Contact geometry, Heegaard Floer homology, and skein theory

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Monash topology seminar
27 March 2024

Outline

1. Introduction to contact geometry and topology
2. “Contact skein theory" — Contact TQFT
3. Floer homology
4. Quantum Pawn Dynamics

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1. Introduction to contact geometry and topology
   - Overview and background
   - Examples of contact manifolds
   - Contact structures on 3-manifolds
2. “Contact skein theory" — Contact TQFT
   - “Inner product” on chord diagrams
   - Bypass surgery
3. Floer homology
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   - A “computation”
4. Quantum Pawn Dynamics

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Contact geometry and topology:

- What it is
- Background, fundamental results
- Examples, questions

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Standing assumptions/warnings:

- All manifolds are smooth, oriented, compact unless otherwise specified.
- All functions smooth unless otherwise specified
- Smooth $= C^\infty$
- Beware sign conventions
- Biased!

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- A major area of research in contemporary low-dimensional geometry and topology
- Connected to many fields of mathematics:
  - symplectic geometry, Gromov-Witten theory, moduli spaces, quantum algebra, foliations, differential equations, mapping class groups, 3-and 4-manifolds, homotopy theory, homological algebra, category theory, knot theory...

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- Connected to many fields of physics:
  - topological quantum field theory, classical mechanics, thermodynamics, optics, string theory, ice skating...

Contact geometry has a sibling: symplectic geometry.

Let M be a manifold.

Definition

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A contact form on M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form.

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- Contact forms only exist in odd dimensions $2n + 1$.
- The kernel of a contact form $\xi = \ker \alpha$ is a codimension-1 plane field on M.

Integrability

Frobenius' theorem:

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A contact form co-orients ξ.

Examples

Standard / “only" examples of symplectic & contact structures:

- $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ and $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$.

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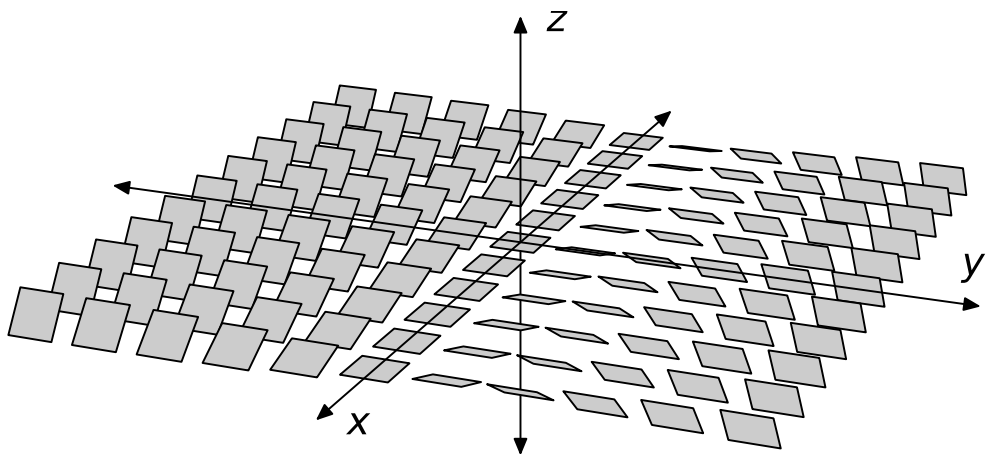

No tangent surfaces!

Only tangent/Legendrian curves! Legendrian knots and links.

Another example:

- Cylindrically symmetric standard contact structure on $\mathbb{R}^3$

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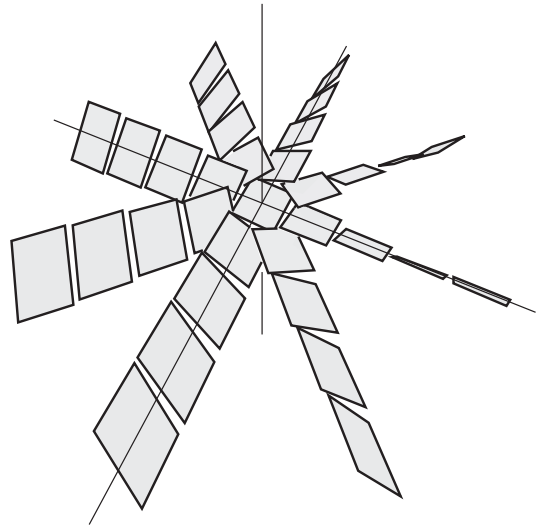

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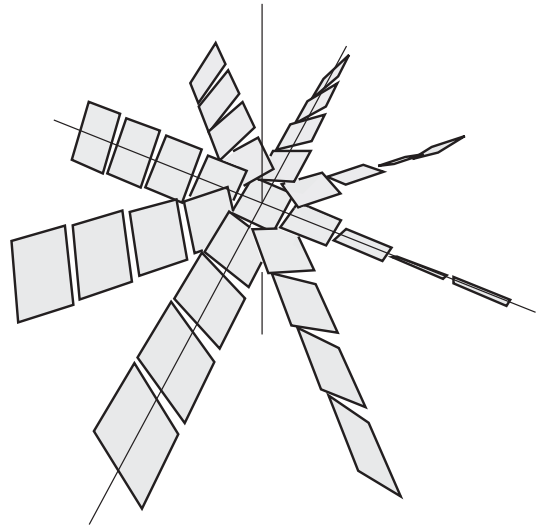


This is equivalent to the standard contact structure.

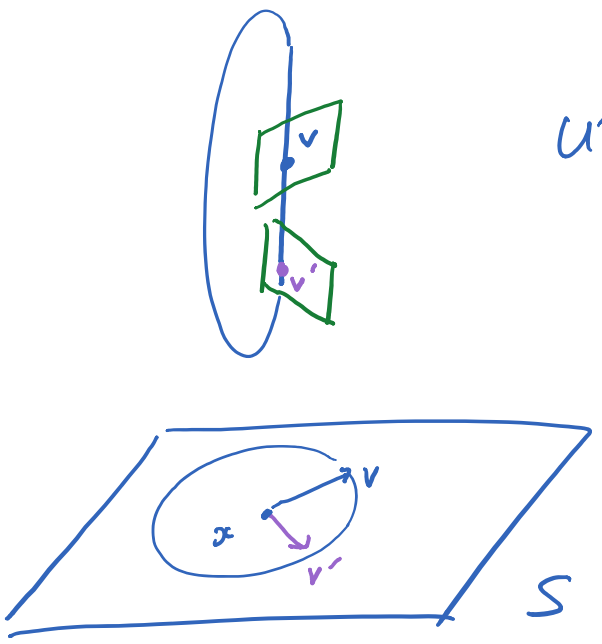
Definition

A contactomorphism is a diffeomorphism f between contact manifolds $(M_1, \xi_1) \longrightarrow (M_2, \xi_2)$ such that $f_*\xi_1 = \xi_2$.

Ice skating

The unit tangent bundle UTS of a smooth surface S has a canonical contact structure ξ:

- Take $(x, v) \in UTS$, with $x \in S$, $v \in T_x S$
- The contact plane there is spanned by v (“tautological direction") and the fibre direction.



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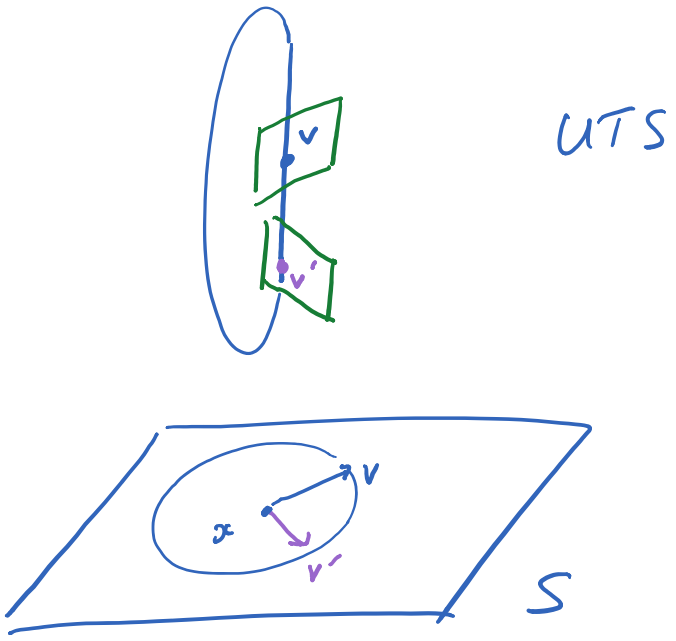

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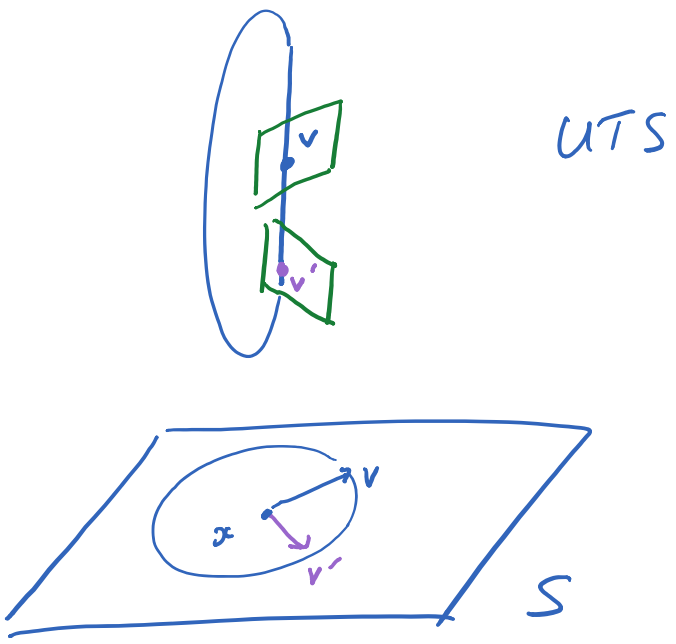

A smooth curve on S lifts uniquely to a Legendrian curve in UTS. Ice skating on an ice rink S:

- Status of skater = (position of skater, direction of skates) $\in UTS$
- Ice skater's path is Legendrian iff she does not skid.

Similar: parking a car, rolling a suitcase.

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Proof.

$$L_{R_\alpha}\alpha = \iota_{R_\alpha} d\alpha + d\iota_{R_\alpha}\alpha = 0 + d1 = 0.$$

□

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Theorem

Let (M, ξ) be a contact 3-manifold with a contact form α. There is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Contact vector fields} \\ \text{on } (M, \xi) \end{array} \right\} \quad \begin{array}{c} \leftrightarrow \\ \mapsto \end{array} \quad \left\{ \begin{array}{c} \text{Smooth functions} \\ H : M \longrightarrow \mathbb{R} \end{array} \right\}$$
$$X \qquad\qquad\qquad\quad \alpha(X)$$

Fundamental theorems in contact topology

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Theorem (Darboux theorem (Darboux 1882))

Let α be a contact form on M^{2n+1}. Near any $p \in M$ there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ such that $p = (0, \ldots, 0)$ and

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Theorem (Gray stability theorem (Gray 1959))

Let ξ_t be a smooth family (an isotopy) of contact structures on M. Then there is an isotopy $\psi_t : M \longrightarrow M$ such that $\psi_{t*}\xi_0 = \xi_t$ for all t.

"If the planes move, the space can follow it."

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Theorem (Taubes '07)

The Weinstein conjecture holds for contact 3-manifolds.

Proof uses Seiberg–Witten Floer homology.

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Generally:

- What invariants are there of contact manifolds, what are their structures and relationships?

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Answer: ∞. (Lutz 1977) — We can perform a Lutz twist on an embedded solid torus arbitrarily many times.

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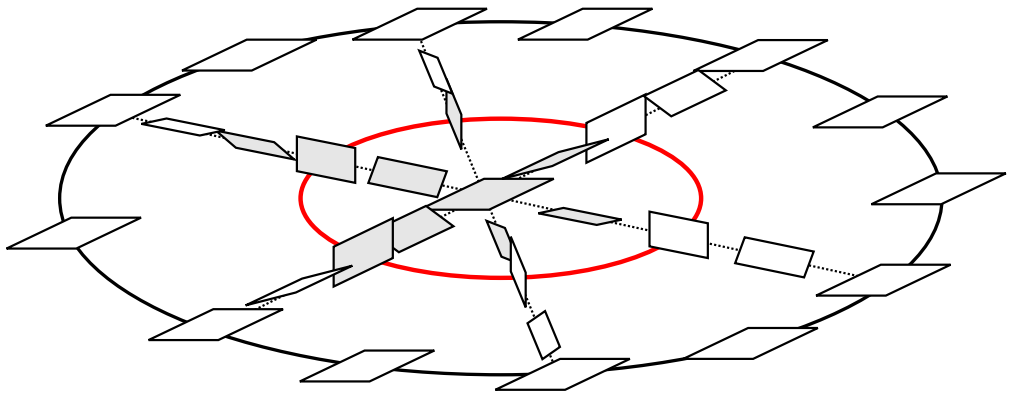
An overtwisted contact structure is one that contains a specific contact disc called an overtwisted disc.
A non-overtwisted contact structure is called tight.



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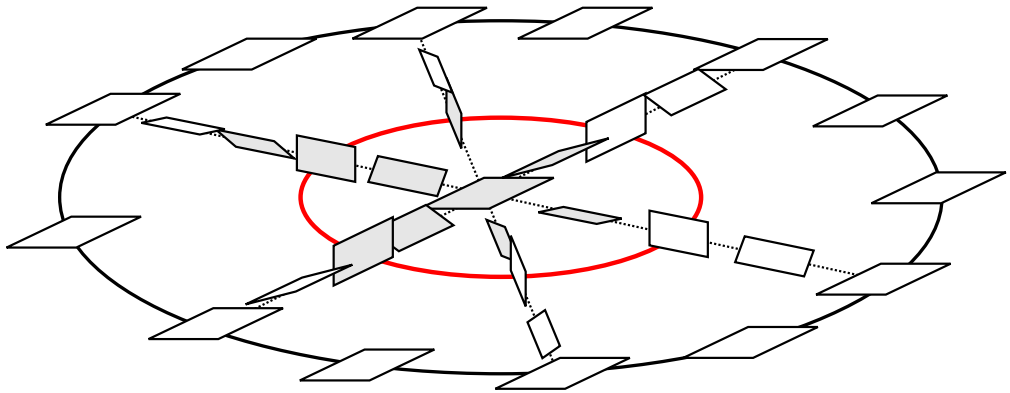

Classifying overtwisted contact structures is homotopy theory!

