Physical aspects of circle packings

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ANZAMP
Gerringong
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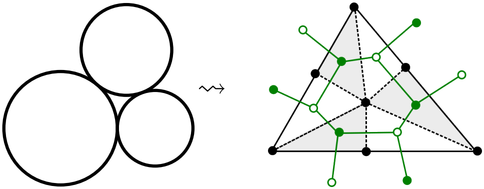

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- Give some background on circle packing
- Discuss relations to spinors and spacetime
- Describe circle packing equations, whose solutions parametrise circle packings
- Discuss relations to statistical mechanics, on-shell diagrams, Grassmannians, amplituhedra, origami.

Overview

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Acknowledgments:

- Includes joint work with Masters student Varsha (MSc) and PhD student Orion Zymaris
- Some work in progress

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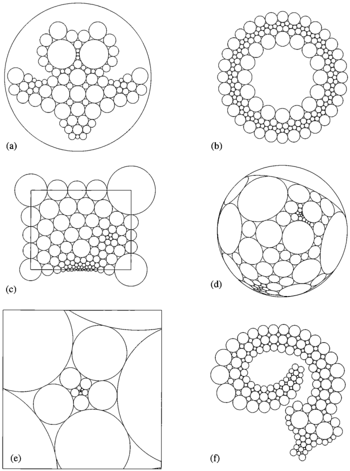

Figure: Stephenson

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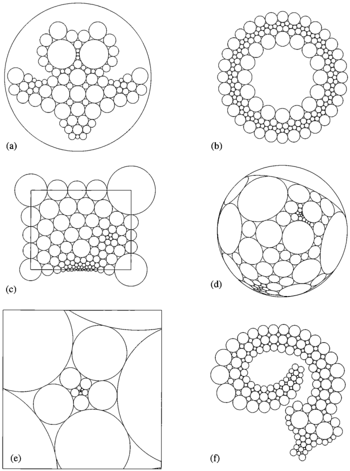

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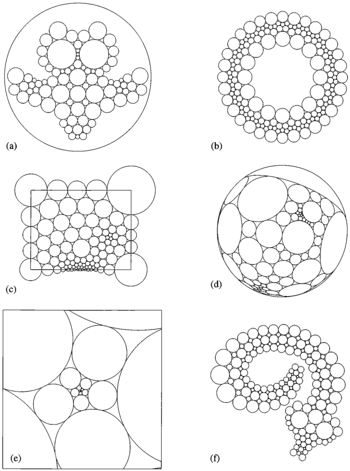

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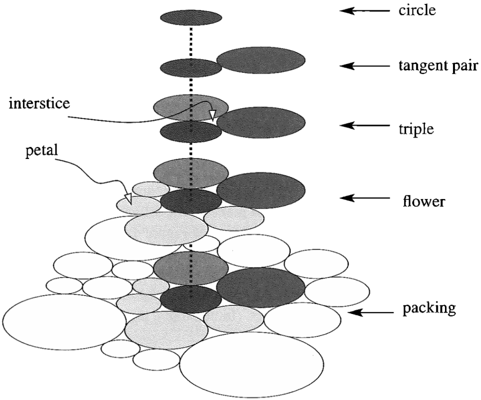
Discrete conformal geom / Riemann s'fces / cx analysis.

Cauchy-Riemann eqns "preserve infinitesimal circles". Here, just preserve circles!

"Quantum geometry"?

Figure: Stephenson

Circle packing terminology



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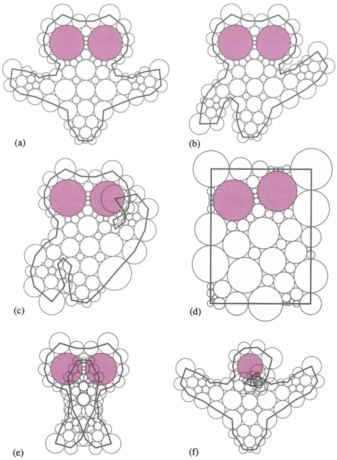

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Bending the owl

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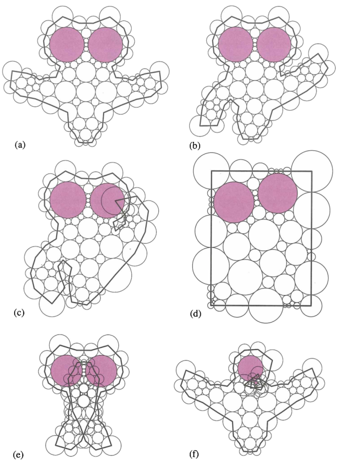
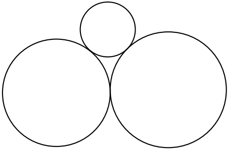


For rigidity, require 3 circles tangent around each interstice.



