 *Let \mathcal{T} be an operation tree for a viable tensor product of diagrams M . Then \mathcal{T} is valid if and only if for every nontight matched pair P of M , \mathcal{T}_P has a twisted vertex label.*

| M | valid operation trees | invalid operation trees |
|-----|-----------------------|-------------------------|
| | | |
| | | |

Table 4: Validity of operation trees on sesqui-occupied local critical tensor products. Red, green, blue and black vertices respectively indicate singular, critical, twisted and tight labels.

Proof By Lemma 7.14, the validity of \mathcal{T} is equivalent to the validity of all the \mathcal{T}_P . Since M is viable, at each matched pair M is tight, twisted, critical or singular. As discussed above, if M_P is tight or twisted at P then \mathcal{T}_P is valid; and clearly if M_P is twisted then \mathcal{T}_P has a twisted vertex label. So it remains to check that when M_P is critical or singular, \mathcal{T}_P is valid if and only if \mathcal{T}_P has a twisted vertex label.

If M_P is critical then, from Tables 4 and 5, \mathcal{T}_P is valid if and only if there is a twisted vertex label. And if M is singular, then \mathcal{T}_P is invalid, and moreover M_P must be an extension of the singular example in Table 2 (Lemma 2.37), so \mathcal{T}_P must be the unique rooted binary planar tree with two leaves; the two leaf labels are tight, and the root label is singular, so there is no twisted vertex label. Thus in each case \mathcal{T}_P is valid if and only if it has a twisted vertex label. \square

| M | valid operation trees | invalid operation trees |
|-----|-----------------------|-------------------------|
| | | |
| | | |

Table 5: Validity of operation trees on doubly occupied local critical tensor products. Red, green, blue and black vertices respectively indicate singular, critical, twisted and tight labels.

7.7 Strong validity

We saw above that when M is valid, then at every nontight $P = \{p, p'\}$, the reduced local operation tree \mathcal{T}_P has a twisted vertex label. But in fact, in almost every case, there is *precisely* one twisted vertex label. The only exception is when M_P is 11 doubly occupied and critical (ie the second row of Table 5), and \mathcal{T}_P is the unique rooted planar binary tree of depth 2 (ie the second valid operation tree shown). This particular operation tree can lead to the multiplication of a diagram crossed at p , with a diagram crossed at p' , producing a diagram in \mathcal{F} . To avoid it, we introduce a “strong” form of validity.

Lemma 7.16 *Let \mathcal{T} be an operation tree for M . The following are equivalent:*

- (i) *For every nontight matched pair P of M , there is a unique lowest vertex of \mathcal{T} among those whose label is twisted at P .*
- (ii) *The operation tree \mathcal{T} is valid, and for each nontight matched pair P of M , there is a unique lowest vertex of \mathcal{T} among those whose label is not tight at P .*
- (iii) *For each nontight matched pair P of M , there is a unique lowest vertex of $\tilde{\mathcal{T}}_P$ among those whose label is twisted.*
- (iv) *For every nontight matched pair P of M , \mathcal{T}_P has a unique twisted vertex label.*

Definition 7.17 The operation tree \mathcal{T} is *strongly valid* if the conditions of Lemma 7.16 hold.

Comparing Lemmas 7.15 and 7.16(iv), it is clear that strong validity implies validity.

Proof of Lemma 7.16 First we show equivalence of (i) and (ii). If \mathcal{T} is not valid, then (i) fails by Lemma 7.15, and (ii) obviously fails. So assume \mathcal{T} is valid. We show that a vertex v of \mathcal{T} , with label M_v , is lowest among those with labels twisted at P if and only if it is lowest among those with labels nontight at P .

If v is lowest among vertices with label twisted at P , then the children of v have labels which are subtensor-products of M_v nontwisted at P . Hence by Lemma 2.39 and Table 3, the labels on these children are tight at P . All descendants of these children have tight labels at P also, again by Lemma 2.39 and Table 3. So v is lowest among vertices of \mathcal{T} with labels nontight at P .

Conversely, if v is lowest among those with labels nontight at P , then all descendants of v have tight labels at P . Then $(M_v)_P$ is the tensor product of the tight labels of its children: it cannot be critical, by Lemma 2.38, and cannot be singular, since \mathcal{T} is valid. So $(M_v)_P$ is twisted, and v is lowest among vertices with label twisted at P . Thus (i) and (ii) are equivalent.

Condition (iii) is just a reformulation of (i).

To see equivalence of (iii) and (iv), recall how \mathcal{T}_P is obtained from $\widetilde{\mathcal{T}}_P$. If the label M_v on a leaf v of $\widetilde{\mathcal{T}}_P$ is idempotent, then we delete v and its incident edge, delete M_v from all labels, and smooth the resulting degree-2 vertex w . Since M_v has only horizontal strands, deleting M_v from a label yields a trivial contraction (Definition 2.11), which does not change the tightness of the label.

Now w has two children v and x in $\widetilde{\mathcal{T}}_P$. Since M_v is tight (being an idempotent), and M_w is an extension of M_x (by the horizontal strands of M_v), it follows M_w and M_x have the same tightness. In particular, neither v nor w can be lowest among those with twisted label. After deleting v and all instances of M_v in labels, the label on w is the same as the label on x . After smoothing w , every remaining vertex has children and descendants with twisted labels if and only if it had them in $\widetilde{\mathcal{T}}_P$. Thus any vertex which was lowest among those with twisted labels was not v or w , so remains as a vertex, and remains lowest among those with twisted labels. So the set of lowest vertices with twisted labels is preserved.

Repeating this process we eventually arrive at \mathcal{T}_P . So $\widetilde{\mathcal{T}}_P$ has a unique lowest vertex among those with twisted labels, if and only if the same is true for \mathcal{T}_P . From the examination of reduced local operation trees in Section 7.6, we observe that a reduced local operation tree has a unique lowest vertex with twisted label if and only if it has a unique vertex with a twisted label. Thus (iii) and (iv) are equivalent. □

The above discussion also immediately implies the following.

Lemma 7.18 *Suppose \mathcal{T} is an operation tree for M which is valid but not strongly valid. Then M has a matched pair which is 11 doubly occupied and critical.* □

By Lemma 7.16(i), the following map is well-defined.

Definition 7.19 Let \mathcal{T} be a strongly valid operation tree for M . The function

$$V_{\mathcal{T}}: \{\text{nontight matched pairs of } M\} \rightarrow \{\text{nonleaf vertices of } \mathcal{T}\}$$

sends a matched pair P to the lowest vertex of \mathcal{T} whose label is twisted at P .

By the argument in the proof of [Lemma 7.16](#) (that (i) and (ii) are equivalent), $V_{\mathcal{T}}(P)$ is also the lowest vertex of \mathcal{T} whose label is not tight at P .

Lemma 7.20 *Let \mathcal{T} be a strongly valid operation tree for M , and let P be a nontight matched pair of M . Then the vertices of \mathcal{T} whose labels are nontight at P are precisely $V_{\mathcal{T}}(P)$ and its ancestors.*

Proof Let the label on $V_{\mathcal{T}}(P)$ be M' . If v is an ancestor of $V_{\mathcal{T}}(P)$, labelled M_v , then M' is a subtensor-product of M_v . As M' is not tight at P , by [Lemma 2.39](#) M_v is not tight at P .

