Geometry of Ptolemy equations for hyperbolic 3-manifolds

Daniel V. Mathews

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Monash Topology seminar
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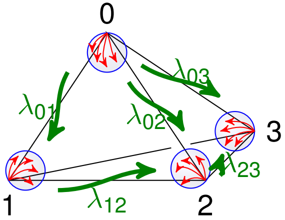

This talk aims to do the following:

- Discuss hyperbolic 3-manifolds and Ptolemy varieties.
- Tell you about how we can use spinors and horospheres in hyperbolic 3-manifolds.
- Tell you how we can use spinors and horospheres to understand the Ptolemy variety.

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Acknowledgments:

- Joint work (in progress) with Jessica Purcell.
- Related to other work with Nicola Harif, Josh Howie, Connie On Yu Hui, Dionne Ibarra, Reef Kitaeff, Lecheng Su, Varsha, Orion Zymaris.

Hyperbolic 3-manifolds and triangulations

Once upon a time there were the following objects:

- M, interior of a compact oriented 3-manifold with nonempty boundary consisting of tori
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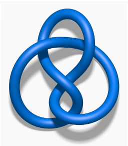

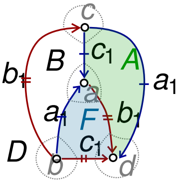

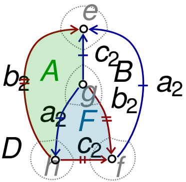

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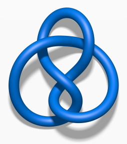

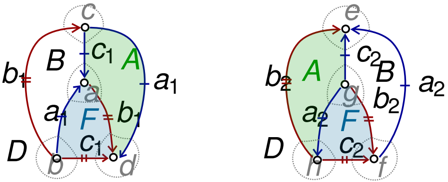

They all lived standardly in $\mathbb{H}^3$, considered via the upper half-space model, with $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ and $\mathrm{Isom}^+ \mathbb{H}^3 \cong PSL_{\mathbb{C}}$ acting by Möbius transformations on $\partial\mathbb{H}^3$.

Definition (Garoufalidis–D. Thurston–Zickert (GTZ) 2015)

A Ptolemy assignment on Δ is a map

$$c\colon \{\text{Oriented edges of } \Delta\} \longrightarrow \mathbb{C}^*, \quad ij \mapsto c_{ij}$$

such that

$$c_{ij} = -c_{ji}$$

and

$$c_{03}c_{12} + c_{01}c_{23} = c_{02}c_{13}.$$

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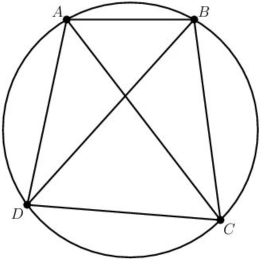
A Ptolemy assignment on $\mathcal{T}$ is a map

$$c\colon \{\text{Oriented edges of } \mathcal{T}\} \longrightarrow \mathbb{C}^*$$

restricting to a Ptolemy assignment on each Δ.

Ptolemy, Almagest ($\sim$ 160 CE): For a cyclic quadrilateral *ABCD* in the Euclidean plane,

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



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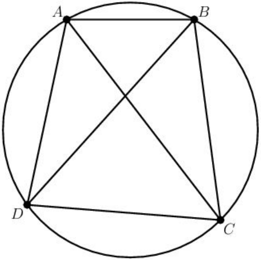

Numbering $(A, B, C, D) \sim (0, 1, 2, 3)$ and denoting distance d_{ij},

$$d_{02} d_{13} = d_{01} d_{23} + d_{03} d_{12}.$$

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Roughly, these may be branched, folded, have flat tetrahedra, but are nondegenerate, complete, spin-decorated at each cusp.

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Other invariants of 3-manifolds and their representations can be found from Ptolemy coordinates:

- Bloch invariant (GTZ 2015)
- Cheeger-Chern-Simons invariant (complex volume)

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- $z(e) = 0, 1, \infty$: Δ degenerate (some vertices coincide).

Hyperbolic structures à la Thurston: gluing equations

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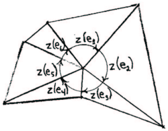
$$z(e_1)z(e_2)\cdots z(e_n) = 1.$$



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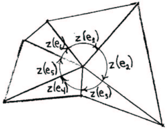

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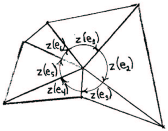

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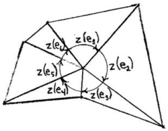

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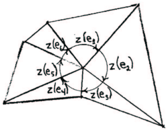

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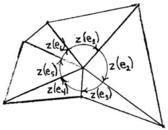

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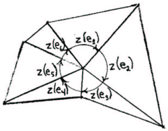

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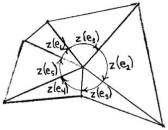

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- Arbitrary $z(e)$: could have <u>degenerate</u> Δ

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- $\mathrm{Hol}(\gamma) = \phi$.