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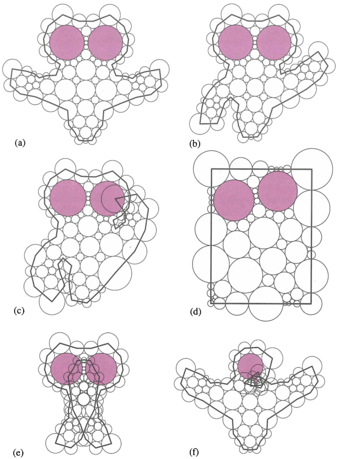

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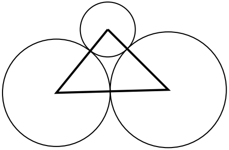


Combinatorially represent this with triangles forming a triangulation.

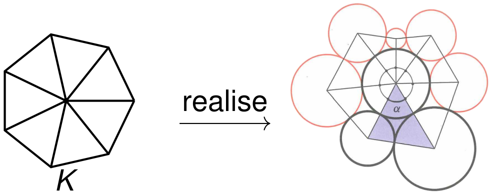
Figure: Stephenson

Circle Packing Formalism: combinatorics to geometry

Let K be a triangulation of an oriented surface (simplicial, possibly w ∂).

A realisation of K is a collection of circles such that

- Vertices = circles
- Edges = tangencies
- Oriented Faces/triangles = oriented tangent triples



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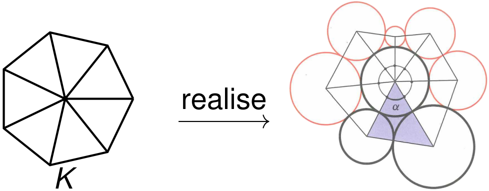

Qns:

- Given K, can K be realised?
- On what metric surfaces are there such circle packings?
- How unique are realisations of K? “Moduli space"?

Koebe/Andreev/Thurston/Beardon/Stephenson/... 1930s–90s:
There are discrete circle packing versions of many theorems of complex analysis (e.g. uniformisation, Riemann mapping).

Circle packing & Discrete complex analysis

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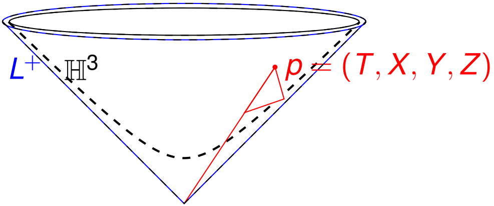
Useful theorem here: discrete boundary value theorem

- $K =$ simplicial complex triangulating a disc
- $\partial K =$ boundary vertices.

Theorem (Discrete boundary value theorem)

Let K be a closed disc, $g \colon \partial K \to \mathbb{R}_+$ any function. Then there exists a circle packing for K in the Euclidean plane, unique up to isometry, with boundary circle radii given by g.

Penrose–Rindler: spinors $\kappa = (\xi, \eta) \in \mathbb{C}^2$ correspond to pointed null flags $\Phi(\kappa)$ on positive light cone L^+ in $\mathbb{R}^{1,3}$.



From Spinors & Spacetime to Circle Packings

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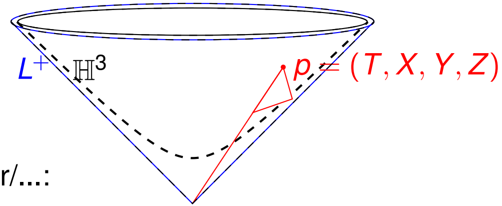

Dirac/Pauli/Wigner/...:

Spinors $\mathbb{C}^2$ $\longrightarrow$ 2×2 Hermitian matrices $\mathcal{H}$ $\longrightarrow$ Minkowski space $\mathbb{R}^{1,3}$

$$\kappa = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \overline{\xi} & \overline{\eta} \end{pmatrix} = \begin{pmatrix} |\xi|^2 & \xi\overline{\eta} \\ \eta\overline{\xi} & |\eta|^2 \end{pmatrix} = \begin{pmatrix} T + Z & X + iY \\ X - iY & T - Z \end{pmatrix}$$

$$\mapsto \phi(\kappa) = (T, X, Y, Z)$$

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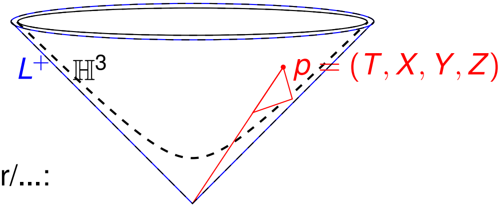

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Flag direction (Penrose, Rindler, M?) $D_\kappa \phi(i\overline{\eta}, -i\overline{\xi})$. (Tangent to L^+.)

- Flag is spin: Multiplying κ by $e^{i\theta}$ rotates flag $\Phi(\kappa)$ by 2θ.

From Spinors to Hyperbolic Geometry & Horospheres

Hyperbolic space $\mathbb{H}^3 \subset \mathbb{R}^{1,3}$ as $T^2 - X^2 - Y^2 - Z^2 = 1$ (hyperboloid model)

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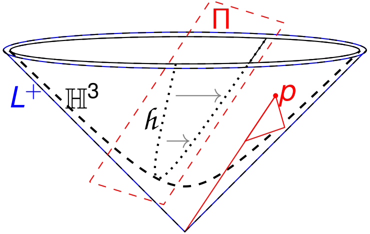

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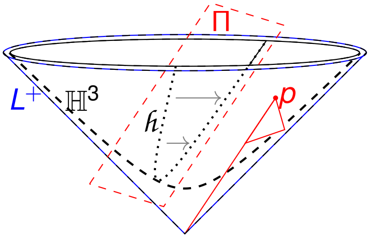
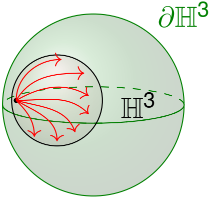
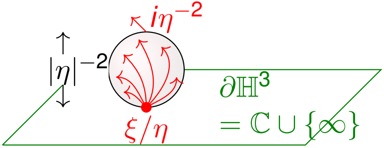




Conformal/“Poincaré" disc model



Upper half space model

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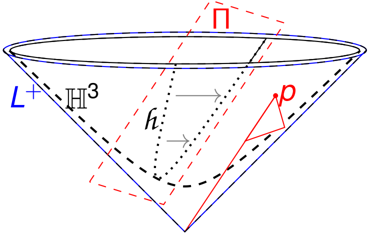

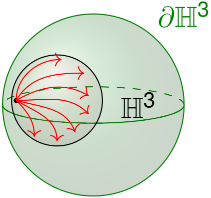

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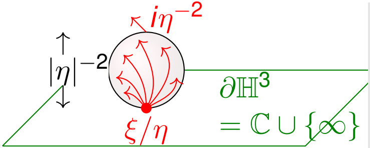

Upper half space model

Theorem (M ’25 Adv. Math.)

$\exists$ *$SL(2,\mathbb{C})$-equivariant bijection*

$$\mathbb{C}^2 \setminus \{0\} \leftrightarrow \left\{ \begin{array}{c} \textit{Horospheres in } \mathbb{H}^3 \\ \textit{with spin directions} \end{array} \right\}$$

From Spinors to Flags to Horospheres to Circles

Thm extends (e.g. $\det(\kappa_1, \kappa_2) = \lambda$-length between horospheres) and generalises:

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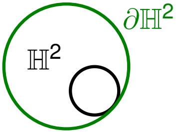
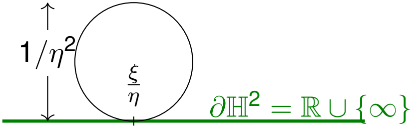


Conformal disc



Upper half plane

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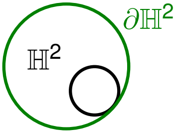