Conversely, suppose a vertex v_0 of \mathcal{T} has label nontight at P . Either v_0 is a lowest such vertex, or v_0 has a child v_1 whose label is also not tight at P . If the latter, then v_1 is either a lowest such vertex, or has a child whose label is nontight at P . In this way, we eventually arrive at a descendant v_* of v_0 which is lowest amongst those whose labels are not tight at P . By the comment after [Definition 7.19](#) then $v_* = V_{\mathcal{T}}(P)$, so v_0 is $V_{\mathcal{T}}(P)$ or one of its ancestors. □

Strong validity shares many of the properties of validity. The following lemmas generalise [Lemmas 7.14](#) and [7.4](#).

Lemma 7.21 *Let \mathcal{T} be an operation tree. The following are equivalent:*

- (i) \mathcal{T} is strongly valid.
- (ii) For all matched pairs P , the local operation tree $\widetilde{\mathcal{T}}_P$ is strongly valid.
- (iii) For all matched pairs P , the reduced local operation tree \mathcal{T}_P is strongly valid.

Proof Characterisation (iii) of [Lemma 7.16](#) only depends on local operation trees $\widetilde{\mathcal{T}}_P$, and (iv) only on reduced local operation trees \mathcal{T}_P . □

Lemma 7.22 *Let \mathcal{T} be a strongly valid operation tree for M , and let v be a vertex of \mathcal{T} labelled by M_v . Let \mathcal{T}_v be the operation subtree of \mathcal{T} below v . Then the following hold:*

- (i) \mathcal{T}_v is a strongly valid operation tree for M_v .
- (ii) The function $V_{\mathcal{T}_v}$ is a restriction of the function $V_{\mathcal{T}}$.

Note that M_v is a subtensor-product of M , so by Lemma 2.40, a matched pair which is nontight in M_v is also nontight in M . Hence the domain of $V_{\mathcal{T}_v}$ is a subset of the domain of $V_{\mathcal{T}}$, so the assertion of (ii) makes sense.

Proof Let P be a matched pair, and consider the local operation trees $\widetilde{\mathcal{T}}_P$ for M_P , and $(\widetilde{\mathcal{T}}_v)_P$ for $(M_v)_P$. To prove (i), we show that if $(M_v)_P$ is not tight, then $(\widetilde{\mathcal{T}}_v)_P$ has a unique lowest vertex with twisted label (Lemma 7.16(iii)); and to prove (ii), we show that this vertex is also the unique lowest vertex with twisted label in $\widetilde{\mathcal{T}}_P$.

So suppose $(M_v)_P$ is not tight. It is also not singular: as \mathcal{T} is strongly valid, \mathcal{T} is valid, so by Lemma 7.4 \mathcal{T}_v is valid; hence M_v is nonsingular, so $(M_v)_P$ is also nonsingular (Lemma 2.36). Thus $(M_v)_P$ is twisted or critical. By Lemma 7.14(ii), $(\widetilde{\mathcal{T}}_v)_P$ is valid; being an operation tree for the nontight $(M_v)_P$, by Lemma 7.15, $(\widetilde{\mathcal{T}}_v)_P$ has a vertex with a twisted label.

Now $(\widetilde{\mathcal{T}}_v)_P$ is the operation subtree of $\widetilde{\mathcal{T}}_P$ below v , with the same vertex labels, consisting of everything in $\widetilde{\mathcal{T}}_P$ from v down. Thus, any lowest vertex with twisted label in $(\widetilde{\mathcal{T}}_v)_P$ is also a lowest vertex in $\widetilde{\mathcal{T}}_P$ with twisted label. As $(M_v)_P$ is twisted or critical, and is a subtensor-product of M_P , then M_P is also twisted or critical (Lemma 2.39). By strong validity of \mathcal{T} and Lemma 7.16(iii), there is a unique lowest vertex in $\widetilde{\mathcal{T}}_P$ with twisted label. As $(\widetilde{\mathcal{T}}_v)_P$ has a vertex with twisted label, the unique lowest vertex in $\widetilde{\mathcal{T}}_P$ with twisted label lies in $(\widetilde{\mathcal{T}}_v)_P$, and it is also the unique lowest vertex in $(\widetilde{\mathcal{T}}_v)_P$ with twisted label. □

Finally, strong validity implies the following nice separation property of nontight matched pairs.

Lemma 7.23 *Let \mathcal{T} be a strongly valid operation tree. Let v and w be vertices of \mathcal{T} , with labels M_v and M_w respectively, such that the operation subtrees \mathcal{T}_v and \mathcal{T}_w below v and w are disjoint.*

For any matched pair P , at least one of M_v and M_w is tight at P .

The disjointness of \mathcal{T}_v and \mathcal{T}_w is equivalent to neither of v and w being a descendant of the other.

Proof Suppose to the contrary that both $(M_v)_P$ and $(M_w)_P$ are not tight. By Lemma 7.22, \mathcal{T}_v and \mathcal{T}_w are strongly valid, so there is a unique lowest vertex x_v in $(\widetilde{\mathcal{T}}_v)_P$ with twisted label, and a unique lowest vertex x_w in $(\widetilde{\mathcal{T}}_w)_P$ with twisted label. But then x_v and x_w are two distinct vertices of $\widetilde{\mathcal{T}}_P$ which are lowest vertices with twisted labels, contradicting strong validity of \mathcal{T} . □

7.8 Transplantation and branch shifts

We now define two further methods to modify operation trees.

The first method, *transplantation*, replaces an operation subtree (Definition 7.2) with another tree.

Definition 7.24 Let \mathcal{T} be an operation tree, and let \mathcal{T}_v be the operation subtree below a nonroot vertex v , labelled M' . Let \mathcal{T}' be another operation tree for M' . Then removing \mathcal{T}_v from \mathcal{T} and replacing it with \mathcal{T}' gives an operation tree \mathcal{U} . We say \mathcal{U} is obtained from \mathcal{T} by *transplanting* \mathcal{T}' for \mathcal{T}_v .

It is easily verified that \mathcal{U} is in fact an operation tree; viability of labels in \mathcal{T} and \mathcal{T}' implies viability of labels in \mathcal{U} . If \mathcal{T} is an operation tree for M , then \mathcal{U} is also an operation tree for M . So \mathcal{T} and \mathcal{U} describe operations on the same inputs, but the operations under v are rearranged.

Note that transplantation is quite different from grafting (Section 7.3). Grafting adds to an operation tree below a leaf, while transplantation replaces part of an operation tree. Grafting adds new leaves with new labels, requiring relabelling throughout the tree, while leaf labels are unchanged under transplantation.

Lemma 7.25 *Suppose that \mathcal{U} is obtained from \mathcal{T} by transplanting \mathcal{T}' for \mathcal{T}_v .*

- (i) *If \mathcal{T} and \mathcal{T}' are valid, then \mathcal{U} is also valid.*
- (ii) *If \mathcal{T} and \mathcal{T}' are strongly valid, then \mathcal{U} is also strongly valid.*

Proof All labels on vertices of \mathcal{U} are inherited from \mathcal{T} or \mathcal{T}' . If both \mathcal{T} and \mathcal{T}' are valid, then all labels are nonsingular, so \mathcal{U} is valid.

Now suppose \mathcal{T} and \mathcal{T}' are strongly valid. Let M and M' be the labels on the root vertex of \mathcal{T} and the vertex v , respectively, and let P be a matched pair at which M is not tight. By strong validity of \mathcal{T} , there is a unique vertex w of \mathcal{T} which is lowest among those with labels nontight at P (Lemma 7.16(ii)). Moreover, the vertices of \mathcal{T} with labels nontight at P are precisely the ancestors of w (Lemma 7.20).

If w is not a vertex of \mathcal{T}_v , then v is not an ancestor of w , so the label M' of v is tight at P . Every vertex label in \mathcal{T}' is a subtensor-product of M' , hence tight at P (Lemma 2.39). So the vertices of \mathcal{U} with labels nontight at P are precisely the vertices of \mathcal{T} with labels nontight at P , and hence there is a unique lowest such vertex, namely w .