Theorem (Eliashberg 1989)

$$\left\{ \begin{array}{c} \textit{Overtwisted contact structures} \\ \textit{on } M \end{array} \right\} \simeq \left\{ \textit{2-plane fields on } M \right\}$$

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Answer (Colin 2001, Honda–Kazez–Matić 2002):

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Answer: Finitely many (Colin–Giroux–Honda 2003)

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  E.g. $L(34, 19)$ has 12 contact structures

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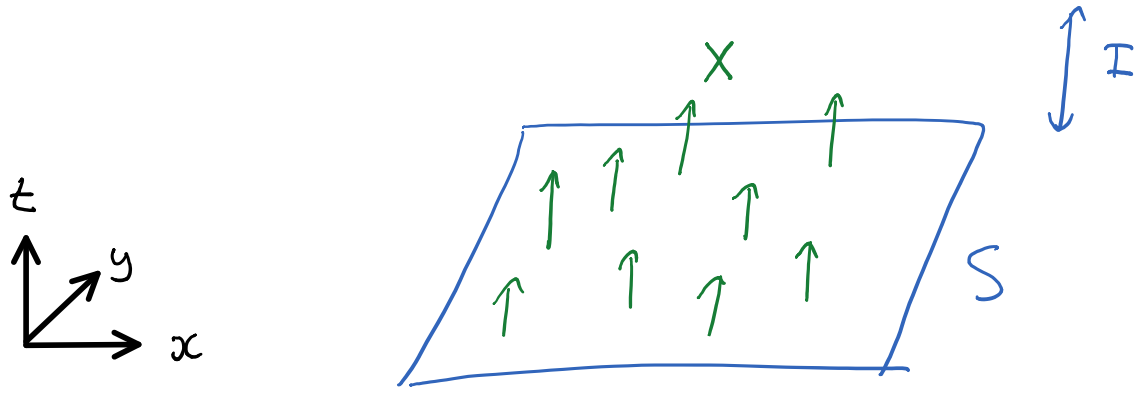
- $M\#\overline{M}$ where M is Poincaré homology sphere:
  NO tight contact structure (Etnyre–Honda 1999).

Definition (Giroux 1991)

An embedded surface S in a contact 3-manifold is convex if there is a contact vector field X transverse to S.



We can take a neighbourhood $S \times I$ ($I = [0, 1]$), with X pointing along I.

- Local coordinates x, y on S, and t on I, so $X = \frac{\partial}{\partial t}$

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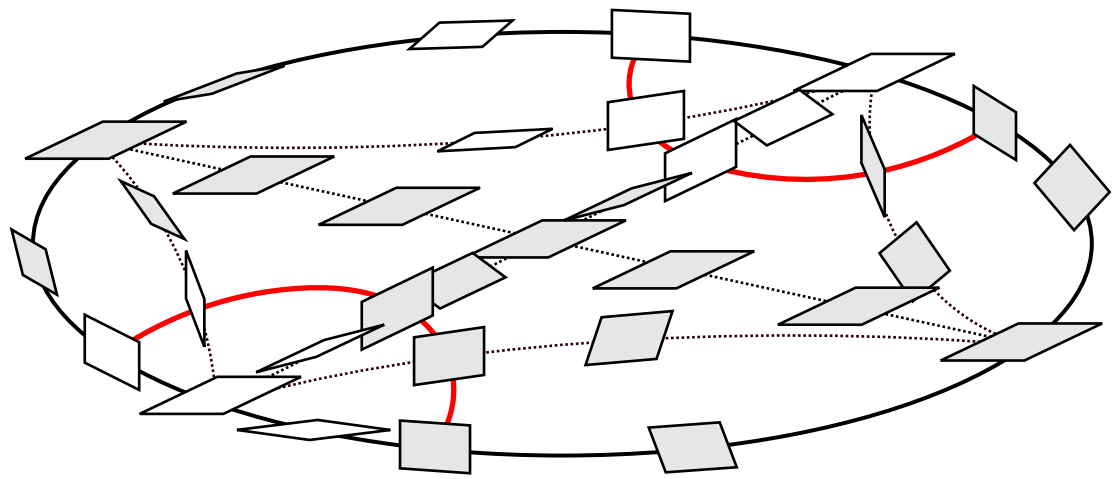

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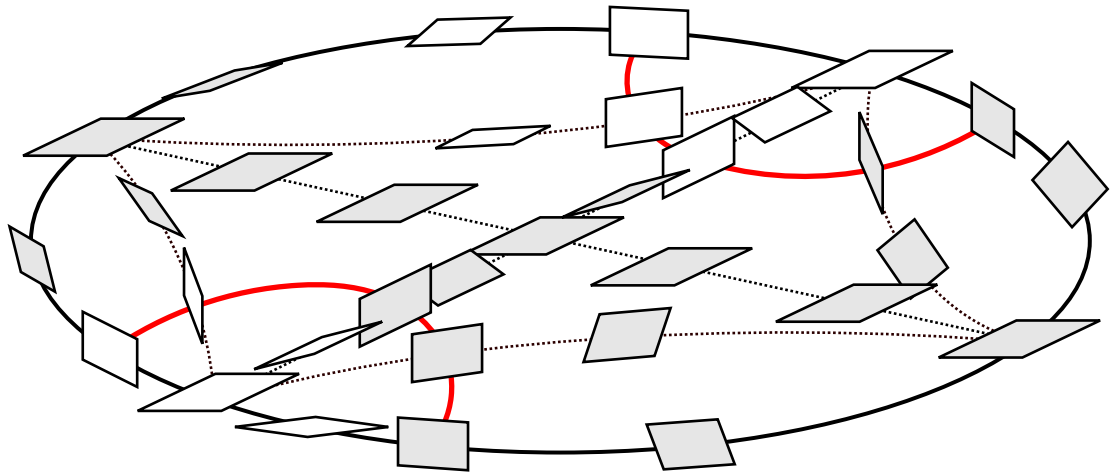

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The dividing set of S is $\Gamma = u^{-1}(0) = \{p \in S : X(p) \in \xi\}$.

One can show that:

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Moral

A contact structure near an embedded surface in a contact 3-manifold can be described by an isotopy class of dividing set.

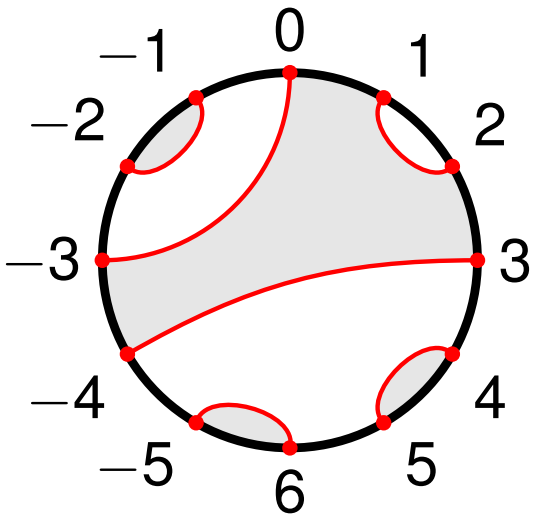
The combinatorial structure of dividing sets and contact structures between them is captured by an algebraic object called the contact category.

Chord diagrams

A chord diagram on a disc D is a collection of non-intersecting curves on D joining points on ∂D.

- A chord diagram describes a contact structure on $D \times I$.

E.g.



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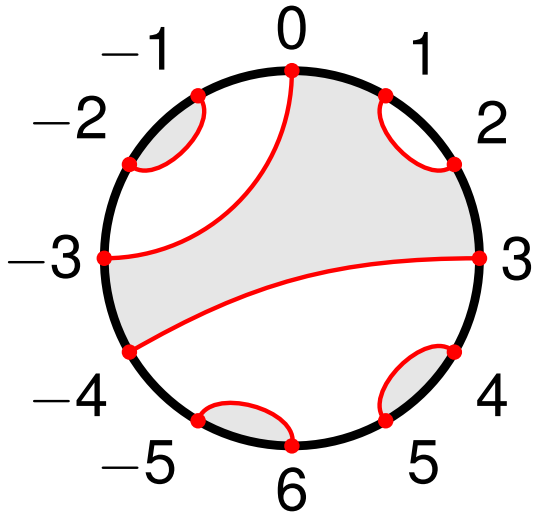

- Boundary assumed Legendrian (=tangent to contact planes).
- Shading = visible side of contact planes.
- Similar to the structure of sutures.

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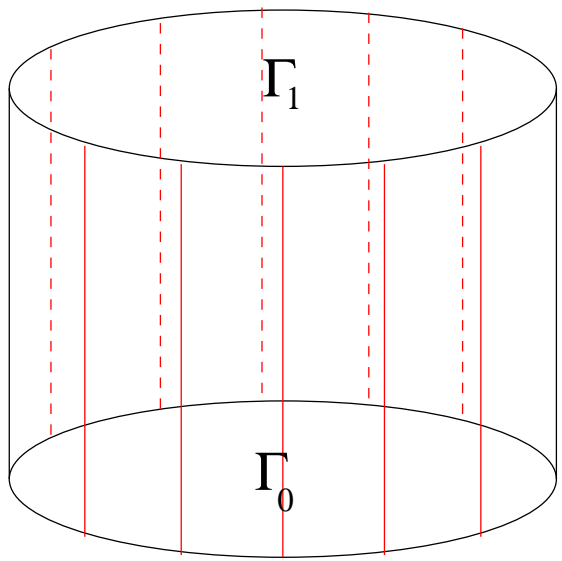

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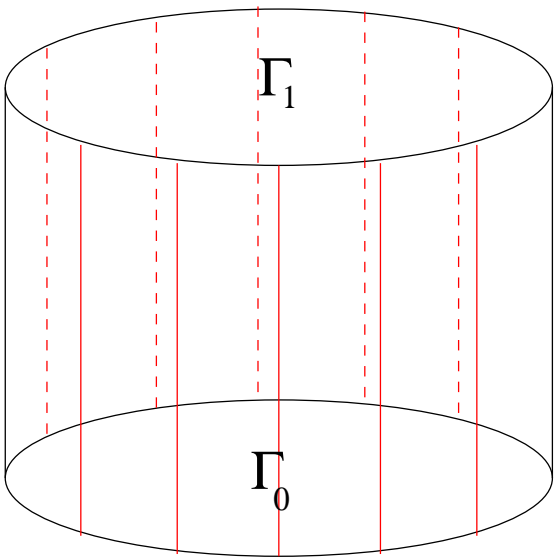
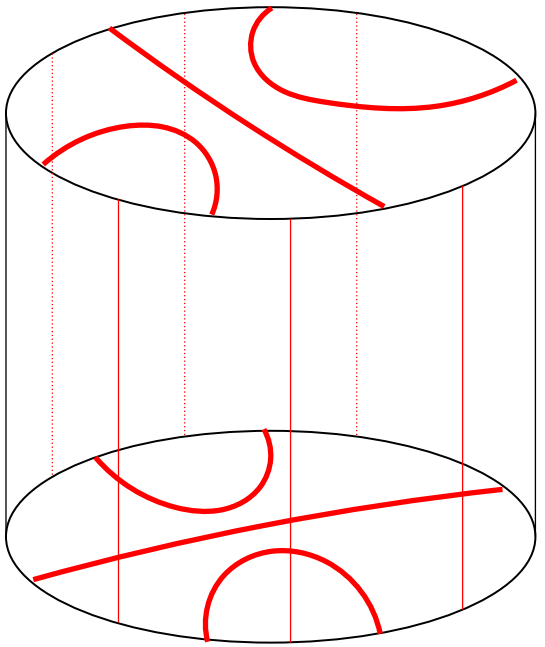


$\rightsquigarrow$



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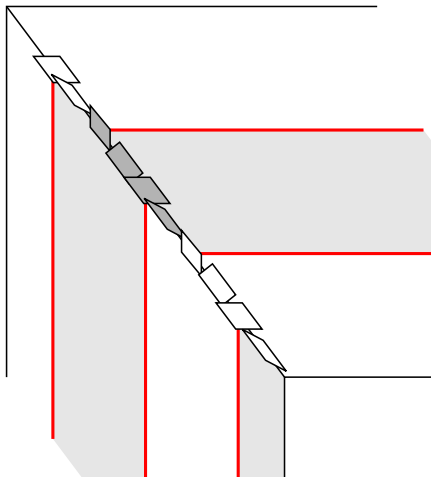

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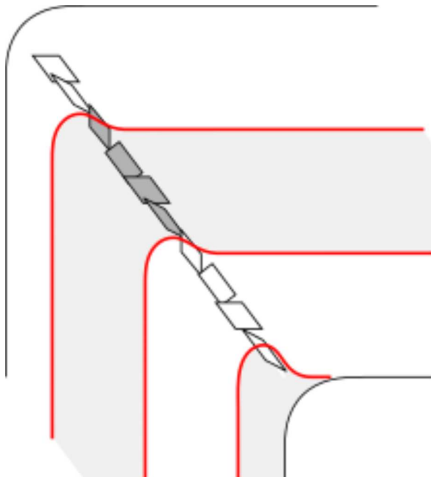

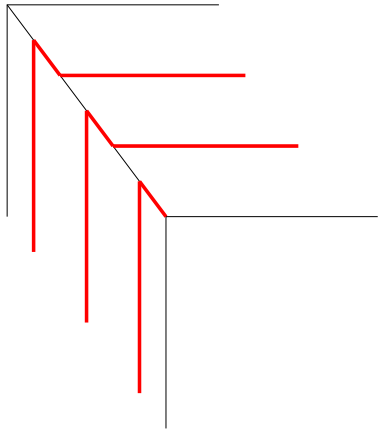

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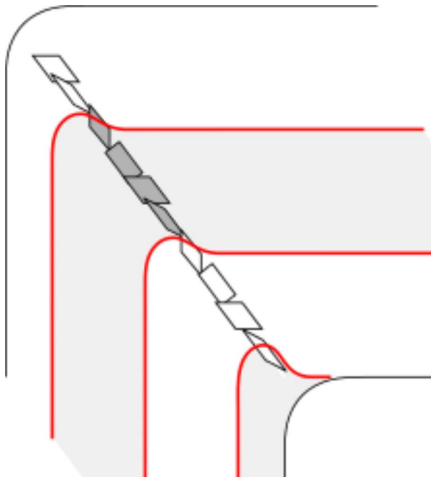

