Spinors and lambda lengths

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National University of Singapore
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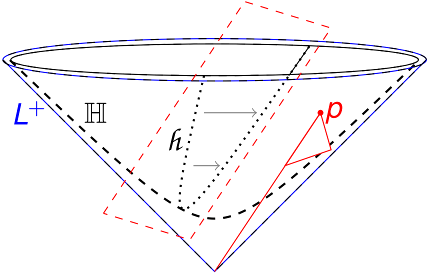

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Acknowledgments:

- This talk discusses work also involving Josh Howie, Dionne Ibarra, Jessica Purcell, Lecheng Su, Varsha, Orion Zymaris.
- Varsha helped draw many of the pictures.

Penrose–Rindler

General ideology: don’t use vectors for geometry/relativity, use spinors for everything!

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Cast of characters:

- Spinors / spin vectors: elements $\kappa = (\xi, \eta)$ of $\mathbb{C}^2$.
- 2×2 Hermitian matrices: $\mathcal{H} = \{A \in M_{2\times 2}(\mathbb{C}) \mid A = A^*\}$.
- Minkowski space $\mathbb{R}^{3,1}$: coordinates (T, X, Y, Z), metric $dT^2 - dX^2 - dY^2 - dZ^2$.

How to do this?

Spinors	$\xrightarrow{\phi_1}$	2×2 Hermitian matrices	$\xrightarrow{\phi_2}$	Minkowski space
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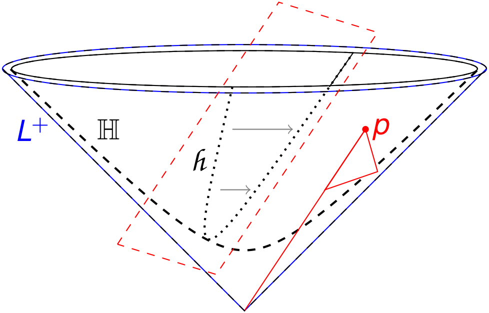
- Pauli matrices, Hopf fibration, stereographic proj. are here

We understand some of the picture now



From $\kappa \in \mathbb{C}^2$, get a point $\phi(\kappa) = p$ on L^+.

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Putting the spin in

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Definition (Penrose–Rindler)

A <u>*pointed null flag*</u> *is an oriented flag* $\mathbb{R}p \subset V$*, where* $p \in L^+$*,* $\mathbb{R}p$ *is future oriented, and* V *is a 2-plane tangent to* L^+*.*

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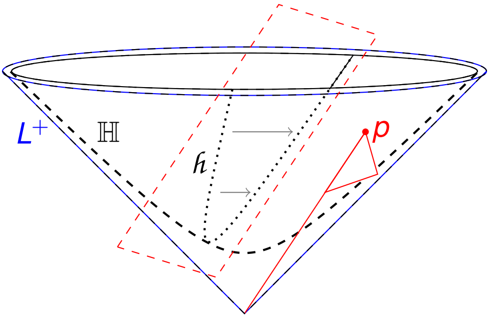

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Spinoriality:

- Take $\kappa \in \mathbb{C}^2$ and consider rotating it: $e^{i\theta}\kappa$.
- $\phi(e^{i\theta}\kappa)$ is constant but $\Phi(e^{i\theta}\kappa)$ is not: plane V rotates.
- As κ rotates by θ, V rotates by 2θ.

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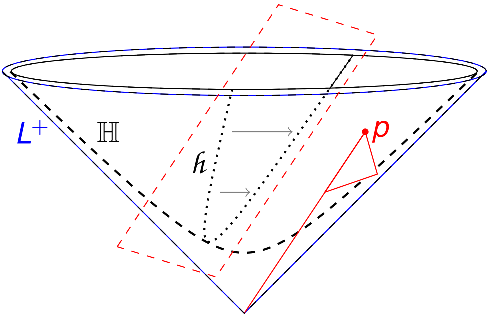