If w is a vertex of \mathcal{T}_v , then v is an ancestor of w , so the label M' of v is nontight at P . Since \mathcal{T}' is strongly valid, there is a unique lowest vertex w' of \mathcal{T}' with label nontight at P (Lemma 7.16(ii) again), and the set of vertices of \mathcal{T}' whose labels are nontight at P are precisely the ancestors of w' (Lemma 7.20 again). Thus in \mathcal{U} , the set of vertices whose labels are nontight at P are the ancestors of w' in \mathcal{T}' , together with the ancestors of v in \mathcal{T} — in other words, the ancestors of w' in \mathcal{U} .

In any case, there is a unique vertex in \mathcal{U} which is lowest among those with labels nontight at P , so by Lemma 7.16(ii) once more, \mathcal{U} is strongly valid. □

The second method, a *branch shift*, rearranges an operation tree in a way corresponding to a modification $((AB)C) \leftrightarrow (A(BC))$.

Given an operation tree \mathcal{T} , denote the left and right children of the root vertex v by v_L and v_R , the left and right children of v_L by v_{LL} and v_{LR} , and generally for any word w in L and R , let v_w denote the descendant of v obtained by successively taking left or right children according to w (if it exists).

Definition 7.26 The operation tree \mathcal{T}' is defined by

$$\mathcal{T}'_L = \mathcal{T}_{LL}, \quad \mathcal{T}'_{RL} = \mathcal{T}_{LR}, \quad \mathcal{T}'_{RR} = \mathcal{T}_R.$$

We say that the operation trees \mathcal{T} and \mathcal{T}' are related by a *branch shift*.

The vertex labels on \mathcal{T}' are either inherited from \mathcal{T} , or determined by the fact that each vertex is labelled with the tensor product of its children’s labels.

Let $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 respectively denote $\mathcal{T}_{LL}, \mathcal{T}_{LR}$ and \mathcal{T}_R ; let N_1, N_2 and N_3 be the vertex labels on v_{LL}, v_{LR} and v_R , respectively; let the root vertex of \mathcal{T}' be v' , and denote its vertices by v'_w for words w in L and R . Then, in \mathcal{T} , v_L is labelled $N_1 \otimes N_2$; and in \mathcal{T}' , v'_R is labelled $N_2 \otimes N_3$. The viability of labels in \mathcal{T} ensures the viability of labels in \mathcal{T}' , so both \mathcal{T} and \mathcal{T}' are operation trees for $N = N_1 \otimes N_2 \otimes N_3$. Observe that upon reversing left and right, \mathcal{T} is obtained from \mathcal{T}' in the same way. See Figure 17.

All labels in \mathcal{T}' appear in \mathcal{T} , with one exception. Thus if \mathcal{T} is valid, then we only have one label to check for validity of \mathcal{T}' , giving the following.

- Lemma 7.27** (i) Suppose \mathcal{T} is valid. Then \mathcal{T}' is valid if and only if $N_2 \otimes N_3$ is nonsingular.
- (ii) Suppose \mathcal{T}' is valid. Then \mathcal{T} is valid if and only if $N_1 \otimes N_2$ is nonsingular. □

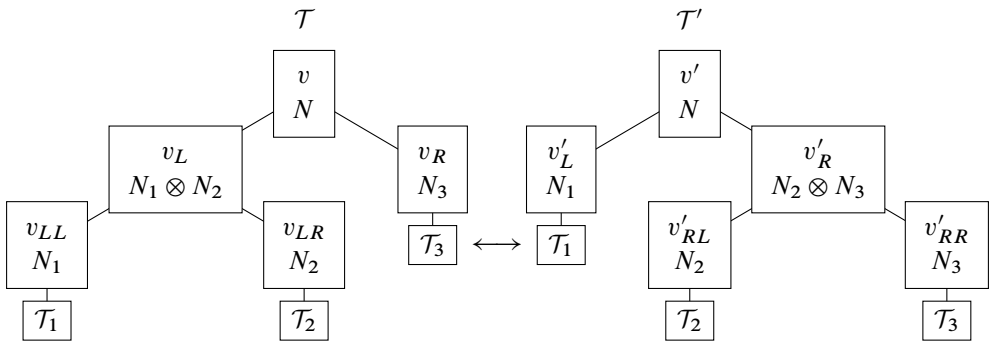


Figure 17: A branch shift.

7.9 Strict distributivity

We now strengthen our notion of distributivity (Definition 7.5).

Definition 7.28 Let \mathcal{T} be a valid operation tree for $M = M_1 \otimes \cdots \otimes M_n$.

- (i) Let v be a vertex of \mathcal{T} with k leaves, labelled M_v . Then v is *strictly distributive* if there are exactly $k - 1$ matched pairs at which M_v is twisted or critical.
- (ii) The tree \mathcal{T} is *strictly f -distributive* if it is strictly distributive at each vertex.
- (iii) The tree \mathcal{T} is *strictly X -distributive* if it is strictly distributive at each nonroot vertex, and there are precisely $n - 2$ matched pairs at which M is twisted or critical.

Recall that distributivity (Definition 7.5) at v requires at least $k - 2$ twisted or critical matched pairs at v ; the strict requirement is that there are precisely $k - 1$ such pairs. Note that Definition 7.28 requires \mathcal{T} to be valid, so no labels are singular.

Lemma 7.29 Let \mathcal{T} be a valid strictly f - or X -distributive operation tree, and let v be a nonroot vertex. Then the operation subtree \mathcal{T}_v of \mathcal{T} below v is strictly f -distributive.

Proof By Lemma 7.4 \mathcal{T}_v is valid, and every vertex of \mathcal{T}_v , being a nonroot vertex of \mathcal{T} , is strictly distributive. □

Strict distributivity imposes strong conditions on the function $V_{\mathcal{T}}$ (Definition 7.19).

Lemma 7.30 *Let \mathcal{T} be an operation tree for $M = M_1 \otimes \cdots \otimes M_n$.*

- (i) *If \mathcal{T} is strongly valid and strictly f -distributive, then $V_{\mathcal{T}}$ is a bijection between nontight matched pairs of M and nonleaf vertices of \mathcal{T} .*
- (ii) *If \mathcal{T} is strongly valid and strictly X -distributive, then $V_{\mathcal{T}}$ is a bijection between nontight matched pairs of M and nonleaf nonroot vertices of \mathcal{T} .*

Since M has n tensor factors, \mathcal{T} has n leaves, hence $n - 1$ nonleaf vertices and $n - 2$ nonleaf nonroot vertices. Strict f -distributivity (resp. X -distributivity) requires that M has precisely $n - 1$ (resp. $n - 2$) nontight matched pairs. So in each case the claimed bijective sets have the same size.

Proof When $n = 1$, if \mathcal{T} is strictly f -distributive, then M has no twisted or critical matched pairs (ie is tight), and \mathcal{T} has no nonleaf vertices. When $n = 2$, if \mathcal{T} is strictly X -distributive, again M is tight, and \mathcal{T} has no nonleaf nonroot vertices. In both cases $V_{\mathcal{T}}$ is a bijection between empty sets.

We now proceed by induction on n . So suppose the result is true for operation trees for $M = M_1 \otimes \cdots \otimes M_k$, where $k < n$, and consider an operation tree \mathcal{T} for $M = M_1 \otimes \cdots \otimes M_n$ which is strongly valid and strictly f -distributive or strictly X -distributive.