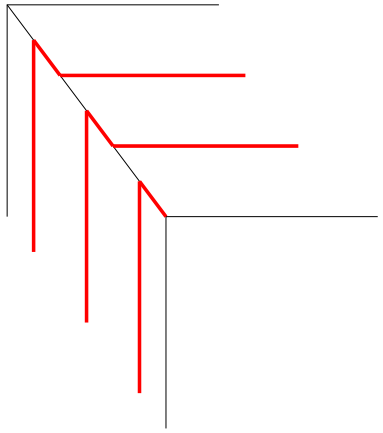

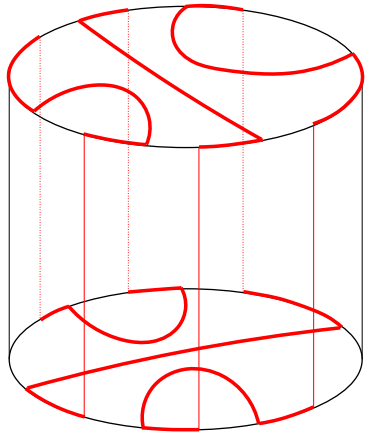

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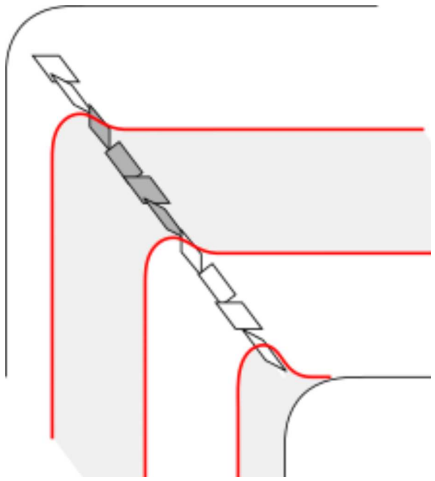
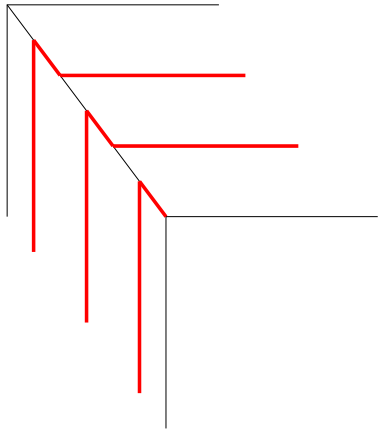
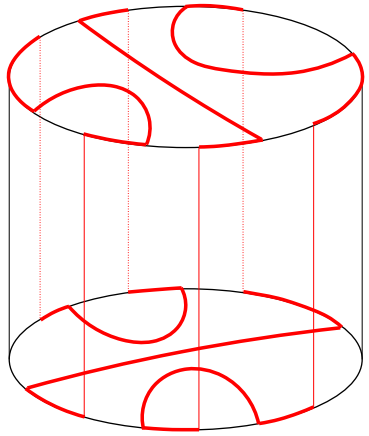




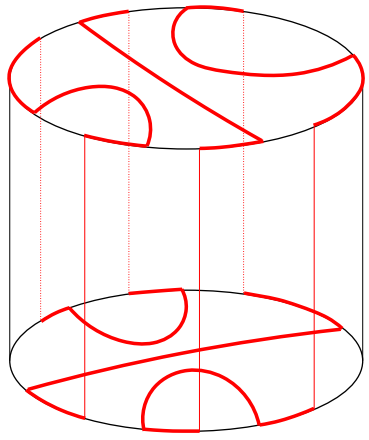


Definition (M.)

$$\langle \Gamma_0 | \Gamma_1 \rangle = \begin{cases} 1 & \textit{if the resulting curves on the cylinder} \\ & \quad \textit{form a single connected curve;} \\ 0 & \textit{if the result is disconnected.} \end{cases}$$

NB: This “inner product” is not symmetric!

Contact meaning of the “inner product”



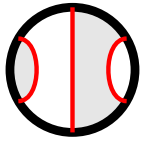
Proposition (Eliashberg)

Let Γ_0, Γ_1 *be chord diagrams. The following are equivalent:*

- $\langle \Gamma_0 | \Gamma_1 \rangle = 1$.
- *The solid cylinder with dividing set* Γ_0 *on the bottom and* Γ_1 *on the top has a tight contact structure.*

Bypass surgery

In a chord diagram on disc D, consider a sub-disc B as shown:



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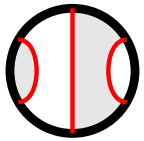
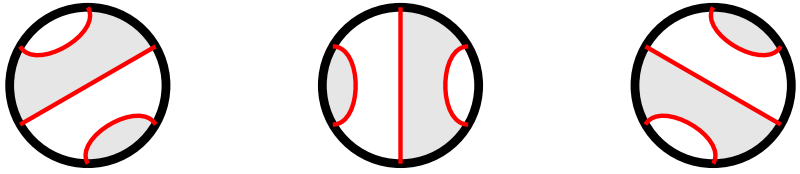


Two natural ways to adjust this chord diagram, consistent with the colours: bypass surgeries.



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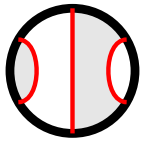

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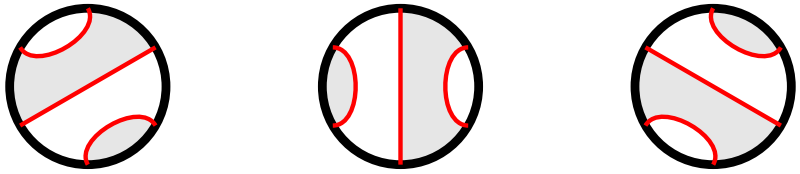

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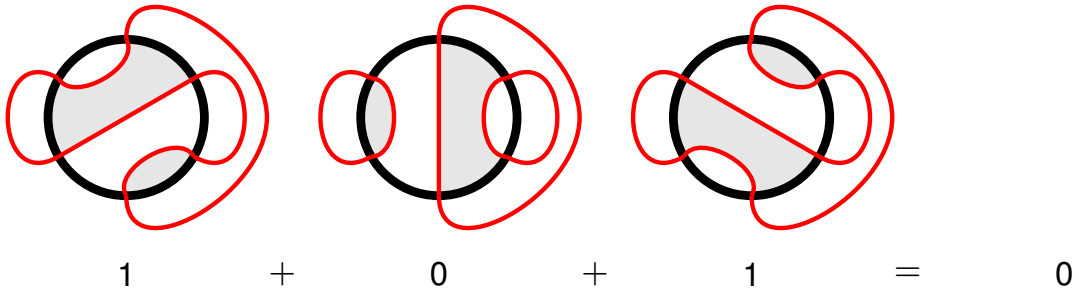
Proposition

With $\Gamma, \Gamma', \Gamma''$ *as above, for any* Γ_1*,*

$$\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma'' | \Gamma_1 \rangle = 0.$$

Bypass surgery

Idea of proof:



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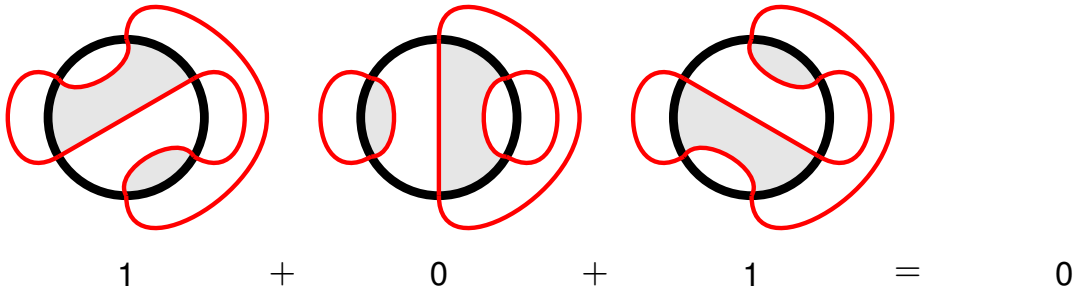
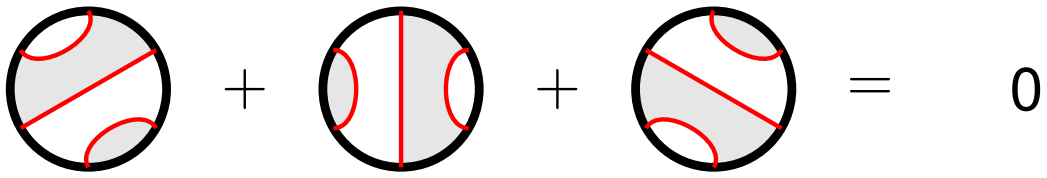


If $\langle \cdot | \cdot \rangle$ is to be nondegenerate, we should have the following bypass relation.



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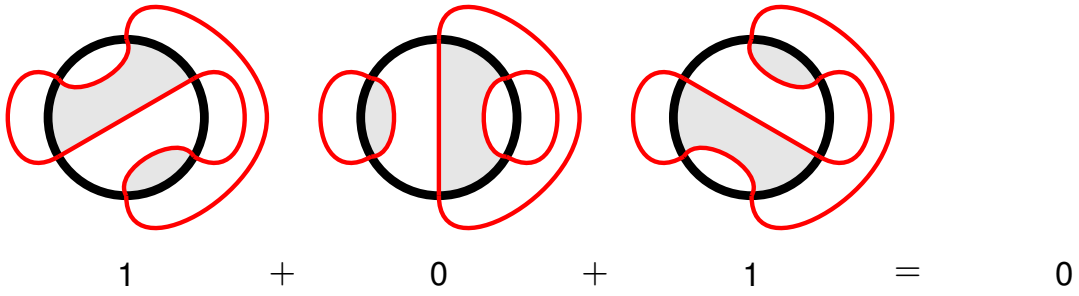

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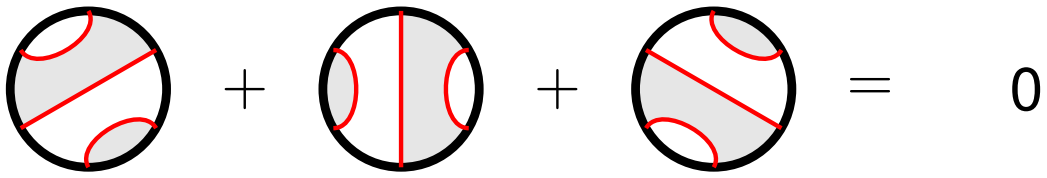

So we define a vector space

$$V_n = \frac{\mathbb{Z}_2\langle \text{Chord diagrams with } n \text{ chords} \rangle}{\text{Bypass relation}}$$

Contact TQFT = QPD = SFH

These definitions give many of the properties of a (2+1)-dimensional topological quantum field theory.

- Contact structure near disc (2-dim) $\rightsquigarrow$ "states" in V_n
- Contact structure over cylinder (2+1-dim) $\rightsquigarrow$ element of $\mathbb{Z}_2$.
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Theorem (M.)

V_n has dimension 2^{n-1} and is isomorphic to

1. *1-dimensional quantum pawn dynamics.*
2. *$SFH(D^2 \times S^1, F_n \times S^1)$ i.e. sutured Floer homology of solid tori*

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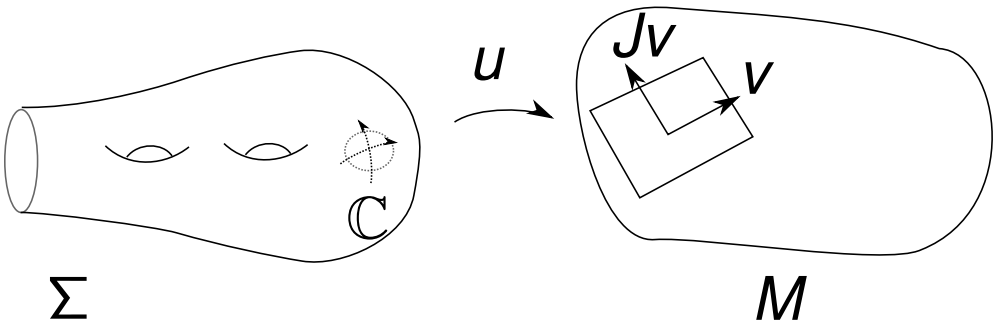

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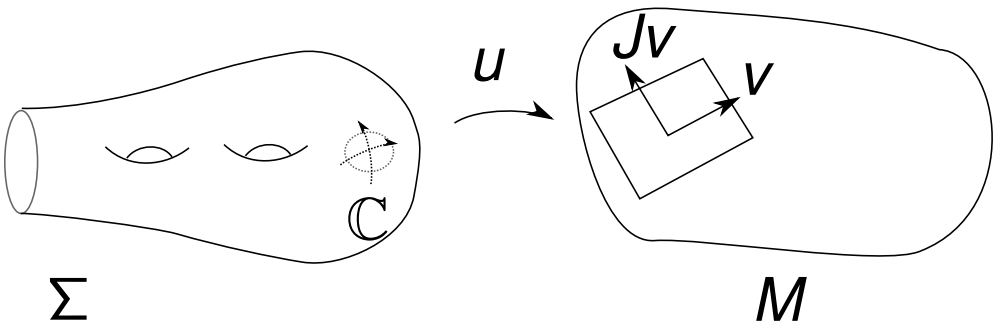
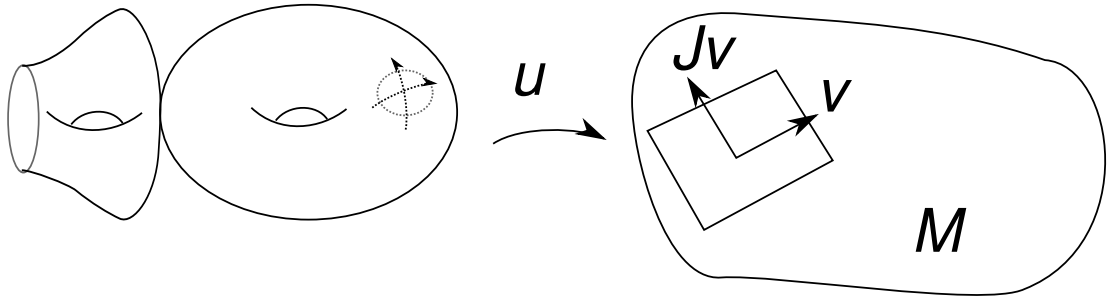


- Given appropriate constraints, the space of holomorphic curves is a finite-dimensional moduli space $\mathcal{M}$.
- Can do analysis using Fredholm/index theory, compactness theorems, etc.

- Boundary of $\overline{\mathcal{M}}$ is stratified: boundary strata are moduli spaces for "degenerate" holomorphic curves (nodal surfaces, etc.)