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- Lift to a path $\widetilde{\gamma}$ in $\widetilde{M}$ (or $\widetilde{\mathcal{T}}$)
- Follow $\widetilde{\gamma}$ under D through hyperbolic ideal tetrahedra.
- Initial/final points of $\widetilde{\gamma}$: related by deck transformation of $\widetilde{M}$.
- Initial/final tetrahedra of $D(\tilde{\gamma})$: related by an isometry ϕ.
- $\mathrm{Hol}(\gamma) = \phi$.
- Since D well defined up to overall isometry, Hol well defined up to conjugation in $PSL_2\mathbb{C}$.

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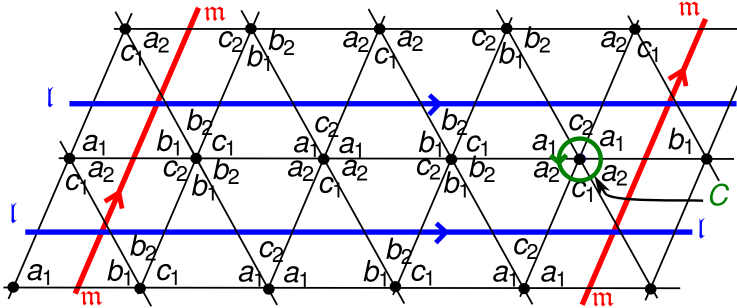

Hyperbolic structures à la Thurston: completeness

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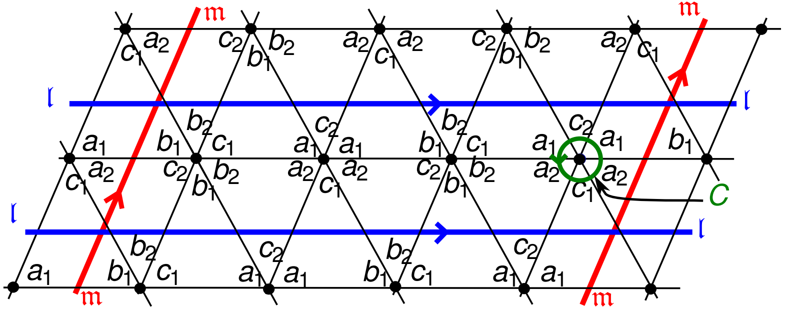

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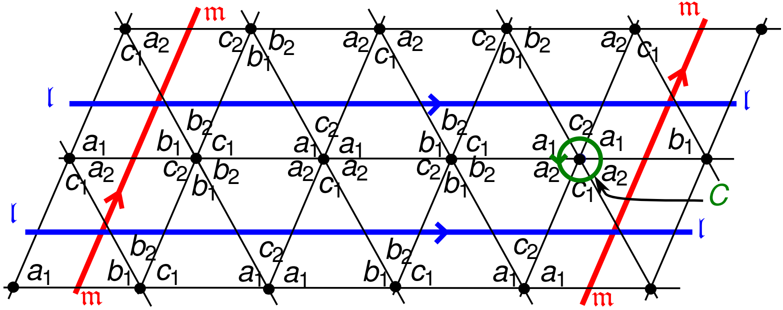

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E.g. here

$$c_1^{+1} b_2^{-1} b_1^{+1} c_2^{-1} c_1^{+1} b_2^{-1} b_1^{+1} c_2^{-1} = 1, \quad b_1^{-1} a_2^{+1} = 1.$$

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Solutions to gluing <u>and</u> completeness equations: also a variety.

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Back to GTZ: WARNING INCOMPREHENSIBLE

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- ρ is generically decorated if it is generic on each Δ.

WTF

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All these concepts mean something in hyperbolic geometry.

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Ptolemy assignments are also in bijection with certain singular nondegenerate spin hyperbolic structures on $\mathcal{T}$:

- *may be branched along edges,*
- *may be folded along faces or have flat tetrahedra,*
- *are complete and spin-decorated at each cusp.*

Rest of this talk: Try to explain this.