Let v_0 be the root vertex of \mathcal{T} , and let v_L and v_R be its left and right children; let their labels be $M_L = M_1 \otimes \cdots \otimes M_i$ and $M_R = M_{i+1} \otimes \cdots \otimes M_n$ respectively. Let \mathcal{T}_L and \mathcal{T}_R be the operation subtrees of \mathcal{T} below v_L and v_R (Definition 7.2). Now \mathcal{T}_L and \mathcal{T}_R are strongly valid (Lemma 7.22) and strictly f -distributive (Lemma 7.29), so by induction we have bijections

$$V_L : \{\text{nontight matched pairs of } M_L\} \rightarrow \{\text{nonleaf vertices of } \mathcal{T}_L\},$$

$$V_R : \{\text{nontight matched pairs of } M_R\} \rightarrow \{\text{nonleaf vertices of } \mathcal{T}_R\},$$

which by Lemma 7.22(ii) are restrictions of $V_{\mathcal{T}}$. Moreover, since \mathcal{T}_L and \mathcal{T}_R are disjoint, and \mathcal{T} is strongly valid, Lemma 7.23 says that the domains of V_L and V_R are disjoint. It's also clear that the ranges of V_L and V_R are disjoint; their union consists of all nonleaf nonroot vertices of \mathcal{T} . The domains (and ranges) of V_L and V_R have cardinalities $i - 1$ and $n - i - 1$ respectively.

Since nontight matched pairs in M_L or M_R are nontight in M (Lemma 2.40), the nontight matched pairs in M_L and M_R form precisely $(i - 1) + (n - i - 1) = n - 2$ nontight matched pairs of M .

If M is strictly X –distributive, then these are all the matched pairs in M , and $V_{\mathcal{T}}$ is the disjoint union of V_L and V_R , hence a bijection as claimed.

If M is strictly f –distributive, then M has precisely $n - 1$ nontight matched pairs. So there is precisely one nontight matched pair P_0 in M which is tight in M_L and M_R . Since P_0 is tight in both M_L and M_R , but nontight in M , v_0 is the lowest vertex of \mathcal{T} whose label is nontight at P_0 , so $V_{\mathcal{T}}(P_0) = v_0$. This, together with V_L and V_R , defines $V_{\mathcal{T}}$; we conclude V is a bijection. \square

7.10 Guaranteed nonzero results

We now show that, in certain cases, X_n and \bar{f}_n must be nonzero, and compute their values.

Theorem 7.31 *Consider an A_{∞} structure on \mathcal{H} arising from a pair ordering \preceq . Suppose M is viable and satisfies the following conditions:*

- (i) *Every valid and distributive operation tree for M is strictly f –distributive, and such a tree exists.*
- (ii) *No matched pair of M is on-on doubly occupied.*

Then $\bar{f}_n(M) \neq 0$. Moreover $\bar{f}_n(M)$ is given by a single diagram D , which is tight at all matched pairs where M is tight or critical, and crossed at all matched pairs where M is twisted.

Theorem 7.31 is a precise version of **Theorem 1.4(i)**. It explicitly describes $\bar{f}_n(M) = D$, which is determined by its H–data, and tightness at each matched pair. There is in fact no choice in constructing D , since choices only exist at 11 doubly occupied pairs, which are explicitly ruled out.

The description of D follows entirely from Maslov index considerations. The existence of a valid and strictly f –distributive tree for M implies that M is twisted or critical at precisely $n - 1$ matched pairs, and tight at all other matched pairs. The Maslov index can only increase by 1 at each nontight matched pair. Since \bar{f}_n has Maslov grading $n - 1$, Maslov grading must increase at every nontight matched pair: from twisted to crossed, and from critical to tight.

When $n = 1$, condition (i) says that $M = M_1$ is tight (there is only one possible operation tree), and the conclusion is that $\bar{f}_1(M)$ is a tight diagram representing M_1 .

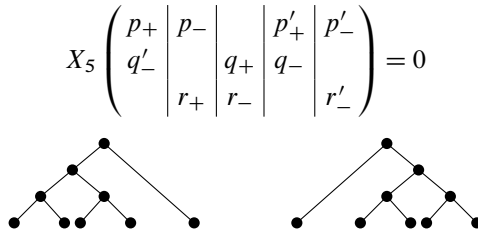


Figure 18: This tensor product $M = M_1 \otimes \cdots \otimes M_5$ (shown in shorthand) has a 11 doubly occupied matched pair $P = \{p, p'\}$, and two valid distributive operation trees as shown, both of which are strongly valid and strictly X -distributive. However $\bar{f}_1 \bar{f}_4 = \bar{f}_4 \bar{f}_1 \neq 0$ and $\bar{f}_2 \bar{f}_3 = \bar{f}_3 \bar{f}_2 = 0$, so $X_5 = 0$.

When $n = 2$, condition (i) says that $M = M_1 \otimes M_2$ has precisely one nontight matched pair P (again there is only one possible operation tree), which must be twisted (Lemma 2.38), and all other matched pairs tight. The conclusion is that $\bar{f}_2(M)$ is a single diagram D twisted at P and elsewhere tight, in agreement with the discussion of Section 4.3.

Theorem 7.32 Consider an A_∞ structure on \mathcal{H} arising from a pair ordering \preceq . Suppose M is viable and satisfies the following conditions.

- (i) Every valid and distributive operation tree for M is strictly X -distributive, and such a tree exists.
- (ii) No matched pair of M is twisted or on-on doubly occupied.

Then $X_n(M)$ is nonzero, and is the homology class of the unique tight diagram with the H -data of M .

Theorem 7.32 is a precise version of Theorem 1.4(ii). The description of $X_n(M)$ follows entirely from the fact that X_n preserves H -data. The uniqueness claim in the theorem makes sense: since M has no 11 doubly occupied pairs, there is only one tight diagram with the same H -data as M .

The exclusion of twisted matched pairs is necessary, since they preclude the existence of a tight diagram (or by Theorem 5.6). The exclusion of 11 doubly occupied pairs is a more heavy-handed assumption, but is necessary for our proof; moreover it cannot be removed because of the example of Figure 18. In this example, $M = M_1 \otimes \cdots \otimes M_5$ is viable, has no twisted matched pairs, and has two valid distributive operation trees, both of which are strongly valid and strictly X -distributive. However it also has a 11 doubly occupied matched pair $P = \{p, p'\}$, and $X_5(M) = 0$.

While the conditions of Theorems 7.31 and 7.32 may seem rather restrictive, they do show that \bar{f}_n and X_n are nonzero in many cases. For instance, in the \bar{f}_3 examples of Lemma 6.2, the first two lines (ie 10 out of 14 examples) can be shown to be nonzero directly from Theorem 7.31. It follows from Theorem 7.32 that X_4 of all the following tensor products are nonzero:

$$\begin{matrix} \left(\begin{array}{c|c|c|c} p_+ & p_- & p'_+ & \\ \hline q_+ & & q_- & q'_+ \end{array} \right) & \left(\begin{array}{c|c|c|c} p_+ & p_- & p'_+ & \\ \hline q_+ & q_- & q_- & q'_+ \end{array} \right) & \left(\begin{array}{c|c|c|c} p_+ & p_- & & p'_+ \\ \hline & q_+ & q_- & q'_+ \end{array} \right) & \left(\begin{array}{c|c|c|c} p_+ & & p_- & p'_+ \\ \hline q_+ & q_- & q'_+ & \end{array} \right) \\ \left(\begin{array}{c|c|c|c} p'_- & p_+ & p_- & p'_+ \\ \hline q_+ & q_- & & q'_+ \end{array} \right) & \left(\begin{array}{c|c|c|c} p'_- & p_+ & p_- & p'_+ \\ \hline q_+ & & q_- & q'_+ \end{array} \right) & \left(\begin{array}{c|c|c|c} p'_- & p_+ & p_- & p'_+ \\ \hline q'_- & q_+ & & q_- \end{array} \right) & \left(\begin{array}{c|c|c|c} p'_- & p_+ & p_- & p'_+ \\ \hline q'_- & & q_+ & q_- \end{array} \right) \end{matrix}$$

The hypotheses of Theorems 7.31 and 7.32 essentially mandate that in each operation described by an operation tree, only one matched pair can be affected.