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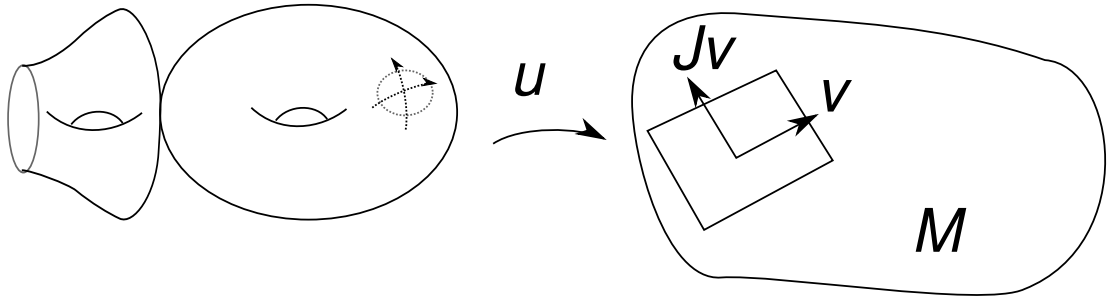

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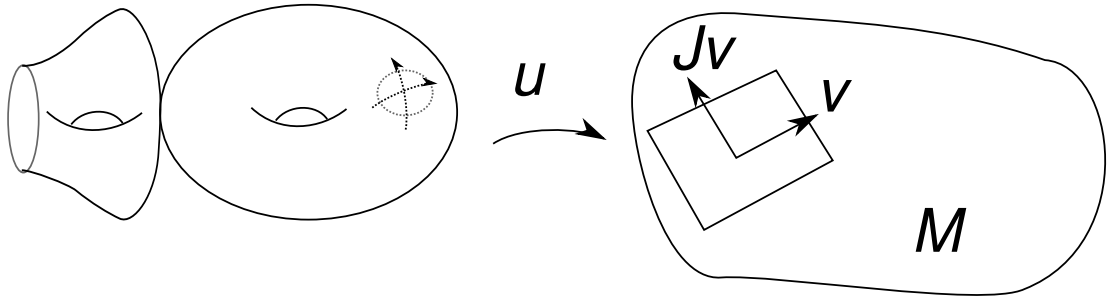

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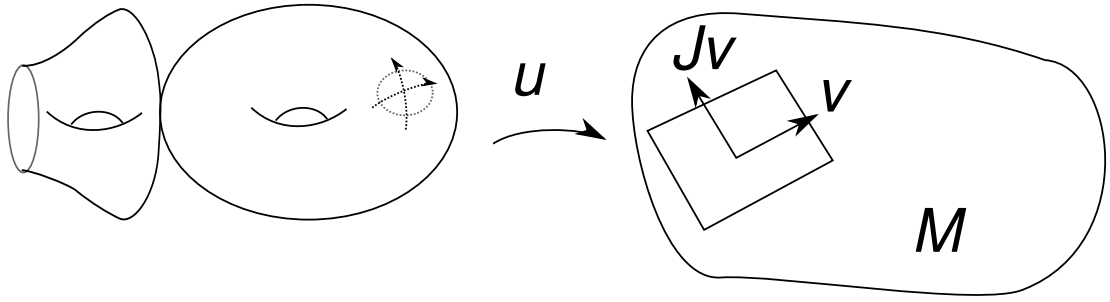

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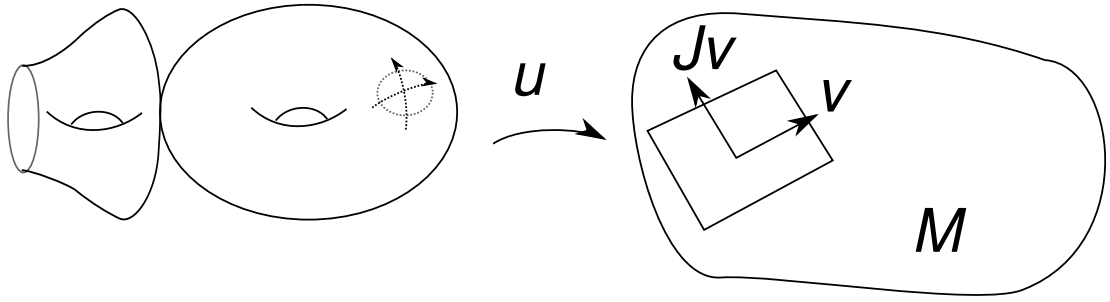

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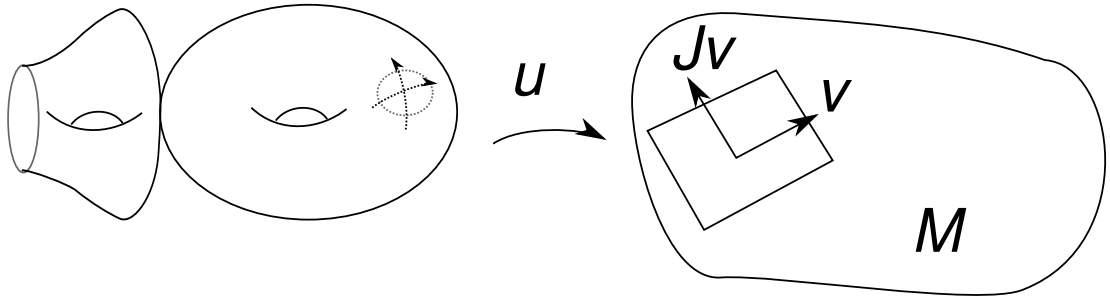

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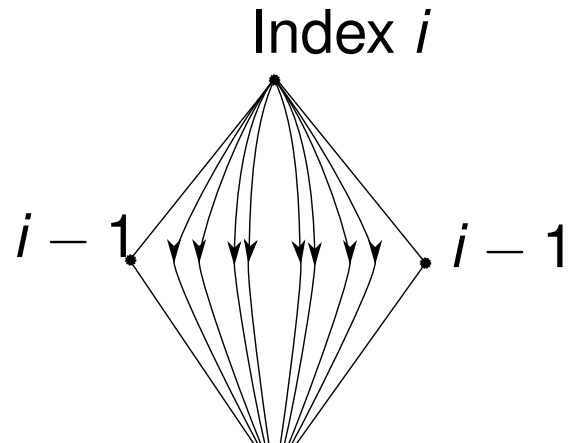
Analogy: singular homology via Morse complex.

- Complex generated by critical points of Morse function f.
- ∂ counts 0-dimensional families of trajectories of ∇f.



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- Sutured Floer homology SFH is a generalisation of $\widehat{HF}$ to sutured manifolds (Juhasz 2006)

$$(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)$$

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- Heegaard Floer homology (hat version) is an invariant of 3-manifolds derived from counting holomorphic curves in a symplectic manifold derived from a Heegaard decomposition (Ozsváth–Szábo 2004).

$$M \rightsquigarrow \widehat{HF}(M)$$

- Sutured Floer homology SFH is a generalisation of $\widehat{HF}$ to sutured manifolds (Juhasz 2006)

$$(M, \Gamma) \rightsquigarrow SFH(M, \Gamma)$$

- A contact structure ξ on M or (M, Γ) determines a contact element $c(\xi) \in \widehat{HF}(M)$ or $SFH(M, \Gamma)$ (Ozsváth–Szabó 2005, Honda–Kazez–Matić 2009).

The holomorphic origin of V_n

We defined V_n as:

- Chord diagrams modulo bypass relation

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Also, V_n is given by Sutured Floer homology:

Theorem (M.)

$$V_n \cong SFH(D^2 \times S^1, F_n \times S^1)$$

Sutured Floer homology

Very roughly... (Ozsváth–Szabó 2004, Juhasz 2006)

- A sutured manifold is a 3-manifold M with boundary, and some curves Γ on ∂M dividing ∂M into alternating positive and negative regions.

Sutured Floer homology

Very roughly... (Ozsváth–Szabó 2004, Juhasz 2006)

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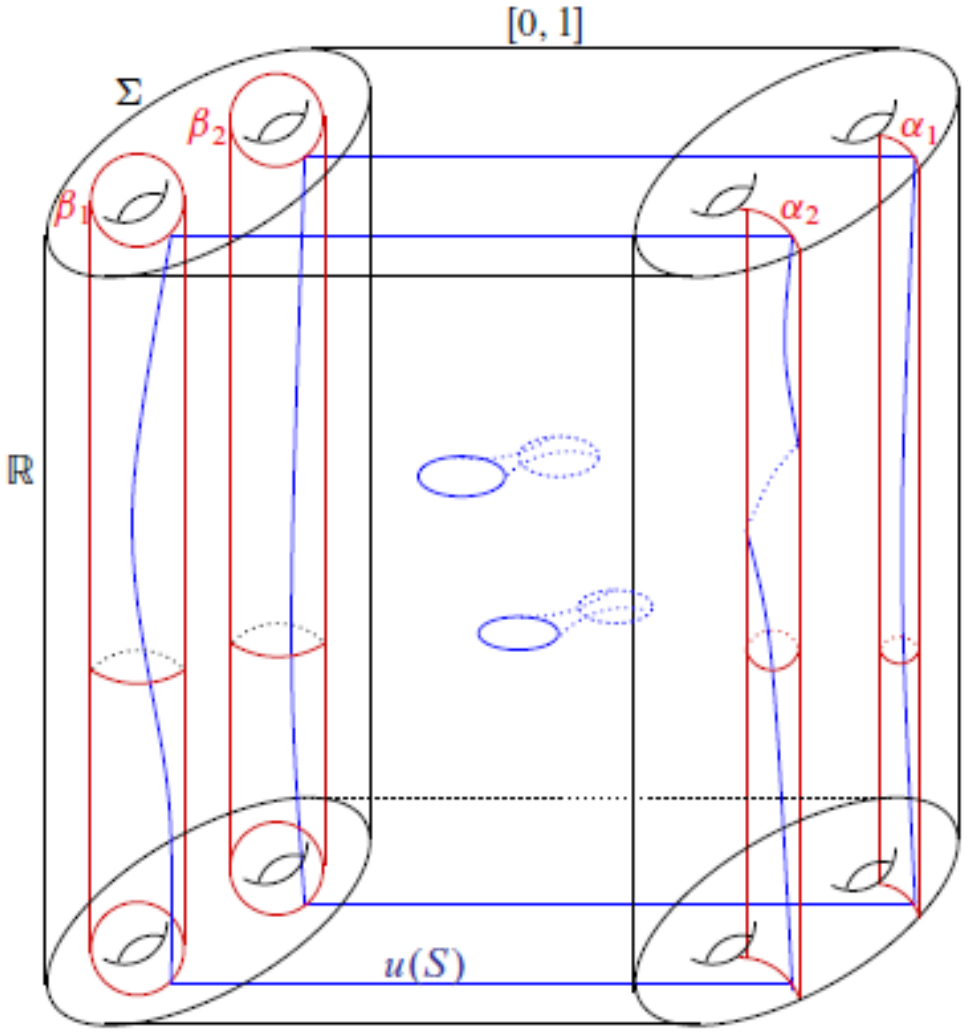
- Consider $\Sigma \times I \times \mathbb{R}$ as a symplectic manifold with an almost complex structure and consider holomorphic curves

$$u \ : \ S \longrightarrow \Sigma \times I \times \mathbb{R}$$

  where S is a Riemann surface.
- Boundary conditions based on Heegaard curves α_i and β_i.

Cylindrical picture of Lipshitz (2006):



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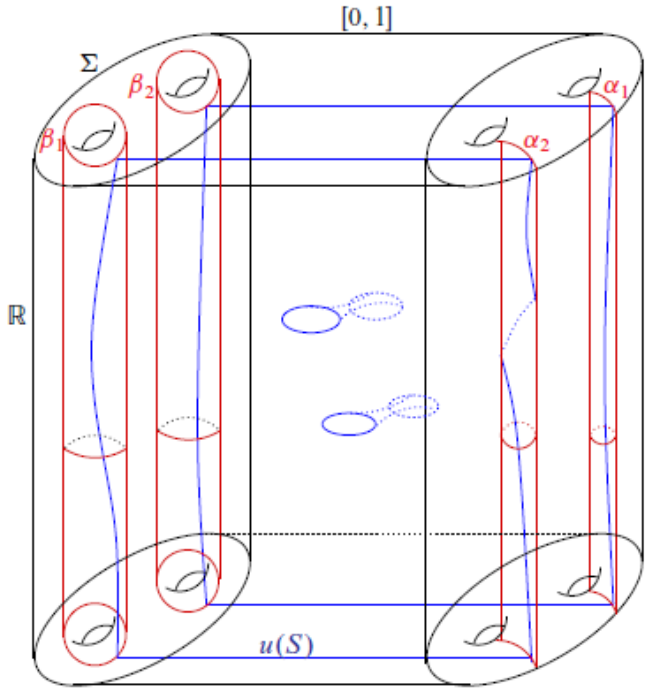


$$\text{ind}\,(D\bar{\partial}) = k - \chi(S) + \sum_{i=1}^{k} \mu(a_i) - \sum_{i=1}^{k} \mu(b_i).$$

Sutured Floer homology

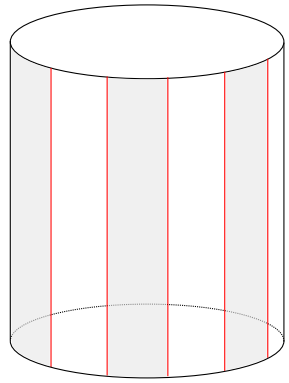
- Chain complex generated by boundary conditions, which are intersections of boundary curves.

$$z_1 \in \alpha_1 \cap \beta_{\sigma(1)},\ z_2 \in \alpha_2 \cap \beta_{\sigma(2)},\ \ldots,\ z_k \in \alpha_k \cap \beta_{\sigma(k)}.$$

- Differential counting index-1 holomorphic curves between boundary conditions.
- Resulting homology is $SFH(M, \Gamma)$.
- Etnyre–Honda (2009): Any contact structure ξ on (M, Γ) defines a natural element $c(\xi) \in SFH(M, \Gamma)$.

Solid tori

We consider the sutured solid torus $D^2 \times S^1$ with $2n$ longitudinal curves $F_n \times S^1$. ($|F_n| = 2n$)



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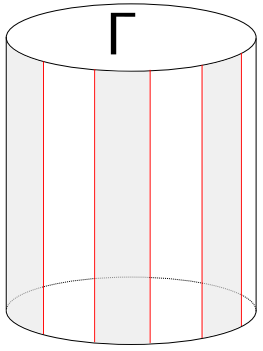
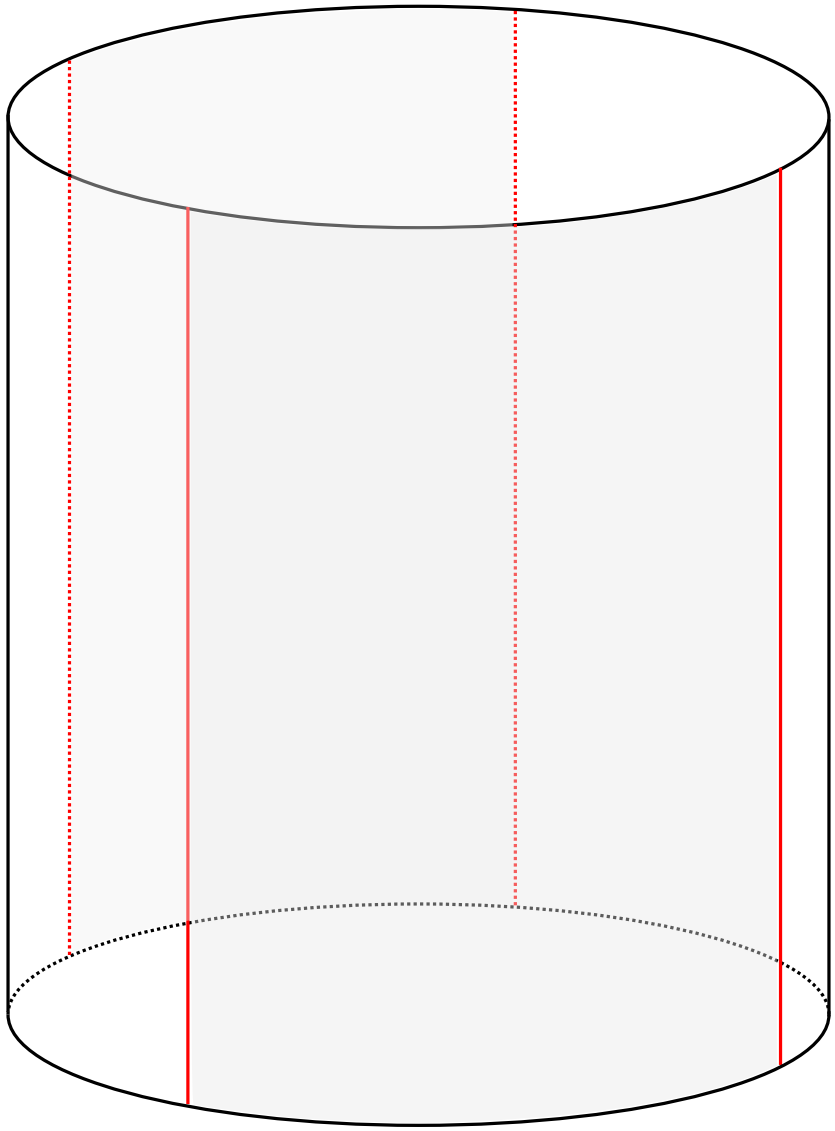


Theorem (M.)

$$SFH(D^2 \times S^1, F_n \times S^1) \cong V_n = \frac{\mathbb{Z}_2\langle \textit{Chord diagrams w/ n chords} \rangle}{\textit{Bypass relation}}$$

Any chord diagram Γ *in* V_n *corresponds to a a contact structure* ξ_Γ *on* $D^2 \times S^1$ *and maps to* $c(\xi_\Gamma)$.