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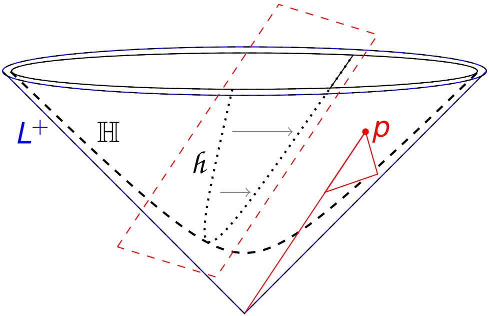

The orientation-preserving linear transformations of $\mathbb{R}^{3,1}$ which preserve L^+ form $SO(3,1)^+ \cong PSL(2,\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$.

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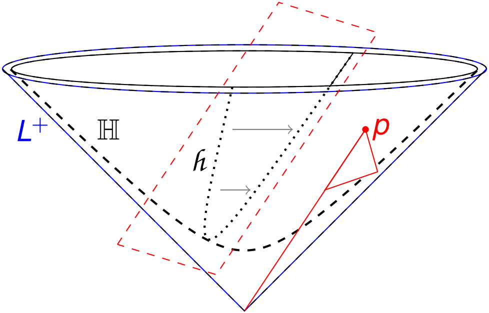

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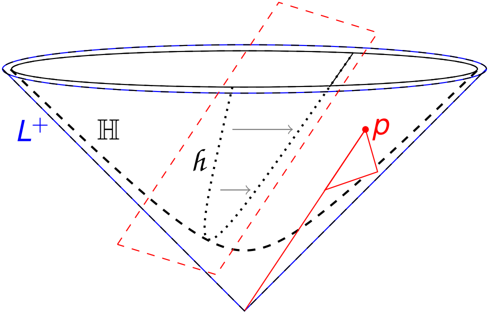

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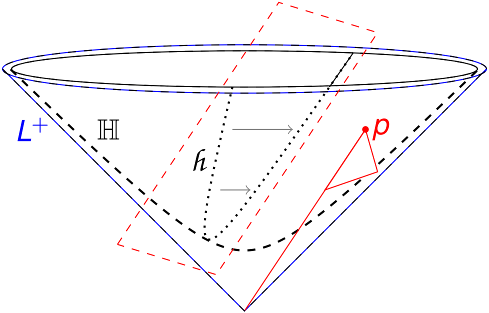

$\kappa \in \mathbb{C}^2 \xrightarrow{\text{Penrose–Rindler}}$ Pointed null flags $(= p \in L^+ \text{ and flag})$ $\xrightarrow{\text{Penner}}$ Horospheres with $\cdots$

New developments

Theorem (M., arxiv:2308.09233)

There is a natural ($SL(2, \mathbb{C})$-equivariant) bijection
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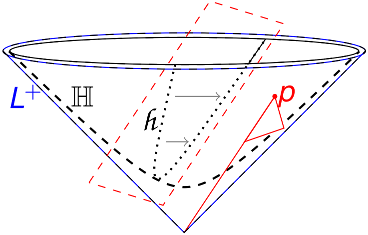

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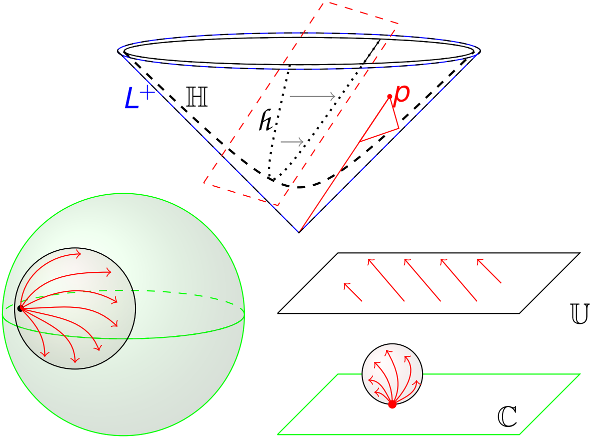

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  (Only makes sense since a horosphere is isometric to the Euclidean plane!)