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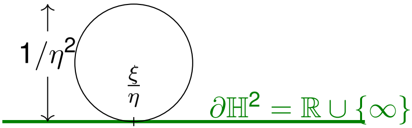
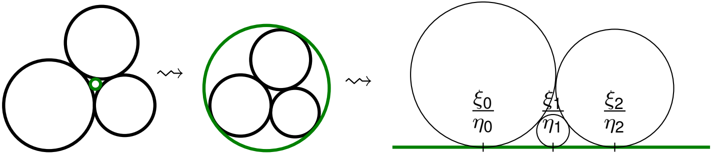


Upper half plane

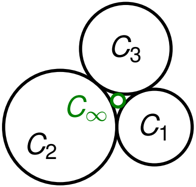
Horocycles in $\mathbb{H}^2$ are tangent circles $\rightsquigarrow$ circle packings!



Horocycles of (ξ_0, η_0), (ξ_1, η_1) tangent $\Leftrightarrow \xi_0\eta_1 - \xi_1\eta_0 = \pm 1$.

Descartes wrote to Elisabeth of the Palatinate about circles.

3-flower:



Descartes 1643: $\kappa_\infty^2 + \kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \frac{1}{2}\left(\kappa_\infty + \kappa_1 + \kappa_2 + \kappa_3\right)^2$

Spinors and Descartes' Circle Theorem

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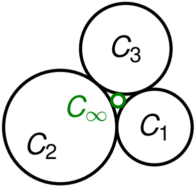

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Theorem (M–Zymaris '25 J.Geom.Phys, arxiv:2504.14593)

Given curvatures κ_∞, κ_j $_{(j \in \mathbb{Z}/n\mathbb{Z})}$, *let* $m_j = \frac{\sqrt{\kappa_{j-1}\kappa_j + \kappa_{j-1}\kappa_\infty + \kappa_j\kappa_\infty}}{\kappa_\infty}$. *Then*

$$\prod_{j=1}^{n}(m_j + i) = \prod_{j=1}^{n}(m_j - i)\,.$$

An integer polynomial equation: $\operatorname{Im}\prod_{j=1}^{n}(m_j + i) = 0$.

Continuing in this vein $\rightsquigarrow$ equations for circle packings.

Circle packing equations

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Theorem (M.–Zymaris arxiv:2504.14593)

For K triangulating a disc/sphere/torus, $\exists$ polynomial eqns s.t.

Variables	$\sim$	*corners of K*
Equations	$\sim$	*int. vertices/edges/faces*
(Pos real) Solution variety	$\sim$	*circle packings realising K*

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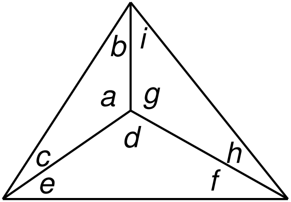
Example:



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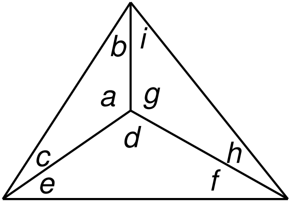

Triangle equations:

$$abc = a + b + c, \quad def = d + e + f$$
$$ghi = g + h + i$$

Edge equations:

$$ae = cd, \quad ai = bg, \quad dh = fg$$

Vertex equation:

$$1 = ad + dg + ga$$

Further physics: discrete holomorphicity, origami

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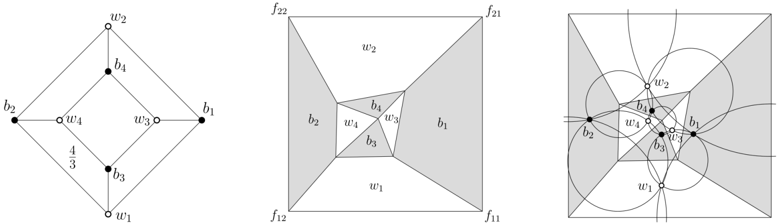
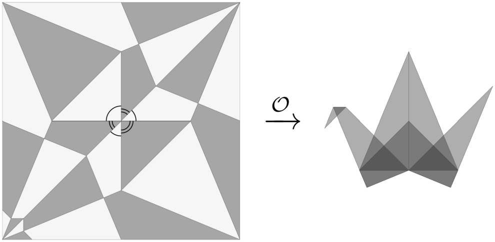