A “computation” of Sutured Floer homology



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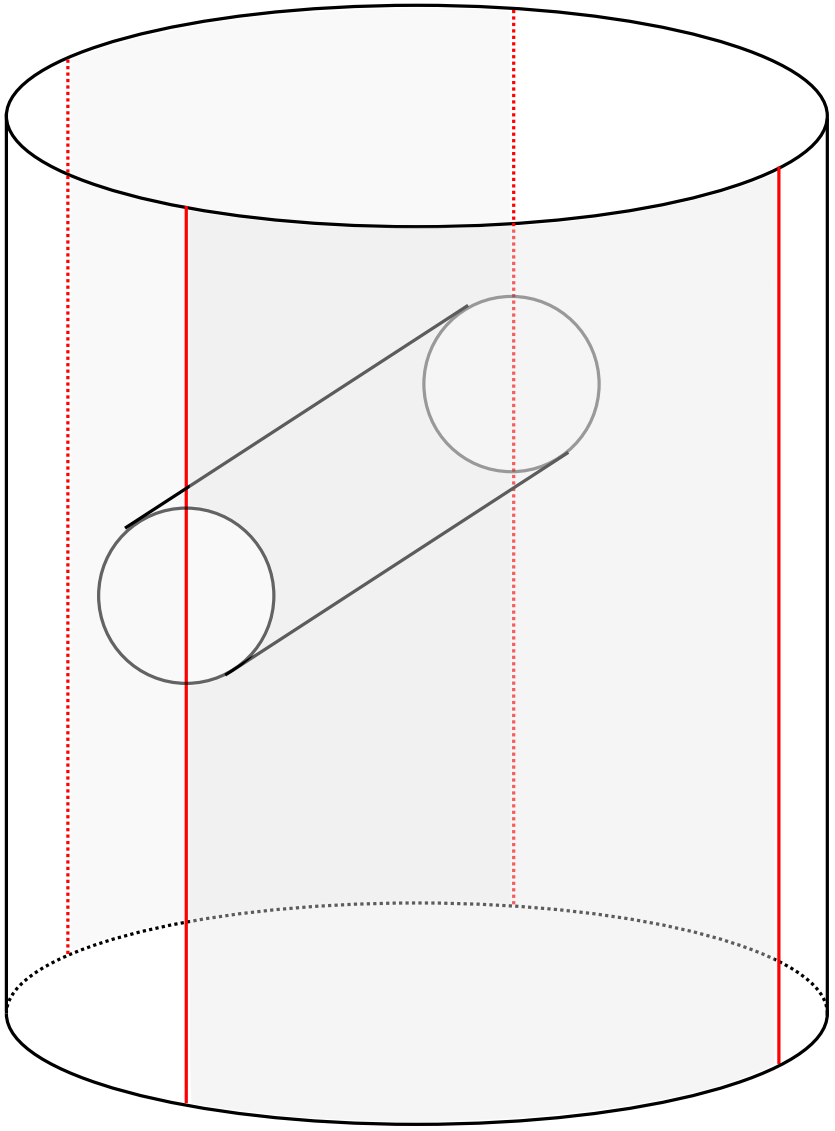

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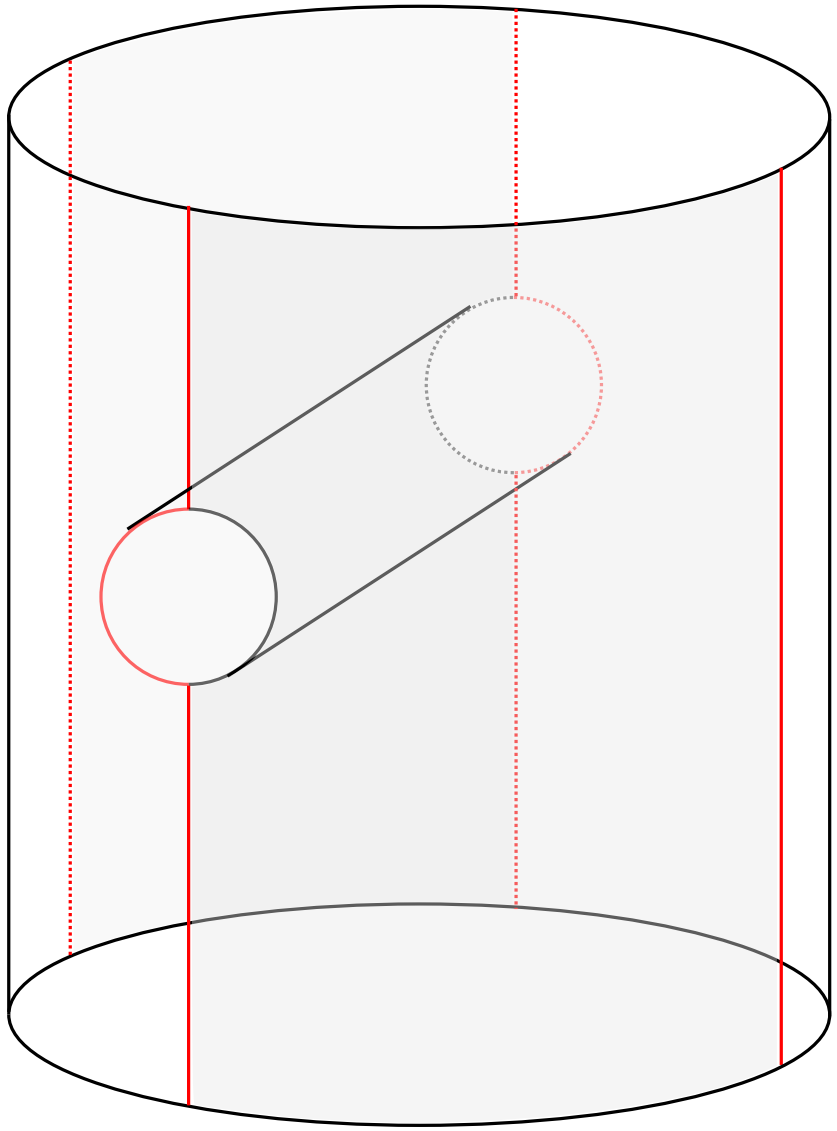

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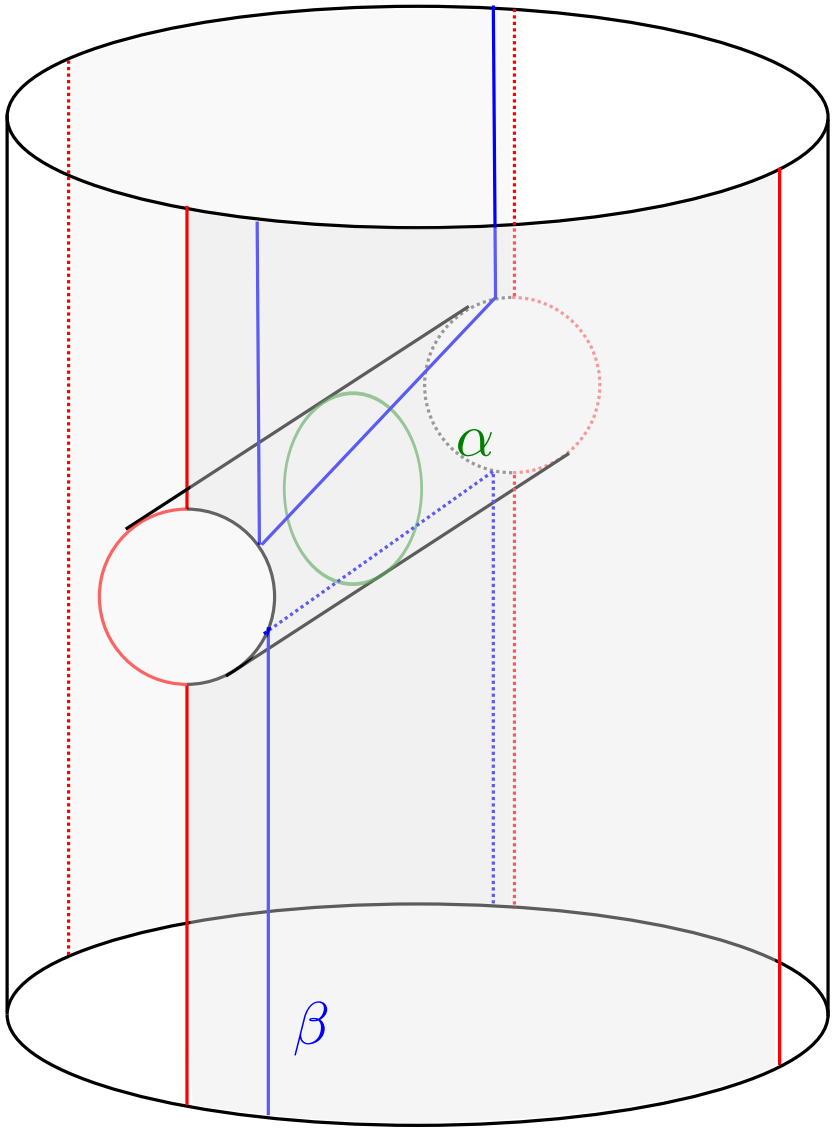

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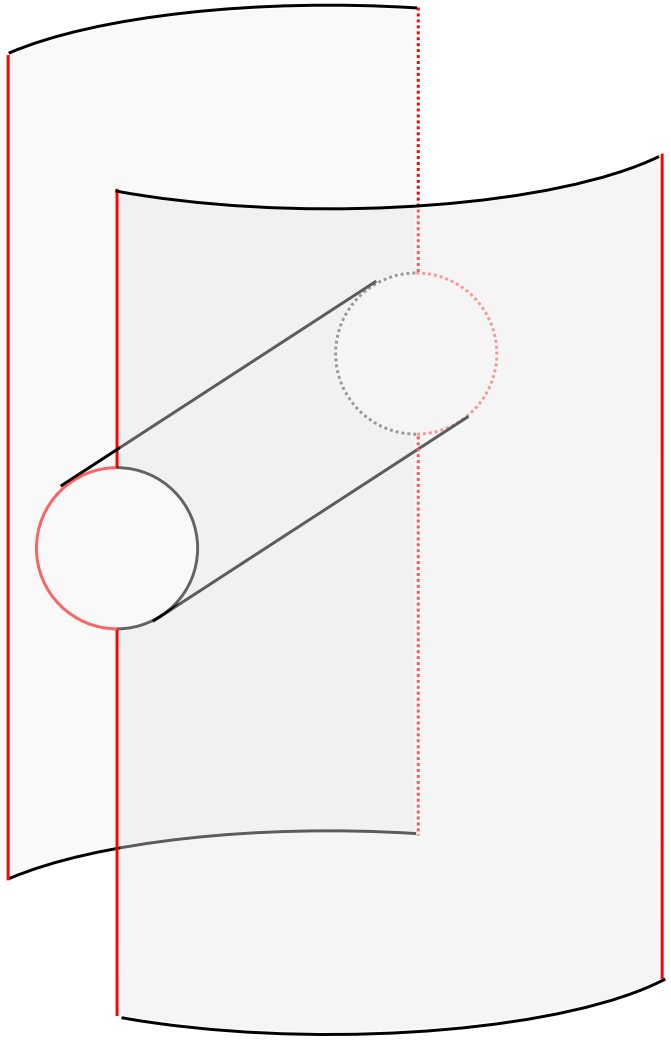

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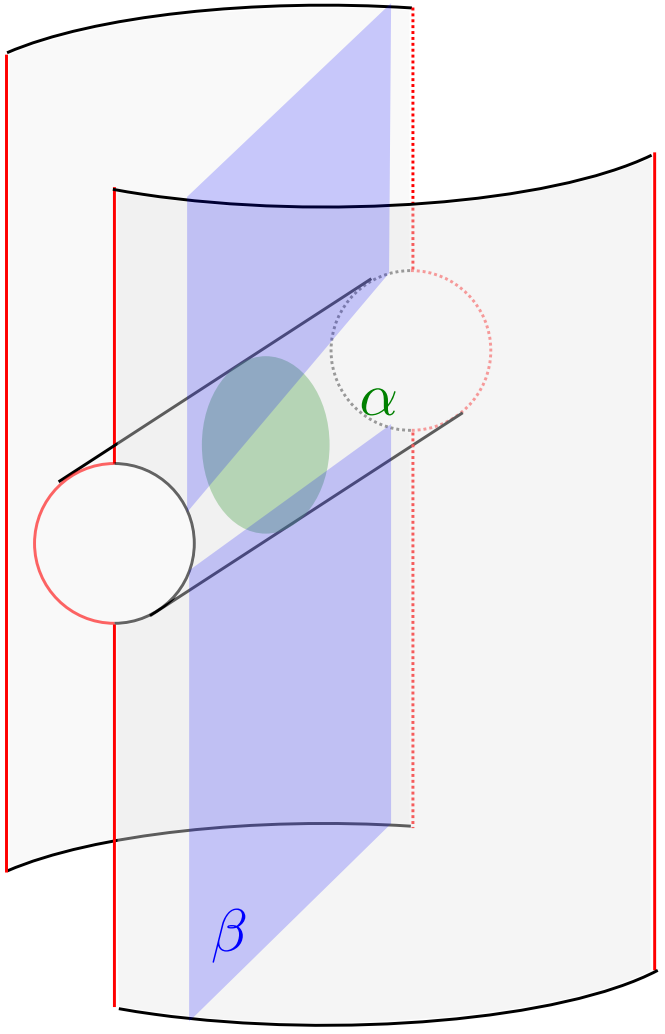

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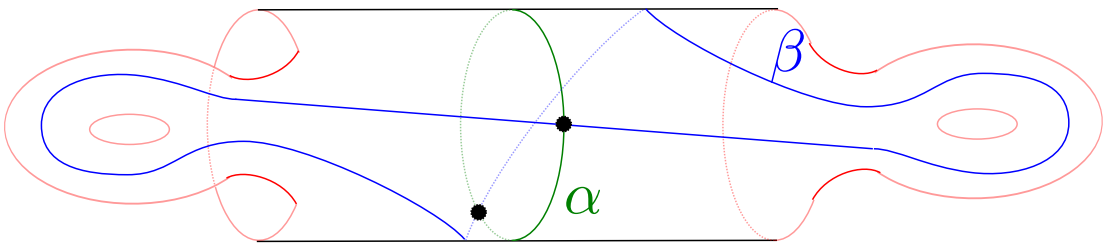

Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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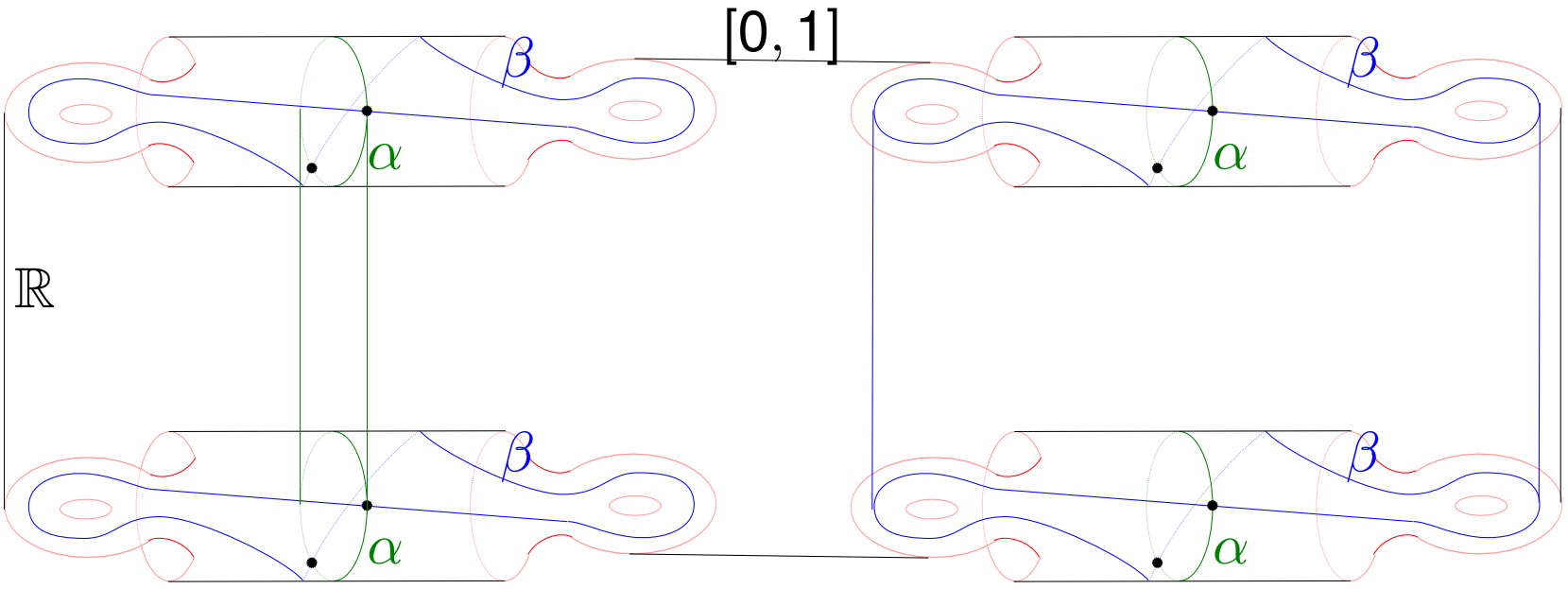

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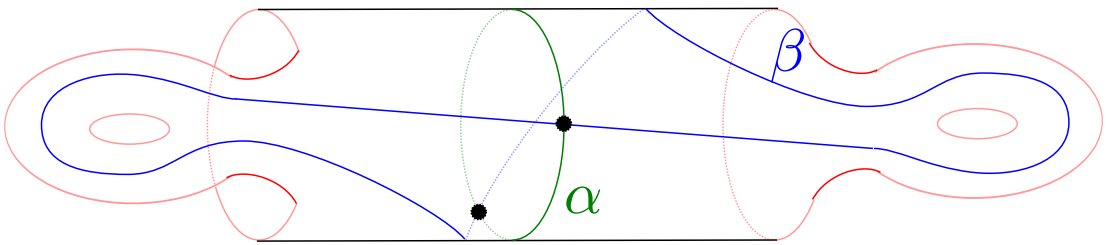


Chain complex $= \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
Nowhere for holomorphic curves to go! $\partial = 0$.

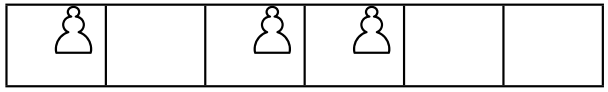
$$SFH = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = V_2$$

Outline

1. Introduction to contact geometry and topology
   - Overview and background
   - Examples of contact manifolds
   - Contact structures on 3-manifolds
2. “Contact skein theory" — Contact TQFT
   - “Inner product” on chord diagrams
   - Bypass surgery
3. Floer homology
   - Floer homology
   - A “computation”
4. Quantum Pawn Dynamics

Quantum Pawn Dynamics

- Pawns on a finite 1-dimensional chessboard.
- A state of the QPD universe:



- Pawns move from left to right, one square at a time.
  (No capturing, no en passant, no double first moves.)

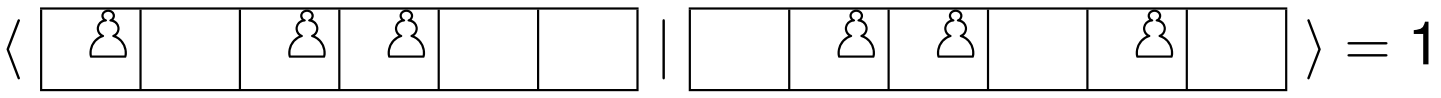
Quantum pawns: "Inner product" $\langle \cdot | \cdot \rangle$ describes the possibility of pawn moves from one state to another.
Valued in $\mathbb{Z}_2$.

Definition (Pawn "inner product")

$$\langle w_0 | w_1 \rangle = \begin{cases} 1 & \textit{if it is possible for pawns to move from } w_0 \textit{ to } w_1 \\ & \quad \textit{(this includes the case } w_0 = w_1\textit{);} \\ 0 & \textit{if not.} \end{cases}$$

Quantum Pawn Dynamics

E.g.



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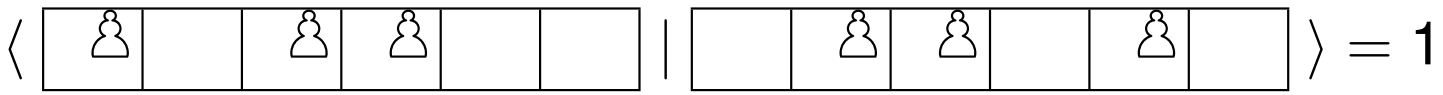
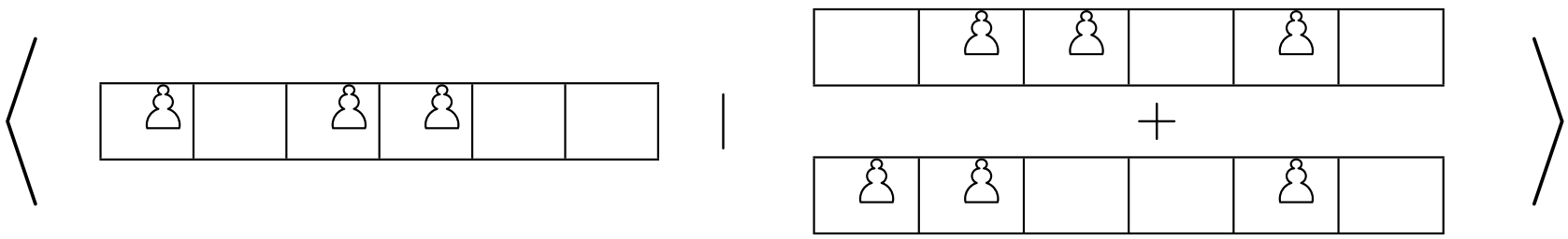


Also, entangled chessboards.

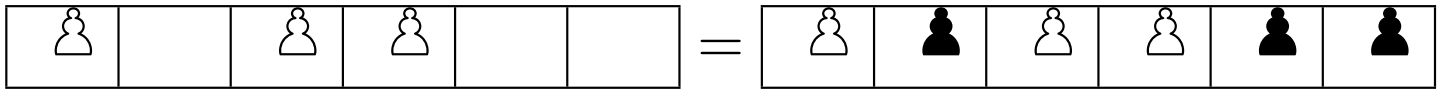


$= 1 + 0 = 1.$

Note asymmetry of $\langle \cdot | \cdot \rangle$.
A “booleanized” partial order. (Complete lattice.)

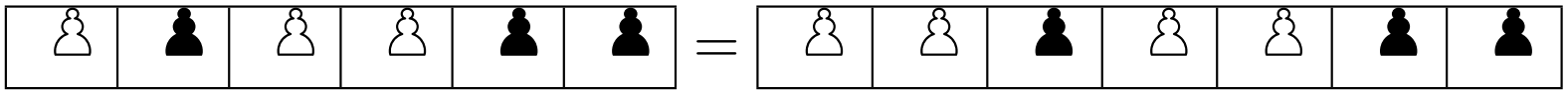
Dirac Pawn Sea

- Think of an “empty” chessboard as a thriving sea of anti-pawns.
  “Anti-pawn” = “absence of pawn”.



Creation and annihilation operators

The initial pawn creation operator $a^*_{p,0}$ adjoins a new initial (leftmost) square to the chessboard, containing a pawn.

$a^*_{p,0}$  $=$

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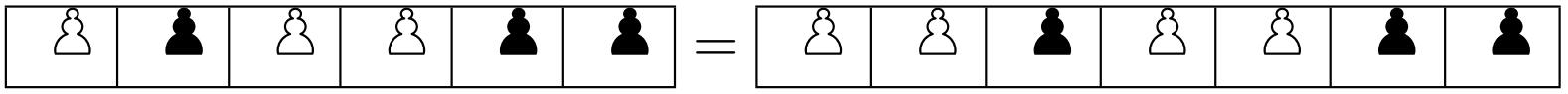
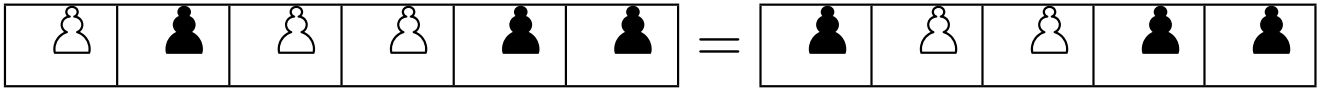
$a^*_{p,0}$ 

The initial pawn annihilation operator $a_{p,0}$ deletes the leftmost square from the chessboard, and a pawn on it.

$a_{p,0}$ 

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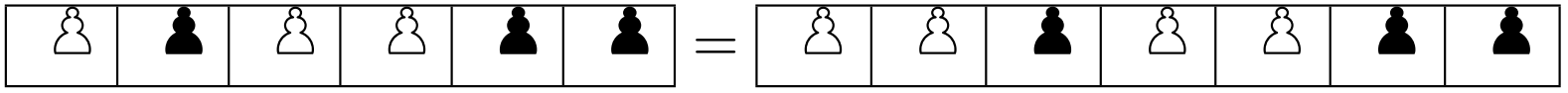
$a^*_{p,0}$ 

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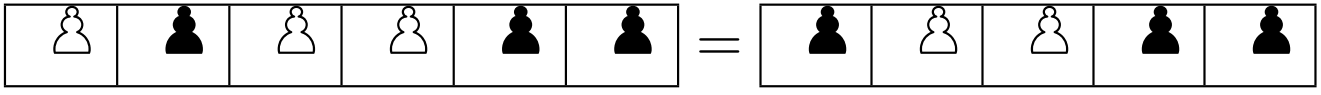
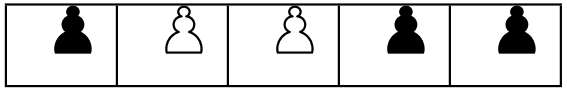
$a_{p,0}$ 

If no pawn (anti-pawn) in the leftmost square, try to delete... “error 404 universe not found” mod 2 = 0.

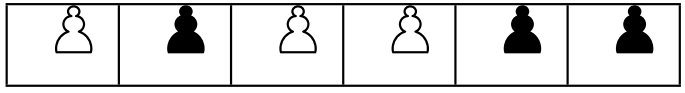
$a_{p,0}$  $= 0$

Similar initial anti-pawn annihilation $a_{q,0}$ and creation $a^\dagger_{q,0}$.

Creation of chessboards

- The vacuum state of the QPD universe is the null chessboard $\emptyset$.
  (Note $\emptyset \neq 0$.)
- Applying initial creation operators to the vacuum can create any chessboard.

$$a^*_{p,0} a^\dagger_{q,0} a^*_{p,0} a^*_{p,0} a^\dagger_{q,0} a^\dagger_{q,0} \quad \emptyset \quad = \quad$$



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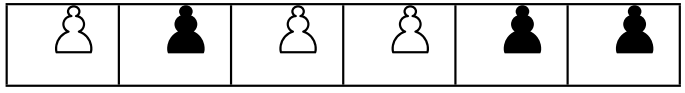

- The $*$ and $\dagger$ refer to adjoints.
  (Galois connections on partial orders.)

Adjoints

- Recall an adjoint f^* of an operator f usually means that

$$\langle fx|y\rangle = \langle x|f^*y\rangle, \quad \langle x|fy\rangle = \langle f^*x|y\rangle.$$

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- As our "inner product" is asymmetric, we have two distinct adjoints $f^*, f^\dagger$ of an operator f.

$$\langle fx|y\rangle = \langle x|f^*y\rangle, \quad \langle x|fy\rangle = \langle f^\dagger x|y\rangle.$$

So $f^{*\dagger} = f^{\dagger *} = f$.

Initial creation and annihilation are adjoint

Proposition

$$\langle a_{p,0} x | y \rangle = \langle x | a^*_{p,0} y \rangle$$

Proof.

$a^*_{p,0} y$ begins with a pawn.
If x begins with an anti-pawn, both sides are 0.
If x begins with a pawn, $\langle x | a^*_{p,0} y \rangle \neq 0$ compares two chessboards with initial pawns.
$a_{p,0}$ removes an initial pawn so $\langle a_{p,0} x | y \rangle$ gives the same result. □

Similarly, initial anti-pawn creation/annihilation †-adjoint.

Adjoining adjoints

What is $a_{p,0}^{**}$? What operator f satisfies

$$\langle a_{p,0}^{*} x | y \rangle = \langle x | f y \rangle ?$$

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Answer: the operator which deletes the first pawn from the left on a chessboard.