We first need a preliminary lemma.

Lemma 7.33 (plenty of trees) *Consider an A_∞ structure on \mathcal{H} defined by a pair ordering. Suppose $M = M_1 \otimes \dots \otimes M_n$ is viable. Further suppose that every valid and distributive operation tree for M is strongly valid and strictly f -distributive, and at least one such tree exists.*

Let $P_0 = \{p_0, p'_0\}$ be a matched pair at which M is twisted. Then there exists a strongly valid, strictly f -distributive operation tree \mathcal{T} for M such that $V_{\mathcal{T}}(P_0)$ is the root vertex v_0 of \mathcal{T} .

Let us say something about what Lemma 7.33 means. At P_0 , M is twisted and hence an extension of the twisted tensor product of Table 2 (Lemma 2.37). So two steps of P_0 are covered, say p_{0+} and p_{0-} , by some M_i and M_j respectively for some $i < j$. Now a subtensor-product M' of M labelling a nonroot vertex of \mathcal{T} is twisted or tight accordingly as M' contains both M_i and M_j , or does not. Lemma 7.33 guarantees the existence of a tree such that all labels on nonroot vertices are tight at P_0 . In other words, M_i and M_j never appear together in any label in \mathcal{T} except at the root vertex v_0 ; as we work our way up the tree, combining tensor factors, M_i and M_j are only combined at the final step, at v_0 . Since P_0 only becomes twisted at v_0 , v_0 is the lowest vertex of \mathcal{T} whose label is twisted at P , and $V_{\mathcal{T}}(P_0) = v_0$.

Since we can find such a tree for each all-on once occupied pair P , this gives us “plenty of trees”, which we need for the proof of Theorem 7.31.

Note that the hypotheses of [Lemma 7.33](#) are weaker than those of [Theorem 7.31](#). If M satisfies the hypotheses of [Theorem 7.31](#), then every valid distributive operation tree for M is strictly f -distributive; but as there are no 11 doubly occupied pairs, any such tree is strongly valid ([Lemma 7.18](#)), so M satisfies the hypotheses of [Lemma 7.33](#).

The following lemma captures an argument we will use repeatedly. The terms in square brackets may be included or not.

Lemma 7.34 *Let M be a viable tensor product of nonzero homology classes of diagrams, which has one of the following two properties:*

- (i) *Every valid and distributive operation tree for M is [strongly valid and] strictly f -distributive, and such a tree exists.*
- (ii) *Every valid and distributive operation tree for M is [strongly valid and] strictly X -distributive, and such a tree exists.*

Let \mathcal{T} be an operation tree for M of the type guaranteed by the condition, and let v be a nonroot vertex of \mathcal{T} , with label M_v . Then M_v satisfies condition (i).

Proof Let \mathcal{T}' be a valid distributive operation tree for M_v . Then we can transplant \mathcal{T}' for the operation subtree \mathcal{T}_v of \mathcal{T} below v to obtain an operation tree \mathcal{U} for M , which is valid ([Lemma 7.25](#)) and distributive (since distributive at each vertex: [Definition 7.5](#)). By assumption then \mathcal{U} is [strongly valid and] strictly f - or X -distributive, so its subtree \mathcal{T}' is also [strongly valid ([Lemma 7.22](#)) and] strictly f -distributive ([Lemma 7.29](#)). Finally, \mathcal{T}_v demonstrates that such a tree exists. \square

Proof of [Lemma 7.33](#) When $n = 1$ the statement is vacuous: $M = M_1$ is tight, the unique operation tree is strongly valid and strictly f -distributive, and $V_{\mathcal{T}}$ is a bijection between empty sets. Proceeding by induction on n , consider a general n , and suppose the result holds for all smaller values of n .

Let \mathcal{T} be a strongly valid and strictly f -distributive operation tree for M , which exists by assumption. By strict f -distributivity at v_0 , there are precisely $n - 1$ matched pairs at which M is nontight (ie twisted or critical). Let the two children of v_0 be v_L and v_R , with labels $M_L = M_1 \otimes \cdots \otimes M_m$ and $M_R = M_{m+1} \otimes \cdots \otimes M_n$ respectively. Let \mathcal{T}_L and \mathcal{T}_R be the operation subtrees of \mathcal{T} below v_L and v_R , respectively. Then \mathcal{T}_L and \mathcal{T}_R are strongly valid ([Lemma 7.22](#)) and strictly f -distributive ([Lemma 7.29](#)) operation trees for M_L and M_R , respectively.

By Lemma 7.30, $V_{\mathcal{T}}$, $V_{\mathcal{T}_L}$ and $V_{\mathcal{T}_R}$ are all bijections, between sets of size $n - 1$, $m - 1$ and $n - m - 1$, respectively; moreover $V_{\mathcal{T}_L}$ and $V_{\mathcal{T}_R}$ are restrictions of $V_{\mathcal{T}}$ (Lemma 7.22) with disjoint domains (Lemma 7.23). Hence there is a unique matched pair P_1 such that $V_{\mathcal{T}}(P_1) = v_0$. Then P_1 is twisted in M (Definition 7.19), but tight in every other tensor product labelling a vertex.

If $P_1 = P_0$ then we are done; so suppose that P_1 and P_0 are distinct. Then $V_{\mathcal{T}}(P_0) \neq V_{\mathcal{T}}(P_1) = v_0$, so $V_{\mathcal{T}}(P_0)$ is a vertex of \mathcal{T}_L or \mathcal{T}_R . Suppose $V_{\mathcal{T}}(P_0)$ lies in \mathcal{T}_L ; the \mathcal{T}_R case is similar.

By Lemma 7.34(i), M_L satisfies the hypotheses of this lemma. By induction there then exists a strongly valid, strictly f -distributive operation tree \mathcal{T}'_L for M_L such that $V_{\mathcal{T}'_L}(P_0) = v_L$. Transplanting this \mathcal{T}'_L for \mathcal{T}_L yields a strongly valid (by Lemma 7.25) and strictly f -distributive (since strictly distributive at each vertex: Definition 7.28) operation tree \mathcal{T}' for M . Moreover, $V_{\mathcal{T}'_L}$ is a restriction of $V_{\mathcal{T}'}$ (Lemma 7.22), so $V_{\mathcal{T}'}(P_0) = v_L$, and since P_1 is tight in M_L and M_R , $V_{\mathcal{T}'}(P_1) = v_0$.

Let the children of v_L be v_{LL} and v_{LR} , and their labels in \mathcal{T}' be $M'_{LL} = M_1 \otimes \cdots \otimes M_k$ and $M'_{LR} = M_{k+1} \otimes \cdots \otimes M_m$. Denote the operation subtrees of \mathcal{T}' (or \mathcal{T}'_L) below v_{LL} and v_{LR} respectively by \mathcal{T}'_{LL} and \mathcal{T}'_{LR} . These are again strongly valid and strictly f -distributive (Lemmas 7.22 and 7.29).