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Matrices and spin-decorated horospheres

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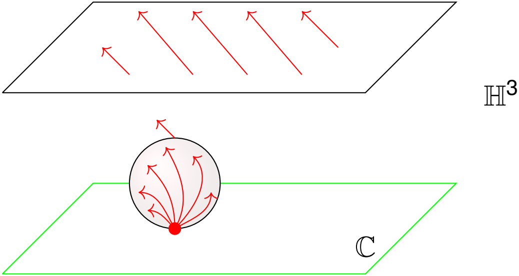

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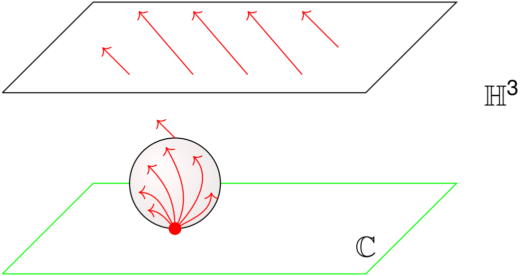

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The correspondence $\mathbb{C}^2_* \cong \{\text{spin-decorated horospheres}\}$ is also achieved by constructions of Penrose–Rindler–Penner–M., in which κ is regarded as a spinor.

Theorem (M.)

There is a natural ($SL(2,\mathbb{C})$-equivariant) bijection
$\mathbb{C}^2_* \leftrightarrow \{$*Spin-decorated horospheres in* $\mathbb{H}^3\}$.

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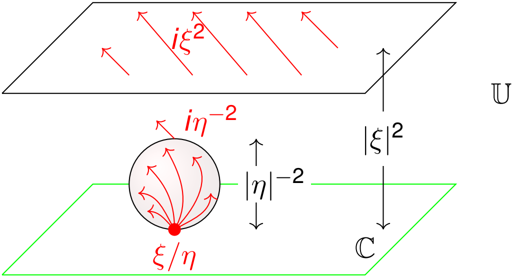

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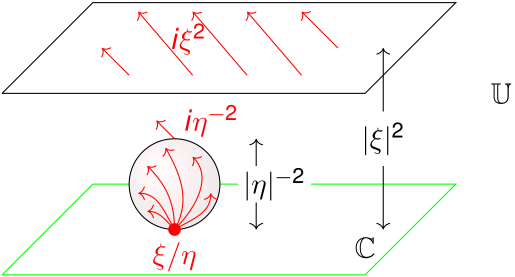
$$(\xi, \eta) \mapsto \left(\begin{array}{c} \text{horosphere centred at } \xi/\eta \\ \text{with Euclidean diameter } \frac{1}{|\eta|^2} \text{ and direction } \frac{i}{\eta^2} \end{array} \right)$$

(when $\eta = 0$, horizontal plane centred at ∞ at height $|\xi|^2$ and direction $i\xi^2$)



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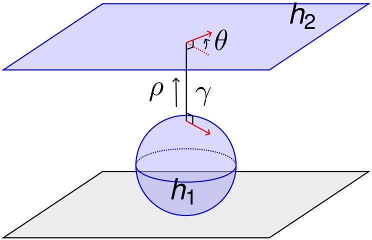

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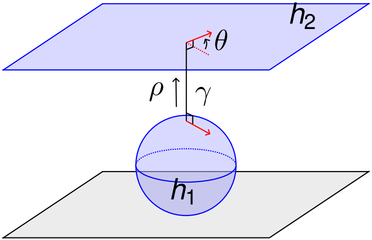

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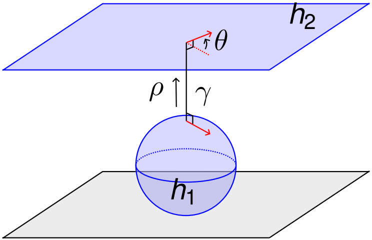

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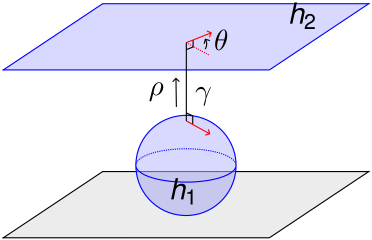

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$\lambda = 0 \Leftrightarrow \rho = -\infty$
$\Leftrightarrow \mathfrak{h}_1, \mathfrak{h}_2$ have same centre.

Recall GTZ: A "decoration on $\rho\colon \pi_1(M) \longrightarrow SL_2\mathbb{C}$" is a ρ-equivariant map $I(\widetilde{T}) \to SL_2\mathbb{C}/\mathcal{P}$.

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So:

- GTZ Decoration on ρ = equivariant choice of spinor / spin-decorated horosphere at each cusp of a developing map.

Recall GTZ: cosets $(g_0\mathcal{P}, g_1\mathcal{P}, g_2\mathcal{P}, g_3\mathcal{P})$ are generic if for each $i < j$, $\det(v_i, v_j) \neq 0$, where v_i is the first column of $g_i \in SL_2\mathbb{C}$.