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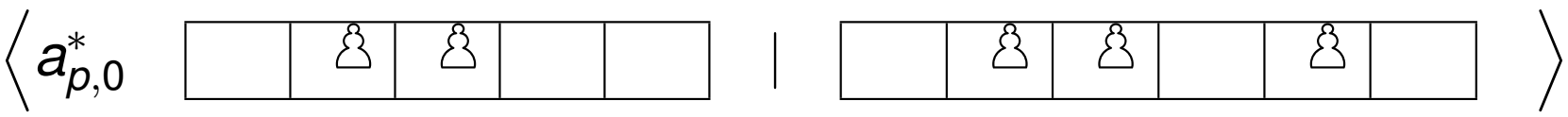

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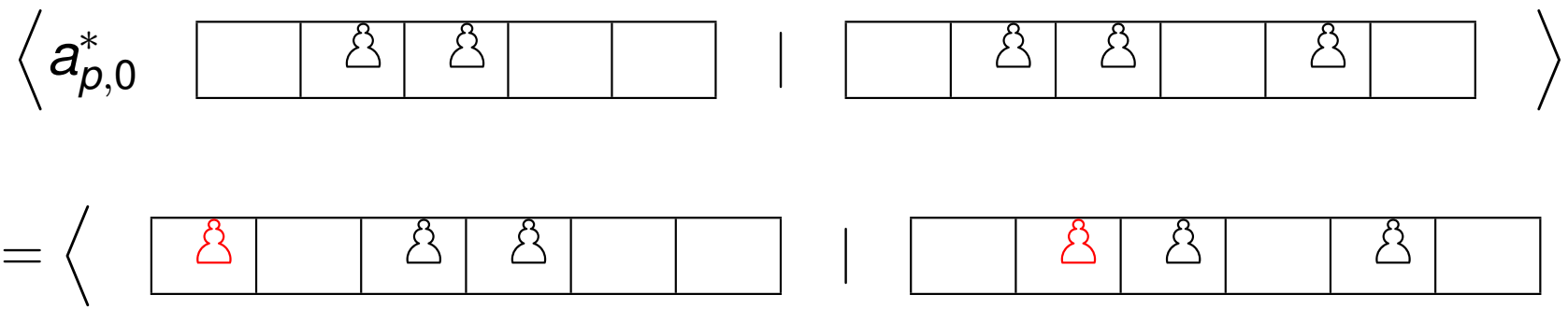

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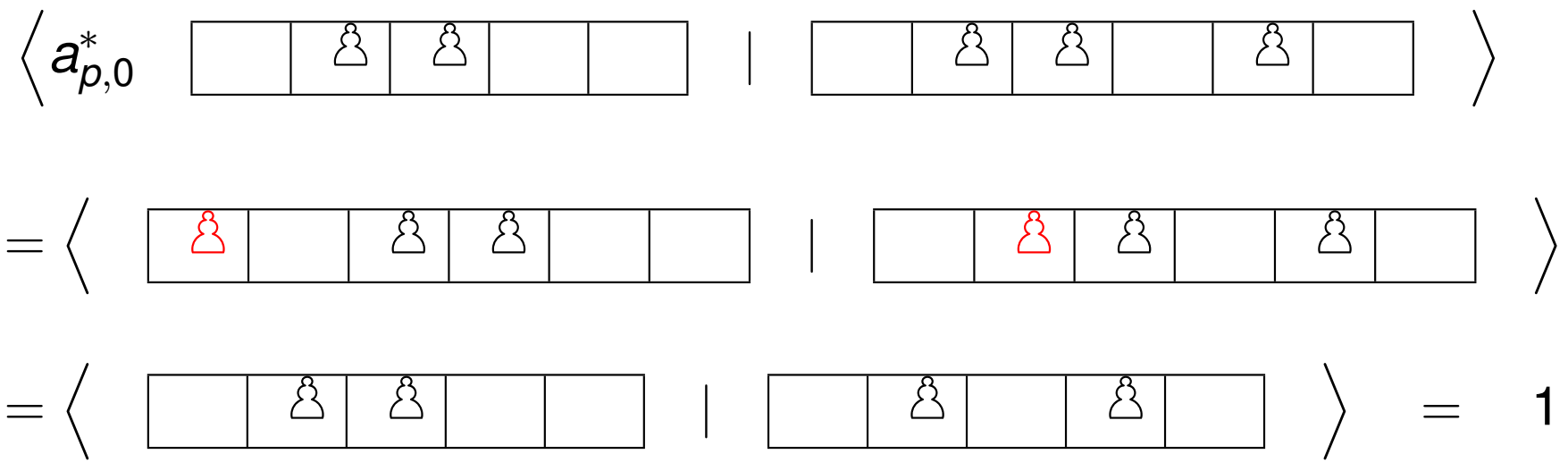


Proposition

The iterated adjoints of $a_{p,0}$ are

$$a_{p,0} \quad \to \quad a^*_{p,0} \quad \to \quad a_{p,1} \quad \to \quad a^*_{p,1} \quad \to \quad a_{p,2} \quad \to \quad \cdots \quad \to \quad a_{p,\Omega} \quad \to \quad a^*_{p,\Omega}$$

where:
$a_{p,i}$ deletes the i'th pawn
$a^*_{p,i}$ doubles the i'th pawn
$a_{p,\Omega}, a^*_{p,\Omega}$ are final pawn creation and annihilation.

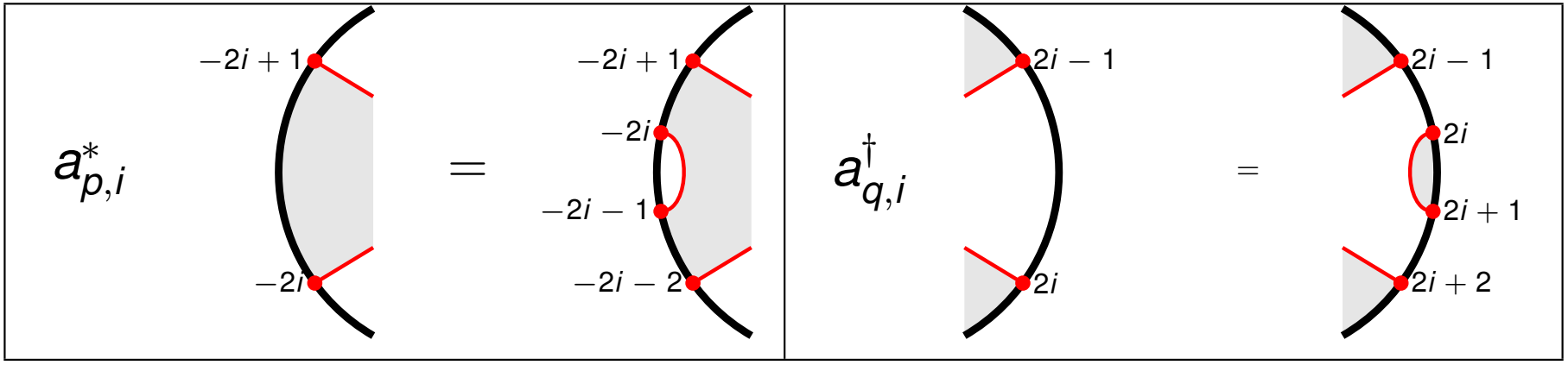
Similarly for anti-pawns in the opposite direction.

$$a^\dagger_{q,\Omega} \quad \to \quad a_{q,\Omega} \quad \to \quad \cdots \quad a_{q,2} \quad \to \quad a^\dagger_{q,1} \quad \to \quad a_{q,1} \quad \to a^\dagger_{q,0} \quad \to \quad a_{q,0}$$

(A simplicial structure.)

Creation and annihilation of chords

Define creation operators $a^*_{p,i}$, $a^\dagger_{q,i}$ to insert a new chord in a specific place in a chord diagram as shown.

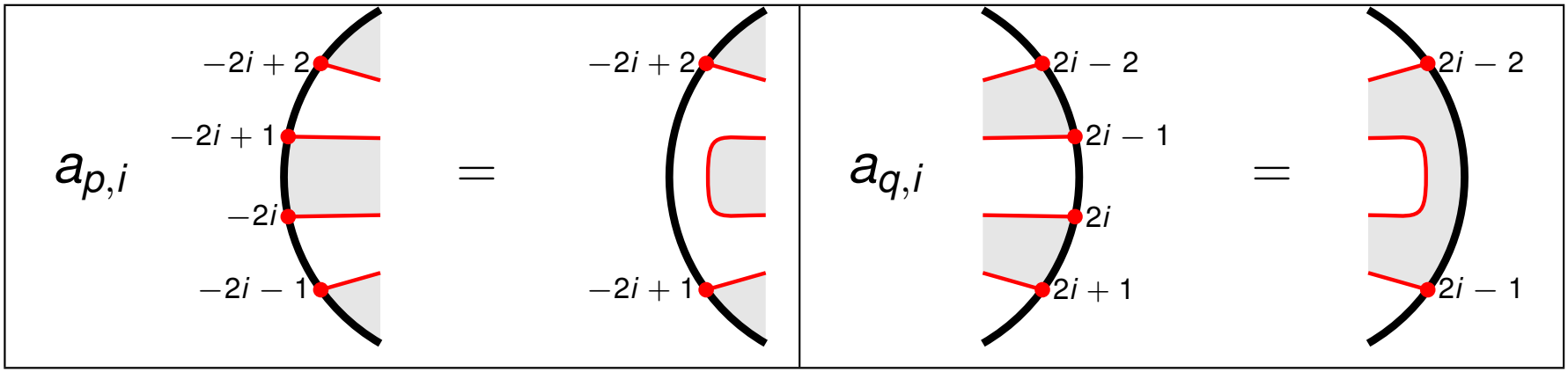


$a^*_{p,i}$ creates a white region i spots down on the left.

$a^\dagger_{q,i}$ creates a black region i spots down on the right.

Creation and annihilation of chords

Define annihilation operators $a_{p,i}$, $a_{q,i}$ to close off chords in a chord diagram as shown.



$a_{p,i}$ closes off a black region i spots down on the left.
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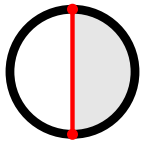
Diagrams of chessboards

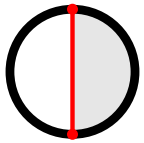
The simplest chord diagram is called the vacuum $\Gamma_\emptyset$.



Build up more complicated diagrams with creation operators.

Diagrams of chessboards

The simplest chord diagram is called the vacuum $\Gamma_\emptyset$.

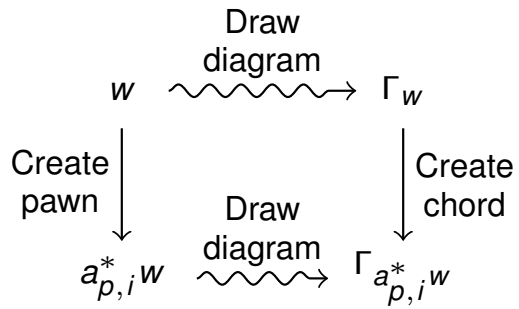


Build up more complicated diagrams with creation operators.

Proposition (M.)

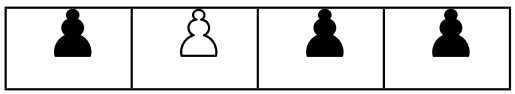
For any chessboard w, there is a chord diagram Γ_w such that creation and annihilation operators agree (are equivariant):

$$\Gamma_{a^*_{p,i}w} = a^*_{p,i}\Gamma_w.$$



Ski slopes

Construction of the slalom skiing chord diagram of a chessboard.

$qpqq \quad \leftrightarrow$ 

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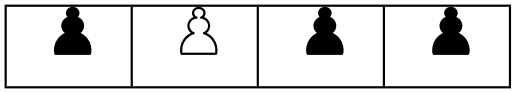
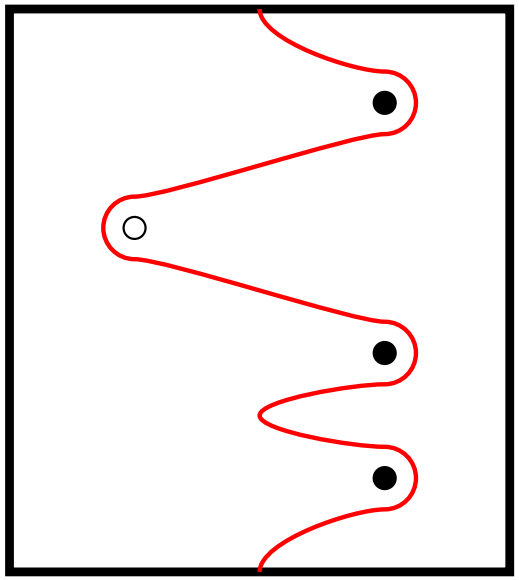
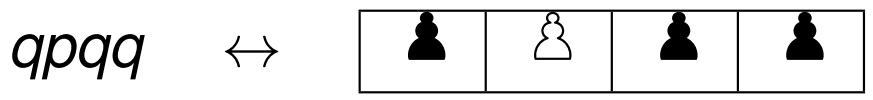
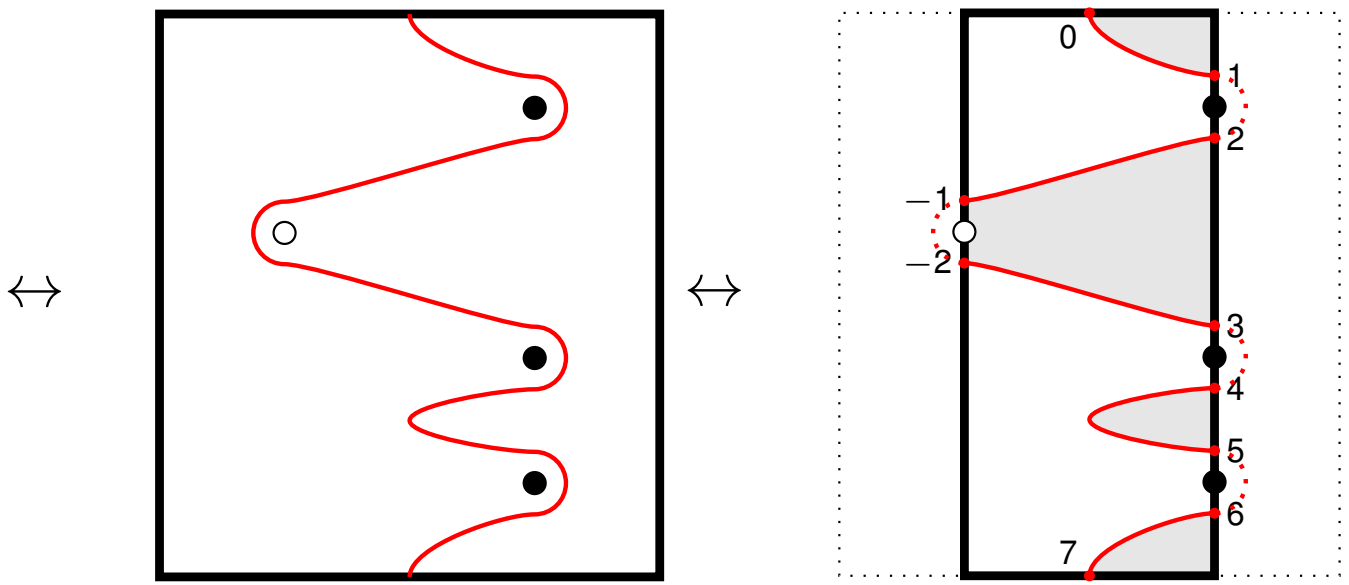
qpqq $\leftrightarrow$ 

$\leftrightarrow$ 

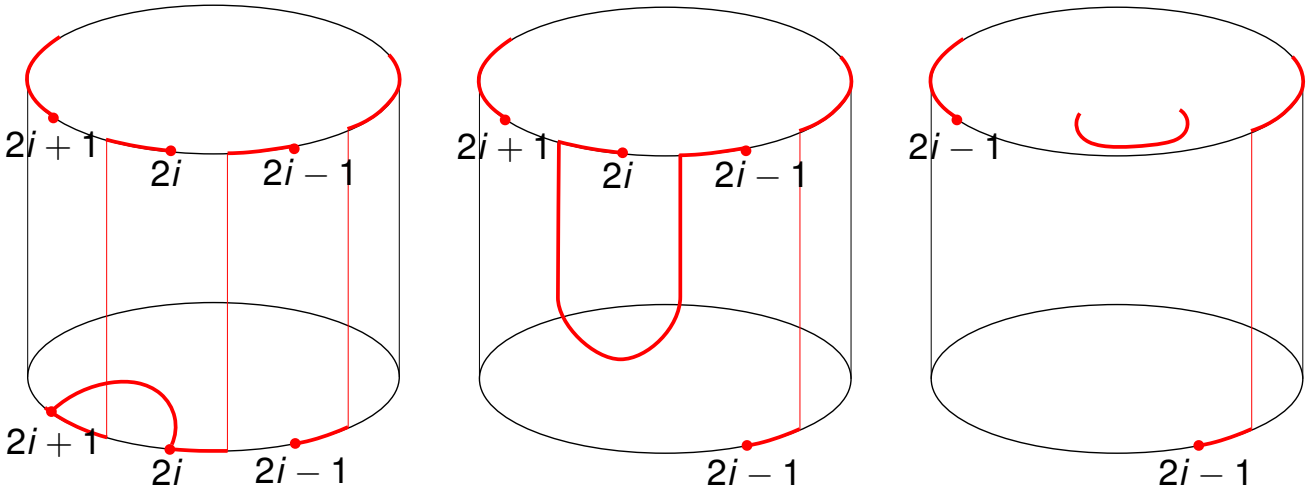
Construction of the slalom skiing chord diagram of a chessboard.