By strict f -distributivity, M'_{LL} , M'_{LR} , M_R and M have precisely $k - 1$, $m - k - 1$, $n - m - 1$ and $n - 1$ nontight matched pairs, respectively. But since \mathcal{T}'_{LL} , \mathcal{T}'_{LR} and \mathcal{T}_R are disjoint subtrees (below v_{LL} , v_{LR} and v_R) of the strongly valid \mathcal{T}' , the sets of matched pairs at which M'_{LL} , M'_{LR} and M_R are nontight are also disjoint (Lemma 7.23). Their union consists of $(k - 1) + (m - k - 1) + (n - m - 1) = n - 3$ matched pairs, which remain nontight in M (Lemma 2.40). The two remaining nontight matched pairs of M are P_0 and P_1 ; these two pairs are tight in each of M'_{LL} , M'_{LR} and M_R since $V_{\mathcal{T}'}(P_0) = v_L$ and $V_{\mathcal{T}'}(P_1) = v_0$.

Now perform a branch shift on \mathcal{T}' (Definition 7.26) to obtain an operation tree \mathcal{T}'' for M . Its root has children $v''_L = v_{LL}$ and v''_R , and the children of v''_R are $v''_{RL} = v_{LR}$ and $v''_{RR} = v_R$. Below v''_L , v''_{RL} and v''_{RR} respectively we have $\mathcal{T}''_L = \mathcal{T}'_{LL}$, $\mathcal{T}''_{RL} = \mathcal{T}'_{LR}$ and $\mathcal{T}''_{RR} = \mathcal{T}_R$. The labels on \mathcal{T}'' are inherited from \mathcal{T}'_{LL} , \mathcal{T}'_{LR} and \mathcal{T}'_R , except that v''_0 is labelled M and v''_R is labelled with $M''_R = M_{k+1} \otimes \cdots \otimes M_n = M'_{LR} \otimes M_R$. In particular, v''_L , v''_{RL} and v''_{RR} are respectively labelled with $M''_L = M'_{LL}$, $M''_{RL} = M'_{LR}$ and $M''_{RR} = M_R$.

We claim \mathcal{T}'' is valid. If P is a matched pair nontight in M , other than P_0 or P_1 , then P is twisted in the label of $V_{\mathcal{T}'}(P)$ (Definition 7.19), which is a vertex of one of $\mathcal{T}'_{LL} = \mathcal{T}'_L$, $\mathcal{T}'_{LR} = \mathcal{T}''_{RL}$ or $\mathcal{T}_R = \mathcal{T}''_{RR}$. And P_0 and P_1 are twisted in M , which is the label of the root. Thus for every matched pair P , there is a vertex of \mathcal{T}'' whose label is twisted at P . By Lemma 7.15 then \mathcal{T}'' is valid.

We also claim \mathcal{T}'' is distributive. Each vertex of \mathcal{T}'' which shares a label with a vertex of distributive tree \mathcal{T}' is distributive. The only remaining vertex is v''_R , which has label $M''_R = M_{k+1} \otimes \cdots \otimes M_n = M'_{LR} \otimes M_R$. Each of the $(m-k-1) + (n-m-1) = n-k-2$ matched pairs P such that $V_{\mathcal{T}'}(P)$ is a vertex of \mathcal{T}'_{LR} or \mathcal{T}_R is nontight in M'_{LR} or M_R , hence also in $M''_R = M'_{LR} \otimes M_R$ (Lemma 2.40). Since there are $n-k$ leaves below v''_R , and there are at least $n-k-2$ matched pairs at which M''_R is twisted or critical, v''_R is distributive, and the claim follows.

Since \mathcal{T}'' is valid and distributive, by assumption then \mathcal{T}'' is strongly valid and strictly f -distributive. Now P_0 is twisted in M and satisfies $V_{\mathcal{T}'}(P_0) = v_L$, so P_0 is twisted in $M_L = M_1 \otimes \cdots \otimes M_m$, but tight in $M'_{LL} = M_1 \otimes \cdots \otimes M_k$ and $M'_{LR} = M_{k+1} \otimes \cdots \otimes M_m$. Supposing without loss of generality that P_0 is twisted at p_0 in M , then the step p_{0+} must be covered by one of M_1, \dots, M_k , and the step p_{0-} must be covered by one of M_{k+1}, \dots, M_m , with no steps of P covered by any of M_{m+1}, \dots, M_n . Thus P_0 is tight in $M''_R = M_{k+1} \otimes \cdots \otimes M_n$, and in $M'_L = M_1 \otimes \cdots \otimes M_k$, the labels of v''_L and v''_R ; but P_0 is twisted in M , the label of v_0 . So $V_{\mathcal{T}''}(P_0) = v_0$, and \mathcal{T}'' is the desired tree. By induction, the proof is complete. □

Proof of Theorem 7.31 We have verified the theorem in small cases, so suppose it is true for all \bar{f}_k with $k < n$, and consider \bar{f}_n .

By Lemma 7.18, since there are no 11 doubly occupied pairs in M , validity and strong validity are equivalent; we use this fact repeatedly. Note that if any subtensor-product M' of M contains a 11 doubly occupied pair, then M would contain one too; so validity and strong validity are also equivalent for operation trees of subtensor-products of M .

Our strategy is to compute $\bar{U}_n(M)$ explicitly, and then compute \bar{f}_n , using the construction of Corollary 4.3. Recall that $\bar{U}_n(M)$ is a sum of terms of the form

$$\bar{f}_i(M_1 \otimes \cdots \otimes M_i) \bar{f}_{n-i}(M_{i+1} \otimes \cdots \otimes M_n)$$

and

$$\bar{f}_{n-j+1}(M_1 \otimes \cdots \otimes M_k \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \otimes \cdots \otimes M_n).$$

The latter type of term is easiest to deal with: we claim they are all zero. Suppose to the contrary that $\bar{f}_{n-j+1}(M_1 \otimes \cdots \otimes M_k) \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j} \otimes \cdots \otimes M_n) \neq 0$. Then by [Proposition 7.10](#) there are valid distributive operation trees \mathcal{T}_X for $M_{k+1} \otimes \cdots \otimes M_{k+j}$ and \mathcal{T}_f for $M_1 \otimes \cdots \otimes M_k \otimes X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \otimes \cdots \otimes M_n$. Grafting \mathcal{T}_X onto \mathcal{T}_f at position $k + 1$ yields ([Lemma 7.9](#)) a valid distributive operation tree \mathcal{T}_{fX} for M . By assumption then \mathcal{T}_{fX} is strictly f -distributive. Applying strict distributivity to the vertex of \mathcal{T}_{fX} corresponding to the root of \mathcal{T}_X , then $M_{k+1} \otimes \cdots \otimes M_{k+j}$ is twisted or critical at precisely $j - 1$ matched pairs. But by [Theorem 5.6](#), since $X_j(M_{k+1} \otimes \cdots \otimes M_{k+j}) \neq 0$ it follows that there are precisely $j - 2$ such pairs. This gives a contradiction, so all such terms are zero.