Figure: Kenyon–Lam–Ramassamy–Russkikh

More physics: origami, Grassmannians, amplituhedra



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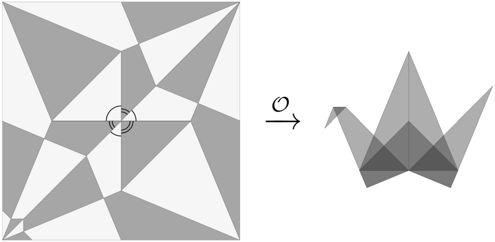

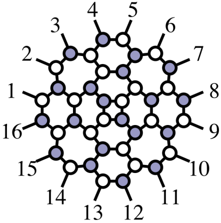

- Arkani-Hamed, Goncharov, T.Lam, Postnikov, Trnka, many (2013–): Scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory can be calculated via bipartite plane on-shell diagrams, using positive Grassmannians from their boundary measurements, and amplituhedra polytopes.

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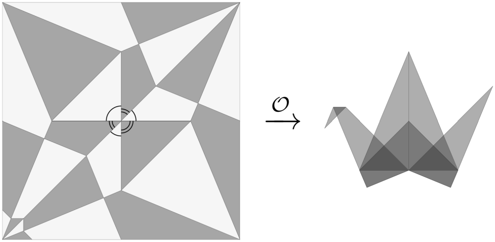
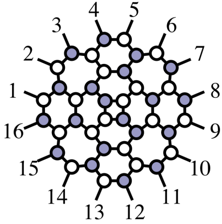




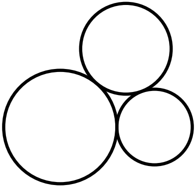
- Arkani-Hamed, Goncharov, T.Lam, Postnikov, Trnka, many (2013–): Scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory can be calculated via bipartite plane on-shell diagrams, using positive Grassmannians from their boundary measurements, and amplituhedra polytopes.
- Galashin (arxiv Oct '24) related amplituhedra to t-immersions and origami crease patterns.

Figures: Hull, Arkani-Hamed–Bourjaily–Cachazo–Goncharov–Postnikov–Trnka

Circle packings from t-immersions, Grassmannians

We can construct circle packings using these techniques!

- Subdivide triangulation $K \rightsquigarrow$ bipartite plane Γ & dual Γ^*



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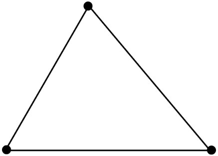

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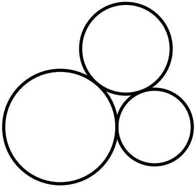

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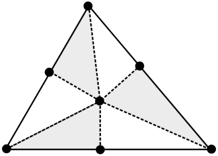

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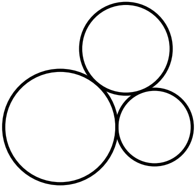

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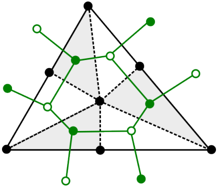

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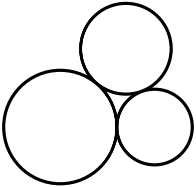

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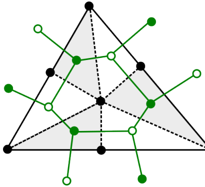

Galashin: t-immersions of Γ are bijective with certain pairs $\lambda, \widetilde{\lambda} \in \mathrm{Gr}(2, n)$ such that $\lambda \perp \widetilde{\lambda}$ and $\lambda \subset C \subset \widetilde{\lambda}^{\perp}$.

- extend $\lambda, \widetilde{\lambda}$ to discrete black/white-holomorphic functions

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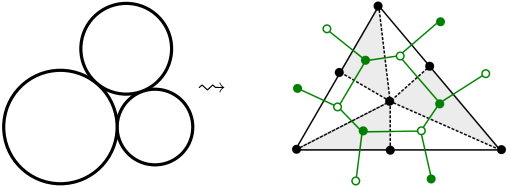

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Work in progress:

- Circle packings arise precisely when “black=white holomorphic" and $\lambda \sim \widetilde{\lambda}$ in an appropriate sense.
- Boundary measurement matrix (given by Γ path weights / dimers) gives trig expressions in realisation of K
- Physics gives new geometric identities in circle packings!

Thanks for listening!

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