Adjoints

Adjoint relations can be seen topologically as “finger moves”.



$$\langle a^{\dagger}_{q,i}\, \Gamma_0 \mid \Gamma_1 \rangle \qquad = \qquad \langle \Gamma_0 \mid a_{q,i}\, \Gamma_1 \rangle$$

Now perhaps believable that adjoint is periodic.

A vector space of chord diagrams

Theorem (M.)

V_n has dimension 2^{n-1} and the diagrams from chessboards of $n-1$ squares form a basis.

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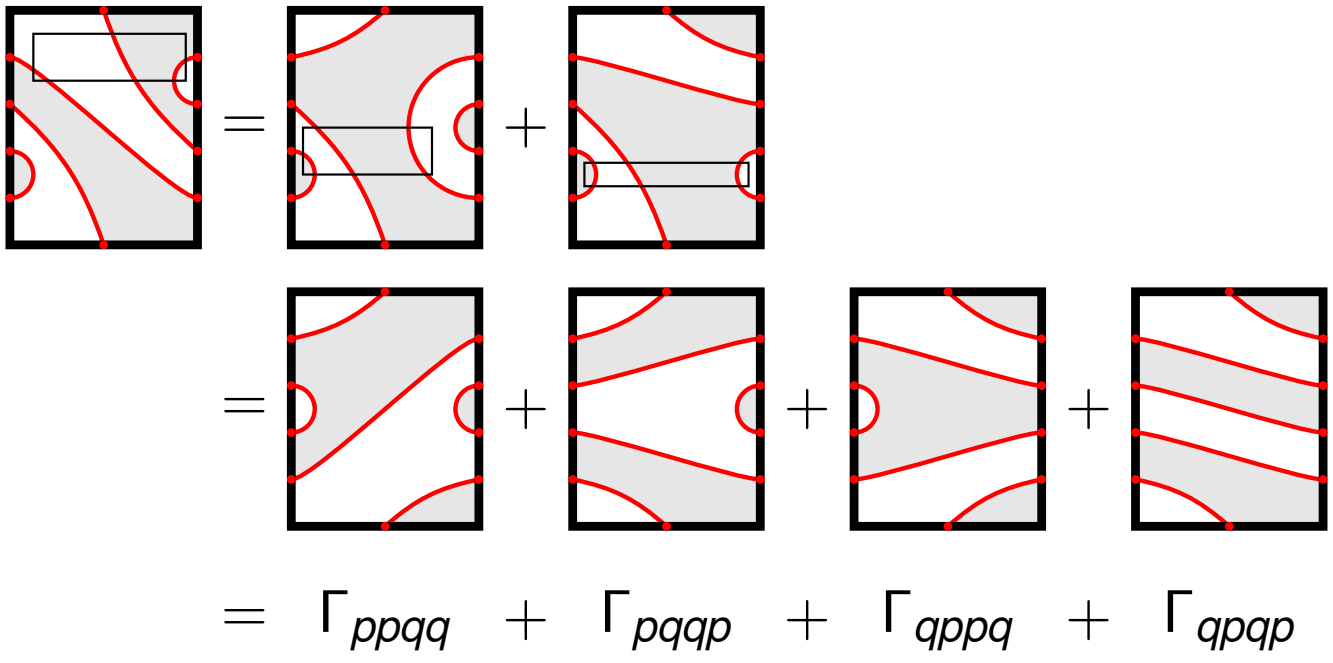
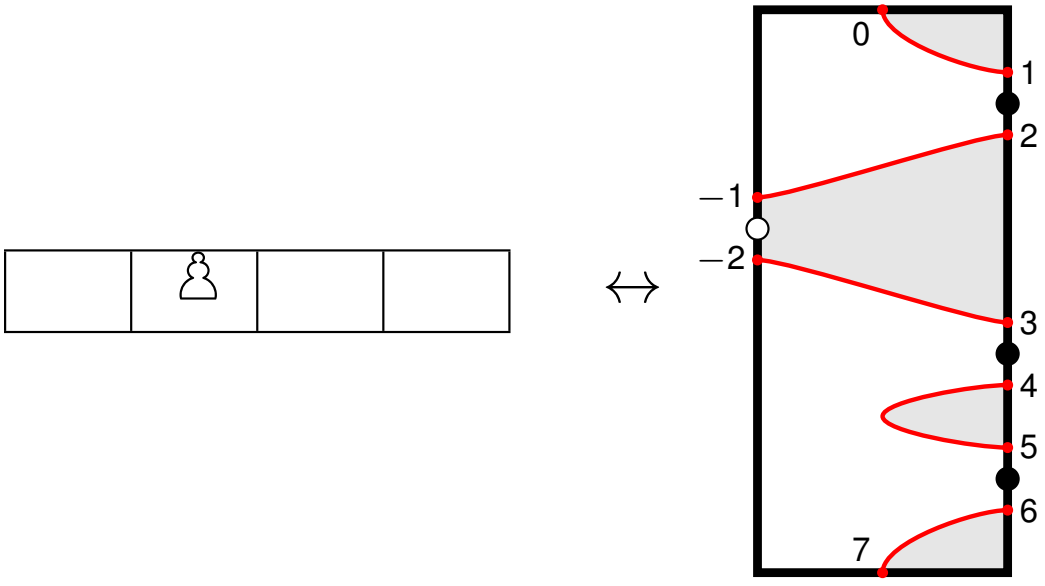


Square decomposition

Pawns correspond to a decomposition of the disc into squares.



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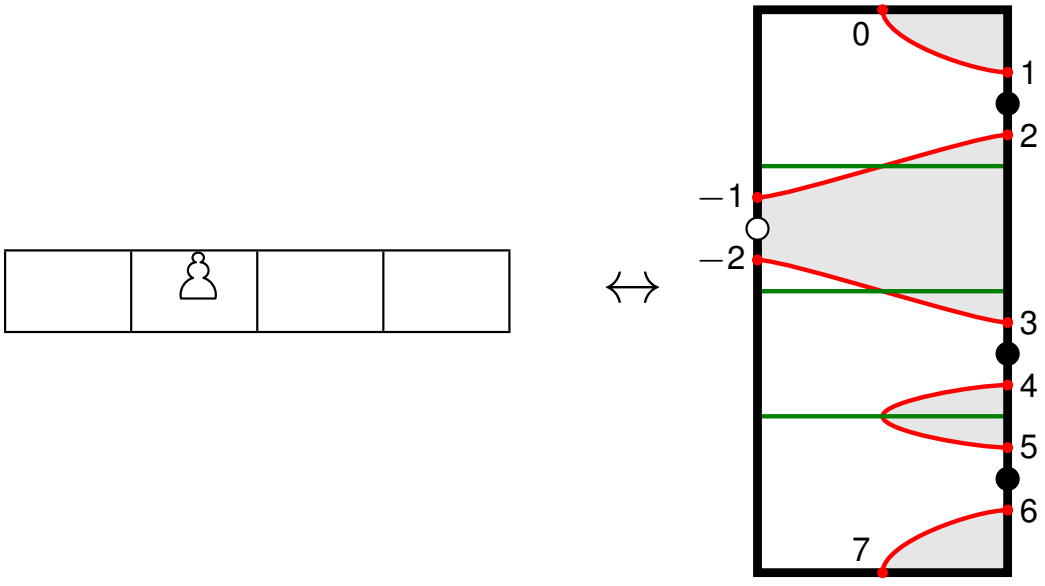

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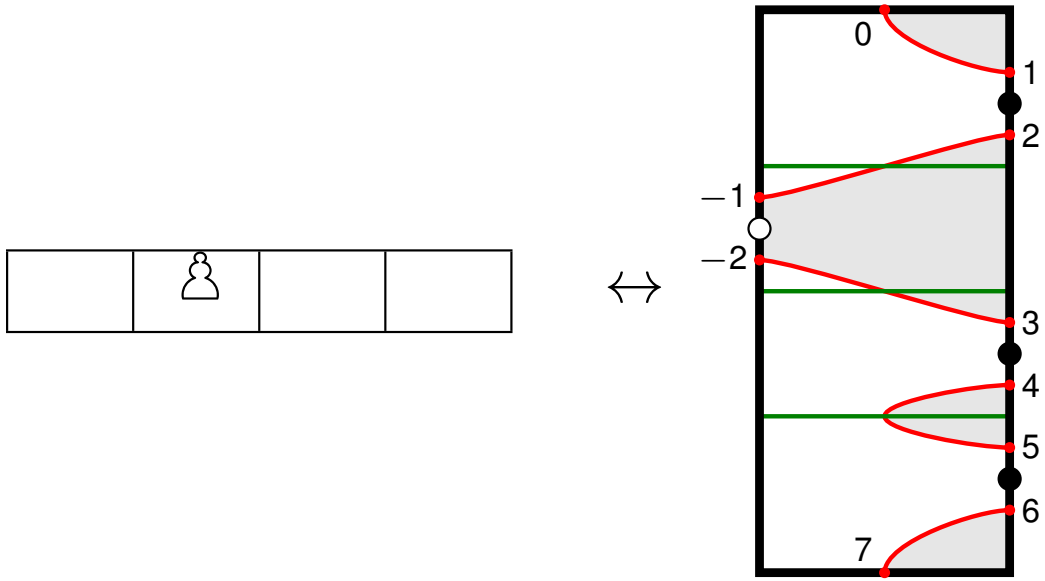
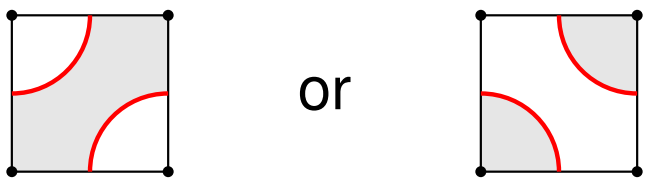


Note after decomposing along green lines, each square is



Behave partly like particles (can be created/annihilated) and partly like qubits (binary). Reminiscent of stat. mech...

Theorem (M.)

V_n has a basis given by the diagrams of chessboards:

$$V_n \cong \mathbb{Z}_2\langle \textit{Chessboards with } n-1 \textit{ squares}\rangle.$$

For any two chessboards w_0, w_1,

$$\langle w_0 | w_1 \rangle = \langle \Gamma_{w_0} | \Gamma_{w_1} \rangle.$$

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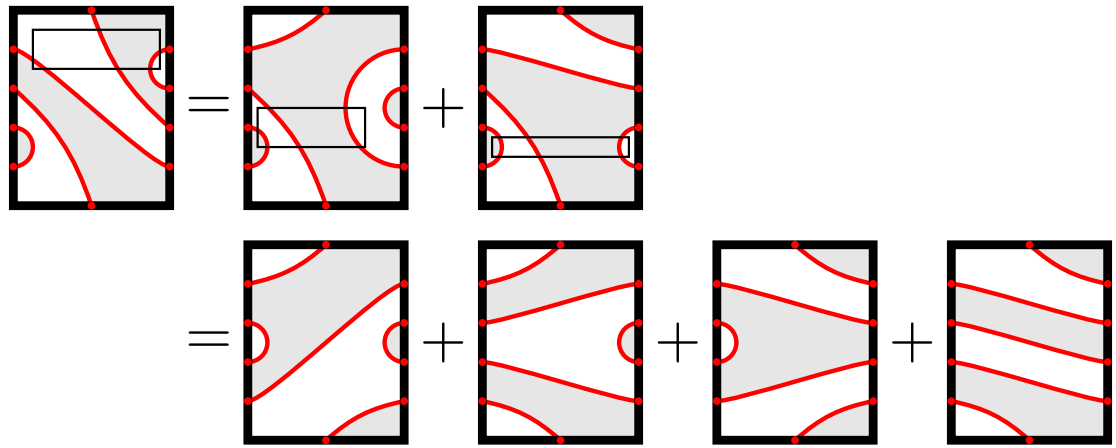

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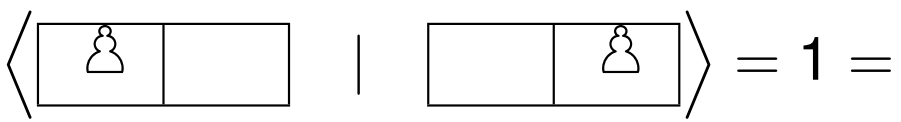
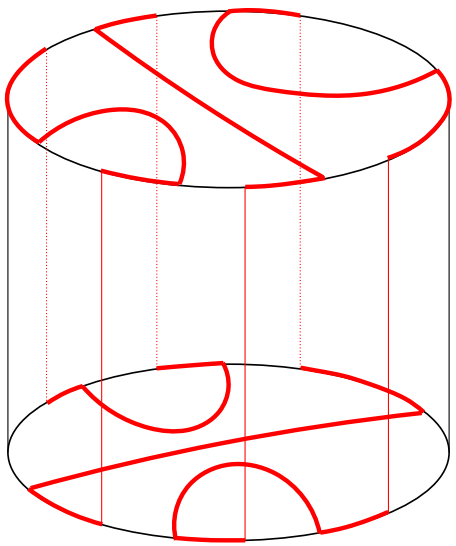


$= 1 =$



Thanks for listening!

Daniel.Mathews@monash.edu