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More rigorously:

- Directions $=$ certain frame fields along horosphere.
- Spin directions $=$ lifts from frame bundle to spin double cover.

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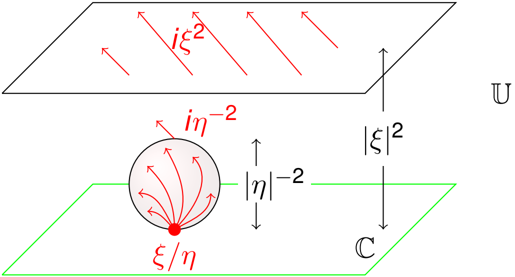

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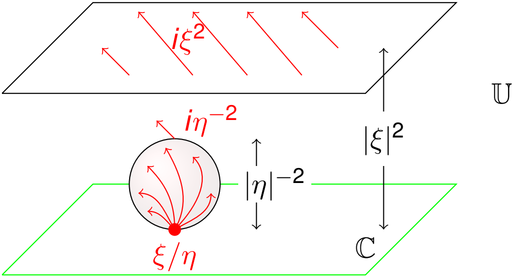
$$(\xi, \eta) \mapsto \left(\begin{array}{c} \text{horosphere centred at } \xi/\eta \\ \text{with Euclidean diameter } \frac{1}{|\eta|^2} \text{ and direction } \frac{i}{\eta^2} \end{array} \right)$$

(when $\eta = 0$, horizontal plane centred at ∞ at height $|\xi|^2$ and direction $i\xi^2$)



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- $SL(2,\mathbb{C})$ acts on $\mathbb{H}^3$ by spin isometries (2π rotation $\neq$ identity, 4π rotation $=$ identity), hence spin directions

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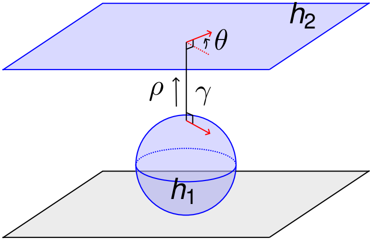
  making $\mathbb{C}^2$ into a complex symplectic vector space.
- Between two corresponding horospheres
  - there is a distance d
  - there is an angle θ between directions (mod 2π).
  - there is an angle θ between spin directions (mod 4π).



Complex 3D lambda lengths

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  making $\mathbb{C}^2$ into a complex symplectic vector space.
- Between two corresponding horospheres
  - there is a distance d
  - there is an angle θ between directions (mod 2π).
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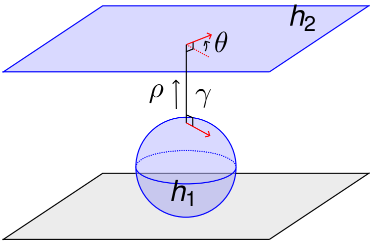

Theorem (M.)

$$\{\kappa, \omega\} = e^{\frac{d+i\theta}{2}}$$

Complex 3D lambda lengths

Now consider two spin vectors and two horospheres.

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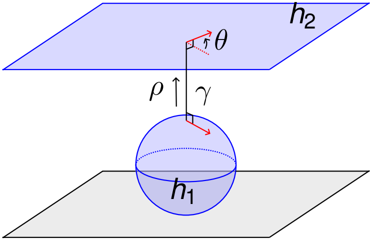


Theorem (M.)

$$\{\kappa, \omega\} = e^{\frac{d+i\theta}{2}}$$

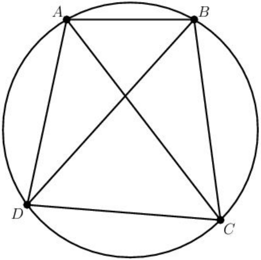
Generalises Penner's λ-lengths in $\mathbb{H}^2$:

$$\lambda = e^{d/2}.$$

Ptolemy equation

Ptolemy, Almagest ($\sim$ 160 CE): For a cyclic quadrilateral *ABCD* in the Euclidean plane,