We now consider terms of the form $\bar{f}_i(M_1 \otimes \cdots \otimes M_i) \bar{f}_{n-i}(M_{i+1} \otimes \cdots \otimes M_n)$ which are nonzero. We will associate to them matched pairs at which M is twisted and eventually obtain a bijection $F: A \rightarrow B$, where

$$A = \{i \mid \bar{f}_i(M_1 \otimes \cdots \otimes M_i) \bar{f}_{n-i}(M_{i+1} \otimes \cdots \otimes M_n) \neq 0\},$$

$$B = \{P \mid M \text{ is twisted at } P\}.$$

So let $M' = M_1 \otimes \cdots \otimes M_i$ and $M'' = M_{i+1} \otimes \cdots \otimes M_n$, and suppose that $\bar{f}_i(M') \bar{f}_{n-i}(M'') \neq 0$. By [Proposition 7.10](#) there are valid (hence strongly valid: M' and M'' are subtensor-products of M , so their validity and strong validity are equivalent) distributive trees \mathcal{T}' for M' and \mathcal{T}'' for M'' . Joining these trees yields an operation tree \mathcal{T} for M ([Definition 7.6](#)), which is valid (hence strongly valid) and distributive ([Lemma 7.7\(iii\)](#)), hence by hypothesis strictly f -distributive. We then have a bijection $V_{\mathcal{T}}$ between nontight matched pairs of M and nonleaf vertices of \mathcal{T} ([Lemma 7.30](#)). Moreover, since \mathcal{T}' and \mathcal{T}'' are subtrees of \mathcal{T} , they are also strictly f -distributive ([Lemma 7.29](#)). Thus M , M' and M'' are twisted or critical at $n - 1$, $i - 1$ and $n - i - 1$ matched pairs, respectively, and tight elsewhere.

By [Lemma 7.34](#), any valid distributive tree for M' is strongly valid and strictly f -distributive; and similarly for M'' . And since M has no 11 doubly occupied matched pairs, neither do the subtensor-products M' or M'' . So the hypotheses of the theorem apply to M' and M'' . By induction then $\bar{f}_i(M')$ and $\bar{f}_{n-i}(M'')$ are given by single diagrams as described in the statement. Moreover, as \mathcal{T}' and \mathcal{T}'' are disjoint subtrees of the strongly valid \mathcal{T} , the matched pairs at which M' and M'' are nontight are disjoint ([Lemma 7.23](#)). This yields $(i - 1) + (n - i - 1) = n - 2$ matched pairs at which M' or M'' is nontight; such pairs are also nontight in M ([Lemma 2.40](#)). So there is precisely one matched pair P_i at which M is nontight but M' and M'' are tight. Then $V_{\mathcal{T}}(P_i)$

is the root vertex v_0 , and P_i is twisted in M (by Definition 7.19, or Lemma 2.38). Indeed, $V_{\mathcal{T}}(P_i)$ is the root vertex for any \mathcal{T} arising as the join of valid distributive operation trees for M' and M'' . Define the function $F: A \rightarrow B$ by $F(i) = P_i$.

By induction $\bar{f}_i(M')$ (resp. $\bar{f}_{n-i}(M'')$) is given by a single diagram which is tight at all matched pairs where M' (resp. M'') is tight or critical, and crossed at all matched pairs where M' (resp. M'') is twisted. We now describe the diagram representing $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ at each matched pair P .

First, suppose P is critical in M . Then $P \neq P_i$, so P is nontight in precisely one of M' or M'' . Considering the known description of $\bar{f}_i(M')$ and $\bar{f}_{n-i}(M'')$, we examine the various cases in the critical column of Table 2, of which M is an extension (Lemma 2.37), and how the P -active factors can be distributed across M' and M'' . We observe that in every case $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ is tight at P .

Second, suppose P is a matched pair at which M is twisted, other than P_i . Then P is nontight in precisely one of M' or M'' . Indeed, there are two P -active factors and they are both in M' , or both in M'' . So $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ at P is the product of an all-on once occupied crossed diagram, and an idempotent, hence is crossed.

Third, suppose P is tight in M . Then P is also tight in M' and M'' (Lemma 2.39), hence also in $\bar{f}_i(M')$ and $\bar{f}_{n-i}(M'')$ (by inductive assumption). So $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ at P is given by multiplying factors in a tight tensor product, hence is tight.

Finally, at P_i , M' and M'' are both tight, but M is twisted. Hence P_i is 11 once occupied by M , with one step covered by M' , and the other by M'' ; by inductive assumption then $\bar{f}_i(M')$ and $\bar{f}_{n-i}(M'')$ are both tight at P_i , so $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ is twisted at P_i .

To summarise: when $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ is nonzero, there is a unique matched pair P_i which is nontight (in fact twisted) in M but tight in M' and M'' ; $V_{\mathcal{T}}(P_i)$ is the root vertex of \mathcal{T} ; and $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ is given by a single diagram which is twisted at P_i , crossed at all other matched pairs which are twisted in M , and tight at all other matched pairs. We set $F(i) = P_i$.

Now we claim that F is injective. Consider another nonzero term

$$\bar{f}_j(M_1 \otimes \cdots \otimes M_j)\bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes M_n), \quad \text{where } i \neq j.$$

We consider the case $i < j$; the case $i > j$ is similar. Applying the same argument as above, we obtain strongly valid and strictly f -distributive trees \mathcal{T}_j , \mathcal{T}'_j and \mathcal{T}''_j for M ,

$M'_j = M_1 \otimes \cdots \otimes M_j$ and $M''_j = M_{j+1} \otimes \cdots \otimes M_n$ respectively. We also obtain the bijection $V_{\mathcal{T}_j}$ between nontight matched pairs of M and nonleaf vertices of \mathcal{T}_j . The matched pair P_j has $V_{\mathcal{T}_j}(P_j)$ as the root of \mathcal{T}_j , and $F(j) = P_j$. We will show that $P_i \neq P_j$.

Now in the valid distributive tree \mathcal{T} constructed above for $\bar{f}_i(M')\bar{f}_{n-i}(M'')$, let v be the lowest common ancestor of the leaves labelled M_j and M_{j+1} . Let P be the matched pair such that $V_{\mathcal{T}}(P) = v$ (well-defined since $V_{\mathcal{T}}$ is bijective). Since $i < j$, v is a vertex of \mathcal{T}'' , hence not the root, so $P \neq P_i$. The label M_v of v is then twisted at P (Definition 7.19), say at the place p . So some M_a with $a \leq j$ covers the step p_+ , and some M_b with $j + 1 \leq b$ covers p_- . As M contains no 11 doubly occupied pairs, any subtensor-product of M which is twisted at P must contain M_a and M_b .

Now consider $V_{\mathcal{T}_j}(P)$, a vertex of \mathcal{T}_j ; call its label $M_{\#}$. Then $M_{\#}$ is twisted at P (Definition 7.19), so $M_{\#}$ contains M_a and M_b as tensor factors. But since $a \leq j$ and $b \geq j + 1$, $M_{\#}$ cannot be a subtensor-product of M'_j or M''_j ; thus $M_{\#} = M$ and $V_{\mathcal{T}_j}(P)$ is the root vertex. Thus $P = P_j$. As $P \neq P_i$ then $P_i \neq P_j$. Thus F is injective.

We now show F is surjective. Take a matched pair P at which M is twisted; we will show $P = P_i$ for some i . By Lemma 7.33 (which, as discussed above, has weaker hypotheses than the present theorem) there is a strongly valid, strictly f -distributive operation tree \mathcal{T}^* for M such that $V_{\mathcal{T}^*}(P)$ is the root vertex v_0^* of \mathcal{T}^* . Let the children of v_0^* be v_L^* and v_R^* , with labels $M_L^* = M_1 \otimes \cdots \otimes M_i$ and $M_R^* = M_{i+1} \otimes \cdots \otimes M_n$ respectively. Then by definition of $V_{\mathcal{T}^*}$, P is tight in M_L^* and M_R^* . By Lemma 7.34, M_L^* and M_R^* satisfy condition (i) of the present theorem; and as M_L^* and M_R^* are subtensor-products of M , which has no 11 doubly occupied pairs, they satisfy condition (ii) also. So by induction $\bar{f}_i(M_L^*)$ and $\bar{f}_{n-i}(M_R^*)$ are both nonzero, given by single diagrams as described in the statement. By Lemma 7.23 they are nontight at disjoint matched pairs. Examining the various possible cases at each matched pair (just as we did for $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ a few paragraphs ago), we conclude that $\bar{f}_i(M_L^*)\bar{f}_{n-i}(M_R^*) \neq 0$. Since P is nontight in M but tight in M_L^* and M_R^* , we have $P = P_i$. So F is surjective, hence a bijection.