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- ρ is generic precisely when it labels the four vertices of every tetrahedron with spin-decorated horospheres at distinct points.

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Theorem (GTZ15)

Ptolemy assignments are in bijection with generically decorated boundary-unipotent hom's $\rho : \pi_1(M) \longrightarrow SL_2\mathbb{C}$ *up to conjugacy.*

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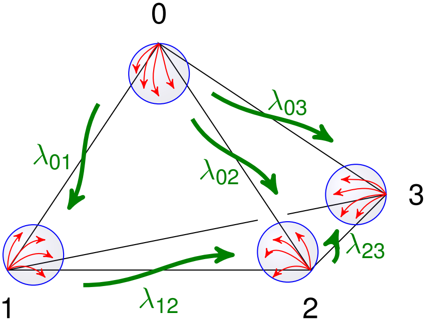
Remaining question:

- How to get from Ptolemy assignments to developing maps, and vice versa?

Theorem (M.)

Given a <u>*spin-decorated ideal tetrahedron*</u> *in* $\mathbb{H}^3$*, (i.e. ideal tetrahedron with spin-decorated horospheres* $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ *at its vertices), the lambda lengths* $\lambda_{ij} \in \mathbb{C}$ *between* $\mathfrak{h}_i$ *and* $\mathfrak{h}_j$ *satisfy*

$$\lambda_{02}\lambda_{13} = \lambda_{01}\lambda_{23} + \lambda_{03}\lambda_{12}.$$



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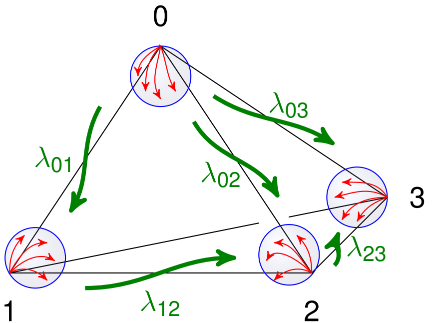

Proof: Plücker relation between 2×2 determinants in a 2×4 matrix.

Even better:

Proposition (M.–Purcell)

Given a Ptolemy assignment $\{c_{ij}\}$ on Δ, up to spin isometry there is a unique nondegenerate spin-decorated hyperbolic ideal tetrahedron, with $\lambda_{ij} = c_{ij}$.

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- These decorated tetrahedra glue together to give a developing map for a spin hyperbolic structure,
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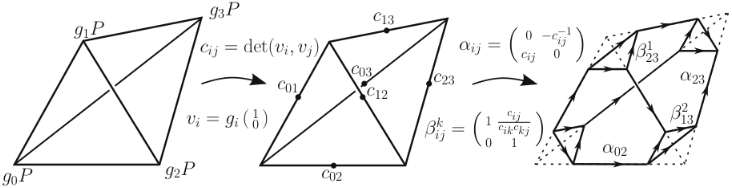
- This developing map has a well-defined holonomy map $\rho\colon \pi_1(M) \longrightarrow SL_2\mathbb{C}$ up to conjugacy.
- This ρ is also the ρ of GTZ.

Further results:

- GTZ also associate matrices to the edges of truncated tetrahedra of the triangulation $\mathcal{T}$: we can show how these "drive frames" around $\widetilde{M}$.



Further results:

- Zickert (2016) defined an enhanced Ptolemy variety $\mathrm{EPt}(\mathcal{T})$ allowing for non-parabolic holonomy at cusps, and showed this corresponded to generically decorated boundary-Borel representations.
- We can also show how points of $\mathrm{EPt}(\mathcal{T})$ correspond to singular hyperbolic structures; more technicalities, relationship with *A*-polynomial...
- Goerner–Zickert (2018) defined a triangulation indendent Ptolemy variety $\overline{\mathrm{Pt}(\mathcal{T})}$; this has a similar hyperbolic interpretation.

Related results:

- M., Varsha, Zymaris showed that spinor–horosphere correspondence applied in arbitrary dimension.
- Howie–Ibarra–M.–Su found all lambda lengths between maximal cusp neighbourhoods in the figure-8 knot complement.
- Harif–Howie–Hui–Ibarra–M.–Su (in progres) are finding many lambda lengths in many arithmetic hyperbolic link complements.

Further questions:

- GTZ and Z have a Ptolemy variety over any $SL_n\mathbb{C}$, not just $SL_2\mathbb{C}$. Similar style geometric meaning?
- GTZ were inspired by work of Fock–Goncharov: further connections?

Thanks for listening!

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