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



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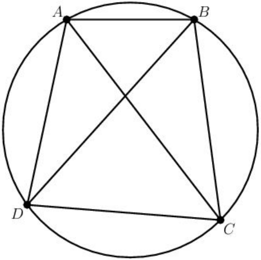
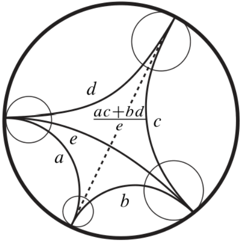


Numbering $(A, B, C, D) \sim (0, 1, 2, 3)$ and denoting distance d_{ij},

$$d_{02} d_{13} = d_{01} d_{23} + d_{03} d_{12}.$$

Penner, 1987: Given four horocycles in the hyperbolic plane with lambda lengths λ_{ij}, $i, j \in \{0, 1, 2, 3\}$,

$$\lambda_{02}\lambda_{13} = \lambda_{01}\lambda_{23} + \lambda_{03}\lambda_{12}.$$



Theorem (M., arxiv:2308.09233)

Given an ideal tetrahedron in $\mathbb{H}^3$, with spin-decorated horospheres $\mathpzc{h}_0, \mathpzc{h}_1, \mathpzc{h}_2, \mathpzc{h}_3$ at its vertices, the lambda lengths $\lambda_{ij} \in \mathbb{C}$ betweeen $\mathpzc{h}_i$ and $\mathpzc{h}_j$ satisfy

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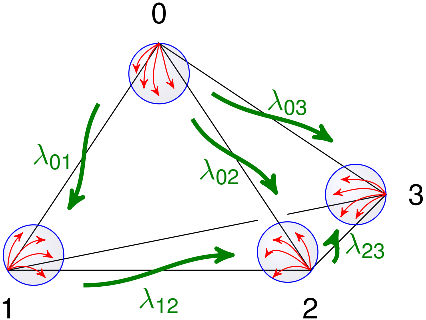

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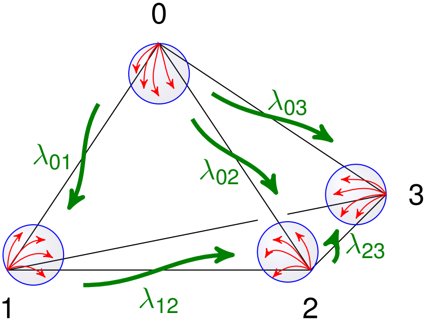

Proof: Plücker relation between 2×2 determinants in a 2×4 matrix.

When spinors $(\xi, \eta) \in \mathbb{R}^2$ have real coordinates, they correspond to horocycles in $\mathbb{H}^2$.

Well-known progression

Dimension *n*	2	3	
Isometries of $\mathbb{H}^n$ $= PSL(2, ?)$	$\mathbb{R}$	$\mathbb{C}$	
Spinors in?	$\mathbb{R}^2$	$\mathbb{C}^2$	

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Well-known progression (& less well known... Ahlfors, Lounesto, Cao, Gongopadhyay, Kellerhals, ...):

Dimension n	2	3	4	many
Isometries of $\mathbb{H}^n$ $= PSL(2, ?)$	$\mathbb{R}$	$\mathbb{C}$	"$\mathbb{H}$"	"Γ_n"
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(Very) recent work with Varsha: all of the above works in 4D hyperbolic space with quaternions.

Ongoing work with Zymaris: some (all?) of the above works in aribtrary dimension with Clifford algebras.

Theorem (M.–Varsha, forthcoming)

There are smooth $SL(2, \$\Gamma)$-equivariant bijections between

1. *quaternionic spinors*
2. *spin multiflags*
3. *spin-deocrated horospheres in $\mathbb{H}^4$.*

Theorem (M.–Varsha, forthcoming)

Given two quaternionic spinors $\kappa_1 = (\xi_1, \eta_1)$, $\kappa_2 = (\xi_2, \eta_2)$, there is a well defined quaternionic lambda length λ_{12} between the corresponding spin-decorated horospheres, and

$$\lambda_{12} = \text{“det”} \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \xi_1^* \eta_2 - \eta_1^* \xi_2.$$