Returning to $\bar{U}_n(M)$, we now see that each nonzero term of $\bar{U}_n(M)$ is of the form $\bar{f}_i(M')\bar{f}_{n-i}(M'')$, and these terms correspond bijectively to the matched pairs P_i at which M is twisted. In fact $\bar{f}_i(M')\bar{f}_{n-i}(M'')$ is twisted at P_i , and crossed at all other matched pairs where M is twisted.

We also observe that $X_n(M) = 0$, since M has precisely $n - 1$ nontight matched pairs, by [Theorem 5.6](#). Thus, following the construction of [Corollary 4.3](#) and the discussion of [Section 3.3](#),

$$\bar{f}_n(M) = \bar{A}_{\mathcal{CR} \leq}^* \bar{U}_n(M).$$

By [Definition 3.18](#), $A_{\mathcal{CR} \leq}^*$ applies a creation operator at P_{\min} , where P_{\min} is the \leq -minimal matched pair among pairs where M is twisted.

We observe that there is precisely one diagram in $\bar{U}_n(M)$ which is twisted at P_{\min} , namely $\bar{f}_i \bar{f}_{n-i}$ where $i = F^{-1}(P_{\min})$, ie where $P_i = P_{\min}$. Applying $\bar{A}_{\mathcal{CR} \leq}^* = \bar{A}_{P_{\min}}^*$ inserts a crossing at P_{\min} to this diagram. All the other diagrams in $\bar{U}_n(M)$ are crossed at P_{\min} , and applying the creation operator gives zero.

We conclude that $\bar{f}_n(M)$ is given by a single diagram, crossed at all matched pairs where M is twisted, and tight elsewhere, as desired. □

Proof of [Theorem 7.32](#) As there are no 11 occupied matched pairs, by [Lemma 7.18](#), validity and strong validity are equivalent.

By [Lemma 5.4](#) (since all the maps f_k in the pair ordering construction are balanced), $X_n(M)$ is represented by the sum of all terms of the form

$$\bar{f}_i(M_1 \otimes \cdots \otimes M_i) \bar{f}_{n-i}(M_{i+1} \otimes \cdots \otimes M_n).$$

Let \mathcal{T} be a valid and strictly X -distributive operation tree for M , which exists by hypothesis. Let its root vertex be v_0 , with children v_L and v_R respectively labelled $M_L = M_1 \otimes \cdots \otimes M_i$ and $M_R = M_{i+1} \otimes \cdots \otimes M_n$. Let \mathcal{T}_L and \mathcal{T}_R be the subtrees below v_L and v_R respectively.

By [Lemma 7.34](#), M_L and M_R satisfy condition (i) of [Theorem 7.31](#); and being subtensor-products of M , which has no 11 doubly occupied pairs, M_L and M_R also satisfy condition (ii). So by [Theorem 7.31](#), $\bar{f}_i(M_L)$ and $\bar{f}_{n-i}(M_R)$ are both nonzero, given by single diagrams. Since \mathcal{T} is strictly X -distributive, M_L and M_R respectively have $i - 1$ and $n - i - 1$ nontight matched pairs. These sets of nontight matched pairs are distinct by [Lemma 7.23](#), and also nontight in M ([Lemma 2.40](#)); hence they provide $n - 2$ distinct nontight matched pairs in M . By strict X -distributivity of \mathcal{T} , M has precisely $n - 2$ nontight matched pairs, so each nontight matched pair of M is nontight in precisely one of M_L or M_R .

By [Theorem 7.31](#), $\bar{f}_i(M_L)$ (resp. $\bar{f}_{n-i}(M_R)$) is crossed at every matched pair where M_L (resp. M_R) is twisted, and elsewhere tight. Thus at every nontight (hence critical;

twisted pairs are ruled out by hypothesis) matched pair of M , precisely one of M_L and M_R is nontight (twisted or critical), and the other is tight. If one of M_L and M_R is critical and the other is tight, then $\bar{f}_i(M_L)$ and $\bar{f}_{n-i}(M_R)$ are tight, and by reference to Table 2 or otherwise, $\bar{f}_i(M_L) \otimes \bar{f}_{n-i}(M_R)$ is tight. If one of M_L and M_R is twisted and the other is tight, then one of $\bar{f}_i(M_L)$ and $\bar{f}_{n-i}(M_R)$ is crossed, and the other is tight, so again by reference to Table 2 or otherwise, $\bar{f}_i(M_L) \otimes \bar{f}_{n-i}(M_R)$ is sublime. Either way, $\bar{f}_i(M_L)\bar{f}_{n-i}(M_R)$ is tight at each nontight matched pair of M . At tight matched pairs of M , $\bar{f}_i(M_L)$ and $\bar{f}_i(M_R)$ are both tight, with tight product. So $\bar{f}_i(M_L)\bar{f}_{n-i}(M_R)$ is the unique tight diagram with the same H-data as M .

Now let P be a nontight matched pair of M . By assumption, P is critical, but not 11 doubly occupied. Thus, by reference to Table 2, P is sesqui-occupied or 00 doubly occupied and M_P is an extension of one of the corresponding critical diagrams shown there (Lemma 2.37). In particular, there is precisely one place p of P such that the steps p_+ and p_- are covered by some M_a and M_b respectively, where $a < b$. We call these the *principal factors* of P . Now if $a \leq i < i + 1 \leq b$, then considering the various cases of Table 2, P is singular in M_L or M_R , contradicting validity of \mathcal{T} . Thus a and b are both $\leq i$, or both $\geq i + 1$. In other words, for any nontight matched pair of M , its principal factors have positions which are both $\leq i$, or both $\geq i + 1$; they do not cross the i^{th} position.

On the other hand, we claim that for any for any $1 \leq j \leq n - 1$ with $j \neq i$, there is a nontight matched pair of P whose principal factors have positions $\leq j$ and $\geq j + 1$; they *do* cross the j^{th} position. To see this, let w be the least common ancestor of the leaves labelled M_j and M_{j+1} . Then w lies in \mathcal{T}_L or \mathcal{T}_R , accordingly as $i > j$ or $i < j$. We suppose $i < j$, so $w \in \mathcal{T}_R$; the \mathcal{T}_L case is similar. Clearly w is neither a leaf nor root, so by Lemma 7.30, there is a unique matched pair P such that $V_{\mathcal{T}}(P) = w$. Let the principal factors of P be M_a and M_b , where $a < b$. Letting M_w denote the label of w , then M_w is twisted at P . Letting w_L and w_R denote the children of v , their labels are tight at P . The label on w_L contains M_a , so by construction $a \leq j$. Similarly the label of w_R contains M_b , and $j + 1 \leq b$. So the two principal factors have positions with are $\leq j$ and $\geq j + 1$ respectively.

Hence, for any $j \neq i$, we must have $\bar{f}_j(M_1 \otimes \cdots \otimes M_j)\bar{f}_{n-j}(M_{j+1} \otimes \cdots \otimes M_n) = 0$. For if this product were nonzero, then we could repeat the argument above and find that no nontight matched pair of M has principal factors whose positions cross the j^{th} position, contradicting the previous paragraph.

We conclude that $X_n(M)$ is the homology class of the single diagram

$$\bar{f}_i(M_L)\bar{f}_{n-i}(M_R),$$

which has the desired properties. □

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