Theorem (M.–Varsha, forthcoming)

Given an ideal tetrahedron with spin-deocrated horospheres at the vertices and lambda lengths λ_{ij},

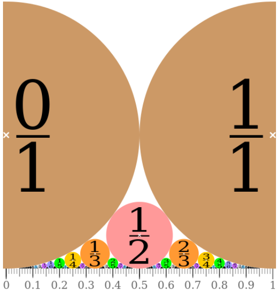
$$\lambda_{02}^{-1}\lambda_{01}\lambda_{31}^{-1}\lambda_{32} + \lambda_{02}^{-1}\lambda_{03}\lambda_{13}^{-1}\lambda_{12} = 1.$$

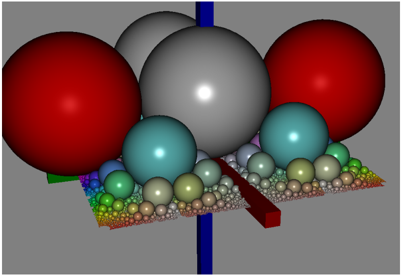
Proofs use work of Ahlfors, Lounesto, Maass, Vahlen on higher-dimensional Möbius transformations and Clifford algebras...

And work of Gel'fand–Retakh on non-commutative determinants...

Taking (ξ, η) to be relatively prime integers yields Ford circles and Farey fractions.

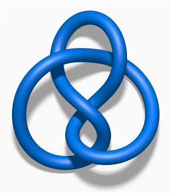
Taking (ξ, η) to be relatively prime Gaussian integers yields Ford spheres.





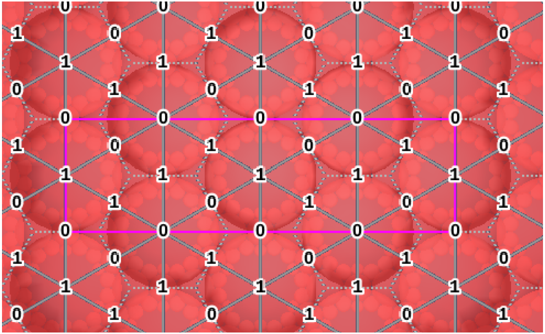
Spinors in knot complements

Let $M = S^3 - K$ where K is the figure-8 knot.



M has a well-known ideal triangulation and complete hyperbolic structure studied by Riley, W. Thurston, many others.

The developing map $\widetilde{M} \longrightarrow \mathbb{H}^3$ can be chosen to have ideal vertices precisely at points of $\mathbb{Q}(\sqrt{3})$.



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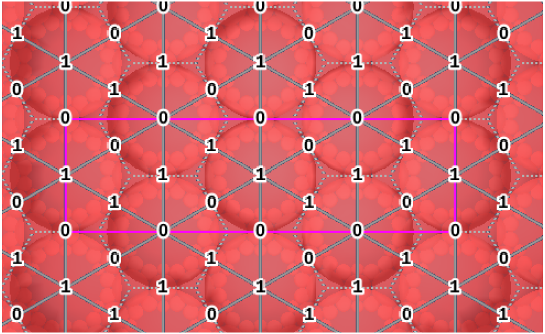

Theorem (Howie–Ibarra–M.–Su, arxiv:2411.06368)

The spinors (ξ, η) consisting of relatively prime Eisenstein integers precisely give the horospheres bounding maximal cusp neighbourhoods in M.

Eisenstein integers = alg. integers in $\mathbb{Q}(\sqrt{3})$
= $\mathbb{Z}[\omega]$ where $\omega^2 + \omega + 1 = 0$.

Theorem (Howie–Ibarra–M.–Su, arxiv:2411.06368)

The λ lengths between spin-decorated horospheres in M are precisely the Eisenstein integers.

Theorem (Howie–Ibarra–M.–Su, arxiv:2411.06368)

The set of hyperbolic distances between maximal cusps in M is precisely

$$\left\{ 2\log|\alpha| \;\middle|\; \alpha \in \mathbb{Z}\left[\frac{1+i\sqrt{3}}{2}\right] \setminus \{0\} \right\}$$

or

$$\left\{ \log n \;\middle|\; \prod_p p^{k_p},\ k_p \text{ even for } p \equiv 2 \bmod 3 \right\}.$$

Garoufalidis–D. Thurston–Zickert (2015) described Ptolemy varieties $\mathcal{P}_N$ for ideally triangulated 3-manifolds M.

$$c_{02}c_{13} = c_{01}c_{23} + c_{03}c_{12}.$$

They showed $\mathcal{P}_N$ describes all boundary-unipotent representations $\pi_1(M) \to SL(N, \mathbb{C})$ up to conjugacy.

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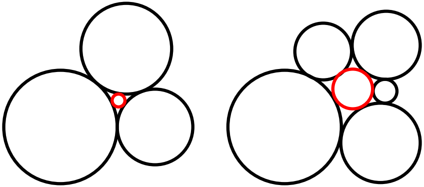
Theorem (M.–Purcell, in progress)

λ-lengths of spin-decorated hyperbolic structures on M satisfy the equations of the $SL(2, \mathbb{C})$ enhanced Ptolemy variety.

Euclidean plane geometry!

Definition

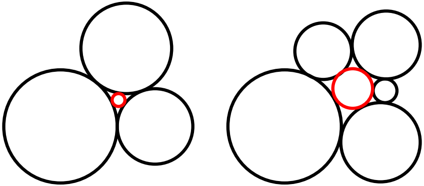
An n-flower consists of a central circle C_∞, and n petal circles C_j (j mod n), externally tangent to each other as shown.



Let $\kappa_\bullet = \frac{1}{r_\bullet^2}$ be the curvature of $C_\bullet$.

General theory of circle packings (Koebe, Andreev, W. Thurston, Beardon, Bowers, Stephenson, ...): if petal curvatures are given, κ_∞ is determined.

Question: For an n-flower, do $\kappa_\infty, \kappa_1, \ldots, \kappa_n$ satisfy an equation?



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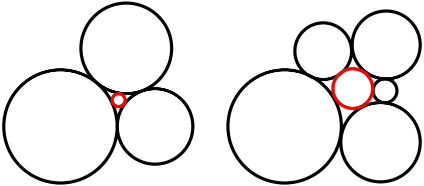

Theorem (Descartes Circle Theorem, Descartes 1643)

Yes, for $n = 3$!

$$(\kappa_\infty + \kappa_1 + \kappa_2 + \kappa_3)^2 = 2\left(\kappa_\infty^2 + \kappa_1^2 + \kappa_2^2 + \kappa_3^2\right).$$

The sum of the squares of all four bends
Is half the square of their sum
– Frederick Soddy, The Kiss Precise (1936)

Theorem (M.–Zymaris, arxiv:2310.11701)

Yes, for all n!

Define m_0 and m_j for $1 \leq j \leq n-1$ as

$$m_0 = \sqrt{\frac{\kappa_0}{\kappa_\infty} + 1}, \quad m_j = \sqrt{\left(\frac{\kappa_j}{\kappa_\infty} + 1\right)\left(\frac{\kappa_{j-1}}{\kappa_\infty} + 1\right) - 1}.$$

Then

$$\frac{m_0^2\, i}{2}\left(\prod_{j=1}^{n-1}(m_j - i) - \prod_{j=1}^{n-1}(m_j + i)\right) - \prod_{j=1}^{\frac{n-1}{2}}\left(m_{2j-1}^2 + 1\right) = 0 \quad \text{for odd } n,$$

$$\frac{i}{2}\left(\prod_{j=1}^{n-1}(m_j - i) - \prod_{j=1}^{n-1}(m_j + i)\right) - \prod_{j=1}^{\frac{n-2}{2}}\left(m_{2j}^2 + 1\right) = 0 \quad \text{for even } n.$$

Thanks for listening!

dan.v.mathews@gmail.com